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LIMIT THEORY FOR JOINT GENERALIZED ORDER STATISTICS

Authors: H.M. Barakat
– Mathematical Department, Faculty of Science, Zagazig University, Zagazig, Egypt
hbarakat2@hotmail.com

E.M. Nigm
– Mathematical Department, Faculty of Science, Zagazig University Zagazig, Egypt
s_nigm@yahoo.com

M.A. Abd Elgawad
– Mathematical Department, Faculty of Science, Benha University, Benha, Egypt
mohamed_salem240@yahoo.com

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Abstract:

- In Kamps [7] generalized order statistics (gos) have been introduced as a unifying theme for several models of ascendingly ordered random variables (rv's). The main aim of this paper is to study the limit joint distribution function (df) of any two statistics in a wide subclass of the gos model known as m-gos. This subclass contains many important practical models of gos such as ordinary order statistics (oos), order statistics with non-integer sample size, and sequential order statistics (sos). The limit df's of lower-lower extreme, upper-upper extreme, lower-upper extreme, central-central and lower-lower intermediate m-gos are obtained. It is revealed that the convergence of the marginals m-gos implies the convergence of the joint df. Moreover, the conditions, under which the asymptotic independence between the two marginals occurs, are derived.

Key-Words:

- *generalized order statistics; generalized extreme order statistics; generalized central order statistics; generalized intermediate order statistics.*

AMS Subject Classification:

- 60F05, 62E20, 62E15, 62G30.

1. INTRODUCTION

Generalized order statistics have been introduced as a unified distribution theoretical set-up which contains a variety of models of ordered rv's. Since Kamps [7] had introduced the concept of gos as a unification of several models of ascendingly ordered rv's, the use of such concept has been steadily growing along the years. This is due to the fact that such concept includes important well-known concepts that have been separately treated in statistical literature. Theoretically, many of the models of ordered rv's contained in the gos model, such as oos, order statistics with non-integral sample size, sos, record values, Pfeifer's record model and progressive type II censored order statistics (pos). These models can be applied in reliability theory. For instance, the sos model is an extension of the oos model and serves as a model describing certain dependencies or interactions among the system components caused by failures of components and the pos model is an important method of obtaining data in lifetime tests. Live units removed early on can be readily used in other tests, thereby saving cost to the experimenter. The concept of gos enables a common approach to structural similarities and analogies. Known results in submodels can be subsumed, generalized, and integrated within a general framework. Kamps [7] defined gos by first defining what he called uniform gos and then using the quantile transformation to obtain the general gos $X(r,n,\tilde{m},k)$, $r=1,2,...,n$, based on a df F, which are defined by their probability density function (pdf)

$$f^{(\tilde{m},k)}_{1,2,...,n:n}(x_1,x_2,...,x_n) = \\ = \left(\prod_{j=1}^{n}\gamma_j\right)\left(\prod_{j=1}^{n-1}\bigl(1-F(x_j)\bigr)^{\gamma_j-\gamma_{j+1}-1}f(x_j)\right)\bigl(1-F(x_n)\bigr)^{\gamma_n-1}f(x_n)\ ,$$

where $F^{-1}(0)\le x_1\le...\le x_n\le F^{-1}(1)$, $\gamma_n=k>0$, $\gamma_r=k+n-r+\sum_{j=r}^{n-1}m_j$, $r=1,2,...,n-1$, and $\tilde{m}=(m_1,\,m_2,...,m_{n-1})\in\mathbb{R}^{n-1}$. Particular choices of the parameters $\gamma_1,\gamma_2,...,\gamma_n$ lead to different models, e.g., m-gos ($\gamma_r=k+(n-r)(m+1)$, $r=1,2,...,n-1$), oos ($k=1,\gamma_r=n-r+1,r=1,2,...,n-1$) and sos ($k=\alpha_n$, $\gamma_r=(n-r+1)\,\alpha_r$, $r=1,2,...,n-1$)[1].

Nasri-Roudsari [10] (see also Barakat [2]) has derived the marginal df of the rth m-gos, $m\ne-1$, in the form $\Phi^{(m,k)}_{r:n}(x)=I_{G_m(x)}(r,N-r+1)$, where $G_m(x)=1-(1-F(x))^{m+1}=1-\bar{F}^{m+1}(x)$, $I_x(a,b)=\frac{1}{\beta(a,b)}\int_o^x t^{a-1}(1-t)^{b-1}dt$ denotes the incomplete beta ratio function and $N=\frac{k}{m+1}+n-1$. By using the well-known relation $I_x(a,b)=1-I_{\bar{x}x}(b,a)$, where $\bar{x}=1-x$, the marginal df of the $(n-r+1)$th m-gos, $m\ne-1$, is given by $\Phi^{(m,k)}_{n-r+1:n}(x)=I_{G_m(x)}(N-R_r+1,R_r)$, where $R_r=\frac{k}{m+1}+r-1$. Moreover, by using the results of Kamps [7], we can write explicitly the joint pdf of the rth and sth m-gos $m\ne-1$, $1\le r<s\le n$,

[1] See, for instance, Kamps ([7]).

as:

$$
\begin{aligned}
f_{r,s:n}^{(m,k)}(x,y) \;=\; & \frac{C_{s-1,n}}{\Gamma(r)\,\Gamma(s-r)}\,\bar{F}^{m}(x)\,g_m^{r-1}(F(x)) \\
& \times\,\big(g_m(F(y)) - g_m(F(x))\big)^{s-r-1}\,\bar{F}^{\gamma_s-1}(y)\,f(x)\,f(y)\;, \qquad (1.1)\\
& \qquad\qquad -\infty < x < y < \infty\;,
\end{aligned}
$$

where $C_{s-1,n} = \prod_{j=1}^{s}\gamma_j$. In the present paper we develop the limit theory for gos, by revealing the asymptotic dependence structural between the members of gos, with fixed and variable ranks. Namely, the limit joint df of the m-gos $X(r,n,m,k)$ and $X(s,n,m,k)$, when $m \neq -1$, is derived in the following three cases:

(**1**) Lower extremes, where r,s are fixed w.r.t. n and upper extremes, where $\grave{r} = n-r+1$, $\grave{s} = n-s+1$, where r,s are fixed w.r.t. n.

(**2**) Central case, where $r,s \to \infty$ and $\frac{r}{N} \to \lambda_1$, $\frac{s}{N} \to \lambda_2$, where $0 < \lambda_1 < \lambda_2 < 1$, as $N \to \infty$ (or equivalently, as $n \to \infty$). A remarkable example of the central oos the pth sample quantile, where $r_n = [np]$, $0 < p < 1$, and $[x]$ denotes the largest integer not exceeding x.

(**3**) Intermediate case, where $r,s \to \infty$ and $\frac{r}{N}, \frac{s}{N} \to 0$, as $N \to \infty$ (or equivalently, as $n \to \infty$). The intermediate oos have many applications, e.g., in the theory of statistics, they can be used to estimate probabilities of future extreme observations and to estimate tail quantiles of the underlying distribution that are extreme relative to the available sample size, see Pickands [12]. Many authors, e.g., Teugels [14] and Mason [9] have also found estimates that are based, in part, on intermediate order statistics.

Everywhere in what follows the symbols ($\underset{n}{\longrightarrow}$) and ($\overset{w}{\underset{n}{\longrightarrow}}$) stand for convergence, as $n \to \infty$ and the weak convergence, as $n \to \infty$.

2. THE JOINT df OF EXTREME m-gos

The following two lemmas, which are originally derived by Nasri-Roudsari [10] and Nasri-Roudsari and Cramer [11] (see also Barakat [2]), extend the well-known results concerning the asymptotic theory of extreme oos to the extreme m-gos. These lemmas can be easily proved by applying the following asymptotic relations, due to Smirnov [13]:

$$\Gamma_r(nA_n) - \delta_{1n} \;\le\; I_{A_n}(r, n-r+1) \;\le\; \Gamma_r(nA_n) - \delta_{2n}\;,$$

if $nA_n \sim A < \infty$, as $n \to \infty$, and

$$1 - \Gamma_r(n\bar{A}_n) - \delta_{2n} \;\le\; I_{A_n}(n-r+1, r) \;\le\; 1 - \Gamma_r(n\bar{A}_n) - \delta_{1n}\;,$$

if $n\bar{A}_n \sim \bar{A} < \infty$, as $n \to \infty$, where $\Gamma_r(x) = \frac{1}{\Gamma(r)} \int_0^x t^{r-1}e^{-t}dt$ is the incomplete gamma function (Gamma df with parameter r), $\delta_{in} > 0$, $\delta_{in} \xrightarrow[n]{} 0$, $i = 1, 2$, and $0 < A_n < 1$.

Lemma 2.1. *Let $m > -1$ and $r \in \{1, 2, ..., n\}$. Then, there exist normalizing constants $c_n > 0$ and d_n, for which*

$$\Phi^{(m,k)}_{r:n}(c_n x + d_n) = I_{G_m(c_n x+d_n)}(r, N - r + 1) \xrightarrow[n]{w} \Phi^{(m,k)}_r(x) , \tag{2.1}$$

where $\Phi^{(m,k)}_r(x)$ is nondegenerate df if, and only if, there exist normalizing constants $\alpha_n > 0$ and β_n, for which $\Phi^{(0,1)}_{r:n}(\alpha_n x + \beta_n) \xrightarrow[n]{w} \Gamma_r(\mathcal{V}_{j,\beta}(x))$, $\beta > 0$. In this case $\Phi^{(m,k)}_r(x) = \Gamma_r(\mathcal{V}_{j,\beta}(x))$, $j \in \{1, 2, 3\}$, where $\mathcal{V}_1(x) = \mathcal{V}_{1;\beta}(x) = e^x$, $\forall\ x$;

$$\mathcal{V}_{2;\beta}(x) = \begin{cases} (-x)^{-\beta}, & x \le 0\,, \\ \infty, & x > 0\,; \end{cases} \qquad \mathcal{V}_{3;\beta}(x) = \begin{cases} 0, & x \le 0\,, \\ x^{\beta}, & x > 0\,. \end{cases}$$

Moreover, c_n and d_n may be chosen such that $c_n = \alpha_{\psi(n)}$ and $d_n = \beta_{\psi(n)}$, where $\psi(n) = n(m + 1)$. Finally, (2.1) holds if, and only if, $NG_m(c_n x + d_n) \xrightarrow[n]{} \mathcal{V}_{j,\beta}(x)$ (note that $N \sim n$, as $n \to \infty$).

Lemma 2.2. *Let $m > -1$ and $r \in \{1, 2, ..., n\}$. Then, there exist normalizing constants $a_n > 0$ and b_n, for which*

$$\Phi^{(m,k)}_{n-r+1:n}(a_n x + b_n) = I_{G_m(a_n x+b_n)}(N - R_r + 1, R_r) \xrightarrow[n]{w} \hat{\Phi}^{(m,k)}_r(x) , \tag{2.2}$$

where $\hat{\Phi}^{(m,k)}_r(x)$ is nondegenerate df if, and only if, there exist normalizing constants $\hat{\alpha}_n > 0$ and $\hat{\beta}_n$, for which $\Phi^{(0,1)}_{n-r+1:n}(\hat{\alpha}_n x + \hat{\beta}_n) \xrightarrow[n]{w} 1 - \Gamma_r(\mathcal{U}_{i,\alpha}(x))$, $\alpha > 0$. In this case $\hat{\Phi}^{(m,k)}_r(x) = 1 - \Gamma_{R_r}(\mathcal{U}^{m+1}_{i,\alpha}(x))$, $i \in \{1, 2, 3\}$, where $\mathcal{U}_1(x) = \mathcal{U}_{1;\alpha}(x) = e^{-x}$, $\forall\ x$;

$$\mathcal{U}_{2;\alpha}(x) = \begin{cases} \infty, & x \le 0\,, \\ x^{-\alpha}, & x > 0\,; \end{cases} \qquad \mathcal{U}_{3;\alpha}(x) = \begin{cases} (-x)^{\alpha}, & x \le 0\,, \\ 0, & x > 0\,. \end{cases}$$

Moreover, a_n and b_n may be chosen such that $a_n = \hat{\alpha}_{\phi(n)}$ and $b_n = \hat{\beta}_{\phi(n)}$, where $\phi(n) = n^{\frac{1}{m+1}}$. Finally, (2.2) holds if, and only if, $N\bar{G}_m(a_n x + b_n) \xrightarrow[n]{} \mathcal{U}^{m+1}_{i,\alpha}(x)$.

We need the following three lemmas proved in the Appendix and individually express interesting and practically useful facts. These lemmas provide us with the asymptotic lower and upper bounds for the joint df's of extreme gos. Therefore, they can be applied to estimate the error committed by the replacement of the exact joint df's of extreme gos by their limiting (see Remark 2.1). Throughout Lemma 2.3, we assume that $1 \le r < s \le n$, while we assume $1 \le s < r \le n$ and $1 \le r, s \le n$, $\grave{s} = n - s + 1$ in Lemma 2.4 and Lemma 2.5, respectively.

Lemma 2.3. *Let $c_n > 0$ and d_n be suitable normalizing constants, for which the limit relations $\Phi^{(m,k)}_{r:n}(x_n) \xrightarrow[n]{w} \Gamma_r(\mathcal{V}_{j,\beta}(x))$ and $\Phi^{(m,k)}_{s:n}(y_n) \xrightarrow[n]{w} \Gamma_s(\mathcal{V}_{j,\beta}(y))$, $j \in \{1,2,3\}$, hold, where $x_n = c_n x + d_n$ and $y_n = c_n y + d_n$. Then the normalized joint df $\Phi^{(m,k)}_{r,s:n}(x_n, y_n)$ of the rth and sth m-gos, $m \neq -1$, satisfies the relations*

$$\begin{aligned}
&\frac{(1-\sigma_N)}{(r-1)!} \int_0^{NG_m(x_n)} \Gamma_{s-r}\big(NG_m(y_n)-u\big)\, u^{r-1} e^{-u}\, du \;\leq \\
(2.3)\qquad &\leq\; \Phi^{(m,k)}_{r,s:n}(x_n, y_n) \\
&\leq\; \frac{(1+\rho_N)}{(r-1)!} \int_0^{NG_m(x_n)} \Gamma_{s-r}(NG_m(y_n)-u) u^{r-1} e^{-u} du\;, \qquad \forall x \leq y\;,
\end{aligned}$$

where $\rho_N, \sigma_N \xrightarrow[n]{} 0$.

Lemma 2.4. *Let $a_n > 0$ and b_n be suitable normalizing constants, for which the limit relations $\Phi^{(m,k)}_{\grave{r}:n}(x_n) \xrightarrow[n]{w} 1-\Gamma_{R_r}(\mathcal{U}^{m+1}_{i,\alpha}(x))$ and $\Phi^{(m,k)}_{\grave{s}:n}(y_n) \xrightarrow[n]{w} 1-\Gamma_{R_s}(\mathcal{U}^{m+1}_{i,\alpha}(y))$, $i \in \{1,2,3\}$, hold, where $x_n = a_n x + b_n$, $y_n = a_n y + b_n$ and $\grave{r} = n-r+1 < n-s+1 = \grave{s}$. Then the joint df of the $\grave{r}$th and $\grave{s}$th m-gos, $m \neq -1$, satisfies the relation*

$$\begin{aligned}
&\frac{\grave{C}_n}{(N+R_s)^{R_r}} \int_{(N+R_s)\frac{\bar{G}_m(x_n)}{G_m(x_n)}}^{(N+R_s)} \int_{(N+R_s)\frac{\bar{G}_m(y_n)}{G_m(y_n)}}^{\phi} e^{-\phi}\theta^{R_s-1} \;\times \\
(2.4)\qquad &\qquad \times \left(1+\frac{\theta}{N+R_s}\right)^{-R_r} (\phi-\theta)^{R_r-R_s-1}\, d\theta\, d\phi \;\leq \\
&\leq\; \Phi^{(m,k)}_{\grave{r},\grave{s}:n}(x_n, y_n) \\
&\leq\; 1-\Gamma_{R_r}\big(N\bar{G}_m(x_n)\big) - \frac{1}{\Gamma(R_r)} \int_{N\bar{G}_m(x_n)}^{N} I_{\frac{N\bar{G}_m(y_n)}{t}}(R_s, R_r-R_s)\, t^{R_r-1} e^{-t}\, dt\;,
\end{aligned}$$

where $\grave{C}_n = \dfrac{\Gamma(N+1)}{\Gamma(N-R_r+1)\;\Gamma(R_r-R_s)\;\Gamma(R_s)}$.

Lemma 2.5. *Let $a_n, c_n > 0$ and b_n, d_n be suitable normalizing constants, for which the limit relations $\Phi^{(m,k)}_{r:n}(x_n) \xrightarrow[n]{w} \Phi^{(m,k)}_r(x) = \Gamma_r(\mathcal{V}_{j,\beta}(x))$, $j \in \{1,2,3\}$, and $\Phi^{(m,k)}_{\grave{s}:n}(y_n) \xrightarrow[n]{w} \hat{\Phi}^{(m,k)}_s(y) = 1-\Gamma_{R_s}(\mathcal{U}^{m+1}_{i,\alpha}(y))$, $i \in \{1,2,3\}$, hold, where $x_n = c_n x + d_n$ and $y_n = a_n y + b_n$. Then, for all large n and for all x and y, for which $\mathcal{V}_{j,\beta}(x) < \infty$, i.e., $\Phi^{(m,k)}_r(x) < 1$ and $\mathcal{U}_{i,\alpha}(y) < \infty$, i.e., $\hat{\Phi}^{(m,k)}_s(y) > 0$, respectively, the joint df of the rth and $\grave{s}$th m-gos, $m \neq -1$, satisfies the relation*

$$\begin{aligned}
(2.5)\qquad \Phi^{(m,k)}_{r:n}(x_n)\, \Phi^{(m,k)}_{\grave{s}:n}(y_n) \;&\leq\; \Phi^{(m,k)}_{r,\grave{s}:n}(x_n, y_n) \\
&\leq\; \Gamma_r\big(NG_m(x_n)\big)\Big(\Gamma_{R_s}(N) - \Gamma_{R_s}\big(N\bar{G}_m(y_n)\big)\Big)\,.
\end{aligned}$$

The first inequality of (2.5) holds for all x, y.

Theorem 2.1. *Under the conditions of Lemma 2.3, 2.4 and 2.5, we get respectively*

$$\text{(2.6)}\quad \Phi^{(m,k)}_{r,s:n}(x_n,y_n) \xrightarrow[n]{w} \begin{cases} \Gamma_s\big(\mathcal{V}_{j,\beta}(y)\big)\,, & x\geq y\,,\\ \dfrac{1}{(r-1)!}\displaystyle\int_0^{\mathcal{V}_{j,\beta}(x)} \Gamma_{s-r}\big(\mathcal{V}_{j,\beta}(y)-u\big)u^{r-1}e^{-u}\,du\,, & x\leq y\,,\end{cases}$$

(2.7)

$$\Phi^{(m,k)}_{\grave r,\grave s:n}(x_n,y_n) \xrightarrow[n]{w} \begin{cases} 1-\Gamma_{R_s}\big(\mathcal{U}^{m+1}_{i,\alpha}(y)\big)\,, & x\geq y\,,\\ 1-\Gamma_{R_r}\big(\mathcal{U}^{m+1}_{i,\alpha}(x)\big)-\dfrac{1}{\Gamma(R_r)}\,\times & \\ \quad\times\int_{\mathcal{U}^{m+1}_{i,\alpha}(x)}^{\infty} I_{\frac{\mathcal{U}^{m+1}_{i,\alpha}(y)}{t}}(R_s,R_r-R_s)\,t^{R_r-1}e^{-t}\,dt\,, & x\leq y\,,\end{cases}$$

and

$$\text{(2.8)}\quad \Phi^{(m,k)}_{r,\grave s:n}(x_n,y_n) \xrightarrow[n]{w} \Phi^{(m,k)}_r(x)\,\hat\Phi^{(m,k)}_s(y) \;=\; \Gamma_r\big(\mathcal{V}_{j,\beta}(x)\big)\Big[1-\Gamma_{R_s}\big(\mathcal{U}^{m+1}_{i,\alpha}(y)\big)\Big].$$

Proof: By noting that $\Phi^{(m,k)}_{r,s:n}(x_n,y_n)=\Phi^{(m,k)}_{s:n}(y_n)$, if $y\leq x$, the relation (2.6) follows by applying Lemmas 2.1 and 2.3. In view of (2.2), (1.1) and the condition of Lemma 2.4, the relation (2.7) follows in the case of $y\leq x$. On the other hand, since both of the lower and upper bounds of (2.4) are equivalent to (as $n\to\infty$) $1-\Gamma_{R_r}(N\bar G_m(x_n))-\frac{1}{\Gamma(R_r)}\int_{N\bar G_m(x_n)}^{N} I_{\frac{N\bar G_m(y_n)}{t}}(R_s,R_r-R_s)\,t^{R_r-1}e^{-t}dt$, then the relation (2.7) in the case $x\leq y$, follows by applying Lemmas 2.2 and 2.4. Finally, by combining Lemmas 2.1, 2.2 and 2.5, the relation (2.8) follows immediately. □

Remark 2.1. One of the referees of the paper suggests a dexterous short proof of Theorem 2.1 based on the result of Cramer [5]. Namely, we get with the notations of Cramer [5] for two lower gos $X_t=X(t,n,m,k)$, $t=r,s$, $r<s$ (Z_j are iid standard exponential rv's; $u(x)=-\log(1-F(x))$; and $\gamma_{j,n}=k+(m+1)(n-j)$)

$$\begin{aligned} P\big(X_r\leq x_n, X_s\leq y_n\big) \;&=\; P\left(\sum_{j=1}^{r}\frac{Z_j}{\gamma_{j,n}}\leq u(x_n)\,,\;\sum_{j=1}^{s}\frac{Z_j}{\gamma_{j,n}}\leq u(y_n)\right)\\ &=\; P\Big(\Lambda_{r,n}\leq n(m+1)\,u(x_n)\,,\;\Lambda_{r,n}+\Delta_{r,s,n}\leq n(m+1)\,u(y_n)\Big)\,,\end{aligned}$$

where $\Lambda_{r,n}=n(m+1)\sum\limits_{j=1}^{r}\frac{Z_j}{\gamma_{j,n}}$ converges to a Gamma distribution with parameter r and $\Delta_{r,s,n}$ converges to a Gamma distribution with parameter $s-r+1$. Moreover, $\Lambda_{r,n}$ and $\Delta_{r,s,n}$ are independent for any n. Provided that F is in the domain of attraction of a minimum-stable distribution we get that $n(m+1)\,u(x_n)$ $(n(m+1)\,u(y_n))$ converges appropriately to some function $\mathcal{V}(x)$ $(\mathcal{V}(y))$. Hence, the limit df is of the type $P(\Lambda_r\leq\mathcal{V}(x),\;\Lambda_r+\Delta_{s-r+1}\leq\mathcal{V}(y))$, where Λ_r and Δ_{s-r+1}

are independent gamma distributed rv's with parameters given above, respectively. This proves the result in (2.6). Similar arguments can be used in proving (2.7) and (2.8). Although, this short method directly results the limit joint df's, but our lengthy method provides more informative results (Lemmas 2.3–2.5), which enable us to estimate the error committed by the replacement of the exact joint df's of extreme gos by their limiting. Actually, in view of the slow rate of convergence of oos (and consequently the gos) (cf. Arnold *et al.* [1], Page 216), Lemmas 2.3–2.5 are of a remarkable practically importance.

Example 2.1 (The limit df's of the generalized range and midrange). Under the conditions of Lemma 2.5 the left and the right extreme m-gos, is asymptotically independent. Therefore, if there exist normalizing constants $a_n, c_n > 0$ and b_n, d_n, for which $a_n/c_n \underset{n}{\longrightarrow} c > 0$ and the limit relations $\Phi^{(m,k)}_{\grave{r}:n}(a_n x + b_n) \xrightarrow[n]{w} 1-\Gamma_{R_r}(\mathcal{U}^{m+1}_{i,\alpha}(x))$, $i \in \{1,2,3\}$, and $\Phi^{(m,k)}_{r:n}(c_n x + d_n) \xrightarrow[n]{w} \Gamma_r(\mathcal{V}_{j,\beta}(x))$, $j \in \{1,2,3\}$, hold, then in view of Lemma 2.9.1 in Galambos [6], the generalized quasi-ranges $R(r,n,m,k) = X(\grave{r},n,m,k) - X(r,n,m,k)$ and the generalized quasi-midranges $M(r,n,m,k) = \frac{1}{2}\big(X(\grave{r},n,m,k) + X(r,n,m,k)\big)$, $r = 1,2,...$, satisfy the relations

$$P\big(R(r,n,m,k) \le a_n x + b_n - d_n\big) \xrightarrow[n]{w} \big[1 - \Gamma_{R_r}(\mathcal{U}^{m+1}_{i,\alpha}(x))\big] \star \big[1 - \Gamma_r(\mathcal{V}_{j,\beta}(-cx))\big]$$

and

$$P\big(2M(r,n,m,k) \le a_n x + b_n + d_n\big) \xrightarrow[n]{w} \big[1 - \Gamma_{R_r}(\mathcal{U}^{m+1}_{i,\alpha}(x))\big] \star \big[\Gamma_r(\mathcal{V}_{j,\beta}(cx))\big] ,$$

respectively, where the symbol $\star$ denotes the convolution operation.

3. LIMIT df's OF THE JOINT CENTRAL m-gos

Consider a variable rank sequence $r = r_n \underset{n}{\longrightarrow} \infty$ and $\sqrt{n}\left(\frac{r}{n} - \lambda\right) \underset{n}{\longrightarrow} 0$, where $0 < \lambda < 1$. Smirnov [13] showed that if there exist normalizing constants $\alpha_n > 0$ and β_n such that

$$\text{(3.1)} \qquad \Phi^{(0,1)}_{r:n}(\alpha_n x + \beta_n) = I_{F(\alpha_n x + \beta_n)}(r, n-r+1) \xrightarrow[n]{w} \Phi^{(0,1)}(x;\lambda) ,$$

where $\Phi^{(0,1)}(x;\lambda)$ is some nondegenerate df, then $\Phi^{(0,1)}(x;\lambda)$ must have one and only one of the types $\mathcal{N}(W_{i;\beta}(x))$, $i = 1,2,3,4$, where $\mathcal{N}(\cdot)$ denotes the standard normal df,

$$W_{1;\beta}(x) = \begin{cases} -\infty\,, & x \le 0\,, \\ c x^\beta\,, & x > 0\,, \end{cases} \qquad W_{2;\beta}(x) = \begin{cases} -c|x|^\beta\,, & x \le 0\,, \\ \infty\,, & x > 0\,, \end{cases}$$

$$W_{3;\beta}(x) = \begin{cases} -c_1|x|^\beta\,, & x \le 0\,, \\ c_2 x^\beta\,, & x > 0\,, \end{cases} \qquad W_{4;\beta}(x) = W_4(x) = \begin{cases} -\infty\,, & x \le -1\,, \\ 0\,, & -1 < x \le 1\,, \\ \infty\,, & x > 1\,, \end{cases}$$

and $\beta, c, c_1, c_2 > 0$. In this case we say that F belongs to the λ-normal domain of attraction of the limit df $\Phi^{(0,1)}(x;\lambda)$, written $F \in \mathcal{D}_\lambda(\Phi^{(0,1)}(x;\lambda))$. Moreover, (3.1) is satisfied with $\Phi^{(0,1)}(x;\lambda) = \mathcal{N}(W_{i;\beta}(x))$, for some $i \in \{1,2,3,4\}$ if, and only if,

$$\sqrt{n}\,\frac{F(\alpha_n x + \beta_n) - \lambda}{C_\lambda} \underset{n}{\longrightarrow} W_{i,\beta}(x)\ ,$$

where $C_\lambda = \sqrt{\lambda(1-\lambda)}$. It is worth to mention that the condition $\sqrt{n}\left(\frac{r}{n} - \lambda\right) \underset{n}{\longrightarrow} 0$ is necessary to have a unique limit law for any two ranks r, r', for which $\lim\limits_{n\to\infty} \frac{r}{n} = \lim\limits_{n\to\infty} \frac{r'}{n}$ (see Smirnov [13]).

Barakat [2], in Theorem 2.2, characterized the possible limit laws of the df $\Phi^{(m,k)}_{n-r+1:n}(x)$. The following corresponding lemma characterizes the possible limit laws of the df $\Phi^{(m,k)}_{r:n}(x)$. The proof of this lemma follows by using the same argument which is applied in the proof of Theorem 2.2 of Barakat [2].

Lemma 3.1. *Let $r = r_n$ be such that $\sqrt{n}\left(\frac{r}{n} - \lambda\right) \underset{n}{\longrightarrow} 0$, where $0 < \lambda < 1$. Furthermore, let $m_1 = m_2 = \ldots = m_{n-1} = m > -1$. Then, there exist normalizing constants $a_n > 0$ and b_n for which*

$$\Phi^{(m,k)}_{r:n}(a_n x + b_n) \underset{n}{\overset{w}{\longrightarrow}} \Phi^{(m,k)}(x;\lambda)\ , \tag{3.2}$$

where $\Phi^{(m,k)}(x;\lambda)$ is a nondegenerate df if, and only if,

$$\sqrt{n}\,\frac{G_m(a_n x + b_n) - \lambda}{C_\lambda} \underset{n}{\longrightarrow} W(x)\ ,$$

where $\Phi^{(m,k)}(x;\lambda) = \mathcal{N}(W(x))$. Moreover, (3.2) is satisfied for some nondegenerate df $\Phi^{(m,k)}(x;\lambda)$ if, and only if, $F \in \mathcal{D}_{\lambda(m)}(\mathcal{N}(W_{i;\beta}(x)))$, for some $i \in \{1,2,3,4\}$, where $\lambda(m) = 1 - \bar{\lambda}^{\frac{1}{m+1}}$ and $\bar{\lambda} = 1 - \lambda$. In this case we have $W(x) = \frac{C^\star_{\lambda(m)}}{C^\star_\lambda}(m+1)\cdot$ $\cdot W_{i;\beta}(x)$, where $C^\star_\lambda = \frac{C_\lambda}{\bar{\lambda}}$ (note that, when $m = 0$, we get $W(x) = W_{i;\beta}(x)$).

We assume that in this section in all time that $r = r_n$, $s = s_n \underset{n}{\longrightarrow} \infty$ and $\sqrt{n}(\frac{r}{n} - \lambda_1), \sqrt{n}(\frac{s}{n} - \lambda_2) \underset{n}{\longrightarrow} 0$, where $0 < \lambda_1 < \lambda_2 < 1$. Moreover, we assume that there are suitable normalizing constants $a_n, c_n > 0$ and b_n, d_n, for which $\Phi^{(m,k)}_{r:n}(a_n x + b_n) \underset{n}{\overset{w}{\longrightarrow}} \Phi^{(m,k)}(x;\lambda_1)$ and $\Phi^{(m,k)}_{s:n}(c_n y + d_n) \underset{n}{\overset{w}{\longrightarrow}} \Phi^{(m,k)}(y;\lambda_2)$, where $\Phi^{(m,k)}(x;\lambda_1)$ and $\Phi^{(m,k)}(y;\lambda_2)$ are nondegenerate df's. Let $\Phi^{(m,k)}_{r,s:n}(x,y)$ be the joint df's of rth and sth m-gos, $m \neq -1$, in view of (1.1) we get $\Phi^{(m,k)}_{r,s:n}(x,y) = \Phi^{(m,k)}_{s:n}(y)$, $y \le x$, and

$$\begin{aligned}\Phi^{(m,k)}_{r,s:n}(x,y) \;=\; & C^\star_n \int_0^{F(x)}\!\!\int_\xi^{F(y)} \bar{\xi}^m\, \bar{\eta}^{\gamma_s - 1}\big(1 - \bar{\xi}^{m+1}\big)^{r-1} \\ & \times \big(\bar{\xi}^{m+1} - \bar{\eta}^{m+1}\big)^{s-r-1} d\eta\, d\xi\ , \qquad x \le y\ ,\end{aligned} \tag{3.3}$$

where $C^\star_n = \frac{(m+1)^2\,\Gamma(N+1)}{\Gamma(N-s+1)\,(r-1)!\,(s-r-1)!}$. The following lemma proved in the Appendix is an essential tool in studying the limit df of the joint central m-gos.

Lemma 3.2. *Let* $\lambda_i = \frac{i}{N+1}$, $\nu_i = 1-\lambda_i$, $\tau_i = \sqrt{\frac{\lambda_i \nu_i}{N+1}}$, $i = r, s$, $0 < R_{rs} = \sqrt{\frac{\lambda_r(1-\lambda_s)}{\lambda_s(1-\lambda_r)}} < 1$, $U_n^{(1)}(x) = \frac{G_m(x_n)-\lambda_r}{\tau_r}$, $U_n^{(2)}(y) = \frac{G_m(y_n)-\lambda_s}{\tau_s}$, $x_n = a_n x + b_n$ *and* $y_n = c_n y + d_n$. *Then*

$$\left|\Phi^{(m,k)}_{r,s:n}(x_n, y_n) - \int_{-\infty}^{U_n^{(1)}(x)}\int_{-\infty}^{U_n^{(2)}(y)} W_{r,s}(\xi,\eta)\, d\xi\, d\eta\right| \xrightarrow[n]{} 0$$

uniformly with respect to x *and* y, *where* $W_{r,s}(\xi,\eta) = \frac{1}{2\pi\sqrt{1-R_{rs}^2}}\, e^{-\frac{(\xi^2+\eta^2-2\xi\eta R_{rs})}{2(1-R_{rs}^2)}}$.

Lemma 3.2 directly yields the following interesting theorem.

Theorem 3.1. *The convergence of the two marginals* $\Phi^{(m,k)}_{r:n}(x_n)$ *and* $\Phi^{(m,k)}_{s:n}(y_n)$ *to nondegenerate df's* $\Phi^{(m,k)}(x;\lambda_1) = \mathcal{N}(W(x))$ *and* $\Phi^{(m,k)}(y;\lambda_2) = \mathcal{N}(\tilde{W}(y))$, *respectively, are necessary and sufficient condition for the convergence of the joint df* $\Phi^{(m,k)}_{r,s:n}(x_n, y_n)$ *to the nondegenerate limit*

$$\Phi^{(m,k)}(x,y;\lambda_1,\lambda_2) = \frac{1}{2\pi\sqrt{1-R^2}} \int_{-\infty}^{W(x)}\int_{-\infty}^{\tilde{W}(y)} e^{-\frac{(\xi^2+\eta^2-2\xi\eta R)}{2(1-R^2)}}\, d\xi\, d\eta\ ,$$

where $R = \sqrt{\frac{\lambda_1(1-\lambda_2)}{\lambda_2(1-\lambda_1)}}$. *Moreover, in view of Lemma 3.1, we deduce that the convergence of the joint df* $\Phi^{(m,k)}_{r,s:n}(x_n, y_n)$, *as well as the convergence of the two marginals* $\Phi^{(m,k)}_{r:n}(x_n)$ *and* $\Phi^{(m,k)}_{s:n}(y_n)$, *occurs if, and only if, with the same normalizing constants, we have* $F \in \mathcal{D}_{\lambda_1(m)}(\mathcal{N}(W_{i;\beta}))$ *and* $F \in \mathcal{D}_{\lambda_2(m)}(\mathcal{N}(W_{j;\beta'}))$, *for some* $i, j \in \{1,2,3,4\}$, *where* $\lambda_t(m) = 1 - \bar{\lambda}_t^{\frac{1}{m+1}}$ *and* $\bar{\lambda}_t = 1-\lambda_t$, $t = 1,2$. *In this case we have* $W(x) = \frac{C^\star_{\lambda_1(m)}}{C^\star_{\lambda_1}}(m+1)\,W_{i;\beta}(x)$ *and* $\tilde{W}(y) = \frac{C^\star_{\lambda_2(m)}}{C^\star_{\lambda_2}}(m+1)\,W_{j;\beta'}(y)$, *where* $C^\star_{\lambda_t} = \frac{C_{\lambda_t}}{\lambda}$, $t = 1,2$.

4. LIMIT df's OF THE JOINT INTERMEDIATE m-gos

Chibisov [4] studied a wide class of intermediate oos, where $r = r_n = \ell^2 n^\alpha(1+\circ(1))$, $0 < \alpha < 1$, and he showed that if there are normalizing constants $\alpha_n > 0$ and β_n such that

$$\Phi^{(0,1)}_{r:n}(\alpha_n x + \beta_n) = \mathrm{I}_{F(\alpha_n x+\beta_n)}(r, n-r+1) \xrightarrow[n]{w} \Phi^{(0,1)}(x)\ , \tag{4.1}$$

where $\Phi^{(0,1)}(x)$ is a nondegenerate df, then $\Phi^{(0,1)}(x)$ must have one and only one of the types $\mathcal{N}(V_i(x))$, $i = 1,2,3$, where $V_1(x) = x$, $\forall x$, and

$$V_2(x) = \begin{cases} -\beta \ln|x|\,, & x \le 0\,, \\ \infty\,, & x > 0\,, \end{cases} \qquad V_3(x) = \begin{cases} -\infty\,, & x \le 0\,, \\ \beta \ln|x|\,, & x > 0\,, \end{cases} \tag{4.2}$$

where β is some positive constant. In this case F belongs to the domain of attraction of the df $\Phi^{(0,1)}(x)$, written $F \in \mathcal{D}(\Phi^{(0,1)}(x))$. Moreover, (4.1) is satisfied with $\Phi^{(0,1)}(x) = \mathcal{N}(V_i(x))$, for some $i \in \{1,2,3\}$ if, and only if,

$$\frac{nF(\alpha_n x + \beta_n) - r_n}{\sqrt{r_n}} \xrightarrow[n]{} V_i(x) \ . \tag{4.3}$$

Wu [15] generalized the Chibisov result for any nondecreasing intermediate rank sequence and proved that the only possible types for the limit df of the intermediate oos are those defined in (4.2).

Barakat [2], in Lemma 2.2 and Theorem 2.3, characterized the possible limit laws of the df of the upper intermediate m-gos. The following corresponding lemma characterizes the possible limit laws of the df of the lower intermediate m-gos. The proof of this lemma follows by using the same argument which is applied in the proof of Lemma 2.2 and Theorem 2.3 of Barakat [2].

Lemma 4.1. *Let* $m_1 = m_2 = ... = m_{n-1} = m > -1$, *and let* r_n *be a nondecreasing intermediate rank sequence. Then, there exist normalizing constants* $a_n > 0$ *and* b_n *such that*

$$\Phi^{(m,k)}_{r_n:n}(a_n x + b_n) \xrightarrow[n]{w} \Phi^{(m,k)}(x) \ , \tag{4.4}$$

where $\Phi^{(m,k)}(x)$ *is a nondegenerate df if, and only if,* $\frac{NG_m(a_n x+b_n)-r_N}{\sqrt{r_N}} \xrightarrow[n]{} V(x)$, *where* $\Phi^{(m,k)}(x) = \mathcal{N}(V(x))$. *Furthermore, let* $r^\star_n$ *be a variable rank sequence defined by* $r^\star_n = r_{\theta^{-1}(N)}$, *where* $\theta(n) = (m+1)N$ *(remember that* $N = \frac{k}{m+1} + n - 1$, *then* $\theta(n) = n$, *if* $m = 0$, $k = 1$, *i.e., in the case of oos). Then, there exist normalizing constants* $a_n > 0$ *and* b_n *for which (4.4) is satisfied for some nondegenerate df* $\Phi^{(m,k)}(x)$ *if, and only if, there are normalizing constants* $\alpha_n > 0$ *and* β_n *for which* $\Phi^{(0,1)}_{r^\star_n:n}(\alpha_n x + \beta_n) \xrightarrow[n]{w} \Phi^{(0,1)}(x)$, *where* $\Phi^{(0,1)}(x)$ *is some nondegenerate df, or equivalently* $\frac{nF(\alpha_n x+\beta_n)-r^\star_n}{\sqrt{r^\star_n}} \xrightarrow[n]{} V_i(x)$, $i \in \{1,2,3\}$, *and* $\Phi^{(0,1)}(x) = \mathcal{N}(V_i(x))$. *In this case, we can take* $a_n = \alpha_{\theta(n)}$ *and* $b_n = \beta_{\theta(n)}$. *Moreover,* $\Phi^{(m,k)}(x)$ *must has the form* $\mathcal{N}(V_i(x))$, *i.e.,* $V(x) = V_i(x)$.

In this section we consider the limit df of the two intermediate m-gos $\eta_r = \frac{X(r,n,m,k)-b_n}{a_n}$ and $\zeta_s = \frac{X(s,n,m,k)-d_n}{c_n}$, where $\frac{r}{n^{\alpha_1}} \xrightarrow[n]{} l_1^2$, $\frac{s}{n^{\alpha_2}} \xrightarrow[n]{} l_2^2$, $0<\alpha_1,\alpha_2<1$, $l_1, l_2 > 0$, and $a_n, c_n > 0$, b_n, d_n are suitable normalizing constants. The main aim of this section is to:

1 – Prove that the weak convergence of the df's of η_r and ζ_s implies the convergence of the joint df of η_r and ζ_s;

2 – Obtain the limit joint df of η_r and ζ_s and derive the condition under which the two statistics η_r and ζ_s are asymptotically independent.

We can distinguish the following distinct and exhausted two cases:

$$A)\ s-r \underset{n}{\longrightarrow} c,\ \ 0\le c<\infty\,, \qquad \text{and} \qquad B)\ s-r \underset{n}{\longrightarrow} \infty\ .$$

Remark 4.1. Under the condition A), we clearly have $l_1{=}\,l_2$, $\alpha_1{=}\,\alpha_2=\alpha$. Moreover $\frac{r}{s}\underset{n}{\longrightarrow} 1$. Finally, under the condition B) we have the following three distinct and exhausted cases:

$B_1)$ $\alpha_2>\alpha_1$, which implies $\frac{r}{s}\underset{n}{\longrightarrow} 0$.

$B_2)$ $\alpha_2=\alpha_1=\alpha$, $l_2>l_1$, which implies $\frac{r}{s}\underset{n}{\longrightarrow}\frac{l_1^2}{l_2^2}$.

$B_3)$ $\alpha_2=\alpha_1=\alpha$, $l_2=l_1$, which implies $\frac{r}{s}\underset{n}{\longrightarrow} 1$.

The following, corresponding lemma (proved in the Appendix) to Lemma 3.2, characterizes the possible limit laws of the joint intermediate m-gos.

Lemma 4.2. *Let* $\Phi^{(m,k)}_{r,s:n}(x_n,y_n){=}P(\eta_r{<}x,\,\zeta_s{<}y)$, $0{<}R_{rs}{=}\sqrt{\frac{\lambda_r(1-\lambda_s)}{\lambda_s(1-\lambda_r)}}{<}1$, $\frac{r}{s}\underset{n}{\longrightarrow} R$, $R_{rs}\underset{n}{\longrightarrow}\sqrt{R}$, $0{\le}R{<}1$, $x_n{=}\,a_nx+b_n$, $y_n{=}\,c_ny+d_n$, $U_n^{(1)}(x)=\frac{G_m(x_n)-\lambda_r}{\tau_r}$, $U_n^{(2)}(y)=\frac{G_m(y_n)-\lambda_s}{\tau_s}$, $\lambda_i=\frac{i}{N+1}$, $\tau_i=\sqrt{\frac{\lambda_i\nu_i}{N+1}}$ *and* $\nu_i=1-\lambda_i$, $i=r,s$. *Then*

$$\left|\Phi^{(m,k)}_{r,s:n}(x_n,y_n)-\frac{1}{2\pi\sqrt{1-R_{rs}^2}}\int_{-\infty}^{U_n^{(1)}(x)}\int_{-\infty}^{U_n^{(2)}(y)} e^{-\frac{(\xi^2+\eta^2-2\,\xi\eta R_{rs})}{2(1-R_{rs}^2)}}\,d\xi\,d\eta\right|$$

converges to zero uniformly with respect to x *and* y.

Lemma 4.2 leads to the following theorem.

Theorem 4.1. *Let* $x_n{=}\,a_nx+b_n$, $y_n{=}\,c_ny+d_n$, $\frac{r}{n},\frac{s}{n}\underset{n}{\longrightarrow} 0$, $\frac{r}{s}\underset{n}{\longrightarrow} R$, *and* $R_{rs}\underset{n}{\longrightarrow}\sqrt{R}$, $0\le R<1$. *Then the convergence of the two marginals* $\Phi^{(m,k)}_{r:n}(x_n)$ *and* $\Phi^{(m,k)}_{s:n}(y_n)$ *to nondegenerate limit df's* $\Phi^{(m,k)}(x)=\mathcal{N}(V(x))$ *and* $\Phi^{(m,k)}(y)=\mathcal{N}(\tilde{V}(y))$, *respectively, are necessary and sufficient condition for the convergence of the joint df* $\Phi^{(m,k)}_{r,s:n}(x_n,y_n)$ *to the nondegenerate limit*

$$\Phi^{(m,k)}_{r,s:n}(x_n,y_n)\ \underset{n}{\overset{w}{\longrightarrow}}\ \frac{1}{2\pi\sqrt{1-R}}\int_{-\infty}^{V(x)}\int_{-\infty}^{\tilde{V}(y)} e^{-\frac{(\xi^2+\eta^2-2\xi\eta\sqrt{R})}{2(1-R)}}\,d\xi\,d\eta\ .$$

Moreover, in view of Lemma 4.1, we deduce that the convergence of the joint df $\Phi^{(m,k)}_{r,s:n}(x_n,y_n)$, *as well as the convergence of the two marginals* $\Phi^{(m,k)}_{r:n}(x_n)$ *and* $\Phi^{(m,k)}_{s:n}(y_n)$, *occurs if, and only if, there are normalizing constants* $\alpha_n,\gamma_n>0$ *and* β_n,δ_n *for which* $\Phi^{(0,1)}_{r^\star_n:n}(\alpha_nx+\beta_n)\underset{n}{\overset{w}{\longrightarrow}}\Phi^{(0,1)}(x)=\mathcal{N}(V_i(x))$ *and* $\Phi^{(0,1)}_{s^\star_n:n}(\gamma_ny+\delta_n)$ $\underset{n}{\overset{w}{\longrightarrow}}\Phi^{(0,1)}(y)=\mathcal{N}(V_j(y))$, *for some* $i,j\in\{1,2,3\}$, *where* $r^\star_n=r_{\theta^{-1}(N)}$, $s^\star_n=s_{\theta^{-1}(N)}$ *and* $\theta(n)=(m{+}1)N$. *In this case, we can take* $a_n{=}\,\alpha_{\theta(n)}$, $c_n{=}\,\gamma_{\theta(n)}$, $b_n{=}\,\beta_{\theta(n)}$ *and* $d_n=\delta_{\theta(n)}$. *Moreover,* $V(x)=V_i(x)$ *and* $\tilde{V}(y)=V_j(y)$. *Finally, the two marginals are asymptotically independent if, and only if,* $\frac{r}{s}\underset{n}{\longrightarrow} 0$, *i.e.,* $R=0$.

APPENDIX

Proof of Lemma 2.3: In (1.1), consider the transformation $\xi = F(u)$, $\eta = F(v)$, we get

$$\text{(A.1)}\quad \begin{aligned} &\Phi^{(m,k)}_{r,s:n}(x_n, y_n) = \\ &= C^{\star}_n \int_0^{F(x_n)} \int_{\xi}^{F(y_n)} \bar{\xi}^m\, \bar{\eta}^{\gamma_s - 1}\, (1 - \bar{\xi}^{m+1})^{r-1}\, (\bar{\xi}^{m+1} - \bar{\eta}^{m+1})^{s-r-1}\, d\eta\, d\xi\ , \end{aligned}$$

where $\bar{\eta} = 1 - \eta$, $\bar{\xi} = 1 - \xi$ and $C^{\star}_n = \frac{C_{s-1,n}}{(m+1)^{s-2}\,(r-1)!\,(s-r-1)!}$. Again, by using the transformation $1 - \bar{\xi}^{m+1} = z$, $1 - \bar{\eta}^{m+1} = w$, we get

$$\text{(A.2)}\quad \Phi^{(m,k)}_{r,s:n}(x_n, y_n) = C^{\star\star}_n \int_0^{G_m(x_n)} \int_z^{G_m(y_n)} (1-w)^{\frac{\gamma_s - m - 1}{m+1}}\, z^{r-1} (w - z)^{s-r-1}\, dw\, dz\,,$$

where $C^{\star\star}_n = \frac{C^{\star}_n}{(m+1)^2}$. On the other hand, we have $\frac{\gamma_s - m - 1}{m+1} = N - s$ and

$$\begin{aligned} \frac{(r-1)!\,(s-r-1)!\,C^{\star\star}_n}{(N-s)^s} &= \frac{\prod_{j=1}^s \gamma_j}{(N-s)^s\,(m+1)^s} = \frac{\prod_{j=1}^s (N-j+1)}{(N-s)^s} = \\ &= \frac{\prod_{j=1}^s (1 - \frac{j-1}{N})}{(1 - \frac{s}{N})^s} = \left(1 + \frac{s^2}{N}\big(1 + o(1)\big)\right)\left(1 - \sum_{j=2}^s \frac{j-1}{N}\big(1 + o(1)\big)\right) = \\ &= 1 + \frac{s^2}{N} - \frac{1}{N}\left(\frac{s^2 - s}{2}\right)\big(1 + o(1)\big) = 1 + \rho_N\ , \end{aligned}$$

where $0 < \rho_N = \frac{1}{2N}\,(s^2 + s)\,(1 + o(1)) \underset{N}{\longrightarrow} 0$. Therefore, by using the transformation $w = \frac{\theta}{N-s}$, $z = \frac{\phi}{N-s}$ and the inequality $(1-z)^n \le e^{-nz}$, $\forall\, 0 \le z \le 1$ (cf. Lemma 1.3.1 in Galambos [6]), we get

$$\begin{aligned} &\Phi^{(m,k)}_{r,s:n}(x_n, y_n) = \\ &= \frac{C^{\star\star}_n}{(N-s)^s} \int_0^{(N-s)G_m(x_n)} \int_{\phi}^{(N-s)G_m(y_n)} \left(1 - \frac{\theta}{N-s}\right)^{N-s} \phi^{r-1} (\theta - \phi)^{s-r-1}\, d\theta\, d\phi \\ &\le \frac{(1 + \rho_N)}{(r-1)!\,(s-r-1)!} \int_0^{NG_m(x_n)} \int_{\phi}^{NG_m(y_n)} e^{-\theta} \phi^{r-1} (\theta - \phi)^{s-r-1}\, d\theta\, d\phi \\ &= \frac{(1 + \rho_N)}{(r-1)!} \int_0^{NG_m(x_n)} \Gamma_{s-r}\big(NG_m(y_n) - u\big)\, u^{r-1}\, e^{-u}\, du\ . \end{aligned}$$

On the other hand, by using the transformation $\frac{w}{1-w} = \frac{\theta}{N+r}$, $\frac{z}{1-z} = \frac{\phi}{N+r}$ in (A.2), and noting that $\frac{(r-1)!\,(s-r-1)!\,C^{\star\star}_n}{(N+r)^s} = \frac{\prod_{j=1}^s (1 - \frac{j-1}{N})}{(1 + \frac{r}{N})^s} = \big(1 - \frac{rs}{N}(1 + o(1))\big)\big(1 - \sum_{j=2}^s \frac{j-1}{N}(1 + o(1))\big) = 1 - \big(\frac{rs}{N} + \sum_{j=2}^s \frac{j-1}{N}\big)\big(1 + o(1)\big) = 1 - \frac{1}{N}\big(rs + \frac{s^2 - s}{2}\big)\big(1 + o(1)\big) =$

$1-\sigma_N^\star$, we get, by using the inequality $e^{-nz}\le(1+z)^{-n}$, $\forall\,0\le z\le 1$,

$$\begin{aligned}
\Phi^{(m,k)}_{r,s:n}(x_n,y_n) &= \\
&= \frac{C_n^{\star\star}}{(N+r)^s}\int_0^{(N+r)G_m(x_n)/\bar G_m(x_n)}\int_\phi^{(N+r)G_m(y_n)/\bar G_m(y_n)}(\theta-\phi)^{s-r-1}\\
&\quad\times\phi^{r-1}\Big(1+\frac{\theta}{N+r}\Big)^{-(N+r)}\Big(1+\frac{\theta}{N+r}\Big)^{2r-1}\Big(1+\frac{\phi}{N+r}\Big)^{-s}d\theta\,d\phi\\
&\ge \frac{(1-\sigma_N^\star)\,\bar F^{(m+1)s}(x_n)}{(r-1)!\,(s-r-1)!}\int_0^{NG_m(x_n)}\int_\phi^{NG_m(y_n)}(\theta-\phi)^{s-r-1}\phi^{r-1}\Big(1+\frac{\theta}{N+r}\Big)^{-(N+r)}d\theta\,d\phi\\
&\ge \frac{(1-\sigma_N)}{(r-1)!\,(s-r-1)!}\int_0^{NG_m(x_n)}\int_\phi^{NG_m(y_n)}(\theta-\phi)^{s-r-1}\,\phi^{r-1}\,e^{-\theta}\,d\theta\,d\phi\\
&= \frac{(1-\sigma_N)}{(r-1)!}\int_0^{NG_m(x_n)}\Gamma_{s-r}\big(NG_m(y_n)-u\big)\,u^{r-1}e^{-u}\,du\,,
\end{aligned}$$

where $\sigma_N=1-(1-\sigma_N^\star)\,\bar F^{(m+1)s}(x_n)\underset{N}{\longrightarrow}0$ (note that $\bar F^{(m+1)s}(x_n)\sim 1$). The lemma is proved. □

Proof of Lemma 2.4: We begin with the relation (A.1), after replacing r and s by $\grave r$ and $\grave s$, respectively. By using the transformation $\bar\xi^{m+1}=z$, $\bar\eta^{m+1}=w$ and noting that $n-r=N-R_r$, $n-s=N-R_s$, $\gamma_{n-s+1}=(m+1)R_s$ and $C_{\grave s-1,n}=C_{N-R_s,n}=(m+1)^{N-R_s+1}\prod_{j=1}^{N-R_s+1}(N-j+1)=(m+1)^{N-R_s+1}\frac{\Gamma(N+1)}{\Gamma(R_s)}$, we get

$$\text{(A.3)}\qquad \Phi^{(m,k)}_{\grave r,\grave s:n}(x_n,y_n)=\grave C_n\int_{\bar G_m(x_n)}^1\int_{\bar G_m(y_n)}^z w^{R_s-1}(1-z)^{N-R_r}(z-w)^{R_r-R_s-1}\,dw\,dz\,,$$

where $\grave C_n=\frac{\Gamma(N+1)}{\Gamma(N-R_r+1)\,\Gamma(R_r-R_s)\,\Gamma(R_s)}$. Again by using the transformation $w=\frac{\theta}{N-R_r}$, $z=\frac{\phi}{N-R_r}$ and the inequality $(1-z)^n\le e^{-nz}$, $\forall\,0\le z\le 1$, we get

$$\begin{aligned}
\Phi^{(m,k)}_{\grave r,\grave s:n}(x_n,y_n) &\le\\
&\le \frac{\grave C_n}{(N-R_r)^{R_r}}\int_{(N-R_r)\bar G_m(x_n)}^{(N-R_r)}\int_{(N-R_r)\bar G_m(y_n)}^{\phi}e^{-\phi}\,\theta^{R_s-1}(\phi-\theta)^{R_r-R_s-1}\,d\theta\,d\phi\,.
\end{aligned}$$

Now, by using Stirling's formula (cf. Lebedev [8]), we have $\frac{\Gamma(R_r-R_s)\,\Gamma(R_s)\,\grave C_n}{(N-R_r)^{R_r}}\sim e^{-R_r}(1-\frac{R_r}{N})^{-(N+\frac12)}\sim 1$, as $N\to\infty$ (i.e., as $n\to\infty$), and noting that $(N-R_r)\cdot\cdot\bar G_m(x_n)\sim N\bar G_m(x_n)$, $(N-R_r)\bar G_m(y_n)\sim N\bar G_m(y_n)$, as $N\to\infty$, we get

$$\begin{aligned}
\Phi^{(m,k)}_{\grave r,\grave s:n}(x_n,y_n) &\le \frac{1}{\Gamma(R_r-R_s)\,\Gamma(R_s)}\int_{N\bar G_m(x_n)}^N\int_{N\bar G_m(y_n)}^\phi e^{-\phi}\,\theta^{R_s-1}(\phi-\theta)^{R_r-R_s-1}d\theta\,d\phi\\
&= \frac{1}{\Gamma(R_r)}\int_{N\bar G_m(x_n)}^N\phi^{R_r-1}e^{-\phi}\Big(1-I_{\frac{N\bar G_m(y_n)}{\phi}}(R_s,R_r-R_s)\Big)\,d\phi\\
&= 1-\Gamma_{R_r}\big(N\bar G_m(x_n)\big)-\frac{1}{\Gamma(R_r)}\int_{N\bar G_m(x_n)}^N I_{\frac{N\bar G_m(y_n)}{t}}(R_s,R_r-R_s)\,t^{R_r-1}e^{-t}\,dt\,.
\end{aligned}$$

On the other hand, by using the transformation $\frac{w}{1-w}=\frac{\theta}{N+R_s}$, $\frac{z}{1-z}=\frac{\phi}{N+R_s}$ in (A.3) and the inequality $e^{-nz}\le(1+z)^{-n}$, $\forall\, 0\le z\le 1$, we get

$$\begin{aligned}\Phi^{(m,k)}_{\grave{r},\grave{s}:n}(x_n,y_n) \;&=\; \frac{\grave{C}_n}{(N{+}R_s)^{R_r}}\int_{(N+R_s)\frac{\bar{G}_m(x_n)}{G_m(x_n)}}^{\infty}\int_{(N+R_s)\frac{\bar{G}_m(y_n)}{G_m(y_n)}}^{\phi}\theta^{R_s-1}\\ &\quad\times\left(1{+}\frac{\phi}{N{+}R_s}\right)^{-(N+R_s)+2R_s-1}\left(1{+}\frac{\theta}{N{+}R_s}\right)^{-R_r}(\phi-\theta)^{R_r-R_s-1}\,d\theta\,d\phi\\ &\ge\; \frac{\grave{C}_n}{(N{+}R_s)^{R_r}}\int_{(N+R_s)\frac{\bar{G}_m(x_n)}{G_m(x_n)}}^{(N+R_s)}\int_{(N+R_s)\frac{\bar{G}_m(y_n)}{G_m(y_n)}}^{\phi}e^{-\phi}\,\theta^{R_s-1}\\ &\quad\times\left(1+\frac{\theta}{N{+}R_s}\right)^{-R_r}(\phi-\theta)^{R_r-R_s-1}\,d\theta\,d\phi\;.\end{aligned}$$

The lemma is proved. □

Proof of Lemma 2.5: The proof of the lower bound follows from the fact that the gos are positively quadrant dependent (see Barakat [3]). To prove the upper bound, in view of (1.1), we have

$$\begin{aligned}&\Phi^{(m,k)}_{r,\grave{s}:n}(x_n,y_n) \;=\\ &\quad=\; D_n\int_0^{F(x_n)}\int_{\xi}^{F(y_n)}\bar{\xi}^m\,\bar{\eta}^{\gamma_{n-s+1}-1}\,(1-\bar{\xi}^{m+1})^{r-1}(\bar{\xi}^{m+1}-\bar{\eta}^{m+1})^{n-s-r}d\eta\,d\xi\,,\end{aligned}\tag{A.4}$$

$\forall\, x_n\le y_n$, where $D_n=\frac{C_{n-s,n}}{(m+1)^{n-s-1}\,(r-1)!\,(n-s-r)!}$. Now, in view of the conditions of the lemma, it is easy to show that $\forall(x,y)$, for which $\mathcal{V}_{j,\beta}(x),\mathcal{U}_{i,\alpha}(y)<\infty$, we have $y_n \xrightarrow[n]{} \omega(F)=\sup\{x: F(x)<1\}>\inf\{x: F(x)>0\}=\alpha(F)\xleftarrow[n]{} x_n$. Therefore, for all large n, the relation (A.4) holds, $\forall x,y$, for which $\mathcal{V}_{j,\beta}(x),\mathcal{U}_{i,\alpha}(y)<\infty$. Now, by using the transformation $1-\bar{\xi}^{m+1}=v$, $\bar{\eta}^{m+1}=u$ and noting that $\frac{\gamma_{n-s+1}-m-1}{m+1}=R_s-1$, we get

$$\Phi^{(m,k)}_{r,\grave{s}:n}(x_n,y_n) \;=\; \frac{D_n}{(m{+}1)^2}\int_0^{G_m(x_n)}\int_{\bar{G}_m(y_n)}^{1-v}u^{R_s-1}\,v^{r-1}\,(1-u-v)^{n-s-r}\,du\,dv\;.$$

Therefore, by using the transformation $u=\frac{w}{N-R_s-r}$, $v=\frac{z}{N-R_s-r}$ and the inequality $(1-z)^n\le e^{-nz}$, $\forall\, 0\le z\le 1$, we get

$$\Phi^{(m,k)}_{r,\grave{s}:n}(x_n,y_n)\;\le\;\tilde{C}_n\int_0^{NG_m(x_n)}\int_{(N-R_s-r)\bar{G}_m(y_n)}^{N}w^{R_s-1}\,z^{r-1}\,e^{-(w+z)}\;dw\,dz\;,$$

where $\tilde{C}_n=\frac{D_n}{(m+1)^2\,(N-R_s-r)^{R_s+r}}$. On the other hand, by using Stirling's formula, we get

$$\begin{aligned}\Gamma(r)\,\tilde{C}_n \;&=\; \frac{C_{N-R_s,n}}{(m{+}1)^{N-R_s+1}\,(N{-}R_s{-}r)^{R_s+r}\;\Gamma(N{-}R_s{-}r{+}1)}\\ &=\;\frac{\Gamma(N{+}1)}{\Gamma(N{-}R_s{-}r{+}1)\,(N{-}R_s{-}r)^{R_s+r}\;\Gamma(R_s)}\;\sim\;\frac{1}{\Gamma(R_s)}\;.\end{aligned}$$

Therefore, since $(N-R_s-r)\,\bar{G}_m(y_n)\sim N\bar{G}_m(y_n)$, we get the upper bound of (2.5). The lemma is proved. □

Proof of Lemma 3.2: For given $\epsilon > 0$, choose T large enough to satisfy the inequalities $\frac{1}{T^2} < \epsilon$ and $\mathcal{N}(-T) < \epsilon$. If $U_n^{(1)}(x) \le -T$. Thus, for sufficiently large n, we get $1 - \bar{F}^{m+1}(x_n) \le \lambda_r - \tau_r T < 1$. Therefore, after routine calculations, we can show that

$$\begin{aligned}
\Phi_{r:n}^{(m,k)}(x_n) &= \frac{1}{\beta(r, N-r+1)} \int_0^{1-\overline{F}^{m+1}(x_n)} \xi^{r-1}(1-\xi)^{N-r}\, d\xi \\
&\le \frac{1}{\beta(r, N-r+1)} \int_0^{\lambda_r-\tau_r T} \xi^{r-1}(1-\xi)^{N-r}\, d\xi \\
&\le \frac{1}{\beta(r, N-r+1)} \int_0^{1} \frac{(\xi-\lambda_r)^2}{\tau_r^2\, T^2}\, \xi^{r-1}(1-\xi)^{N-r} d\xi \\
&= \frac{N+1}{(N+2)\, T^2} < \frac{1}{T^2} < \epsilon\ .
\end{aligned}$$

Since $\Phi_{r,s:n}^{(m,k)}(x_n, y_n) \le \Phi_{r:n}^{(m,k)}(x_n)$, then $\Phi_{r,s:n}^{(m,k)}(x_n, y_n) < \epsilon$. Similarly, if $U_n^{(2)}(y) \le -T$, we can prove that $\Phi_{r,s:n}^{(m,k)}(x_n, y_n) \le \Phi_{s:n}^{(m,k)}(y_n) < \epsilon$. On the other hand, we have

$$\int_{-\infty}^{U_n^{(1)}(x)} \int_{-\infty}^{U_n^{(2)}(y)} W_{r,s}(\xi, \eta)\, d\xi\, d\eta \le \min\Big(\mathcal{N}\big(U_n^{(1)}(x)\big), \mathcal{N}\big(U_n^{(2)}(y)\big)\Big) < \epsilon\ .$$

Therefore, if $U_n^{(1)}(x) \le -T$ or $U_n^{(2)}(y) \le -T$, we get

$$\left| \Phi_{r,s:n}^{(m,k)}(x_n, y_n) - \int_{-\infty}^{U_n^{(1)}(x)} \int_{-\infty}^{U_n^{(2)}(y)} W_{r,s}(\xi, \eta)\, d\xi\, d\eta \right| \le 2\,\epsilon\ .$$

Now, if $U_n^{(1)}(x) \ge T$, then $1 - \bar{F}^{m+1}(x_n) \ge \lambda_r + \tau_r T$. Therefore, after routine calculations, we get

$$\begin{aligned}
1 - \Phi_{r:n}^{(m,k)}(x_n) &\le \frac{1}{\beta(r, N-r+1)} \int_{\lambda_r+\tau_r T}^{1} \xi^{r-1}(1-\xi)^{N-r} d\xi \\
&\le \frac{1}{\beta(r, N-r+1)} \int_0^{1} \frac{(\xi-\lambda_r)^2}{\tau_r^2\, T^2}\, \xi^{r-1}(1-\xi)^{N-r} d\xi \\
&= \frac{N+1}{(N+2)\, T^2} < \frac{1}{T^2} < \epsilon\ .
\end{aligned}$$

Thus, we also get

$$\text{(A.5)} \qquad \Phi_{s:n}^{(m,k)}(y_n) - \Phi_{r,s:n}^{(m,k)}(x_n, y_n) \le 1 - \Phi_{r:n}^{(m,k)}(x_n) < \epsilon\ .$$

On the other hand, in view of our assumptions and Lemma 3.1, we get

$$\begin{aligned}
\mathcal{N}\big(U_n^{(2)}(y)\big) - \int_{-\infty}^{U_n^{(1)}(x)} \int_{-\infty}^{U_n^{(2)}(y)} W_{r,s}(\xi, \eta)\, d\xi\, d\eta &= \\
\text{(A.6)} \qquad &= \int_{U_n^{(1)}(x)}^{\infty} \int_{-\infty}^{U_n^{(2)}(y)} W_{r,s}(\xi, \eta)\, d\xi\, d\eta \\
&\le \frac{1}{\sqrt{2\pi}} \int_{U_n^{(1)}(x)}^{\infty} e^{\frac{-\xi^2}{2}}\, d\xi \le \frac{1}{\sqrt{2\pi}} \int_{T}^{\infty} e^{\frac{-\xi^2}{2}}\, d\xi < \epsilon\ ,
\end{aligned}$$

for sufficiently large n, and

$$\left|\Phi_{s:n}^{(m,k)}(y_n) - \mathcal{N}\big(U_n^{(2)}(y)\big)\right| < \epsilon\ , \tag{A.7}$$

for sufficiently large n. The relations (A.5), (A.6) and (A.7) show that when $U_n^{(1)}(x) \geq T$, we have $|\Phi_{r,s:n}^{(m,k)}(x_n, y_n) - \int_{-\infty}^{U_n^{(1)}(x)} \int_{-\infty}^{U_n^{(2)}(y)} W_{r,s}(\xi,\eta)\, d\xi\, d\eta| < 3\,\epsilon$. Similarly, we can show that the last inequality holds for sufficiently large n, if $U_n^{(2)}(y) \geq T$. In order to complete the proof of the lemma, we have to consider the case $|U_n^{(1)}(x)|, |U_n^{(2)}(y)| < T$. First, we note that, since $G_m(x_n) \underset{n}{\longrightarrow} \lambda_1 < \lambda_2 \underset{n}{\longleftarrow} G_m(y_n)$, we have $x_n \leq y_n$, for sufficiently large n. Therefore, for sufficiently large n, $\Phi_{r,s:n}^{(m,k)}(x_n, y_n)$ is given by (3.3). Moreover, in this case we have $1 - \bar{F}^{m+1}(x_n) > \lambda_r - \tau_r T \geq 0$ and $1 - \bar{F}^{m+1}(y_n) > \lambda_s - \tau_s T \geq 0$. Thus,

$$\begin{aligned}
\Phi_{r,s:n}^{(m,k)}(x_n, y_n) &= \int_0^{1-\bar{F}^{m+1}(x_n)} \int_z^{1-\bar{F}^{m+1}(y_n)} \varphi_{r,s:n}^{(m,k)}(w,z)\; dw\, dz \\
&= \int_0^{\lambda_r - \tau_r T} \int_z^{1-\bar{F}^{m+1}(y_n)} \varphi_{r,s:n}^{(m,k)}(w,z)\; dw\, dz \\
&\quad + \int_{\lambda_r-\tau_r T}^{1-\bar{F}^{m+1}(x_n)} \int_z^{\lambda_s-\tau_s T} \varphi_{r,s:n}^{(m,k)}(w,z)\; dw\, dz \\
&\quad + \int_{\lambda_r-\tau_r T}^{1-\bar{F}^{m+1}(x_n)} \int_{\lambda_s-\tau_s T}^{1-\bar{F}^{m+1}(y_n)} \varphi_{r,s:n}^{(m,k)}(w,z)\; dw\, dz\ ,
\end{aligned} \tag{A.8}$$

where $\varphi_{r,s:n}^{(m,k)}(w,z) = \frac{C_n^\star}{(m+1)^2}\, z^{r-1}(1-w)^{N-s}\,(w-z)^{s-r-1}$. We shall separately consider, each of the integrals in the summation (A.8):

$$\begin{aligned}
&\int_0^{\lambda_r-\tau_r T} \int_z^{1-\bar{F}^{m+1}(y_n)} \varphi_{r,s:n}^{(m,k)}(w,z)\; dw\, dz \;\leq \\
&\qquad \leq \int_0^{\lambda_r-\tau_r T} \int_z^1 \varphi_{r,s:n}^{(m,k)}(w,z)\; dw\, dz \\
&\qquad = \frac{C_n^\star}{(m+1)^2} \int_0^{\lambda_r-\tau_r T} \int_z^1 z^{r-1}(1-w)^{N-s}\,(w-z)^{s-r-1}\, dw\, dz \\
&\qquad = \frac{\Gamma(N+1)}{\Gamma(N-r+1)\,\Gamma(r)} \int_0^{\lambda_r-\tau_r T} z^{r-1}(1-z)^{N-r}\, dz \;<\; \frac{1}{T^2} \;<\; \epsilon\ ,
\end{aligned} \tag{A.9}$$

$$\begin{aligned}
&\int_{\lambda_r-\tau_r T}^{1-\bar{F}^{m+1}(x_n)} \int_z^{\lambda_s-\tau_s T} \varphi_{r,s:n}^{(m,k)}(w,z)\; dw\, dz \;\leq \\
&\qquad \leq \int_0^{\lambda_s-\tau_s T} \int_z^{\lambda_s-\tau_s T} \varphi_{r,s:n}^{(m,k)}(w,z)\; dw\, dz \\
&\qquad = \frac{\Gamma(N+1)}{\Gamma(N-s+1)\,(s-1)!} \int_0^{\lambda_s-\tau_s T} w^{s-1}(1-w)^{N-s}\, dw \;<\; \frac{1}{T^2} \;<\; \epsilon\ ,
\end{aligned} \tag{A.10}$$

and by using the transformation $z = \lambda_r + \xi\,\tau_r$, $w = \lambda_s + \eta\,\tau_s$, the third integral takes the form

$$\int_{\lambda_r-\tau_r T}^{1-\bar{F}^{m+1}(x_n)}\int_{\lambda_s-\tau_s T}^{1-\bar{F}^{m+1}(y_n)}\varphi_{r,s:n}^{(m,k)}(w,z)\,dw\,dz = \\ = A_{r,s:n}\int_{-T}^{U_n^{(1)}(x)}\int_{-T}^{U_n^{(2)}(y)} g_{r,s:n}(\xi,\eta)\,d\eta\,d\xi \ ,$$

where

$$A_{r,s:n} = \frac{\Gamma(N+1)\,\tau_r\,\tau_s\,\lambda_r^{r-1}\,\nu_s^{N-s}(\lambda_s-\lambda_r)^{s-r-1}}{\Gamma(N-s+1)\,(r-1)!\,(s-r-1)!}$$

and

$$g_{r,s:n}(\xi,\eta) = \left(1+\frac{\xi\tau_r}{\lambda_r}\right)^{r-1}\left(1+\frac{\eta\tau_s-\xi\tau_r}{\lambda_s-\lambda_r}\right)^{s-r-1}\left(1-\frac{\eta\tau_s}{\nu_s}\right)^{N-s} .$$

On the other hand, by using Stirling's formula $\Gamma(M+1)=e^{-M}\sqrt{2\pi M}\,\cdot$ $\cdot\, M^M(1+\circ(1))$, as $M\to\infty$, we get

$$\begin{aligned} A_{r,s:n} &= \frac{(N+1)^2\,\Gamma(N+1)\,\tau_r\,\tau_s\,\lambda_r^{r}\,\nu_s^{N-s}(\lambda_s-\lambda_r)^{s-r}}{\Gamma(N-s+1)\,r!\,(s-r)!} \\ &= \frac{1+\circ(1)}{2\pi\sqrt{\frac{(N+1)\,(s-r)}{s(N-r+1)}}} = \frac{1+\circ(1)}{2\pi\sqrt{1-R_{rs}^2}} \ . \end{aligned}$$

Also, it is easy to show that

$$\begin{aligned} g_{r,s:n}(\xi,\eta) &= \left(1+\frac{\xi\tau_r}{\lambda_r}\right)^{r}\left(1+\frac{\eta\tau_s-\xi\tau_r}{\lambda_s-\lambda_r}\right)^{s-r}\left(1-\frac{\eta\tau_s}{\nu_s}\right)^{N-s} \\ &\quad\times\left[\left(1+\frac{\xi\tau_r}{\lambda_r}\right)^{-1}\left(1+\frac{\eta\tau_s-\xi\tau_r}{\lambda_s-\lambda_r}\right)^{-1}\right] \\ &= \left(1+\frac{\xi\tau_r}{\lambda_r}\right)^{r}\left(1+\frac{\eta\tau_s-\xi\tau_r}{\lambda_s-\lambda_r}\right)^{s-r}\left(1-\frac{\eta\tau_s}{\nu_s}\right)^{N-s} \qquad \text{(A.11)} \\ &\quad\times\left[\left(1-\frac{\xi\tau_r}{\lambda_r}\,(1+o(1))\right)\left(1-\frac{\eta\tau_s-\xi\tau_r}{\lambda_s-\lambda_r}\,(1+\circ(1))\right)\right] \\ &= \left(1+\frac{\xi\tau_r}{\lambda_r}\right)^{r}\left(1+\frac{\eta\tau_s-\xi\tau_r}{\lambda_s-\lambda_r}\right)^{s-r}\left(1-\frac{\eta\tau_s}{\nu_s}\right)^{N-s}\big(1+\rho_n(\xi,\eta)\big)\,, \end{aligned}$$

where $\rho_n(\xi,\eta)\xrightarrow[n]{}0$, uniformly in any finite interval $(-T,T)$ of the value ξ and η. Furthermore, we have

$$\begin{aligned} r\ln\left(1+\frac{\xi\tau_r}{\lambda_r}\right) &= r\left(\frac{\xi\tau_r}{\lambda_r}-\frac{\xi^2\tau_r^2}{2\lambda_r^2}+\frac{\xi^3\tau_r^3}{3\lambda_r^3}+\cdots\right) \qquad \text{(A.12)} \\ &= \xi\,\tau_r(N+1)-\frac{\xi^2\nu_r}{2}+\circ\left(\frac{T^3}{\sqrt{r}}\right), \end{aligned}$$

$$(A.13)\qquad \begin{aligned}(s-r)\,\ln\!\left(1+\frac{\eta\tau_s-\xi\tau_r}{\lambda_s-\lambda_r}\right) &= \\ &= (\eta\tau_s-\xi\tau_r)\,(N+1)-\frac{1}{2}\,\frac{(\eta\tau_s-\xi\tau_r)^2}{\lambda_s-\lambda_r}\,(N+1)+\circ\!\left(\frac{T^3}{\sqrt{s}}\right)\end{aligned}$$

and

$$(A.14)\qquad (N-s)\,\ln\!\left(1-\frac{\eta\tau_s}{\nu_s}\right) = -\eta\tau_s(N+1)-\frac{1}{2}\,\eta^2\lambda_s+\circ\!\left(\frac{\lambda_s^{\frac{3}{2}}\,T^3}{\sqrt{N}}\right).$$

Therefore, by combining (A.11)–(A.14), as $n\to\infty$ (or equivalently as $N\to\infty$), we get

$$\begin{aligned}\ln g_{r,s:n}(\xi,\eta) &= r\,\ln\!\left(1+\frac{\xi\tau_r}{\lambda_r}\right)+(s-r)\,\ln\!\left(1+\frac{\eta\tau_s-\xi\tau_r}{\lambda_s-\lambda_r}\right)+(N-s)\,\ln\!\left(1-\frac{\eta\tau_s}{\nu_s}\right)\\ &\sim -\frac{\xi^2\nu_r}{2}-\frac{\eta^2\tau_s^2-2\,\xi\,\eta\,\tau_r\tau_s+\xi^2\tau_r^2}{2(\lambda_s-\lambda_r)}\,(N+1)-\frac{1}{2}\,\eta^2\lambda_s\\ &= -\frac{\xi^2\nu_r}{2}\left(1+\frac{\lambda_r}{\lambda_s-\lambda_r}\right)-\frac{1}{2}\,\eta^2\lambda_s\left(1+\frac{\nu_s}{\lambda_s-\lambda_r}\right)-\frac{1}{2}\left(-2\,\xi\,\eta\,\frac{\tau_r\,\tau_s}{\lambda_s-\lambda_r}\right)\\ &= -\frac{1}{2}\,\frac{\lambda_s(1-\lambda_r)}{\lambda_s-\lambda_r}\left(\xi^2+\eta^2-2\,\xi\,\eta\sqrt{\frac{\lambda_r(1-\lambda_s)}{\lambda_s(1-\lambda_r)}}\right),\end{aligned}$$

which implies $g_{r,s:n}(\xi,\eta)=e^{-\frac{\left(\xi^2+\eta^2-2\,\xi\eta R_{rs}\right)}{2(1-R_{rs}^2)}}\big(1+\circ(1)\big)$. Therefore, for sufficiently large n (or equivalently for large N), we get

$$\left|\int_{\lambda_r-\tau_rT}^{1-\bar F^{m+1}(x_n)}\int_{\lambda_s-\tau_sT}^{1-\bar F^{m+1}(y_n)}\varphi^{(m,k)}_{r,s:n}(w,z)\;dw\,dz-\int_{-T}^{U_n^{(1)}(x)}\int_{-T}^{U_n^{(2)}(y)}W_{r,s}(\xi,\eta)\;d\xi\,d\eta\right|<\epsilon\,.$$

Since,

$$\int_{-\infty}^{-T}\int_{-T}^{U_n^{(2)}(y)}W_{r,s}(\xi,\eta)\;d\xi\,d\eta+\int_{-\infty}^{U_n^{(1)}(x)}\int_{-\infty}^{-T}W_{r,s}(\xi,\eta)\;d\xi\,d\eta\;<\;2\mathcal{N}(-T)\;<\;2\,\epsilon$$

and

$$\begin{aligned}\int_{-\infty}^{U_n^{(1)}(x)}\int_{-\infty}^{U_n^{(2)}(y)}W_{r,s}(\xi,\eta)\;d\xi\,d\eta &= \\ &= \int_{-\infty}^{U_n^{(1)}(x)}\int_{-\infty}^{-T}W_{r,s}(\xi,\eta)\;d\xi\,d\eta+\int_{-\infty}^{-T}\int_{-T}^{U_n^{(2)}(y)}W_{r,s}(\xi,\eta)\;d\xi\,d\eta\\ &\quad+\int_{-T}^{U_n^{(1)}(x)}\int_{-T}^{U_n^{(2)}(y)}W_{r,s}(\xi,\eta)\;d\xi\,d\eta\;,\end{aligned}$$

then

$$\left|\int_{\lambda_r-\tau_rT}^{1-\bar F^{m+1}(x_n)}\int_{\lambda_s-\tau_sT}^{1-\bar F^{m+1}(y_n)}\varphi^{(m,k)}_{r,s:n}(w,z)\;dw\,dz-\int_{-\infty}^{U_n^{(1)}(x)}\int_{-\infty}^{U_n^{(2)}(y)}W_{r,s}(\xi,\eta)\;d\xi\,d\eta\right|<3\,\epsilon\,.$$

By combining the last inequality with (A.9) and (A.10) we get, for sufficient large n, the inequality

$$\left|\Phi^{(m,k)}_{r,s:n}(x_n, y_n) - \int_{-\infty}^{U_n^{(1)}(x)}\int_{-\infty}^{U_n^{(2)}(y)} W_{r,s}(\xi,\eta)\, d\xi\, d\eta\right| < 5\,\epsilon\ ,$$

which proves the lemma in the case $\left|U_n^{(1)}(x)\right|, \left|U_n^{(2)}(y)\right| < T$. This completes the proof. □

Proof of Lemma 4.2: Under the condition of the lemma $(0 \le R < 1)$, we consider only the cases $B_1)$ and $B_2)$. On the other hand, the proof is very close to the proof of Lemma 3.2. Therefore, we only show the necessary changes in the proof of Lemma 3.2. For given $\epsilon > 0$, we choose T, large enough to satisfy both of the inequalities $\frac{1}{T^2} < \epsilon$, and $\mathcal{N}(-T) < \epsilon$. In this case it is easy to see that the proof of the two lemmas coincides in the cases $U_n^{(t)}(\cdot) \le -T$ and $U_n^{(t)}(\cdot) \ge T,\ t = 1,2$. Therefore, we only prove the lemma under the case $|U_n^{(1)}(x)| < T$ and $|U_n^{(2)}(y)| < T$. In this case we have $1-\bar{F}^{m+1}(x_n) > \lambda_r - \tau_r T \ge 0$ and $1-\bar{F}^{m+1}(y_n) > \lambda_s - \tau_s T \ge 0$. Thus, we get

$$\text{(A.15)}\qquad \begin{aligned}\Phi^{(m,k)}_{r,s:n}(x_n, y_n) \ &= \int_0^{\lambda_r-\tau_r T}\int_z^{1-\bar{F}^{m+1}(y_n)} \varphi^{(m,k)}_{r,s:n}(w,z)\, dw\, dz \\ &+ \int_{\lambda_r-\tau_r T}^{1-\bar{F}^{m+1}(x_n)}\int_z^{\lambda_s-\tau_s T} \varphi^{(m,k)}_{r,s:n}(w,z)\, dw\, dz \\ &+ \int_{\lambda_r-\tau_r T}^{1-\bar{F}^{m+1}(x_n)}\int_{\lambda_s-\tau_s T}^{1-\bar{F}^{m+1}(y_n)} \varphi^{(m,k)}_{r,s:n}(w,z)\, dw\, dz\ ,\end{aligned}$$

where $\varphi^{(m,k)}_{r,s:n}(w,z) = \frac{C_n^\star}{(m+1)^2}\, z^{r-1}(1-w)^{N-s}(w-z)^{s-r-1}$. We shall separately consider, each of the integrals in the summation (A.15).

$$\begin{aligned}\int_0^{\lambda_r-\tau_r T}\int_z^{1-\bar{F}^{m+1}(y_n)} \varphi^{(m,k)}_{r,s:n}(w,z)\, dw\, dz \ &\le \int_0^{\lambda_r-\tau_r T}\int_z^{1} \varphi^{(m,k)}_{r,s:n}(w,z)\, dw\, dz\ = \\ &= \frac{\Gamma(N+1)}{\Gamma(N-r+1)\,(r-1)!}\int_0^{\lambda_r-\tau_r T} z^{r-1}(1-z)^{N-r}\, dz\ <\ \frac{1}{T^2}\ <\ \epsilon\ .\end{aligned}$$

Since $|U_n^{(1)}(x)| < T$, for large N, we get

$$\text{(A.16)}\qquad 1-\bar{F}^{m+1}(x_n) < \lambda_r + \tau_r T\ .$$

On the other hand, we have

$$\text{(A.17)}\qquad \frac{\lambda_r+\tau_r T}{\lambda_s-\tau_s T} \underset{n}{\longrightarrow} \begin{cases} 0\,, & \text{in the case } B_1)\,,\\ \dfrac{l_l^2}{l_2^2}\,, & \text{in the case } B_2)\,.\end{cases}$$

Therefore, for large N, the relations (A.16) and (A.17) imply the inequality $1-\bar{F}^{m+1}(x_n) < \lambda_s - \tau_s T$, which in turn leads to the following estimate for the 2nd integral in (A.15):

$$\begin{aligned}
\int_{\lambda_r-\tau_r T}^{1-\bar{F}^{m+1}(x)} \int_z^{\lambda_s-\tau_s T} & \varphi_{r,s:n}^{(m,k)}(w,z)\, dw\, dz \le \\
&\le \int_0^{\lambda_s-\tau_s T}\int_z^{\lambda_s-\tau_s T} \varphi_{r,s:n}^{(m,k)}(w,z)\, dw\, dz \\
&= \int_0^{\lambda_s-\tau_s T}\int_0^{w} \varphi_{r,s:n}^{(m,k)}(w,z)\, dz\, dw \\
&= \frac{\Gamma(N+1)}{\Gamma(N-s+1)\,(s-1)!}\int_0^{\lambda_s-\tau_s T} w^{s-1}(1-w)^{N-s}\, dw \;<\; \frac{1}{T^2} \;<\; \epsilon\;.
\end{aligned}$$

It is easy to show that, under the cases B_1) and B_2), the mathematical treatments of the third integral of the summation, as well as the remaining part of the proof, is exactly the same as in the proof of Lemma 3.2. This completes the proof. □

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ON THE IMPACT OF FALSELY ASSUMING I.I.D. OUTPUT IN THE PROBABILITY OF MISLEADING SIGNALS

Authors: Manuel Cabral Morais
– Department of Mathematics and CEMAT, Instituto Superior Técnico, University of Lisbon, Portugal
maj@math.ist.utl.pt

Patrícia Ferreira Ramos
– CEMAT, Instituto Superior Técnico, University of Lisbon, Portugal
patriciaferreira@ist.utl.pt

António Pacheco
– Department of Mathematics and CEMAT, Instituto Superior Técnico, University of Lisbon, Portugal
apacheco@math.ist.utl.pt

Wolfgang Schmid
– Department of Statistics, European University Viadrina, Germany
schmid@euv-frankfurt-o.de

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Abstract:

- Misleading signals (MS) are valid alarms which correspond to the misinterpretation of a shift in the process mean (resp. variance) as a shift in the process variance (resp. mean), when we deal with simultaneous schemes for these two parameters. MS can be fairly frequent, as reported by some authors, and occur for instance when:
  - the individual chart for the mean triggers a signal before the one for the variance, even though the process mean is on-target and the variance is off-target; or
  - the individual chart for the variance triggers a signal before the one for the mean, although the variance is in-control and the process mean is out-of-control.

  This paper illustrates how (un)reliable are the traditional simultaneous Shewhart- and EWMA-type schemes in identifying which parameter has changed, under the false assumption of independence, namely when the output process within each sample follows AR(1), AR(2) or ARMA (1,1) models. This is done by means of Monte Carlo simulation and the estimation of the probability of a misleading signal (PMS).
  Finally, we go on to compare these estimates of PMS with the values of the PMS of simultaneous Shewhart- and EWMA-type residual schemes whose control statistics take into account the autocorrelation structure of the output process.

Key-Words:

- *statistical process control; misleading signals; time series; simultaneous residual schemes.*

AMS Subject Classification:

- 62P30, 60G99.

1. THE PHENOMENON OF MISLEADING SIGNALS

In most monitoring applications, we assume that the quality characteristic is an absolutely continuous random variable with a normal distribution with mean μ and variance σ^2. Quality control charts are graphical SPC tools whose main purpose is to detect (removable) special or assignable causes responsible for changes in μ and σ^2. Standard practice is to run two individual charts at the same time, one for μ and another one for σ^2. The resulting scheme is known as a simultaneous scheme and it provides a way to satisfy Shewhart's dictum that proper process control implies monitoring both location and dispersion.

When we use a simultaneous scheme, the quality characteristic is deemed to be out-of-control whenever a signal is triggered by either individual chart: a signal suggests a potential change in μ, in σ^2 or in both μ and σ^2. Moreover, it is expected that the chart for the mean will help us detect increases or decreases in μ from a target value μ_0 and that the chart for the variance will assist us in the detection of increases in σ^2 from an in-control value σ_0^2. However, it has been pointed out by some authors (e.g. [21], [10] and [18]) that the misidentification of the parameter that has changed can occur frequently, which means that a shift in μ can be misinterpreted as a shift in σ^2 and vice-versa. [21] termed these two events as misleading signals (MS) and [10] systematized them and only considered MS of types III and IV:

- the individual chart for μ triggers a signal before the one for σ^2, although the process mean is on-target and the variance is off-target; and
- the individual chart for σ^2 triggers a signal before the one for μ, even though the process variance is on-target and the mean is off-target.

Now, note that special or assignable causes on the chart for μ can differ from those on the chart for σ^2: for instance, cyclic patterns in $\bar{X}$-charts may result from systematic changes in temperature or regular rotation of operators/machines, whereas S^2-charts reveal cycles because of maintenance schedules or tool wear ([9, pp. 189–190]). Furthermore, the diagnostic and correction procedures that follow a signal can differ depending on which chart triggers the alarm, as mentioned by [11] and [8]. Therefore, the occurrence of a MS can lead to an inappropriate diagnose and to unnecessary correction measures and hence to an increase in production and inspection costs.

2. EXISTING WORK

The main question regarding misleading signals should not be whether they happen or not, but rather how frequently they occur, as pointed out by [11].

Unsurprisingly, the probability of a misleading signal (PMS) should be considered as an additional performance measure of simultaneous schemes for μ and σ^2.

The behavior of the PMS of types III and IV has been addressed for i.i.d. and Gaussian output by a few authors ([10], [18], [12], [19] and [11]). For example, the numerical results in [12] and [11] suggest that simultaneous Shewhart-type schemes compare unfavorably to their EWMA counterparts and that the values of both PMS are far from negligible, specially for small and moderate shifts in μ and σ^2.

The study of the phenomenon of MS has been extended by [2], [8], [14] and [15] to the following change point model, proposed by [7] and [6] and dealing with autocorrelated output. Let us denote by $\{Y_{i,j}\}$ the target process, where i represents the sample number and j the number of the observation within the sample. Samples have fixed size n, are independent and represented by $(Y_{i,1}, ..., Y_{i,n})$. However, we shall assume that $\{Y_{i,1}, ..., Y_{i,n}\}$ follows a (weakly) stationary Gaussian process with known mean μ_0 and known autocovariance function $\{\gamma_0, \gamma_1, ..., \gamma_{n-1}\}$, for every i. The observed process, $\{X_{i,j}\}$, is related to the target process as follows:

$$X_{i,j} \;=\; \mu_0 + \delta\,\gamma_0 + \theta\,(Y_{i,j} - \mu_0)\;, \qquad i = 1, 2, \ldots\;, \tag{2.1}$$

where $\delta = [E(X_{i,j}) - \mu_0]/\sqrt{\gamma_0}$ (resp. $\theta = \sqrt{V(X_{i,j})/\gamma_0}$) represents the magnitude of the shift in the process mean (resp. standard deviation). As put by [8], the assumption of independent samples but autocorrelated output within each sample is rather reasonable in SPC because the intervals between successive samples are significantly large when compared to the time required to take a sample, resulting in negligible correlation between samples and considerable correlation within each sample.

There are essentially three approaches to monitor shifts in the mean and variance of the observed process and they play a major role in the performance of the simultaneous schemes and, obviously, on the PMS. We could plot the sample mean and variance of each of the original data in a traditional simultaneous scheme, however, with readjusted control limits to account for the autocorrelation; the resulting scheme is called a modified simultaneous scheme ([7] and [6]). Alternatively, we could plot the sample mean and variance of the residuals instead of the original data, in a traditional simultaneous scheme, i.e., use what is called a simultaneous residual scheme ([7], [6] and [8]). Lastly, we could ignore the autocorrelation structure and assume the output is i.i.d. within each sample and use the traditional simultaneous schemes.

Results by [8], [2], [14] and [15] suggest that the presence of autocorrelation can have a significant impact in the PMS of simultaneous Shewhart and EWMA residual schemes for the process mean and variance of stationary processes.

[8] and [2] showed that the PMS of Type III is not affected by the autoregressive parameter and larger nonnegative values of this parameter are associated to more frequent MS of Type IV, when dealing with simultaneous Shewhart- and EWMA-type residual schemes for the mean and the variance of AR(1) output. [14] used stochastic ordering to prove that the PMS of Type IV of simultaneous Shewhart (resp. EWMA) residual schemes for the process mean and variance of stationary AR(1) output increases with the autoregressive parameter in the interval $(-1, 1)$ (resp. (0,1)). [15] is an obvious extension of [14] to general stationary Gaussian processes, such as AR(2) and ARMA(1,1) models, and it also identified regions where the PMS of Type IV is a monotonous function of the parameters of these models. In addition to this, [8] showed how unreliable are the traditional simultaneous Shewhart and EWMA schemes in identifying which parameter has changed under the false assumption of i.i.d. output, when we are in fact dealing with stationary AR(1) output.

In the present paper, we recall some of these results for AR(1) output and we extend these investigations to AR(2) and ARMA(1,1) processes with a few unexpected results. All the estimates of PMS were obtained via an extensive Monte Carlo simulation study and we go on to compare them to the PMS values associated with simultaneous residual schemes. But before proceeding to this study of the impact of falsely assuming i.i.d. output on the PMS, we shall briefly describe simultaneous residual schemes for autocorrelated output in the next section.

3. SIMULTANEOUS RESIDUAL SCHEMES AND PMS

Residual charts ([1]) can prevent the process mean and variance of autocorrelated output from wandering too far from their targets. Besides that, these charts are theoretically very appealing because their control statistics take the autocorrelation explicitly into account, and reduce the monitoring problem to the well-known case of detecting shift in the mean and variance of i.i.d. output ([23, p. 63]). Moreover, since control charts are ultimately used by non-statisticians, we favor "one fits all" procedures, such as residual charts, that are easily understood and can be applied to most industrial processes.

The control statistics of the individual residuals charts for the process mean and variance of a stationary Gaussian process may be defined in terms of standardized residuals ([8]), such as the following ones

$$\hat{\varepsilon}_{i,j} = \frac{X_{i,j} - \hat{X}_{i,j}}{\sqrt{V_{\delta=0,\theta=1}(X_{i,j} - \hat{X}_{i,j})}} = \theta\, \hat{\epsilon}_{i,j} + \delta \sqrt{\gamma_0}\, b_j \ , \qquad (3.1)$$

where: $V_{\delta=0,\theta=1}(X_{i,j}-\hat{X}_{i,j})$ represents the in-control variance of the residuals of the output process; $\hat{\epsilon}_{i,j} \sim_{i.i.d.} \mathcal{N}(0,1)$ are the standardized residuals of the target process; $\mathbf{b}=(b_1,...,b_n)$ is the vector of the b_js, which are functions of $\gamma_0,\gamma_1,...,\gamma_{n-1}$ that can be recursively obtained by using the Durbin–Levinson algorithm ([3, p. 169]). If the (fitted) model is valid, the standardized residuals are independent normal r.v. and the sample mean and variance of these residuals,

$$\bar{\hat{\varepsilon}}_i = \frac{1}{n}\sum_{j=1}^{n}\hat{\varepsilon}_{i,j} \,, \tag{3.2}$$

$$\hat{S}_i^2 = \frac{1}{n-1}\sum_{j=1}^{n}(\hat{\varepsilon}_{i,j}-\bar{\hat{\varepsilon}}_i)^2 \,, \tag{3.3}$$

are independent r.v. such that

$$\bar{\hat{\varepsilon}}_i \sim_{i.i.d.} \mathcal{N}\left(\frac{\delta\sqrt{\gamma_0}}{n}\sum_{j=1}^{n}b_j\,,\,\frac{\theta^2}{n}\right), \tag{3.4}$$

$$\frac{(n-1)\,\hat{S}_i^2}{\theta^2} \sim_{i.i.d.} \chi^2_{n-1,\nu}\,, \tag{3.5}$$

where $\chi^2_{n-1,\nu}$ denotes the noncentral χ^2-distribution with $n-1$ degrees of freedom and noncentrality parameter equal to

$$\nu = \left(\frac{\delta}{\theta}\right)^2\gamma_0\left(\sum_{j=1}^{n}b_j^2 - n\bar{b}^2\right). \tag{3.6}$$

The control limits of the individual charts for μ and σ^2 that constitute the simultaneous residual scheme do not depend on the underlying in-control observed process — be it i.i.d. or autocorrelated —, coincide with the ones of traditional individual charts for the mean and variance of i.i.d. processes, and are listed in Table 1 for convenience and were previously adopted by [8].

By capitalizing on the distributional properties of $\bar{\hat{\epsilon}}_i$ and $\hat{S}_i^2$ we can conclude that the run lengths of the individual Shewhart-type residual charts for μ and σ^2, $RL_{S-\mu}(\delta,\theta,\mathbf{b})$ and $RL_{S-\sigma}(\delta,\theta,\mathbf{b})$, and the run length of the simultaneous Shewhart residual scheme, $RL_{S-\mu,\sigma}(\delta,\theta,\mathbf{b})$, have geometric distributions with parameters say $\xi_{S-\mu}(\delta,\theta,\mathbf{b})$, $\xi_{S-\sigma}(\delta,\theta,\mathbf{b})$ and $\xi_{S-\mu,\sigma}(\delta,\theta,\mathbf{b})$, where $\xi_{S-\mu,\sigma}(\delta,\theta,\mathbf{b}) = \xi_{S-\mu}(\delta,\theta,\mathbf{b}) + \xi_{S-\sigma}(\delta,\theta,\mathbf{b}) - \xi_{S-\mu}(\delta,\theta,\mathbf{b}) \times \xi_{S-\sigma}(\delta,\theta,\mathbf{b})$ because a simultaneous residual scheme triggers a signal as soon as a signal is observed on either constituent charts. The Markov chain approach ([4]) provides approximations to the distributions of the run lengths $RL_{E-\mu}(\delta,\theta,\mathbf{b})$, $RL_{E-\sigma}(\delta,\theta,\mathbf{b})$ and $RL_{E-\mu,\sigma}(\delta,\theta,\mathbf{b})$. As a consequence we can provide exact expressions (resp. approximate values) for the average run length (ARL) or any other RL related performance measure, such as the PMS of simultaneous Shewhart (resp. EWMA)

Table 1: Control statistics and limits of the individual Shewhart- $(S-\mu, S-\sigma)$ and EWMA-type $(E-\mu, E-\sigma)$ residual charts for μ and σ^2.

Control statistics	**Control limits**
$\bar{\hat{\varepsilon}}_i$	$LCL_{S-\mu} = -\frac{\gamma_{S-\mu}}{\sqrt{n}}$ $UCL_{S-\mu} = -LCL_{S-\mu}$
$\hat{S}_i^2$	$LCL_{S-\sigma} = 0$ $UCL_{S-\sigma} = 1 + \gamma_{S-\sigma}\sqrt{\frac{2}{n-1}}$
$Z_{\bar{\hat{\varepsilon}},i} = \begin{cases} E(\bar{\hat{\varepsilon}}) = 0\,, & i=0\,, \\ (1-\lambda_\mu)\,Z_{\bar{\hat{\varepsilon}},i-1} + \lambda_\mu\,\bar{\hat{\varepsilon}}_i\,, & i=1,\ldots \end{cases}$	$LCL_{E-\mu} = -\gamma_{E-\mu}\sqrt{\frac{\lambda_\mu}{n(2-\lambda_\mu)}}$ $UCL_{E-\mu} = -LCL_{E-\mu}$
$Z_{\hat{S}^2,i} = \begin{cases} E(\hat{S}^2) = 1\,, & i=0\,, \\ (1-\lambda_\sigma)\,Z_{\hat{S}^2,i-1} + \lambda_\sigma\,\hat{S}_i^2\,, & i=1,\ldots \end{cases}$	$LCL_{E-\sigma} = 0$ $UCL_{E-\sigma} = 1 + \gamma_{E-\sigma}\sqrt{\frac{2\,\lambda_\sigma}{(n-1)\,(2-\lambda_\sigma)}}$

residual schemes. In fact, if we focus on the detection of downward and upward shifts in μ and upward shifts in σ^2, then the two PMS can be simply written as

$$\begin{aligned} \text{PMS}_{\text{III}}(\theta,\mathbf{b}) &= P\big[RL_\mu(0,\theta,\mathbf{b}) < RL_\sigma(0,\theta,\mathbf{b})\big] \\ &= \sum_{i=1}^{+\infty} P\big[RL_\mu(0,\theta,\mathbf{b}) = i\big] \times P\big[RL_\sigma(0,\theta,\mathbf{b}) > i\big]\,, \qquad \theta > 1\,, \end{aligned} \tag{3.7}$$

$$\begin{aligned} \text{PMS}_{\text{IV}}(\delta,\mathbf{b}) &= P\big[RL_\sigma(\delta,1,\mathbf{b}) < RL_\mu(\delta,1,\mathbf{b})\big] \\ &= \sum_{i=1}^{+\infty} P\big[RL_\sigma(\delta,1,\mathbf{b}) = i\big] \times P\big[RL_\mu(\delta,1,\mathbf{b}) > i\big]\,, \qquad \delta \neq 0\,, \end{aligned} \tag{3.8}$$

where RL_μ and RL_σ denote the RL of the individual Shewhart or EWMA-type charts for μ and σ^2, respectively. We ought to mention that a relative error of 10^{-6} is considered in the truncation of the series defining $\text{PMS}_{\text{III}}(\theta,\mathbf{b})$ and $\text{PMS}_{\text{IV}}(\delta,\mathbf{b})$, whenever we need to calculate approximate values of these two performance measures. For more details on the exact and approximate distributions of these RL and on the exact and approximate values of the PMS, please refer to [8].

4. THE IMPACT OF FALSELY ASSUMING INDEPENDENCE ON THE PMS

In this section, we shall ignore the autocorrelation structure, assume that the output is i.i.d. within each sample and use traditional individual charts to detect shifts in μ and upward shifts in σ^2. The control limits of these charts coincide with the ones of the individual residual charts (see Table 2). However, the control statistics depend on the sample mean and variance of the standardized output,

$$\bar{X}_i^* = \frac{1}{n}\sum_{j=1}^{n}\frac{X_{i,j}-\mu_0}{\sqrt{\gamma_0}}~, \tag{4.1}$$

$$(S_i^*)^2 = \frac{1}{n-1}\sum_{j=1}^{n}\frac{(X_{i,j}-\bar{X}_i)^2}{\gamma_0}~, \tag{4.2}$$

not on the sample mean and variance of the standardized residuals. Suffice to say that $\bar{X}_i^*$ and $(S_i^*)^2$ are the control statistics of the traditional Shewhart-type charts for μ and σ^2 ($S^*-\mu$ and $S^*-\sigma$). As for the traditional EWMA-type charts ($E^*-\mu$ and $E^*-\sigma$), they make use of the statistics

$$Z_{\bar{X}^*,i} = \begin{cases} 0\,, & i=0~, \\ (1-\lambda_\mu)\,Z_{\bar{X}^*,i-1}+\lambda_\mu\bar{X}_i^*\,, & i=1,\ldots~, \end{cases} \tag{4.3}$$

$$Z_{(S^*)^2,i} = \begin{cases} 1\,, & i=0~, \\ (1-\lambda_\sigma)\,Z_{(S^*)^2,i-1}+\lambda_\sigma\,(S_i^*)^2\,, & i=1,\ldots~. \end{cases} \tag{4.4}$$

Should the output be i.i.d. or simultaneous residual schemes for the mean and variance of autocorrelated output are at use, we would be able to provide exact expressions (resp. approximations) for the PMS in the Shewhart (resp. EWMA) case, as seen in the previous section. Be that as it may, in the presence of autocorrelation, the statistics $\bar{X}_i^*$ and $(S_i^*)^2$ are no longer independent r.v., and therefore we have to rely on Monte Carlo simulation to obtain estimates of the PMS, when the output process within each sample, follows an AR(1), AR(2) or an ARMA(1,1) model.

For illustration purposes, we considered the target process, $(Y_{i,1},...,Y_{i,n})$ for each i ($i=1,...,rep$), drawn from a Gaussian stationary process with zero mean ($\mu_0=0$) and unit variance ($\gamma_0=1$), where the number of replications is equal to $rep=10^6$ for each set of parameter values. Furthermore, we simulated samples of size $n=5$ of this in-control process, obtained the out-of-control process and the observed values of the control statistics, compared the latter with the control limits and counted the number of misleading signals and the number of signals triggered by the simultaneous schemes and estimated the corresponding PMS. In addition to that, we have taken: $\lambda_\mu=\lambda_\sigma=\lambda=1, 0.05$ (allowing the comparison between Shewhart- and EWMA-type schemes); $\theta=1.02, 1.10, 1.20$ (PMS of Type III); $\delta=0.05, 0.50, 1.00$ (PMS of Type IV). Moreover, the critical values $\gamma_{S-\mu}$, $\gamma_{S-\sigma}$, $\gamma_{E-\mu}$ and $\gamma_{E-\sigma}$ were calculated in such way that the in-control average run length (ARL) of both the individual traditional charts for μ and σ are approximately the same, i.e. $ARL_\mu(0,1,\mathbf{b})=ARL_\sigma(0,1,\mathbf{b})$, and the ARL of the simultaneous scheme is approximately equal to 500 samples, that is $ARL_{\mu,\sigma}(0,1,\mathbf{b})=500$; the resulting critical values and the corresponding in-control ARL are summarized in Table 2 and coincide with the ones in [8]. Please bear in mind that, when dealing with Markov approximations, we considered 101 transient states to determine these critical values and all the RL related measures.

Table 2: Critical values for the individual Shewhart ($\lambda = 1$) and EWMA charts.

γ_μ	γ_σ	$ARL_\mu(0,1,\mathbf{b})$	$ARL_\mu(0,1,\mathbf{b})$	$ARL_{\mu,\sigma}(0,1,\mathbf{b})$	λ
3.2904	5.1144	999.550	999.495	500.011	1
2.8817	2.9103	986.202	986.162	499.641	0.05

4.1. AR(1) model

The AR(1) model is usually reported as the most frequently encountered in practice ([23, p. 10]). The process $\{Y_{i,j}\}$ follows a stationary Gaussian AR(1) model with mean μ_0, variance $\gamma_0 = \sigma_0^2$ and autoregressive parameter ϕ, for each i, if

$$Y_{i,j} = \mu_0 + \phi(Y_{i,j-1} - \mu_0) + \varepsilon_{i,j} \;, \tag{4.5}$$

where: ϕ is a constant satisfying $-1 < \phi < 1$; and $\{\varepsilon_{i,j}\}$ is a sequence of disturbances such that $\varepsilon_{i,j} \sim_{i.i.d.} \mathcal{N}(0, \sigma_\varepsilon^2)$, with $\sigma_\varepsilon^2 = (1-\phi^2) \times \sigma_0^2$.

If we use simultaneous Shewhart- and EWMA-type residual schemes then we can provide exact and approximate values of PMS of Type III (resp. IV); these results can be found in Table 3 (resp. in the center of Table 4). As previously noted by [8] and illustrated by Table 3, the PMS of Type III does not depend on ϕ. In fact, a close inspection of the noncentrality parameter ν, the probabilities $\xi_{S-\mu}(\delta,\theta,\mathbf{b})$, $\xi_{S-\sigma}(\delta,\theta,\mathbf{b})$, etc. leads to the conclusion that these parameters do not depend on $\mathbf{b}$ — when $\delta = 0$ —, thus $\text{PMS}_{\text{III}}(\theta,\mathbf{b}) := \text{PMS}_{\text{III}}(\theta)$ for any Gaussian stationary model. Table 3 (resp. 4) also shows that $\text{PMS}_{\text{III}}(\theta)$ (resp. $PMS_{\text{IV}}(\delta,\phi)$) can be larger than 0.47 (resp. 0.49), for very small shifts in σ^2 (resp. μ), while at the same time reinforcing that the simultaneous Shewhart residual scheme seems to have larger PMS of Type III (resp. Type IV) than its EWMA analog. It should also be noted that $\text{PMS}_{\text{IV}}(\delta,\phi)$ appears to increase with $\phi \in (0,1)$, as already referred by [8].

Now, we investigate what happens to both PMS if the autocorrelation structure is not recognized or ignored and traditional control charts are used when $\phi \in (-1,1)$. A reasonably large set of estimates of the PMS of types III and IV when autocorrelation is disregarded can be found in Table 4, along with values of $\text{PMS}_{\text{IV}}(\delta,\phi)$ when adequate simultaneous Shewhart and EWMA residual schemes were used instead of the traditional ones. Even though the values in Table 4 refer to $\pm\phi = 0, 0.3, 0.5, 0.7, 0.9, 0.95$, figures 1 through 4 were drawn considering $\pm\phi = 0(0.05)0.95(0.01)0.99$; these estimates will be made available to those who are interested and request them from the authors. As in [8], we obtained estimates of PMS of types III and IV that are close to the corresponding values of PMS when simultaneous residual schemes are at use, for $\phi = 0$, as illustrated by Table 4 and by the grey and black lines intersecting at $\phi = 0$ in figures 1–4.

Table 3: PMS of Type III of simultaneous Shewhart ($\lambda = 1$) and EWMA residual schemes.

θ	$PMS_{III}(\theta)$	λ
1.02	0.475786	1
	0.343880	0.05
1.10	0.397714	1
	0.100265	0.05
1.20	0.331373	1
	0.042865	0.05

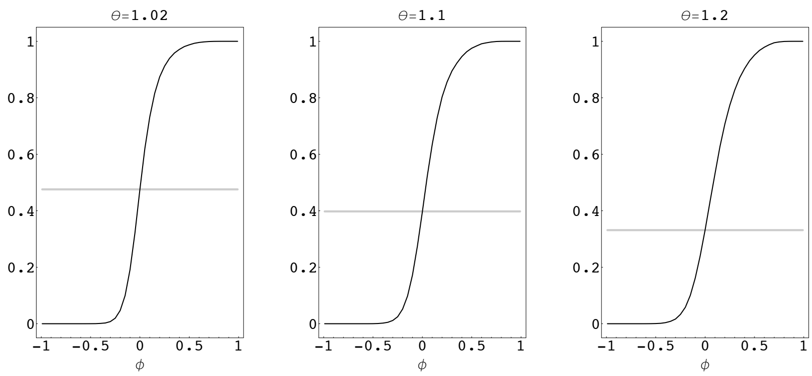

Figure 1: AR(1) model, Shewhart — $PMS_{III}(\theta)$ (simultaneous residual scheme, grey line) and estimates of PMS of Type III (traditional simultaneous scheme, black line).

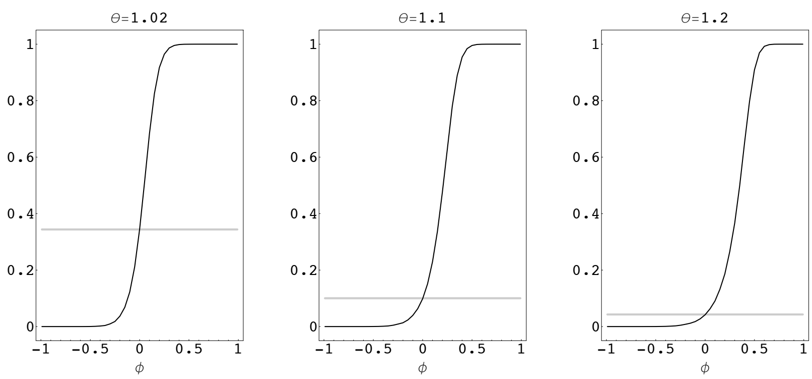

Figure 2: AR(1) model, EWMA — $PMS_{III}(\theta)$ (simultaneous residual scheme, grey line) and estimates of PMS of Type III (traditional simultaneous scheme, black line).

Table 4: AR(1) model — estimates of PMS of Type III of traditional simultaneous scheme; $\mathrm{PMS_{IV}}(\delta,\phi)$ of simultaneous residual scheme; estimates of PMS of Type IV of traditional simultaneous scheme.

		$\phi \in (-1,1)$											
		−0.95	−0.90	−0.70	−0.50	−0.30	0	0.30	0.50	0.70	0.90	0.95	λ
θ	1.02	0.000000	0.000000	0.000000	0.000110	0.007510	0.475890	0.939740	0.987590	0.999240	1.000000	1.000000	1
		0.000000	0.000000	0.000000	0.000250	0.009400	0.344270	0.986560	0.999830	1.000000	1.000000	1.000000	0.05
	1.10	0.000000	0.000000	0.000000	0.000170	0.011470	0.396250	0.895130	0.975500	0.997660	1.000000	1.000000	1
		0.000000	0.000000	0.000010	0.000230	0.004680	0.098620	0.780360	0.995280	0.999970	1.000000	1.000000	0.05
	1.20	0.000000	0.000000	0.000000	0.000580	0.016570	0.332010	0.827120	0.951150	0.994980	1.000000	1.000000	1
		0.000000	0.000000	0.000000	0.000140	0.002480	0.041700	0.366390	0.909000	0.999800	1.000000	1.000000	0.05
δ	0.05	0.245605	0.331756	0.429811	0.457404	0.470828	0.481977	0.488703	0.492034	0.494900	0.497702	0.498534	1
		0.021403	0.040098	0.109687	0.169903	0.221933	0.288653	0.346313	0.381892	0.417014	0.455573	0.467676	0.05
	0.50	0.000411	0.003322	0.009211	0.017601	0.030661	0.061967	0.113516	0.165244	0.240161	0.363509	0.414297	1
		0.004417	0.002008	0.000750	0.000930	0.001422	0.002907	0.006195	0.010799	0.020925	0.055933	0.085086	0.05
	1.00	0.000000	0.000000	0.000620	0.001398	0.002359	0.005825	0.016431	0.035020	0.081256	0.229454	0.323318	1
		0.002409	0.012133	0.000422	0.000158	0.000141	0.000262	0.000794	0.002010	0.006279	0.032854	0.064217	0.05
δ	0.05	1.000000	1.000000	1.000000	0.999940	0.992360	0.483350	0.050420	0.009680	0.000500	0.000000	0.000000	1
		1.000000	1.000000	0.999770	0.992400	0.919140	0.288740	0.003400	0.000050	0.000010	0.000000	0.000000	0.05
	0.50	0.999960	0.999980	0.998590	0.953870	0.577890	0.061520	0.011340	0.003560	0.000350	0.000000	0.000000	1
		0.180890	0.158220	0.090350	0.043940	0.017100	0.003350	0.000370	0.000010	0.000000	0.000000	0.000000	0.05
	1.00	0.788930	0.745620	0.419080	0.137540	0.036130	0.005740	0.002070	0.000840	0.000120	0.000000	0.000000	1
		0.061330	0.049940	0.021770	0.007360	0.002170	0.000310	0.000040	0.000010	0.000000	0.000000	0.000000	0.05

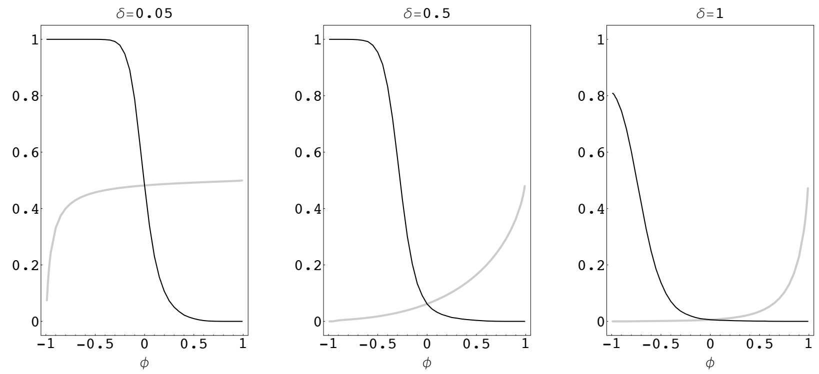

Figure 3: AR(1) model, Shewhart — $\mathrm{PMS_{IV}}(\delta,\phi)$ (simultaneous residual scheme, grey line) and estimates of PMS of Type IV (traditional simultaneous scheme, black line).

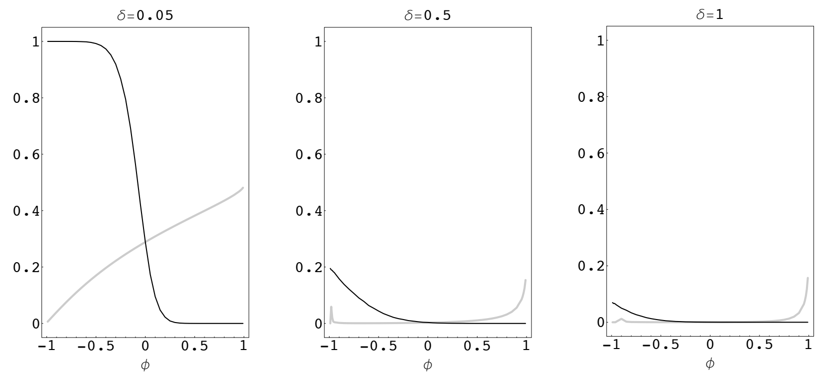

Figure 4: AR(1) model, EWMA — $\mathrm{PMS_{IV}}(\delta,\phi)$ (simultaneous residual scheme, grey line) and estimates of PMS of Type IV (traditional simultaneous scheme, black line).

When we neglect the autocorrelation structure, the estimates of the PMS of Type III increase from 0 to 1 with ϕ, even though $\text{PMS}_{\text{III}}(\theta)$ does not exceed 0.5 or depend on ϕ when simultaneous residual schemes are at use, as figures 1 and 2 portray quite vividly. Besides that, it is apparent from figures 3 and 4 that the estimates of PMS of Type IV seem to decrease with ϕ, whereas for simultaneous residual schemes $\text{PMS}_{\text{IV}}(\delta,\phi)$ tends to increase with ϕ (see Table 4). Additionally, the PMS of types III and IV are very sensitive to autocorrelation, for instance, we got for the simultaneous EWMA scheme:

- $\text{PMS}_{\text{III}}(1.02) = 0.343880$, still the corresponding estimates are 0.000110 and 0.987590, for $\phi = -0.5$ and $\phi = 0.5$;
- $\text{PMS}_{\text{IV}}(0.05,-0.5) = 0.169903$ and $\text{PMS}_{\text{IV}}(0.05,0.5) = 0.381892$, while the estimated values are 0.992400 and 0.000050.

It should be also added that the values in tables 3 and 4 and the graphs in figures 1–4 suggest that replacing the traditional Shewhart with traditional EWMA charts only offers improvement with regard to MS of Type IV (for all values of ϕ), even though both $\text{PMS}_{\text{III}}(\theta)$ and $\text{PMS}_{\text{IV}}(\delta,\phi)$ seem to decrease when a simultaneous EWMA residual scheme takes the place of a simultaneous Shewhart residual scheme.

4.2. AR(2) model

The AR(2) process was originally used by G.U. Yule in 1927 to describe the behavior of a simple pendulum and since then it has been widely used to describe a variety of phenomena, namely occurring in engineering and other related fields ([22]) such as industry. Let us recall that the process $\{Y_{i,j}\}$ follows a stationary AR(2) model with mean μ_0, variance $\gamma_0 = \sigma_0^2$ and parameters ϕ_1 and ϕ_2, for each i, if

$$Y_{i,j} = \mu_0 + \phi_1(Y_{i,j-1} - \mu_0) + \phi_2(Y_{j,i-2} - \mu_0) + \varepsilon_{i,j} \, , \tag{4.6}$$

where: the parameters ϕ_1 and ϕ_2 lie in a triangular region restricted by $-1<\phi_2<1$, $\phi_1+\phi_2 < 1$ and $\phi_2-\phi_1 < 1$; and the innovations satisfy $\varepsilon_{i,j} \sim_{i.i.d.} \mathcal{N}(0,\sigma_\varepsilon^2)$, with $\sigma_\varepsilon^2 = \frac{(1+\phi_2)\,[(1-\phi_2)^2-\phi_1^2]}{1-\phi_2} \times \sigma_0^2$.

The investigations on the impact of falsely assuming i.i.d. output — instead of an AR(2) model — in the PMS of types III and IV led to some interesting results.

Firstly, note that the graphs in Figure 5 (resp. 6) were restricted to the EWMA scheme and to $\theta = 1.02$ (resp. $\delta = 0.05$) because similar ones were obtained for the Shewhart scheme or most of the other values of θ (resp. δ) and

ϕ_1 and ϕ_2; however, tables 5 and 6 provide results for a wider constellation of parameters. Moreover, since the family of AR(2) processes includes the i.i.d. process and the sub-family of AR(1) processes: when $\phi_1 = \phi_2 = 0$, the estimates of the PMS of Type III (resp. Type IV) in tables 5 and 6 are close to the values of $\text{PMS}_{\text{III}}(\theta)$ (resp. the corresponding values of $\text{PMS}_{\text{IV}}(\delta, \phi_1, \phi_2)$) in Table 3 (resp. tables 5 and 6); when $\phi_2 = 0$, the values of $\text{PMS}_{\text{IV}}(\delta, \phi_1, \phi_2)$ in Table 5 obviously coincide with the ones of $\text{PMS}_{\text{IV}}(\delta, \phi)$; finally, when $\phi_2 = 0$, the estimated results of the PMS of types III and IV in Table 5 are comparable to the ones we obtained for the AR(1) model in Table 4.

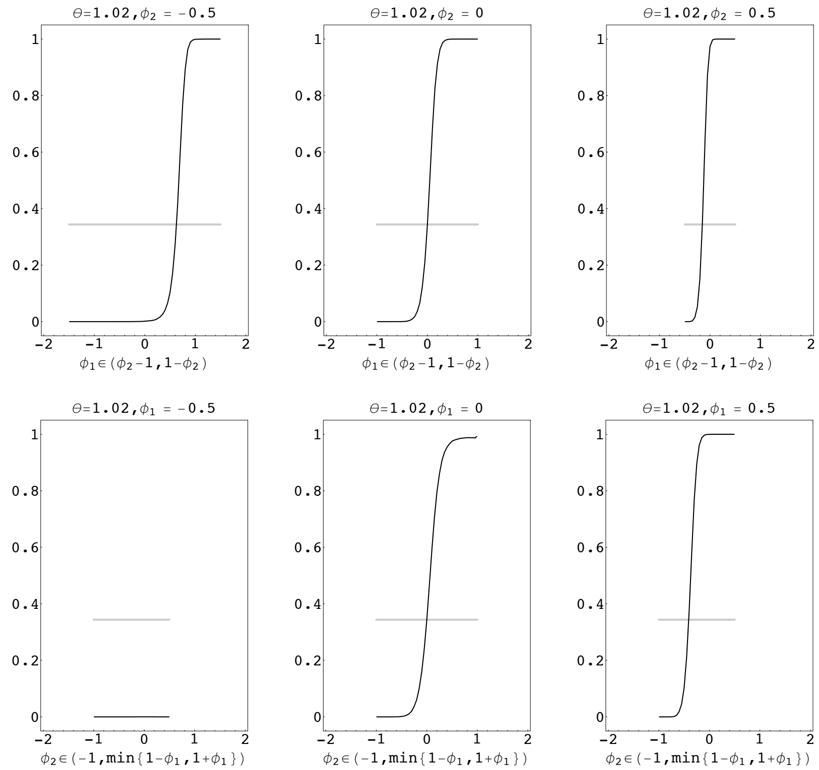

Figure 5: AR(2) model, EWMA — $\text{PMS}_{\text{III}}(\theta)$ (simultaneous residual scheme, grey line) and estimates of PMS of Type III (traditional simultaneous scheme, black line), for $\phi_1 \in (\phi_2 - 1, 1 - \phi_2)$ [top] and $\phi_2 \in (-1, \min\{1 - \phi_1, 1 + \phi_1\})$ [bottom].

Secondly, when ϕ_2 takes a fixed value in $(-1, 1)$ such as $\phi_2 = -0.5, 0, 0.5$, the estimates of the PMS of Type III (resp. Type IV) increase (resp. decrease) with $\phi_1 \in (\phi_2 - 1, 1 - \phi_2)$ instead of being constant (resp. increasing), as shown by Figure 5 (resp. 6); analogously, when $\phi_1 = -0.5, 0, 0.5$, the estimates of the PMS of Type III (resp. Type IV) also increase (resp. tend to decrease) with $\phi_2 \in (-1, \min\{1 - \phi_1, 1 + \phi_1\})$ when they should not vary (resp. should increase). Curiously enough when $\phi_2 = -0.5$ (resp. $\phi_1 = -0.5$) and $\phi_1 \in (\phi_2 - 1, 0]$ (resp. $\phi_2 \in (-1, \min\{1 - \phi_1, 1 + \phi_1\})$) the estimates of the PMS of Type III are all very close to zero, as Figure 5 and Table 5 suggest, i.e., the individual EWMA chart for the process variance tends to signal earlier than the one for the process mean most of the time, when there is a small upward shift in σ^2. A comparable result was obtained for the estimates of the PMS of Type IV: when $\phi_2 = -0.5$ (resp. $\phi_1 = -0.5$) and $\phi_1 \in (\phi_2 - 1, 0]$ (resp. $\phi_2 \in (-1, \min\{1-\phi_1, 1+\phi_1\})$), these estimates are very close to 1 (see Figure 6 or tables 5 and 6), certainly because the individual EWMA chart for σ^2 tends to trigger alarms sooner than the one for μ most of the time, when there is a small shift in the process mean.

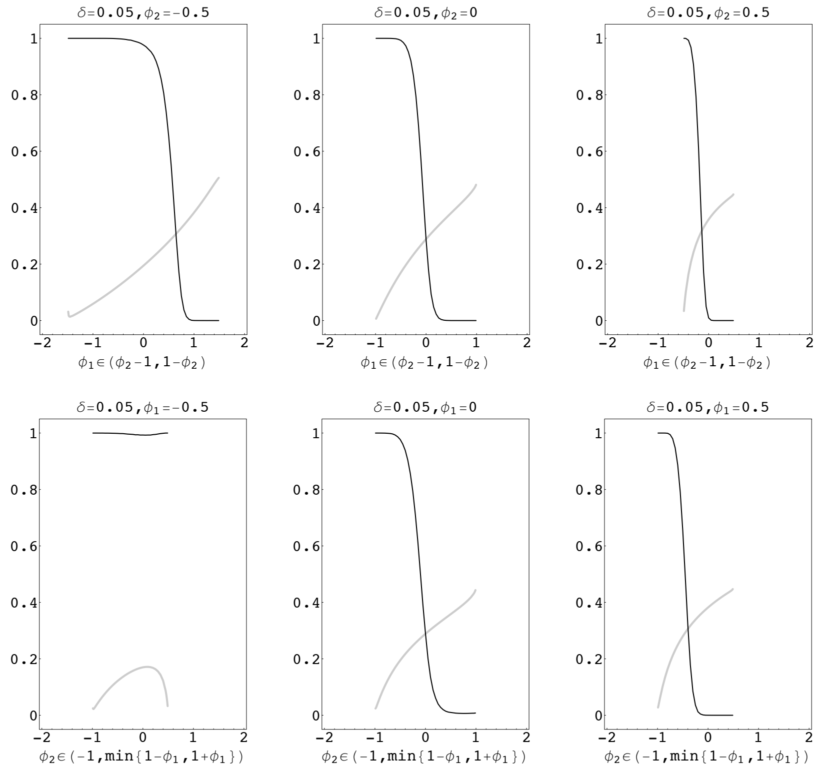

Figure 6: AR(2) model, EWMA — $\text{PMS}_{\text{IV}}(\delta, \phi_1, \phi_2)$ (simultaneous residual scheme, grey line) and estimates of PMS of Type IV (traditional simultaneous scheme, black line), for $\phi_1 \in (\phi_2 - 1, 1 - \phi_2)$ [top] and $\phi_2 \in (-1, \min\{1 - \phi_1, 1 + \phi_1\})$ [bottom].

Thirdly, we ought to refer that the discrepancies between the estimates of both PMS and their corresponding values are all too apparent not only in figures 5 and 6, but also in tables 5 and 6. In fact, if we (un)conscientiously disregard the autocorrelation structure of the output and adopt traditional simultaneous schemes for the process mean and variance instead of simultaneous residual schemes, we are bound to overestimate or underestimate the PMS depending on the values of the parameters ϕ_1 and ϕ_2. For example, for the simultaneous Shewhart residual scheme, we got:

Table 5: AR(2) model, $\phi_2 = -0.5, 0, 0.5$ — estimates of PMS of Type III of traditional simultaneous scheme; $\text{PMS}_{\text{IV}}(\delta, \phi_1, \phi_2)$ of simultaneous residual scheme; estimates of PMS of Type IV of traditional simultaneous scheme.

		$\phi_2 = -0.5,\ \phi_1 \in (\phi_2 - 1, 1 - \phi_2)$											
		−1.45	−1.40	−0.90	−0.50	−0.30	0	0.30	0.50	0.90	1.40	1.45	λ
θ	1.02	0.000000	0.000000	0.000000	0.000000	0.000010	0.000720	0.011150	0.071770	0.842410	1.000000	1.000000	1
		0.000000	0.000000	0.000000	0.000020	0.000080	0.001620	0.016380	0.104460	0.988900	1.000000	1.000000	0.05
	1.10	0.000000	0.000000	0.000000	0.000010	0.000110	0.001220	0.015700	0.085110	0.793730	0.999990	1.000000	1
		0.000000	0.000000	0.000000	0.000010	0.000030	0.001010	0.008140	0.044860	0.884560	1.000000	1.000000	0.05
	1.20	0.000000	0.000000	0.000000	0.000020	0.000260	0.002620	0.024310	0.098430	0.738910	0.999950	1.000000	1
		0.000000	0.000000	0.000000	0.000010	0.000050	0.000540	0.004940	0.024110	0.564330	1.000000	1.000000	0.05
δ	0.05	0.138986	0.197873	0.393962	0.437392	0.450454	0.464818	0.475377	0.481078	0.490475	0.500429	0.501417	1
		0.013758	0.017160	0.070914	0.121760	0.149613	0.194614	0.243968	0.279727	0.360091	0.483677	0.496673	0.05
	0.50	0.000000	0.000001	0.007995	0.012968	0.017160	0.027299	0.045670	0.066361	0.154214	0.556766	0.630473	1
		0.131107	0.084971	0.001655	0.001145	0.001214	0.001594	0.002504	0.003731	0.011401	0.271185	0.510216	0.05
	1.00	0.000000	0.000000	0.000139	0.001325	0.001960	0.003169	0.005704	0.009563	0.041267	0.666765	0.811956	1
		0.000000	0.000000	0.002232	0.000810	0.000516	0.000404	0.000503	0.000768	0.004035	0.450430	0.775576	0.05
δ	0.05	1.000000	1.000000	1.000000	1.000000	0.999950	0.999360	0.988830	0.922900	0.139220	0.000000	0.000000	1
		1.000000	1.000000	0.999980	0.998540	0.994410	0.976350	0.891690	0.650750	0.004600	0.000000	0.000000	0.05
	0.50	1.000000	1.000000	0.999990	0.990560	0.959620	0.848090	0.600330	0.349410	0.033800	0.000000	0.000000	1
		0.871920	0.675190	0.034890	0.036680	0.031800	0.028900	0.023530	0.017040	0.000930	0.000000	0.000000	0.05
	1.00	0.943180	0.952440	0.518420	0.159430	0.104840	0.072870	0.051150	0.035160	0.006500	0.000000	0.000000	1
		0.418460	0.223430	0.005860	0.004660	0.004220	0.004350	0.003760	0.003240	0.000320	0.000000	0.000000	0.05

		$\phi_2 = 0,\ \phi_1 \in (\phi_2 - 1, 1 - \phi_2)$											
		−0.95	−0.90	−0.70	−0.50	−0.30	0	0.30	0.50	0.70	0.90	0.95	λ
θ	1.02	0.000000	0.000000	0.000000	0.000030	0.007440	0.475730	0.937650	0.987950	0.999320	1.000000	1.000000	1
		0.000000	0.000000	0.000010	0.000240	0.009180	0.344010	0.986890	0.999840	1.000000	1.000000	1.000000	0.05
	1.10	0.000000	0.000000	0.000000	0.000190	0.011080	0.398280	0.894630	0.976030	0.997670	1.000000	1.000000	1
		0.000000	0.000000	0.000000	0.000190	0.004460	0.099910	0.777120	0.995430	0.999990	1.000000	1.000000	0.05
	1.20	0.000000	0.000000	0.000000	0.000680	0.017310	0.330990	0.827600	0.951130	0.993990	1.000000	1.000000	1
		0.000000	0.000000	0.000010	0.000110	0.002510	0.043050	0.367860	0.907220	0.999770	1.000000	1.000000	0.05
δ	0.05	0.245605	0.331756	0.429811	0.457404	0.470828	0.481977	0.488703	0.492034	0.494900	0.497702	0.498534	1
		0.021403	0.040098	0.109687	0.169903	0.221933	0.288653	0.346313	0.381892	0.417014	0.455573	0.467676	0.05
	0.50	0.000411	0.003322	0.009211	0.017601	0.030661	0.061967	0.113516	0.165244	0.240161	0.363509	0.414297	1
		0.004417	0.002008	0.000750	0.000930	0.001422	0.002907	0.006195	0.010799	0.020925	0.055933	0.085086	0.05
	1.00	0.000000	0.000000	0.000620	0.001398	0.002359	0.005824	0.016431	0.035020	0.081256	0.229454	0.323318	1
		0.002409	0.012133	0.000422	0.000158	0.000141	0.000262	0.000794	0.002010	0.006279	0.032854	0.064217	0.05
δ	0.05	1.000000	1.000000	1.000000	0.999930	0.992470	0.482494	0.048980	0.009800	0.000680	0.000000	0.000000	1
		1.000000	1.000000	0.999830	0.992790	0.920050	0.287590	0.003720	0.000020	0.000000	0.000000	0.000000	0.05
	0.50	0.999980	0.999960	0.998730	0.955350	0.580780	0.059860	0.011580	0.003230	0.000270	0.000000	0.000000	1
		0.179330	0.160950	0.090900	0.043180	0.017730	0.003100	0.000300	0.000020	0.000000	0.000000	0.000000	0.05
	1.00	0.789070	0.744350	0.413940	0.134390	0.036730	0.005790	0.002280	0.000990	0.000050	0.000000	0.000000	1
		0.061170	0.050720	0.021960	0.006970	0.002150	0.000270	0.000020	0.000010	0.000000	0.000000	0.000000	0.05

		$\phi_2 = 0.5,\ \phi_1 \in (\phi_2 - 1, 1 - \phi_2)$											
		−0.45	−0.40	−0.30	−0.20	−0.10	0	0.10	0.20	0.30	0.40	0.45	λ
θ	1.02	0.000000	0.000010	0.002740	0.077290	0.454160	0.866230	0.984990	0.999460	1.000000	1.000000	1.000000	1
		0.000000	0.000140	0.014690	0.150070	0.632900	0.973230	0.999810	1.000000	1.000000	1.000000	1.000000	0.05
	1.10	0.000000	0.000020	0.006050	0.096260	0.438920	0.819820	0.971280	0.997940	0.999980	1.000000	1.000000	1
		0.000000	0.000140	0.009360	0.071180	0.297150	0.774640	0.992040	0.999980	1.000000	1.000000	1.000000	0.05
	1.20	0.000000	0.000030	0.011040	0.121060	0.422070	0.761270	0.942220	0.993600	0.999860	1.000000	1.000000	1
		0.000000	0.000190	0.006330	0.039050	0.143380	0.419480	0.873390	0.999000	1.000000	1.000000	1.000000	0.05
δ	0.05	0.425908	0.455393	0.474771	0.482647	0.487085	0.490010	0.492133	0.493786	0.495153	0.496379	0.497022	1
		0.103984	0.163775	0.241873	0.292843	0.329739	0.358237	0.381300	0.400664	0.417437	0.432431	0.439677	0.05
	0.50	0.008753	0.016719	0.039676	0.068293	0.100647	0.135836	0.1735485	0.214001	0.258226	0.309794	0.343080	1
		0.000772	0.000920	0.001907	0.003438	0.005566	0.008450	0.012366	0.017820	0.025858	0.039377	0.051647	0.05
	1.00	0.000549	0.001379	0.003606	0.007808	0.014985	0.026349	0.043585	0.069296	0.108180	0.171820	0.225618	1
		0.000500	0.000181	0.000221	0.000428	0.000858	0.001675	0.003188	0.006037	0.011775	0.025701	0.043189	0.05
δ	0.05	1.000000	0.999990	0.996360	0.919740	0.528740	0.116840	0.011320	0.000460	0.000000	0.000000	0.000000	1
		0.999520	0.993190	0.907280	0.612750	0.171630	0.009410	0.000100	0.000000	0.000000	0.000000	0.000000	0.05
	0.50	0.999360	0.989080	0.814290	0.413890	0.135410	0.030830	0.003830	0.000210	0.000000	0.000000	0.000000	1
		0.073170	0.069410	0.050980	0.025950	0.008120	0.001250	0.000100	0.000000	0.000000	0.000000	0.000000	0.05
	1.00	0.494000	0.286850	0.121670	0.054140	0.021170	0.005860	0.001270	0.000100	0.000000	0.000000	0.000000	1
		0.025000	0.021770	0.014870	0.006890	0.002250	0.000370	0.000010	0.000000	0.000000	0.000000	0.000000	0.05

Table 6: AR(2) model, $\phi_1 = -0.5, 0, 0.5$ — estimates of PMS of Type III of traditional simultaneous scheme; $\text{PMS}_{\text{IV}}(\delta, \phi_1, \phi_2)$ of simultaneous residual scheme; estimates of PMS of Type IV of traditional simultaneous scheme.

		$\phi_1 = -0.5,\ \phi_2 \in (-1, \min\{1-\phi_1, 1+\phi_1\})$											
		−0.95	−0.90	−0.70	−0.50	−0.30	0	0.10	0.20	0.30	0.40	0.45	λ
θ	1.02	0.000000	0.000000	0.000000	0.000000	0.000010	0.000100	0.000040	0.000030	0.000000	0.000000	0.000000	1
		0.000000	0.000000	0.000000	0.000000	0.000040	0.000280	0.000330	0.000230	0.000050	0.000000	0.000000	0.05
	1.10	0.000000	0.000000	0.000000	0.000000	0.000020	0.000170	0.000160	0.000040	0.000010	0.000000	0.000000	1
		0.000000	0.000000	0.000000	0.000010	0.000040	0.000180	0.000290	0.000190	0.000080	0.000020	0.000000	0.05
	1.20	0.000000	0.000000	0.000000	0.000020	0.000140	0.000530	0.000590	0.000300	0.000040	0.000000	0.000000	1
		0.000000	0.000000	0.000000	0.000000	0.000040	0.000130	0.000190	0.000120	0.000080	0.000000	0.000000	0.05
δ	0.05	0.265973	0.331442	0.412628	0.437392	0.449385	0.457404	0.457915	0.456777	0.452369	0.437529	0.412047	1
		0.028191	0.041403	0.087209	0.121760	0.147475	0.169903	0.171484	0.167913	0.155284	0.122515	0.087187	0.05
	0.50	0.001453	0.012770	0.012890	0.012968	0.014656	0.017601	0.017971	0.017452	0.015354	0.010649	0.007313	1
		0.087440	0.032186	0.002498	0.001145	0.000918	0.000930	0.000947	0.000942	0.000887	0.000765	0.000832	0.05
	1.00	0.000000	0.000000	0.000784	0.001325	0.001320	0.001398	0.001437	0.001430	0.001289	0.000815	0.000293	1
		0.002662	0.238867	0.004256	0.000810	0.000272	0.000158	0.000160	0.000173	0.000204	0.000327	0.000685	0.05
δ	0.05	1.000000	1.000000	1.000000	1.000000	0.999960	0.999960	0.999990	1.000000	1.000000	1.000000	1.000000	1
		0.999970	0.999980	0.999700	0.998780	0.996370	0.993010	0.992950	0.994340	0.996680	0.999220	0.999960	0.05
	0.50	0.999850	0.999730	0.998040	0.990070	0.969240	0.954960	0.962810	0.976340	0.988620	0.998010	0.999780	1
		0.064170	0.061210	0.045270	0.035980	0.030520	0.045120	0.055290	0.071140	0.085320	0.090880	0.087500	0.05
	1.00	0.522970	0.457320	0.270600	0.161080	0.108920	0.137290	0.170260	0.219500	0.293710	0.422840	0.568690	1
		0.013980	0.013680	0.008090	0.004810	0.003540	0.007430	0.011440	0.017860	0.024520	0.029830	0.028010	0.05

		$\phi_1 = 0,\ \phi_2 \in (-1, \min\{1-\phi_1, 1+\phi_1\})$											
		−0.95	−0.90	−0.70	−0.50	−0.30	0	0.30	0.50	0.70	0.90	0.95	λ
θ	1.02	0.000000	0.000000	0.000010	0.000580	0.021740	0.475280	0.829100	0.867240	0.874730	0.874670	0.875560	1
		0.000000	0.000000	0.000050	0.001520	0.020380	0.344220	0.907420	0.975770	0.986270	0.987750	0.987190	0.05
	1.10	0.000000	0.000000	0.000010	0.001230	0.028860	0.398150	0.762310	0.820770	0.837850	0.844030	0.842710	1
		0.000000	0.000000	0.000050	0.001080	0.009290	0.099020	0.484720	0.777010	0.902690	0.940200	0.945050	0.05
	1.20	0.000000	0.000000	0.000050	0.002830	0.034870	0.330540	0.679990	0.763560	0.793910	0.803910	0.804720	1
		0.000000	0.000000	0.000020	0.000550	0.004910	0.041310	0.197860	0.415300	0.653040	0.794970	0.818310	0.05
δ	0.05	0.325587	0.384272	0.446882	0.464818	0.473945	0.481977	0.487213	0.49001	0.492604	0.495445	0.496394	1
		0.040491	0.064109	0.139835	0.194614	0.237379	0.288653	0.331564	0.358237	0.385490	0.418047	0.429451	0.05
	0.50	0.015391	0.022880	0.021202	0.027299	0.037796	0.061967	0.099469	0.135836	0.188234	0.278600	0.319136	1
		0.072186	0.016166	0.002189	0.001594	0.001818	0.002907	0.005309	0.008450	0.014691	0.032342	0.044485	0.05
	1.00	0.000000	0.000113	0.003602	0.003169	0.003391	0.005824	0.013624	0.026349	0.055378	0.141787	0.197402	1
		0.326726	0.062125	0.002158	0.000404	0.000212	0.000262	0.000693	0.001675	0.004892	0.021400	0.037551	0.05
δ	0.05	1.000000	1.000000	0.999990	0.999160	0.975660	0.483339	0.147560	0.116520	0.108380	0.108150	0.108120	1
		1.000000	0.999960	0.998380	0.977150	0.856280	0.290070	0.027930	0.009950	0.006680	0.007060	0.007340	0.05
	0.50	0.999900	0.999640	0.987880	0.847910	0.412620	0.061730	0.027550	0.030070	0.036960	0.045330	0.047820	1
		0.097810	0.089420	0.053190	0.030550	0.012830	0.002850	0.001220	0.001230	0.001760	0.003140	0.003420	0.05
	1.00	0.554140	0.467040	0.197720	0.073030	0.024560	0.005800	0.004050	0.006100	0.008850	0.012780	0.013620	1
		0.022550	0.018790	0.009510	0.004170	0.001590	0.000290	0.000220	0.000340	0.000730	0.001490	0.001790	0.05

		$\phi_1 = 0.5,\ \phi_2 \in (-1, \min\{1-\phi_1, 1+\phi_1\})$											
		−0.95	−0.90	−0.70	−0.50	−0.30	0	0.10	0.20	0.30	0.40	0.45	λ
θ	1.02	0.000000	0.000000	0.000270	0.073120	0.633280	0.988150	0.997440	0.999790	0.999980	1.000000	1.000000	1
		0.000000	0.000000	0.001210	0.103060	0.768940	0.999810	1.000000	1.000000	1.000000	1.000000	1.000000	0.05
	1.10	0.000000	0.000000	0.000760	0.086300	0.576680	0.975240	0.994340	0.999220	0.999970	1.000000	1.000000	1
		0.000000	0.000000	0.000990	0.043900	0.359030	0.994990	0.999880	1.000000	1.000000	1.000000	1.000000	0.05
	1.20	0.000000	0.000000	0.001740	0.097770	0.515500	0.951820	0.985120	0.997400	0.999900	1.000000	1.000000	1
		0.000000	0.000000	0.000700	0.023350	0.158150	0.907670	1.000000	1.000000	1.000000	1.000000	1.000000	0.05
δ	0.05	0.379857	0.426024	0.469467	0.481078	0.486905	0.492034	0.493287	0.494409	0.495447	0.496467	0.497046	1
		0.061691	0.101246	0.211369	0.279727	0.328298	0.381892	0.396572	0.410115	0.422683	0.434450	0.440385	0.05
	0.50	0.035274	0.034565	0.043821	0.066361	0.098203	0.165244	0.193828	0.226430	0.264326	0.311321	0.343262	1
		0.041673	0.011949	0.003422	0.003731	0.005337	0.010799	0.014136	0.018945	0.026321	0.039280	0.051442	0.05
	1.00	0.000045	0.005096	0.009032	0.009563	0.014028	0.035020	0.049987	0.072778	0.108633	0.170302	0.224296	1
		0.209786	0.038736	0.002032	0.000768	0.000771	0.002010	0.003244	0.005635	0.010750	0.024221	0.042037	0.05
δ	0.05	1.000000	1.000000	0.999740	0.923440	0.340790	0.009680	0.001720	0.000120	0.000000	0.000000	0.000000	1
		1.000000	1.000000	0.979870	0.653430	0.084830	0.000100	0.000000	0.000000	0.000000	0.000000	0.000000	0.05
	0.50	1.000000	1.000000	0.921130	0.352000	0.070190	0.003450	0.000840	0.000080	0.000000	0.000000	0.000000	1
		0.078310	0.072090	0.042240	0.016830	0.003880	0.000060	0.000000	0.000000	0.000000	0.000000	0.000000	0.05
	1.00	0.995080	0.821420	0.127650	0.035080	0.010130	0.000970	0.000150	0.000000	0.000000	0.000000	0.000000	1
		0.017940	0.016050	0.007670	0.002640	0.000800	0.000060	0.000000	0.000000	0.000000	0.000000	0.000000	0.05

- $PMS_{III}(1.1) = 0.397714$, while the estimates can take values from 0.000000 ($\phi_1 = -0.9, \phi_2 = -0.5$) to 1.000000 ($\phi_1 = 0.5, \phi_2 = 0.45$), but also values in between, such as 0.438920 ($\phi_1 = -0.1, \phi_2 = 0.5$) and 0.011080 ($\phi_1 = -0.3, \phi_2 = 0$);
- $PMS_{IV}(0.5, -0.5, -0.5) = 0.012968$, $PMS_{IV}(0.5, 1.45, -0.5) = 0.630473$, $PMS_{IV}(0.5, -0.1, 0.5) = 0.100647$ and $PMS_{IV}(0.5, 0, 0.5) = 0.135836$, nevertheless, the estimated PMS are equal to 0.990070, 0.000000, 0.135410 and 0.030070, respectively.

To sum up, these results and the ones in tables 5 and 6 are in accordance to the ones we reported in the previous subsection for the AR(1) model — when we fail to recognize an AR(2) process and mistakenly design a simultaneous scheme assuming i.i.d. output, the estimates of the PMS of Type III (resp. Type IV) tend to increase (resp. decrease) with parameters ϕ_1 and ϕ_2. As a consequence, only simultaneous residual schemes will give protection to both types of MS.

4.3. ARMA(1,1) model

Autocorrelated output from stable (continuous) processes frequently follow ARMA models of low order ([13, p. 2]), such as ARMA(1,1). The process $\{Y_{i,j}\}$ follows a stationary and invertible ARMA(1,1) model with mean μ_0, variance $\sigma_0^2 = \gamma_0$, autoregressive parameter ϕ and moving average parameter α, for every i, if

$$Y_{i,j} = \mu_0 + \phi(Y_{i,j-1} - \mu_0) + \varepsilon_{i,j} - \alpha\,\varepsilon_{i,j-1} , \tag{4.7}$$

where $-1 < \phi, \alpha < 1$ and $\varepsilon_{i,j} \sim_{i.i.d.} \mathcal{N}(0, \sigma_\varepsilon^2)$, with $\sigma_\varepsilon^2 = \frac{1-\phi^2}{1+\alpha^2-2\phi\alpha} \times \sigma_0^2$.

Now it is time to investigate the impact of falsely assuming i.i.d. output — rather than recognizing the ARMA(1,1) nature of the output — in the PMS of both types III and IV.

Once again we restricted ourselves to the EWMA scheme, $\theta = 1.02$ and $\delta = 0.05$ when it comes to graphical illustrations because the graphs we obtained for the Shewhart scheme or most of the other values of θ, δ, ϕ and α are similar to the ones in figures 7 and 8; tables 7 and 8 provide complementary results. In addition to this, we should remind the reader that the sub-family of AR(1) processes and the i.i.d. process are particular cases of the ARMA(1,1) processes. As a consequence we were able to check the values we got for $PMS_{IV}(\delta, \phi, \alpha)$ in Table 7 (resp. Table 8) when $\alpha = 0$ (resp. $\phi = \alpha = 0$), with the ones in Table 4. Unsurprisingly, the estimates of the PMS of Type III (resp. Type IV) in tables 7 and 8 are close to the values of $PMS_{III}(\theta)$ (resp. $PMS_{IV}(\delta, \phi, \alpha)$) in Table 3 (resp. tables 7 and 8); moreover, when $\alpha = 0$, the estimates of the PMS of types III and IV in Table 7 are comparable to the ones we obtained for the AR(1) model in Table 4.

Tables 7 and 8 and Figure 8 lead us to state that $\mathrm{PMS_{IV}}(\delta, \phi, \alpha)$ seems to: decrease with α, for varying or fixed ϕ, unlike $\mathrm{PMS_{IV}}(\delta, \phi)$ and $\mathrm{PMS_{IV}}(\delta, \phi_1, \phi_2)$ that tend to increase with the model parameter(s); increase with the autoregressive parameter ϕ like in the two previous models. Once again the adoption of a simultaneous EWMA residual scheme in place of a simultaneous Shewhart residual scheme yields a decrease of the $\mathrm{PMS_{IV}}(\delta, \phi, \alpha)$, for most values of δ, ϕ and α of this specific output process.

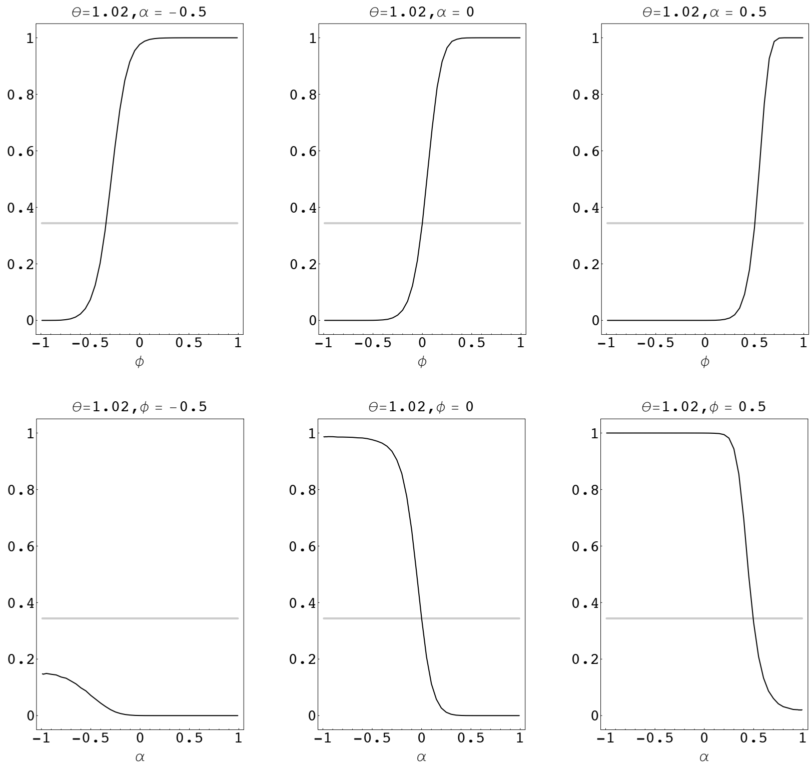

Figure 7: ARMA(1,1) model, EWMA — $\mathrm{PMS_{III}}(\theta)$ (simultaneous residual scheme, grey line) and estimates of PMS of Type III (traditional simultaneous scheme, black line), for $\phi \in (-1, 1)$ [top] and $\alpha \in (-1, 1)$ [bottom].

When we ignore that the output follows an ARMA$(1,1)$ model, the estimates of the PMS of Type III increase (resp. decrease) from 0 to 1 with ϕ (resp. α), although $\mathrm{PMS_{III}}(\theta)$ is constant when we adopt simultaneous residual schemes, as depicted by figures 7 and 8; curiously, when $\phi = -0.5$ the estimates of the PMS of Type III estimates are in general smaller than $\mathrm{PMS}(\theta)$ in the EWMA case for $\theta = 1.02$, as shown in Figure 7, but also for the Shewhart case and $\theta = 1.1, 1.2$. Furthermore, Figure 8 suggests that the estimates of PMS of Type IV decrease (resp. increase) with ϕ (resp. α); however, the values of $\mathrm{PMS_{IV}}(\delta, \phi, \alpha)$ tend to increase (resp. tend to decrease) with ϕ (resp. α) for fixed α (resp. ϕ), as portrayed by Table 7 (resp. 8). It goes without saying that correlation has quite an impact on the PMS of types III and IV. For example, for the simultaneous EWMA residual scheme, we got:

- $\mathrm{PMS_{III}}(1.2) = 0.042865$, whereas the estimates take values from 0.000000 ($\phi = -0.5, \alpha = 0.5$) to 1.000000 ($\phi = 0.9, \alpha = 0.5$) and values in the interval $(0, 1)$, such as 0.371910 $\phi = 0, \alpha = -0.5$) and 0.992420 ($\phi = 0.5, \alpha = -0.3$);
- $\mathrm{PMS_{IV}}(1.0, -0.5, 0.6) = 0.045997$, $\mathrm{PMS_{IV}}(1.0, 0.9, 0.5) = 0.0121289$, $\mathrm{PMS_{IV}}(1.0, 0, -0.6) = 0.002448$ and $\mathrm{PMS_{IV}}(1.0, 0.5, -0.3) = 0.007234$, however, the estimated PMS are equal to 0.013120, 0.000000, 0.000180 and 0.000000, respectively.

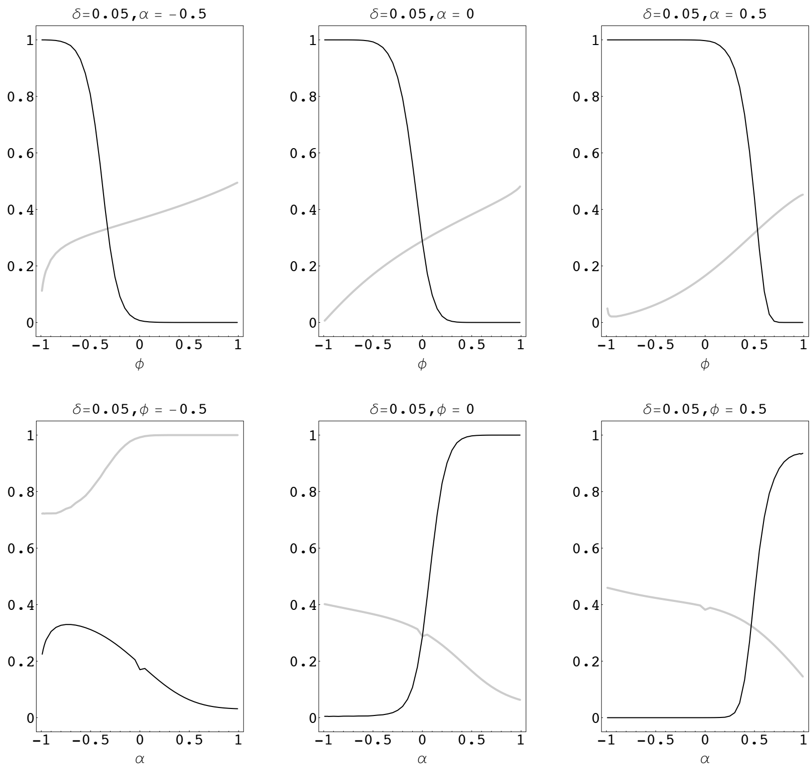

Figure 8: ARMA(1,1) model, EWMA — $\mathrm{PMS_{IV}}(\delta, \phi, \alpha)$ (simultaneous residual scheme, grey line) and estimates of PMS of Type IV (traditional simultaneous scheme, black line), for $\phi \in (-1, 1)$ [top] and $\alpha \in (-1, 1)$ [bottom].

Once again, the numerical results in tables 7 and 8 revealed that substituting the traditional Shewhart by a traditional EWMA chart can be frequently followed by an increase of the estimates of the PMS of types III and IV, even though both $\mathrm{PMS_{III}}(\theta)$ and $\mathrm{PMS_{IV}}(\delta, \phi, \alpha)$ decrease (in general) when a simultaneous EWMA residual scheme replaces a simultaneous Shewhart residual scheme.

Table 7: ARMA(1,1) model, $\alpha = -0.5, 0, 0.5$ — estimates of PMS of Type III of traditional simultaneous scheme; $\text{PMS}_{\text{IV}}(\delta, \phi, \alpha)$ of simultaneous residual scheme; estimates of PMS of Type IV of traditional simultaneous scheme.

		$\alpha = -0.5,\ \phi \in (-1,1)$											
		−0.95	−0.90	−0.70	−0.50	−0.30	0	0.30	0.50	0.70	0.90	0.95	λ
θ	1.02	0.000000	0.000020	0.014870	0.237770	0.663090	0.897370	0.975290	0.994530	0.999690	1.000000	1.000000	1
		0.000000	0.000060	0.005400	0.073120	0.462260	0.976270	0.999600	0.999990	1.000000	1.000000	1.000000	0.05
	1.10	0.000000	0.000090	0.021090	0.218110	0.578860	0.846470	0.956710	0.988500	0.999110	1.000000	1.000000	1
		0.000000	0.000090	0.003500	0.033410	0.165390	0.749910	0.993510	0.999860	1.000000	1.000000	1.000000	0.05
	1.20	0.000000	0.000250	0.030950	0.206270	0.490990	0.777930	0.926250	0.976410	0.997130	1.000000	1.000000	1
		0.000000	0.000010	0.002650	0.018190	0.077920	0.371910	0.911080	0.996570	0.999960	1.000000	1.000000	0.05
δ	0.05	0.461709	0.470959	0.481183	0.484953	0.487537	0.490736	0.493693	0.495669	0.497736	0.500100	0.500867	1
		0.181982	0.221266	0.282392	0.311712	0.334569	0.366417	0.399620	0.423733	0.450097	0.479688	0.488022	0.05
	0.50	0.028599	0.035284	0.059992	0.080718	0.102717	0.145310	0.211128	0.278578	0.378206	0.532869	0.588638	1
		0.002089	0.001917	0.002903	0.004068	0.005543	0.009165	0.017346	0.030600	0.066607	0.223590	0.346540	0.05
	1.00	0.004555	0.003995	0.006088	0.009456	0.014463	0.028959	0.066950	0.131011	0.281473	0.614743	0.731917	1
		0.001019	0.000378	0.000313	0.000464	0.000743	0.001773	0.005713	0.016133	0.062199	0.362196	0.581421	0.05
δ	0.05	1.000000	0.999870	0.984940	0.742730	0.299680	0.085390	0.019830	0.004300	0.000260	0.000000	0.000000	1
		0.999830	0.999230	0.980050	0.807800	0.266820	0.006970	0.000160	0.000030	0.000000	0.000000	0.000000	0.05
	0.50	0.995940	0.978720	0.702140	0.190550	0.054190	0.020570	0.006460	0.001870	0.000120	0.000000	0.000000	1
		0.317100	0.337220	0.151560	0.028600	0.005020	0.000720	0.000090	0.000000	0.000000	0.000000	0.000000	0.05
	1.00	0.517520	0.399720	0.116810	0.021890	0.007260	0.003600	0.001280	0.000500	0.000020	0.000000	0.000000	1
		0.109900	0.105250	0.027830	0.003170	0.000570	0.000210	0.000050	0.000000	0.000000	0.000000	0.000000	0.05

		$\alpha = 0,\ \phi \in (-1,1)$											
		−0.95	−0.90	−0.70	−0.50	−0.30	0	0.30	0.50	0.70	0.90	0.95	λ
θ	1.02	0.000000	0.000000	0.000000	0.000070	0.007220	0.474190	0.938290	0.988060	0.999320	1.000000	1.000000	1
		0.000000	0.000000	0.000000	0.000210	0.009000	0.344310	0.987430	0.999840	1.000000	1.000000	1.000000	0.05
	1.10	0.000000	0.000000	0.000000	0.000210	0.011400	0.398590	0.895530	0.975530	0.997850	1.000000	1.000000	1
		0.000000	0.000000	0.000000	0.000210	0.004680	0.100180	0.779820	0.995310	1.000000	1.000000	1.000000	0.05
	1.20	0.000000	0.000000	0.000000	0.000630	0.016890	0.331170	0.828060	0.950540	0.994370	0.999990	1.000000	1
		0.000000	0.000000	0.000000	0.000070	0.002620	0.041820	0.366710	0.909390	0.999690	1.000000	1.000000	0.05
δ	0.05	0.245605	0.331756	0.429811	0.457404	0.470828	0.481977	0.488703	0.492034	0.494900	0.497702	0.498534	1
		0.021403	0.040098	0.109687	0.169903	0.221933	0.288653	0.346313	0.381892	0.417014	0.455573	0.467676	0.05
	0.50	0.000411	0.003322	0.009211	0.017601	0.030661	0.061967	0.113516	0.165244	0.240161	0.363509	0.414297	1
		0.004417	0.002008	0.000750	0.000930	0.001422	0.002907	0.006195	0.010799	0.020925	0.055933	0.085086	0.05
	1.00	0.000000	0.000000	0.000620	0.001398	0.002359	0.005824	0.016431	0.035020	0.081256	0.229454	0.323318	1
		0.002409	0.012133	0.000422	0.000158	0.000141	0.000262	0.000794	0.002010	0.006279	0.032854	0.064217	0.05
δ	0.05	1.000000	1.000000	1.000000	0.999960	0.991760	0.484249	0.051580	0.009020	0.000510	0.000000	0.000000	1
		1.000000	1.000000	0.999770	0.992870	0.918780	0.286200	0.003850	0.000050	0.000000	0.000000	0.000000	0.05
	0.50	1.000000	0.999970	0.998620	0.954720	0.578830	0.062320	0.010940	0.003100	0.000200	0.000000	0.000000	1
		0.178620	0.159300	0.091340	0.043710	0.017720	0.003220	0.000380	0.000040	0.000000	0.000000	0.000000	0.05
	1.00	0.789230	0.746060	0.414820	0.134970	0.036230	0.005680	0.001970	0.000800	0.000110	0.000000	0.000000	1
		0.061530	0.048890	0.021810	0.007440	0.002110	0.000300	0.000130	0.000000	0.000000	0.000000	0.000000	0.05

		$\alpha = 0.5,\ \phi \in (-1,1)$											
		−0.95	−0.90	−0.70	−0.50	−0.30	0	0.30	0.50	0.70	0.90	0.95	λ
θ	1.02	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.047350	0.695910	0.983310	0.999980	1.000000	1
		0.000000	0.000000	0.000000	0.000000	0.000000	0.000040	0.019940	0.326950	0.986850	1.000000	1.000000	0.05
	1.10	0.000000	0.000000	0.000000	0.000000	0.000000	0.000070	0.051120	0.599140	0.965840	0.999960	1.000000	1
		0.000000	0.000000	0.000000	0.000000	0.000000	0.000060	0.009000	0.130160	0.847100	1.000000	1.000000	0.05
	1.20	0.000000	0.000000	0.000000	0.000000	0.000000	0.000120	0.056460	0.506320	0.928280	0.999870	1.000000	1
		0.000000	0.000000	0.000000	0.000000	0.000000	0.000010	0.005150	0.065750	0.531280	1.000000	1.000000	0.05
δ	0.05	0.179105	0.215087	0.322009	0.382990	0.421192	0.456249	0.476720	0.485569	0.491830	0.496163	0.496981	1
		0.021150	0.021172	0.038021	0.063325	0.097070	0.165267	0.252675	0.316783	0.379364	0.433895	0.445274	0.05
	0.50	0.000000	0.000017	0.007394	0.010726	0.012774	0.021023	0.046081	0.085810	0.161890	0.287884	0.325411	1
		0.381255	0.220021	0.015474	0.003924	0.001904	0.001426	0.002305	0.004426	0.010519	0.030728	0.041320	0.05
	1.00	0.000000	0.000000	0.000000	0.000077	0.001070	0.002618	0.004777	0.010706	0.034000	0.126378	0.172331	1
		0.000000	0.000000	0.126200	0.009655	0.002860	0.000553	0.000314	0.000545	0.001976	0.012129	0.019974	0.05
δ	0.05	1.000000	1.000000	1.000000	1.000000	1.000000	0.999990	0.948060	0.266680	0.010810	0.000000	0.000000	1
		1.000000	1.000000	1.000000	1.000000	0.999980	0.997240	0.896750	0.440600	0.004400	0.000000	0.000000	0.05
	0.50	1.000000	1.000000	1.000000	1.000000	0.999980	0.973430	0.327820	0.053930	0.005320	0.000000	0.000000	1
		0.162440	0.137210	0.083930	0.062050	0.046100	0.031970	0.020810	0.009870	0.000580	0.000000	0.000000	0.05
	1.00	0.850880	0.877470	0.892810	0.772690	0.476960	0.114260	0.022080	0.008290	0.001590	0.000000	0.000000	1
		0.054780	0.044580	0.022150	0.012550	0.007630	0.003710	0.001600	0.000880	0.000090	0.000000	0.000000	0.05

Table 8: ARMA(1,1) model, $\phi = -0.5, 0, 0.5$ — estimates of PMS of Type III of traditional simultaneous scheme; $\mathrm{PMS_{IV}}(\delta, \phi, \alpha)$ of simultaneous residual scheme; estimates of PMS of Type IV of traditional simultaneous scheme.

		$\phi = -0.5,\ \alpha \in (-1,1)$											
		−0.95	−0.90	−0.70	−0.50	−0.30	0	0.30	0.50	0.70	0.90	0.95	λ
	1.02	0.457660	0.452470	0.400090	0.238130	0.039670	0.000050	0.000000	0.000000	0.000000	0.000000	0.000000	1
		0.149230	0.146220	0.122350	0.071460	0.021110	0.000270	0.000000	0.000000	0.000000	0.000000	0.000000	0.05
θ	1.10	0.408010	0.406120	0.357100	0.216580	0.046720	0.000220	0.000000	0.000000	0.000000	0.000000	0.000000	1
		0.071460	0.072440	0.058790	0.034090	0.010020	0.000250	0.000000	0.000000	0.000000	0.000000	0.000000	0.05
	1.20	0.359950	0.358600	0.318480	0.200150	0.055320	0.000480	0.000000	0.000000	0.000000	0.000000	0.000000	1
		0.042440	0.041090	0.032110	0.018770	0.005870	0.000190	0.000000	0.000000	0.000000	0.000000	0.000000	0.05
	0.05	0.480395	0.484236	0.486991	0.484953	0.479980	0.457404	0.425647	0.382990	0.333826	0.300133	0.295870	1
		0.275450	0.305104	0.329699	0.311712	0.273922	0.169903	0.103053	0.063325	0.041691	0.033141	0.032201	0.05
δ	0.50	0.061584	0.078360	0.096743	0.080718	0.055258	0.017601	0.012169	0.010726	0.009900	0.006023	0.005301	1
		0.003321	0.004088	0.005077	0.004068	0.002669	0.000930	0.001503	0.003924	0.016522	0.036310	0.036225	0.05
	1.00	0.008038	0.009944	0.012695	0.009456	0.005499	0.001398	0.001089	0.000077	0.000000	0.000000	0.000000	1
		0.000588	0.000564	0.000622	0.000464	0.000294	0.000158	0.001836	0.009655	0.138196	0.128850	0.106785	0.05
	0.05	0.511620	0.516940	0.573440	0.745730	0.958320	0.999910	1.000000	1.000000	1.000000	1.000000	1.000000	1
		0.722820	0.722860	0.745010	0.805760	0.902680	0.992200	1.000000	1.000000	1.000000	1.000000	1.000000	0.05
δ	0.50	0.140790	0.138420	0.144850	0.189830	0.416760	0.955190	0.999800	1.000000	1.000000	1.000000	1.000000	1
		0.033610	0.035080	0.030420	0.028340	0.030000	0.044150	0.058530	0.060210	0.063270	0.063950	0.065000	0.05
	1.00	0.022700	0.021930	0.020800	0.021210	0.036800	0.137390	0.487070	0.772930	0.905330	0.941310	0.943090	1
		0.004240	0.004660	0.003900	0.002900	0.003760	0.007430	0.011170	0.013190	0.014090	0.014060	0.013580	0.05

		$\phi = 0,\ \alpha \in (-1,1)$											
		−0.95	−0.90	−0.70	−0.50	−0.30	0	0.30	0.50	0.70	0.90	0.95	λ
	1.02	0.908920	0.910160	0.906110	0.898620	0.868240	0.473770	0.003420	0.000020	0.000000	0.000000	0.000000	1
		0.987130	0.986940	0.984640	0.976120	0.935300	0.345880	0.004750	0.000040	0.000000	0.000000	0.000000	0.05
θ	1.10	0.864100	0.864520	0.859050	0.846670	0.801580	0.398260	0.006000	0.000040	0.000000	0.000000	0.000000	1
		0.849400	0.847540	0.825910	0.752340	0.548550	0.100580	0.002220	0.000000	0.000000	0.000000	0.000000	0.05
	1.20	0.801860	0.802200	0.799270	0.775030	0.712870	0.333620	0.010280	0.000130	0.000000	0.000000	0.000000	1
		0.499590	0.495990	0.463250	0.371320	0.224130	0.042200	0.001450	0.000040	0.000000	0.000000	0.000000	0.05
	0.05	0.494277	0.493794	0.492182	0.490736	0.488951	0.481977	0.472243	0.456249	0.430660	0.397957	0.389504	1
		0.399002	0.395418	0.381231	0.366417	0.348203	0.288653	0.228148	0.165267	0.109878	0.073722	0.067526	0.05
δ	0.50	0.253890	0.232569	0.176869	0.145310	0.118661	0.061967	0.035647	0.021023	0.015768	0.014755	0.014389	1
		0.029515	0.023790	0.013012	0.009165	0.006755	0.002907	0.001799	0.001426	0.002150	0.004869	0.005802	0.05
	1.00	0.130325	0.101906	0.047069	0.028959	0.018979	0.005824	0.003498	0.002618	0.001933	0.000368	0.000175	1
		0.024256	0.015134	0.003745	0.001773	0.001023	0.000262	0.000262	0.000553	0.003228	0.010105	0.014359	0.05
	0.05	0.076440	0.077440	0.079140	0.085810	0.113210	0.480624	0.995560	1.000000	1.000000	1.000000	1.000000	1
		0.004430	0.005020	0.005360	0.007330	0.017850	0.286010	0.946410	0.997210	0.999960	0.999980	0.999970	0.05
δ	0.50	0.021250	0.021240	0.020560	0.021970	0.021780	0.062320	0.635820	0.972370	0.999010	0.999890	0.999860	1
		0.000840	0.000870	0.000550	0.000780	0.000890	0.003330	0.017170	0.032600	0.043610	0.046410	0.047700	0.05
	1.00	0.004280	0.004370	0.004510	0.003810	0.003410	0.005260	0.035820	0.114660	0.277980	0.425890	0.440990	1
		0.000230	0.000260	0.000160	0.000200	0.000150	0.000220	0.002000	0.003800	0.005520	0.006120	0.005910	0.05

		$\phi = 0.5,\ \alpha \in (-1,1)$											
		−0.95	−0.90	−0.70	−0.50	−0.30	0	0.30	0.50	0.70	0.90	0.95	λ
	1.02	0.995020	0.994880	0.994800	0.994460	0.993490	0.988250	0.936280	0.693570	0.272970	0.118170	0.107720	1
		0.999980	0.999980	1.000000	0.999980	0.999960	0.999820	0.943680	0.328860	0.060820	0.021760	0.020650	0.05
θ	1.10	0.990200	0.989810	0.989520	0.988890	0.986900	0.975600	0.889050	0.601510	0.238770	0.110700	0.104630	1
		0.999870	0.999850	0.999820	0.999870	0.999590	0.995420	0.600210	0.131810	0.029260	0.011710	0.010760	0.05
	1.20	0.979300	0.978080	0.978060	0.976880	0.972870	0.952220	0.813460	0.506270	0.207750	0.106540	0.099790	1
		0.997670	0.997770	0.997420	0.996350	0.992420	0.908190	0.278260	0.064790	0.015310	0.006500	0.005850	0.05
	0.05	0.499689	0.499118	0.497066	0.495669	0.494662	0.492034	0.489918	0.485569	0.477396	0.461660	0.455367	1
		0.456381	0.452179	0.436525	0.423733	0.412621	0.381892	0.358381	0.316783	0.256556	0.182519	0.162304	0.05
δ	0.50	0.517509	0.482040	0.354280	0.278578	0.236756	0.165244	0.130423	0.085810	0.048758	0.025838	0.021982	1
		0.195365	0.154296	0.058931	0.030600	0.021007	0.010799	0.007627	0.004426	0.002485	0.001683	0.001633	0.05
	1.00	0.601221	0.528941	0.258189	0.131011	0.083377	0.035020	0.022134	0.010706	0.005346	0.003420	0.003090	1
		0.399714	0.300459	0.063075	0.016133	0.007234	0.002010	0.001179	0.000545	0.000364	0.000598	0.000806	0.05
	0.05	0.003560	0.003520	0.003890	0.003850	0.004640	0.009530	0.050440	0.268780	0.694960	0.865420	0.875450	1
		0.000020**	0.000020	0.000000	0.000020	0.000010	0.000070	0.000850	0.009530	0.037110	0.058900	0.060270	0.05
δ	0.50	0.001580	0.001420	0.001600	0.001820	0.002090	0.003150	0.013320	0.053700	0.171200	0.282310	0.295480	1
		0.000020*	0.000020*	0.000000*	0.000020*	0000000	0000000	0000000	0000000	0000000	0000000	0000000	0.05
	1.00	0.000400	0.000440	0.000610	0.000520	0.000580	0.000860	0.002340	0.007770	0.019470	0.030020	0.031220	1
		0.000030	0.000000	0.000000	0.000000	0.000000	0.000010	0.000160	0.000930	0.003890	0.006170	0.006440	0.05

5. CONCLUDING REMARKS

The introduction of automatic measuring devices and the subsequent increase in the frequency of the measurements led to autocorrelated output, a major issue in the process industries, as [13, p. iii] felt bound to point out.

This paper confirms that autocorrelation can cause traditional simultaneous control schemes to produce misleading signals either more or less frequently than simultaneous residual schemes, depending on the type of autocorrelation. In fact, if we ignore or neglect the autocorrelation structure of the output then we can obtain estimates of PMS of types III and IV smaller than the ones we would obtain if we adopted simultaneous residual schemes to monitor the mean and variance of AR(1), AR(2) and ARMA(1,1) processes; furthermore, the regions where these schemes are superseded by the traditional ones tend to be symmetric for the PMS of types III and IV, thus, only simultaneous residual schemes will give the necessary protection to MS of both types, as previously mentioned by [8] for the AR(1) process. Some monotonicity properties of the real PMS of Type IV for AR(2) and ARMA(1,1), in terms of the model parameters, surface pointedly in this study, adding up to one already enunciated by [8] for the AR(1) model — $\text{PMS}_{\text{IV}}(\delta, \phi_1, \phi_2)$ (resp. $\text{PMS}_{\text{IV}}(\delta, \phi, \alpha)$) appears to increase with both ϕ_1 and ϕ_2 (resp. increase with ϕ and decrease with α), when simultaneous residual schemes are used to control the mean and variance of AR(2) (resp. ARMA(1,1)) output. This paper also reaffirms that simultaneous EWMA residuals schemes should be preferred to the Shewhart-type if we plan to anticipate a few dramatic reductions of the PMS of types III and IV.

Misleading signals deserve further investigation while using other simultaneous schemes for μ and σ^2 suchlike the ones pertinently proposed by [5], simultaneous EWMA schemes with the following characteristics: their constituent charts for σ^2 are able to detect both upward and downward shifts in the process variance; the maximum of ARL for fixed $\mu = \mu_0$ and for varying σ^2 is attained at $\sigma^2 = \sigma_0^2$. Future research can also be done in the following direction: assess the impact on MS of falsely assuming a simpler model, e.g., an AR(1) model, when the output is better described by a slightly more complex process, e.g., an AR(2) process or an ARMA(1,1) model.

Since MS can be rather frequent and the general assumption of independence can have a meaningful effect in the ability of a simultaneous scheme for the process mean and variance to identify which one of these two parameters has changed, it is convenient to implement additional procedures for use as diagnostic aids to determine which parameters changes, as recommended by [20]. Although investigation on these diagnostic procedures is beyond the scope of this paper, this issue will be certainly considered in future work and we shall take into account that [18] suggest the use of the pattern of the points beyond the control limits of the constituent charts in the identification of the parameter that has effectively changed (a plausible justification for this diagnostic aid stems from the fact that changes in μ and σ^2 have different impacts in those patterns).

Finally, let us remind the reader that the phenomenon of MS can also arise in other settings, such as multivariate control schemes for the mean vector and the covariance matrix of i.i.d. output as investigated by [17] and [16].

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A REPARAMETERIZED BIRNBAUM–SAUNDERS DISTRIBUTION AND ITS MOMENTS, ESTIMATION AND APPLICATIONS

Authors: Manoel Santos-Neto
– Departamento de Estatística, Universidade Federal de Campina Grande, Brazil
manoel.ferreira@ufcg.edu.br

Francisco José A. Cysneiros
– Departamento de Estatística, Universidade Federal de Pernambuco, Brazil
cysneiros@de.ufpe.br

Víctor Leiva
– Instituto de Estadística, Universidad de Valparaíso,
Facultad de Ingeniería y Ciencias, Universidad Adolfo Ibáñez,
Chile
victorleivasanchez@gmail.com www.victorleiva.cl

Michelli Barros
– Departamento de Estatística, Universidade Federal de Campina Grande, Brazil
michelli.karinne@gmail.com

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Abstract:

- The Birnbaum–Saunders (BS) distribution is a model that is receiving considerable attention due to its good properties. We provide some results on moments of a reparameterized version of the BS distribution and a generation method of random numbers from this distribution. In addition, we propose estimation and inference for the mentioned parameterization based on maximum likelihood, moment, modified moment and generalized moment methods. By means of a Monte Carlo simulation study, we evaluate the performance of the proposed estimators. We discuss applications of the reparameterized BS distribution from different scientific fields and analyze two real-world data sets to illustrate our results. The simulated and real data are analyzed by using the R software.

Key-Words:

- *data analysis; maximum likelihood and moment estimation; Monte Carlo method; random number generation; statistical software.*

AMS Subject Classification:

- 62F86, 60E05.

1. INTRODUCTION

The Birnbaum–Saunders (BS) distribution is being widely considered. This distribution is unimodal and positively skewed, has positive support and two parameters corresponding to its shape and scale; see Birnbaum & Saunders (1969a), Johnson *et al.* (1995) and Athayde *et al.* (2012). Interest in the BS distribution is due to its physical theoretical arguments, its attractive properties and its relationship with the normal model. Although the BS distribution has its genesis from material fatigue, it has been used for applications in: agriculture, business, contamination, engineering, finance, food, forest and textile industries, informatics, insurance, medicine, microbiology, mortality, nutrition, pharmacology, psychology, quality control, queue theory, toxicology, water quality and wind energy; see Leiva *et al.* (2007, 2008c, 2010a,b, 2011, 2012, 2014a,b,d), Ahmed *et al.* (2008), Barros *et al.* (2008), Balakrishnan *et al.* (2009a,b, 2011), Bhatti (2010), Kotz *et al.* (2010), Vilca *et al.* (2010), Sanhueza *et al.* (2011), Santana *et al.* (2011), Villegas *et al.* (2011), Azevedo *et al.* (2012), Ferreira *et al.* (2012), Paula *et al.* (2012), Fierro *et al.* (2013), Marchant *et al.* (2013a,b) and Saulo *et al.* (2013).

One of the most studied topics in the BS distribution is its estimation and inference. Several types of estimators for its original parameterization have been proposed. Birnbaum & Saunders (1969b) found its maximum likelihood (ML) estimators. Bhattacharyya & Fries (1982) mentioned that the lack of an exponential family structure for the BS distribution complicates the statistical inference of its parameters. Engelhardt *et al.* (1981), Achcar (1993), Chang & Tang (1994) and Dupuis & Mills (1998) proposed other types of estimators of the original parameters. However, in all of these cases, it is not possible to find explicit expressions for its estimators, so that numerical procedures must be used. Ng *et al.* (2003) introduced a modified moment (MM) method for estimating the BS model parameters, which provides simple analytical expressions to compute them. From & Li (2006) presented and summarized several estimation methods for the BS distribution. Results about improved inference for this distribution are attributed to Lemonte *et al.* (2007) and Cysneiros *et al.* (2008). Thus, different estimation aspects related to the BS distribution have been considered by a number of authors. Nevertheless, not much attention has been paid to parameterizations that are different from that originally proposed by Birnbaum & Saunders (1969a), which was based on the physics of materials. Some works on reparameterizations of the BS distribution were proposed by Volodin & Dzhungurova (2000), Ahmed *et al.* (2008), Lio *et al.* (2010) and Santos-Neto *et al.* (2012). The present work is focused on Santos-Neto *et al.* (2012)'s reparameterization.

Our main motivation for studying this reparameterization of the BS distribution is based on the search of estimators with good statistical properties. Such a reparameterization is useful, because, first, moment estimates for the original parameterization of the BS distribution do not have a closed-form, but this is possible with Santos-Neto *et al.* (2012)'s reparameterization and, second, it allows a response variable to be modeled in its original scale (see Leiva *et al.*, 2014c), which is not possible with the parameterizations proposed until now.

The objectives of this paper are:

- **(i)** to provide some results on moments of a reparameterized version of the BS distribution and a generator of random numbers;
- **(ii)** to propose estimators for this reparameterization;
- **(iii)** to study the performance of these estimators;
- **(iv)** to apply the results to real-world data.

The proposed estimators are based on generalized moment (GM), ML, MM and moment methods.

The article is organized as follows. In Section 2, we present some results of the reparameterized version of the BS distribution that include a shape analysis, a generator of random numbers, its characteristic function (CF) and its moments. In Section 3, we develop estimation and inference for this reparameterization based on the GM, ML, MM and moment methods. In Section 4, we evaluate the performance of the proposed estimators through Monte Carlo (MC) simulations. In Section 5, we conduct an application with two real-world data sets, one from engineering and another from economics, which is a new application of the BS distribution. In Sections 4 and 5, computational aspects based on packages in the `R` software are discussed. In Section 6, we sketch some conclusions of this study.

2. BS DISTRIBUTIONS

In this section, we present some results of a reparameterized version of the BS distribution, including a shape analysis, a generator of random numbers and its moments.

2.1. The original parameterization

The first parameterization of the BS distribution was proposed by Birnbaum & Saunders (1969a) based on the physics of materials in terms of shape (α) and scale (β) parameters. Thus, if a random variable (RV) Y follows the BS distribution with parameters $\alpha > 0$ and $\beta > 0$, the notation $Y \sim \mathrm{BS}(\alpha, \beta)$ is used and the corresponding probability density function (PDF) is given by

$$(2.1)\qquad f(y;\alpha,\beta) = \frac{1}{\sqrt{2\pi}}\,\exp\!\left(-\frac{1}{2\alpha^2}\left[\frac{y}{\beta}+\frac{\beta}{y}-2\right]\right)\frac{[y+\beta]}{2\alpha\sqrt{\beta y^3}}\,, \qquad y > 0\,.$$

2.2. A reparameterized version of the BS distribution

Recently, Santos-Neto *et al.* (2012) proposed a reparameterized version of the BS distribution, given, with respect to the original parameterization, by $\alpha = \sqrt{2/\delta}$ and $\beta = \delta\mu/[\delta+1]$, such that $\delta = 2/\alpha^2$ and $\mu = \beta[1+\alpha^2/2]$, where $\delta > 0$ and $\mu > 0$ are shape and mean parameters, respectively. For details about motivations and justifications for this reparameterized version, see Santos-Neto *et al.* (2012) and Leiva *et al.* (2014c).

Thus, the PDF of $Y \sim \mathrm{BS}(\mu, \delta)$ is given by

$$
\begin{aligned}
f(y;\mu,\delta) \;&=\; \frac{\exp(\delta/2)\,\sqrt{\delta+1}}{4\sqrt{\pi\mu}\,y^{3/2}}\left[y+\frac{\delta\mu}{\delta+1}\right] \\
&\quad\times\exp\!\left(-\frac{\delta}{4}\left[\frac{y\{\delta+1\}}{\delta\mu}+\frac{\delta\mu}{y\{\delta+1\}}\right]\right), \qquad y>0\,.
\end{aligned}
\tag{2.2}
$$

From (2.1) and considering the indicated reparameterization, one can note that BS and standard normal RVs are related by

$$
\begin{aligned}
Y \;&=\; \frac{\delta\mu}{\delta+1}\left[\frac{Z}{\sqrt{2\delta}}+\sqrt{\left\{\frac{Z}{\sqrt{2\delta}}\right\}^2+1}\,\right]^2 \quad \text{and} \\
Z \;&=\; \sqrt{\frac{\delta}{2}}\left[\sqrt{\frac{\{\delta+1\}Y}{\mu\delta}}-\sqrt{\frac{\mu\delta}{\{\delta+1\}Y}}\,\right].
\end{aligned}
\tag{2.3}
$$

Hence, from (2.3), the cumulative distribution function (CDF) and the quantile function (QF) of $Y \sim \mathrm{BS}(\mu, \delta)$ are, respectively, given by

$$
F(y;\mu,\delta) \;=\; \Phi\!\left(\sqrt{\frac{\delta}{2}}\left[\sqrt{\frac{\{\delta+1\}y}{\mu\delta}}-\sqrt{\frac{\mu\delta}{\{\delta+1\}y}}\,\right]\right), \qquad y>0\,,
$$

and

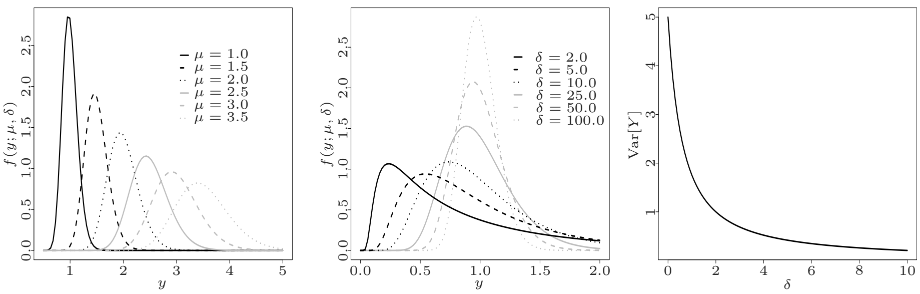
$$
y(q;\mu,\delta) \;=\; F^{-1}(q) \;=\; \frac{\delta\mu}{\delta+1}\left[\frac{z(q)}{\sqrt{2\delta}}+\sqrt{\left\{\frac{z(q)}{\sqrt{2\delta}}\right\}^2+1}\,\right]^2, \qquad 0<q<1\,,
$$

where $z(q)$ is the qth quantile of the standard normal distribution and F^{-1} is the inverse CDF of Y. The hazard rate function of Y is defined by

$$
\begin{aligned}
h(y;\mu,\delta) \;&=\; \frac{f(y;\mu,\delta)}{1-F(y;\mu,\delta)} \;=\; \frac{\exp(\delta/2)\,\sqrt{\delta+1}}{4\sqrt{\pi\mu y^3}}\left[y+\frac{\delta\mu}{\delta+1}\right] \\
&\quad\times\frac{\exp\left(-\frac{\delta}{4}\left[\frac{y\{\delta+1\}}{\delta\mu}+\frac{\delta\mu}{y\{\delta+1\}}\right]\right)}{\Phi\left(-\sqrt{\frac{\delta}{2}}\left[\sqrt{\frac{\{\delta+1\}y}{\mu\delta}}-\sqrt{\frac{\mu\delta}{\{\delta+1\}y}}\right]\right)}, \qquad y>0\,.
\end{aligned}
$$

2.3. Shape analysis

Figures 1(a)–1(b) show shapes for the PDF of $Y \sim \mathrm{BS}(\mu, \delta)$ considering different values of μ, when δ is fixed, and different values of δ, when μ is fixed. From Figure 1(a), note that the parameter μ controls the scale of the PDF, so that it is a scale parameter and also the mean of the distribution. This aspect can be formally verified because $b\,Y \sim \mathrm{BS}(b\,\mu, \delta)$, with $b > 0$. From Figure 1(b), notice that the parameter δ controls the shape of the PDF, making it more platykurtic as δ increases. Figure 1(c) shows a graphical plot of δ versus $\mathrm{Var}[Y]$, for $\mu = 1.0$. This figure allows the effect exerted by δ on the variance of the distribution to be detected. Note that such a variance decreases as δ increases, and it converges to 5.0, when δ goes to zero. Then, by means of this graphical analysis, we note that δ is a precision parameter.



(a) (b) (c)

Figure 1: PDF plots of a reparameterized BS distribution for different values of μ with $\delta = 100.0$ (a) and of δ with $\mu = 1.0$ (b), and plot of δ versus $\mathrm{Var}[Y]$ (c).

2.4. Number generation

Random numbers from the reparameterized BS distribution can be obtained by using the generator described in Algorithm 1.

Algorithm 1 – Generator of BS random numbers
1: Generate a random number z from a RV $Z \sim \mathrm{N}(0,1)$;
2: Set values for μ and δ of $Y \sim \mathrm{BS}(\mu, \delta)$;
3: Compute a random number y from $Y \sim \mathrm{BS}(\mu, \delta)$, using formula given in (2.3);
4: Repeat steps 1 to 3 until the required amount of numbers to be completed.

2.5. Moments

Another way to characterize a distribution is by using its CF, which allows us to obtain its moments. Here, we provide some results on the CF and moments of the reparameterized BS distribution. Moments for the original parameterization of the BS distribution can be found in Leiva *et al.* (2008a) and Balakrishnan *et al.* (2009a). In the literature on the BS distribution, the CF is practically not studied. From the PDF given in (2.2), we obtain the CF of $Y \sim \mathrm{BS}(\mu, \delta)$ in the following theorem.

Theorem 2.1. *Let $Y \sim \mathrm{BS}(\mu, \delta)$. Then, the CF $\varphi\colon \mathbb{R} \to \mathbb{C}$ of Y is*

$$\begin{aligned}\varphi(t) &= \mathrm{E}\Big[\exp(itY)\Big] \\ &= \frac{1}{2}\left[\left\{1 + \frac{\sqrt{\delta+1}}{\sqrt{1+\delta-4ti\mu}}\right\} \exp\left(\frac{\delta\left\{\sqrt{\delta+1} - \sqrt{1+\delta-4ti\mu}\right\}}{2\sqrt{\delta+1}}\right)\right], \qquad t \in \mathbb{R}\,,\end{aligned}$$

where $i = \sqrt{-1}$ is the imaginary unit.

Proof: The result is obtained using algebraic and integration methods. □

Corollary 2.1. *Let $Y \sim \mathrm{BS}(\mu, \delta)$ with CF φ as given in Theorem 2.1. Then, the rth derivative of φ with respect to t, evaluated at the point $t = 0$, is*

$$\begin{aligned}\varphi(0)^{(r)} &= \frac{\mathrm{d}^r \varphi(t)}{\mathrm{d}t^r}\bigg|_{t=0} \\ &= i^r\,\mathrm{E}\Big[Y^r \exp(itY)\Big]\Big|_{t=0} \\ &= \frac{1}{2\sqrt{\pi}\,[\delta+1]^{\frac{3}{2}}}\left[i^r \mu^r \delta^2 \exp\left(\frac{\delta}{2}\right)\right. \\ &\quad \left.\times \left\{\left(\delta^{r-\frac{1}{2}} + \delta^{r-\frac{3}{2}}\right)(\delta+1)^{\frac{1}{2}-r} K_{r+\frac{1}{2}}\left(\frac{\delta}{2}\right) + \delta^{r-\frac{3}{2}}(\delta+1)^{\frac{3}{2}-r} K_{r-\frac{1}{2}}\left(\frac{\delta}{2}\right)\right\}\right],\end{aligned}$$

where K_v is the modified Bessel function of second type.

Table 1 displays the values of the function K_v (see Abramowitz & Stegun, 1972) for some values of v, which are useful for calculating the moments around zero of the BS distribution.

Table 1: Values of $K_v(\delta/2)$ for the indicated values of v.

v	$K_v(\delta/2)$
$\frac{1}{2}$	$\dfrac{\sqrt{\pi}\,\exp(-\frac{1}{2}\,\delta)}{\sqrt{\delta}}$
$\frac{3}{2}$	$K_{\frac{1}{2}}\left(\frac{\delta}{2}\right)\left[1+\frac{2}{\delta}\right]$
$\frac{5}{2}$	$K_{\frac{1}{2}}\left(\frac{\delta}{2}\right)\left[1+\frac{6}{\delta}+\frac{12}{\delta^2}\right]$
$\frac{7}{2}$	$K_{\frac{1}{2}}\left(\frac{\delta}{2}\right)\left[1+\frac{12}{\delta}+\frac{60}{\delta^2}+\frac{120}{\delta^3}\right]$
$\frac{9}{2}$	$K_{\frac{1}{2}}\left(\frac{\delta}{2}\right)\left[1+\frac{20}{\delta}+\frac{180}{\delta^2}+\frac{840}{\delta^3}+\frac{1680}{\delta^4}\right]$

By means of Theorem 2.1 and Corollary 2.1, it is possible to obtain the moments around zero of $Y \sim \mathrm{BS}(\mu,\delta)$. By using the fact that $\varphi(0)^{(r)} = i^r \mathrm{E}[Y^r]$, we can easily find, for example, the four first moments of Y as

$$\begin{aligned}
\mathrm{E}\big[Y\big] &= \mu\,, \qquad \mathrm{E}\big[Y^2\big] = \mu^2\,\frac{[\delta^2+4\,\delta+6]}{[\delta+1]^2}\,,\\
\mathrm{E}\big[Y^3\big] &= \mu^3\,\frac{[\delta^3+9\,\delta^2+36\,\delta+60]}{[\delta+1]^3} \qquad \text{and}\\
\mathrm{E}\big[Y^4\big] &= \mu^4\,\frac{[\delta^4+16\delta^3+120\delta^2+460\delta+840]}{[\delta+1]^4}\,.
\end{aligned} \tag{2.4}$$

The rth central moment of $Y \sim \mathrm{BS}(\mu,\delta)$, which we denote by μ_r, is given by

$$\mu_r = \mathrm{E}\big[Y-\mu\big]^r = \sum_{j=0}^{r}\binom{r}{j}(-1)^{r-j}\,\mathrm{E}\big[Y^j\big]\,\mu^{r-j}\,, \qquad r=2,3,... \tag{2.5}$$

From (2.4) and (2.5), we have that the variance of Y is $\mathrm{Var}[Y]=\mu^2[2\,\delta+5]/[\delta+1]^2$, which allows the parameter δ to be interpreted as a precision parameter because, for μ fixed, the variance of Y decreases when δ increases. In addition, we can rewrite this variance as $\mathrm{Var}[Y]=V(\mu)/\phi$, where $\phi=[\delta+1]^2/[2\,\delta+5]$ and $V(\mu)=\mu^2$, with $V(\mu)$ acting as a "variance function", such as in generalized linear models.

Another interesting result is that the reparameterized BS distribution preserves the reciprocation property of the original BS distribution, that is, $1/Y$ is in the same family of distributions of Y. Thus, if $Y \sim \mathrm{BS}(\mu,\delta)$, then $1/Y \sim \mathrm{BS}([\delta+1]^2/\mu\,\delta^2,\,\delta)$ and, consequently,

$$\mathrm{E}\big[1/Y\big] = \frac{[\delta+1]^2}{\mu\,\delta^2} \qquad \text{and} \qquad \mathrm{Var}\big[1/Y\big] = \frac{[2\,\delta+5]\,[\delta+1]^2}{\mu^2\,\delta^4}\,.$$

3. ESTIMATION

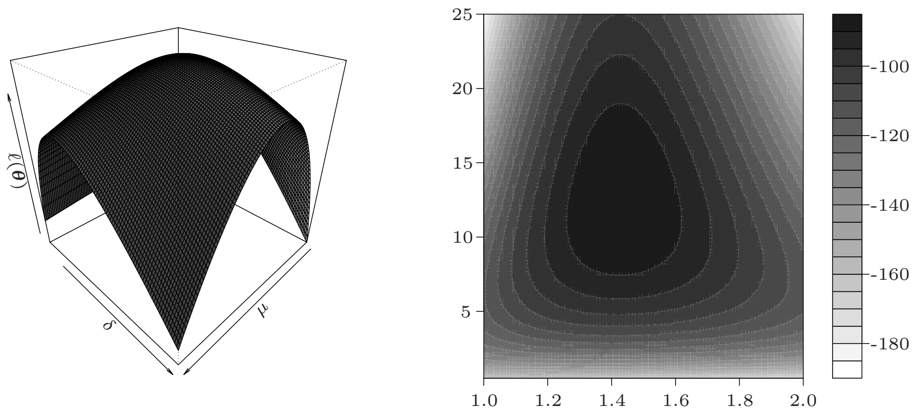
In this section, we derive estimation and inference for the parameters, in the sequel denoted by $\boldsymbol{\theta} = [\mu, \delta]^\top$, of the reparameterized BS distribution based on the GM, ML, MM and moment methods.

3.1. Maximum likelihood estimation

Let $\boldsymbol{Y} = [Y_1, ..., Y_n]^\top$ be a random sample of size n from $Y \sim \mathrm{BS}(\mu, \delta)$. Then, the log-likelihood function for $\boldsymbol{\theta}$ is

$$\ell(\boldsymbol{\theta}) = \sum_{j=1}^{n} \ell_j(\boldsymbol{\theta}) \ , \tag{3.1}$$

where $\ell_j(\boldsymbol{\theta})$ is the logarithm of the PDF given in (2.2) replacing y by y_j. Figure 2 displays graphical plots of the log-likelihood function and its respective contours, considering, as illustration, a sample from $Y \sim \mathrm{BS}(\mu{=}1.5, \delta{=}10)$. In this figure, note that the shape of the log-likelihood function is well behaved and, through its contours, it is easy to see the region where the values that maximize the function $\ell(\boldsymbol{\theta})$ given in (3.1) are located.



(a) (b)

Figure 2: Plots of the log-likelihood function (a) and its respective contours (b), for the $\mathrm{BS}(\mu{=}1.5, \delta{=}10)$ distribution.

As is well-known, to obtain the ML estimates of the parameters, we must equal the score functions to zero. In the case of the reparameterized BS distribution, the score vector for $\boldsymbol{\theta}$ is given by $U(\boldsymbol{\theta}) = [U_\mu, U_\delta]^\top$, where

$$U_\mu = \frac{\partial \ell(\boldsymbol{\theta})}{\partial \mu} = \sum_{j=1}^{n}\left[\frac{\delta}{\delta y_j + y_j + \delta\mu} + \frac{y_j\{\delta+1\}}{4\mu^2} - \frac{\delta^2}{4y_j\{\delta+1\}} - \frac{1}{2\mu}\right]$$

and

$$U_\delta = \frac{\partial \ell(\boldsymbol{\theta})}{\partial \delta} = \sum_{j=1}^{n}\left[\frac{y_j+\mu}{\delta y_j + y_j + \delta\mu} - \frac{y_j}{4\mu} - \frac{\delta\{\delta+2\}\mu}{4\{\delta+1\}^2 y_j} + \frac{\delta}{2\{\delta+1\}}\right].$$

Such as in the case of the original BS parameterization, for the reparameterized version, it is not possible to find closed-form estimators for its parameters. Then, we must use an iterative numerical method to optimize the function $\ell(\boldsymbol{\theta})$ given in (3.1). For example, a Newton–Raphson type algorithm can be used in this case.

The corresponding expected Fisher information matrix, denoted by $\mathcal{K}(\boldsymbol{\theta}) = [\mathcal{K}_{\theta_j\theta_k}]$, has elements

$$\begin{aligned}
\mathcal{K}_{\mu\mu} &= -\mathrm{E}\left[\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \mu^2}\right] = n\left[\frac{\delta}{2\mu^2} + \frac{\delta^2}{\{\delta+1\}^2}\, I(\boldsymbol{\theta})\right], \\
\mathcal{K}_{\delta\mu} &= -\mathrm{E}\left[\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \mu\,\partial \delta}\right] = n\left[\frac{1}{2\mu\{\delta+1\}} + \frac{\delta\mu}{\{\delta+1\}^3}\, I(\boldsymbol{\theta})\right] \quad \text{and} \\
\mathcal{K}_{\delta\delta} &= -\mathrm{E}\left[\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \delta^2}\right] = n\left[\frac{\delta_j^2 + 3\delta_j + 1}{2\delta_j^2\{\delta_j+1\}^2} + \frac{\mu_j^2}{\{\delta_j+1\}^4}\, I(\boldsymbol{\theta})\right],
\end{aligned} \tag{3.2}$$

where $\mathcal{K}_{\delta\mu} = \mathcal{K}_{\mu\delta}$ and

$$I(\boldsymbol{\theta}) = \int_0^\infty \left[y + \frac{\mu\delta}{\delta+1}\right]^{-2} f(y;\boldsymbol{\theta})\,\mathrm{d}y\,.$$

Under regularity conditions (see Cox & Hinkley, 1974), we have that the corresponding variance-covariance matrix is $\mathrm{Cov}[\widehat{\mu},\widehat{\delta}] = \mathcal{K}(\boldsymbol{\theta})^{-1}$, whose elements of $\mathcal{K}(\boldsymbol{\theta})$ are given in (3.2). In addition, in general, as is well-known, ML estimators have an asymptotic bivariate normal joint distribution. Thus, in our case, $[\hat{\mu},\hat{\delta}]^\top$ approximately follows the distribution

$$\mathrm{N}_2\left(\begin{bmatrix}\mu\\ \delta\end{bmatrix},\, \mathcal{K}(\boldsymbol{\theta})^{-1}\right).$$

3.2. Moment estimation

Moment conditions are needed to estimate parameters by using the moment method; see Mátyás (1999). Next, we define these conditions.

Definition 3.1. Let $\boldsymbol{Y} = [Y_1, ..., Y_n]^\top$ be a random sample of size n from any distribution. We want to estimate an unknown $p\times 1$ parameter vector $\boldsymbol{\theta}$, with true value $\boldsymbol{\theta}_0$. Let $g(Y_j, \boldsymbol{\theta})$ be a $q \times 1$ vector, which is a continuous function of $\boldsymbol{\theta}$, and assume that $\mathrm{E}[g(Y_j, \boldsymbol{\theta})]$ exists and it is finite for all j and $\boldsymbol{\theta}$. Then, the moment conditions to estimate $\boldsymbol{\theta}$ are that $\mathrm{E}[g(Y_j, \boldsymbol{\theta}_0)] = \mathbf{0}$.

We want to estimate the vector $\boldsymbol{\theta}$ using the moment conditions given in Definition 3.1. First, we consider the case when $p = q$, that is, when $\boldsymbol{\theta}$ is exactly identified by the moment conditions. Thus, these conditions represent a set of p equations, with p unknown parameters. Solving these equations, we find the true value of $\boldsymbol{\theta}$, $\boldsymbol{\theta}_0$ say, which satisfies the mentioned moment conditions. However, it is not possible to observe $\mathrm{E}[g(Y_j, \boldsymbol{\theta})]$, but only $g(y_j, \boldsymbol{\theta})$. In this way, a natural procedure is to define the sample moments of $g(Y_j, \boldsymbol{\theta})$, given by

(3.3) $$g_n(\boldsymbol{\theta}) = \frac{1}{n} \sum_{j=1}^{n} g(Y_j, \boldsymbol{\theta}) \ .$$

If the sample moments are estimators of the population moments with good properties, we then hope that the estimator $\widetilde{\boldsymbol{\theta}}$ holding the sample moment conditions $g_n(\boldsymbol{\theta}) = \mathbf{0}$ is a good estimator of the true value $\boldsymbol{\theta}_0$, which holds the population moment conditions $\mathrm{E}[g(Y_j, \boldsymbol{\theta})] = \mathbf{0}$. Hence, $\widetilde{\boldsymbol{\theta}}$ is a moment estimator of $\boldsymbol{\theta}$.

Theorem 3.1. *Let $\boldsymbol{Y} = [Y_1, ..., Y_n]^\top$ be a random sample of size n from $Y \sim \mathrm{BS}(\mu, \delta)$. Then, the moment estimators of μ and δ are, respectively,*

$$\widetilde{\mu} = \bar{Y} \qquad \text{and} \qquad \widetilde{\delta} = \frac{\bar{Y}^2 - S^2 + \sqrt{\bar{Y}^4 + 3\bar{Y}^2 S^2}}{S^2} \ ,$$

where $\bar{Y} = [1/n] \sum_{j=1}^{n} Y_j$ and $S^2 = [1/n] \sum_{j=1}^{n} [Y_j - \bar{Y}]^2$.

Proof: Recall from (2.4) and (2.5) that $\mathrm{E}[Y - \mu]^2 = \mu^2 [2\delta + 5]/[\delta + 1]^2$ and $\mathrm{E}[Y] = \mu$. Also, recall $\boldsymbol{\theta} = [\mu, \delta]^\top$ and define the vector of functions

$$g(Y_j, \boldsymbol{\theta}) = \left[Y_j - \mu, \ \ \{Y_j - \mu\}^2 - \frac{\mu^2 \{2\delta + 5\}}{\{\delta + 1\}^2} \right]^\top .$$

Then, the moment conditions are $\mathrm{E}[g(Y_j, \boldsymbol{\theta}_0)] = \mathbf{0}$. We have that $g_n(\widetilde{\boldsymbol{\theta}}) = \mathbf{0}$, with g_n defined in (3.3), implies that

$$\frac{1}{n} \sum_{j=1}^{n} Y_j - \widetilde{\mu} = 0 \qquad \text{and} \qquad \frac{1}{n} \sum_{j=1}^{n} [Y_j - \widetilde{\mu}]^2 - \frac{\widetilde{\mu}^2 [2\widetilde{\delta} + 5]}{[\widetilde{\delta} + 1]^2} = 0 \ ,$$

which, after some algebraic manipulations, result to be

(3.4) $$\widetilde{\mu} = \bar{Y} \qquad \text{and} \qquad \widetilde{\delta} = \frac{1 - \widetilde{\kappa}^2 + \sqrt{3\widetilde{\kappa}^2 + 1}}{\widetilde{\kappa}^2} \ ,$$

where $\widetilde{\kappa} = \sqrt{S^2}/\bar{Y}$ is the sample coefficient of variation (CV), with $0 < \widetilde{\kappa} < \sqrt{5}$. Therefore, we have that (3.4) can be rewritten as

$$\widetilde{\mu} = \bar{Y} \qquad \text{and} \qquad \widetilde{\delta} = \frac{\bar{Y}^2 - S^2 + \sqrt{\bar{Y}^4 + 3\bar{Y}^2 S^2}}{S^2} \ . \qquad \square$$

Theorem 3.2. *Let* $\boldsymbol{Y} = [Y_1, ..., Y_n]^\top$ *be a random sample of size* n *from* $Y \sim \mathrm{BS}(\mu, \delta)$. *Then,* $\widetilde{\mu}$ *and* $\widetilde{\delta}$ *have an asymptotic bivariate normal joint distribution, that is,* $[\widetilde{\mu}, \widetilde{\delta}]^\top$ *approximately follows the distribution*

$$\mathrm{N}_2\left(\begin{bmatrix}\mu\\ \delta\end{bmatrix}, \frac{1}{n}\begin{bmatrix}\frac{\mu^2\{2\delta+5\}}{\{\delta+1\}^2} & -\frac{\mu\{2\delta^2+8\delta-3\}}{\{\delta+1\}\,\{\delta+4\}}\\ -\frac{\mu\{2\delta^2+8\delta-3\}}{\{\delta+1\}\,\{\delta+4\}} & \frac{2\delta^4+28\delta^3+122\delta^2+126\delta+57}{\{\delta+4\}^2}\end{bmatrix}\right).$$

Proof: Let $\boldsymbol{Y} = [Y_1, ..., Y_n]^\top$ be independent identically distributed (IID) RVs according to $Y \sim \mathrm{BS}(\mu, \delta)$ and $\mathrm{E}[Y^4]$ given in (2.4) be finite. In addition, let $\widetilde{\mu} = f_1(\bar{Y}, S^2)$ and $\widetilde{\delta} = f_2(\bar{Y}, S^2)$ be the moment estimators of the parameters μ and δ, respectively. Assume that the random vector

$$\sqrt{n}\begin{bmatrix}\bar{Y} - \mathrm{E}[Y]\\ S^2 - \mathrm{E}[Y-\mu]^2\end{bmatrix}$$

approximately follows the distribution

$$\mathrm{N}_2\left(\begin{bmatrix}0\\ 0\end{bmatrix}, \boldsymbol{\Sigma}\right), \qquad \text{where } \boldsymbol{\Sigma} = \begin{bmatrix}\nu & \mu_3\\ \mu_3 & \mu_4-\nu^2\end{bmatrix},$$

with

$$\nu = \mathrm{Var}[Y] = \frac{\mu^2[2\delta+5]}{[\delta+1]^2}, \quad \mu_3 = \frac{4[3\delta+11]\mu^3}{[\delta+1]^3} \quad \text{and} \quad \mu_4-\nu^2 = \frac{8\mu^4[\delta^2+20\delta+76]}{[\delta+1]^4}.$$

We want to determine the asymptotic joint distribution of the estimators $\widetilde{\mu} = f_1(\bar{Y}, S^2)$ and $\widetilde{\delta} = f_2(\bar{Y}, S^2)$. These estimators can be expressed as

$$f_1(x,y) = x \qquad \text{and} \qquad f_2(x,y) = \frac{x^2 - y + \sqrt{x^4 + 3x^2y}}{y}.$$

By using the delta method (see Rao, 1965), we obtain that the random vector

$$\sqrt{n}\begin{bmatrix}\widetilde{\mu} - \mu\\ \widetilde{\delta} - \delta\end{bmatrix}$$

approximately follows the distribution

$$\mathrm{N}_2\left(\begin{bmatrix}0\\ 0\end{bmatrix}, \boldsymbol{\Sigma}\right),$$

where

$$\boldsymbol{\Sigma} = \begin{bmatrix}\frac{\mu^2\{2\delta+5\}}{\{\delta+1\}^2} & -\frac{\mu\{2\delta^2+8\delta-3\}}{\{\delta+1\}\,\{\delta+4\}}\\ -\frac{\mu\{2\delta^2+8\delta-3\}}{\{\delta+1\}\,\{\delta+4\}} & \frac{\{2\delta^4+28\delta^3+122\delta^2+126\delta+57\}}{\{\delta+4\}^2}\end{bmatrix}. \qquad \square$$

3.3. Modified moment estimation

Ng *et al.* (2003) used the fact that the BS distribution satisfies the reciprocation property to propose MM estimates for its parameters. The MM estimation method is a variation of the moment estimation method, substituting the expression that equates the second population and sample moments by equating the expected value of $1/Y$ with $[1/n]\sum_{j=1}^{n} 1/Y_j$. Because the reparameterized BS distribution preserves the reciprocation property, once again, the MM estimates of its parameters μ and δ can be easily obtained.

Theorem 3.3. *Let $\boldsymbol{Y} = [Y_1, ..., Y_n]^\top$ be a random sample of size n from $Y \sim \mathrm{BS}(\mu, \delta)$. Then, the MM estimators of μ and δ are, respectively,*

$$\breve{\mu} = \bar{Y} \qquad \text{and} \qquad \breve{\delta} = \left[\sqrt{\frac{\bar{Y}}{\bar{Y}_h}} - 1\right]^{-1},$$

where $\bar{Y}_h = \left[\{1/n\}\sum_{j=1}^{n}\{1/Y_j\}\right]^{-1}$.

Proof: Let $\boldsymbol{Y} = [Y_1, ..., Y_n]^\top$ be a random sample of size n from $Y \sim \mathrm{BS}(\mu, \delta)$. Then, $\mathrm{E}[Y] = \mu$ and $\mathrm{E}[1/Y] = [\delta+1]^2/[\mu\delta^2]$. Thus,

$$g(Y_j, \boldsymbol{\theta}) = \left[Y_j - \mu\,, \; \frac{1}{Y_j} - \frac{\{\delta+1\}^2}{\mu\,\delta^2}\right]^\top.$$

Recall the moment conditions are $\mathrm{E}[g(Y_j, \boldsymbol{\theta}_0)] = \mathbf{0}$. We have that $g_n(\breve{\boldsymbol{\theta}}) = \mathbf{0}$, with g_n defined in (3.3), implies that

$$\text{(3.5)} \qquad \frac{1}{n}\sum_{j=1}^{n} Y_j - \breve{\mu} = 0 \qquad \text{and} \qquad \frac{1}{n}\sum_{j=1}^{n}\frac{1}{Y_j} - \frac{[\breve{\delta}+1]^2}{\breve{\mu}\,\breve{\delta}^2} = 0\,.$$

Hence, solving (3.5), we obtain the MM estimators

$$\breve{\mu} = \bar{Y} \qquad \text{and} \qquad \breve{\delta} = \left[\sqrt{\frac{\bar{Y}}{\bar{Y}_h}} - 1\right]^{-1},$$

where $\bar{Y}_h$ is defined in Theorem 3.3. In addition, we have that $\breve{\delta}$ is well-defined for $\bar{Y}_h \neq \bar{Y}$, when $\bar{Y}_h < \bar{Y}$. □

Theorem 3.4. *Let $\boldsymbol{Y} = [Y_1, ..., Y_n]^\top$ be a random sample of size n from $Y \sim \mathrm{BS}(\mu, \delta)$. Then, $\breve{\mu}$ and $\breve{\delta}$ have an asymptotic bivariate normal joint distribution, that is, $[\breve{\mu}, \breve{\delta}]^\top$ approximately follows the distribution*

$$\mathrm{N}_2\left(\begin{bmatrix}\mu\\ \delta\end{bmatrix}, \; \frac{1}{n}\begin{bmatrix}\frac{\mu^2\{2\delta+5\}}{\{\delta+1\}^2} & -\frac{2\mu\delta}{\delta+1}\\ -\frac{2\mu\delta}{\delta+1} & 2\,\delta^2\end{bmatrix}\right).$$

Proof: Let $\boldsymbol{Y} = [Y_1, ..., Y_n]^\top$ be IID RVs according to $Y \sim \mathrm{BS}(\mu, \delta)$ and $\mathrm{E}[Y_j^4] < \infty$. Then, the vector $[\bar{Y}, \bar{Y}_h^{-1}]^\top$ follows an asymptotic bivariate normal distribution, which implies that

$$\sqrt{n}\begin{bmatrix} \bar{Y} - \mathrm{E}[Y] \\ \bar{Y}_h^{-1} - \mathrm{E}[Y^{-1}] \end{bmatrix} \;\dot{\sim}\; \mathrm{N}_2\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \boldsymbol{\Sigma}\right),$$

where "$\dot{\sim}$" means "approximately follows the distribution" and

$$\boldsymbol{\Sigma} = \begin{bmatrix} \mathrm{Var}[Y] & \mathrm{Cov}[Y, Y^{-1}] \\ \mathrm{Cov}[Y, Y^{-1}] & \mathrm{Var}[Y^{-1}] \end{bmatrix},$$

with

$$\mathrm{Var}[Y] = \frac{\mu^2[2\delta+5]}{[\delta+1]^2}, \quad \mathrm{Cov}[Y,Y^{-1}] = 1 - \frac{[\delta+1]^2}{\delta^2} \quad \text{and} \quad \mathrm{Var}[Y^{-1}] = \frac{[2\delta+5]\,[\delta+1]^2}{\mu^2\,\delta^4}.$$

However, our interest is to find the asymptotic joint distribution of $\breve{\mu} = f_1(\bar{Y}, \bar{Y}_h^{-1})$ and $\breve{\delta} = f_2(\bar{Y}, \bar{Y}_h^{-1})$. For these estimators, consider $f_1(x,y) = x$, $f_2(x,y) = [\sqrt{xy} - 1]^{-1}$ and the delta method. Then,

$$\sqrt{n}\begin{bmatrix} \breve{\mu} - \mu \\ \breve{\delta} - \delta \end{bmatrix} \;\dot{\sim}\; \mathrm{N}_2\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \boldsymbol{\Sigma}\right),$$

where

$$\boldsymbol{\Sigma} = \begin{bmatrix} \frac{\mu^2\{2\delta+5\}}{\{\delta+1\}^2} & -\frac{2\mu\delta}{\delta+1} \\ -\frac{2\mu\delta}{\delta+1} & 2\,\delta^2 \end{bmatrix}.$$

□

3.4. Generalized moment estimation

The GM method provides estimators that are in general consistent, but in general not efficient. The GM method is an extension of the usual moment estimation method; see details in Mátyás (1999) and in the following definition.

Definition 3.2. Let $\boldsymbol{Y} = [Y_1, ..., Y_n]^\top$ be a random sample of size n from any distribution. We want to estimate an unknown $p \times 1$ parameter vector $\boldsymbol{\theta}$, with true value $\boldsymbol{\theta}_0$. Let $\mathrm{E}[g(Y_j, \boldsymbol{\theta}_0)] = \boldsymbol{0}$ be a set of q moment conditions and $g_n(\boldsymbol{\theta})$ be the corresponding sample moments given in (3.3). Define the criterion function

$$Q_n(\boldsymbol{\theta}) = g_n(\boldsymbol{\theta})^\top \boldsymbol{A}_n^{-1} g_n(\boldsymbol{\theta}),$$

where $\boldsymbol{A}_n$ is a $O_p(1)$ stochastic positive definite matrix. Then, the GM estimator of $\boldsymbol{\theta}$ is

$$\breve{\boldsymbol{\theta}} = \underset{\boldsymbol{\theta}}{\mathrm{argmin}}\; Q_n(\boldsymbol{\theta}). \tag{3.6}$$

As mentioned, in general, the GM method provides consistent estimators, but $\boldsymbol{\theta}$ must be the unique solution of $\mathrm{E}[g(Y_j, \boldsymbol{\theta})]$ and an element of a compact space. Some assumptions on high order moments of $g(Y_j, \boldsymbol{\theta})$ also are needed. However, there are no restrictions on the model that generates the data, except for the case of dependent data.

Considering $q > p$ in Definition 3.2, we can perform the $\mathcal{J}$ test (see Hansen, 1982) to assess the moment conditions and/or the specification of model, because it acts as an omnibus test for model misspecification. In this case, the null hypothesis H_0: $\mathrm{E}[g(Y_j, \boldsymbol{\theta}_0)] = \mathbf{0}$ can be tested by using the statistic $n g_n(\breve{\boldsymbol{\theta}})^\top \breve{\boldsymbol{A}}_n^{-1} g_n(\breve{\boldsymbol{\theta}})$, which approximately follows the χ^2_{q-p} distribution under H_0; see Mátyás (1999). If the model is misspecified and/or some of the moment conditions do not hold, then the $\mathcal{J}$ statistic will have a small p-value.

Theorem 3.5. *Let $\boldsymbol{Y} = [Y_1, ..., Y_n]^\top$ be a random sample of size n from $Y \sim \mathrm{BS}(\mu, \delta)$. Then, the GM estimators of μ and δ, $\breve{\mu}$ and $\breve{\delta}$ say, can be obtained in a general setting from* (3.6).

Proof: The result is direct from (3.6). □

Theorem 3.6. *Let $\boldsymbol{Y} = [Y_1, ..., Y_n]^\top$ be a random sample of size n from $Y \sim \mathrm{BS}(\mu, \delta)$. Then, $\breve{\mu}$ and $\breve{\delta}$ have an asymptotic bivariate normal joint distribution, that is, $[\breve{\mu}, \breve{\delta}]^\top$ approximately follows the distribution*

$$\mathrm{N}_2\left(\begin{bmatrix}\mu \\ \delta\end{bmatrix}, \frac{1}{n}\boldsymbol{V}\right),$$

where

$$\boldsymbol{V} = \mathrm{E}\left[\frac{\partial g(Y_j, \boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right]^\top \boldsymbol{A}_n^{-1} \mathrm{E}\left[\frac{\partial g(Y_j, \boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right].$$

Proof: Given some regularity conditions (see Mátyás, 1999, Section 1.3.2), as n goes to infinity, the GM estimator converges to a bivariate normal distribution and so the random vector $\sqrt{n}[\breve{\boldsymbol{\theta}} - \boldsymbol{\theta}] \dot{\sim} \mathrm{N}_2(\mathbf{0}, \boldsymbol{V})$, where

$$\boldsymbol{V} = \mathrm{E}\left[\frac{\partial g(Y_j, \boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right]^\top \boldsymbol{A}_n^{-1} \mathrm{E}\left[\frac{\partial g(Y_j, \boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right]. \qquad \square$$

To obtain point and interval estimates of the parameters of the BS distribution, we can use the `gmm` package (see Chaussé, 2010) of the `R` software (`www.R-project.org`). The matrix $\boldsymbol{A}_n$, which produces efficient estimators for $\boldsymbol{\theta}$, can be estimated by an heteroskedasticity and autocorrelation consistent covariance matrix; see Newey & West (1987) and Chaussé (2010). To obtain the corresponding estimates, we run the `gmm` function using as starting values $\mu_0 = \breve{\mu}$ and $\delta_0 = \breve{\delta}$. To test the specification of estimated model, we use the $\mathcal{J}$ test through of the `specTest()` function also available in the `gmm` package.

4. SIMULATION

In this section, we conduct a study based on MC simulations to evaluate the performance of the GM, ML, MM and moment estimators for the reparameterized BS distribution.

MC replications are based on Algorithm 1.For each replication generated by this algorithm, we calculate GM, ML, MM and moment estimates. The algorithm and estimation methods are implemented in the `R` software by using the `gamlss` (see Stasinopoulos & Rigby, 2007) and `gmm` packages. For details about generation of numbers from the BS distribution, see Leiva *et al.* (2008b) and Barros *et al.* (2009). Then, the mean, bias, standard error (SE) and squared root of the mean squared error ($\sqrt{\text{MSE}}$) of these estimators are empirically computed. We obtain point estimates, confidence intervals (CIs) and their coverage probabilities (CPs) of 95% level, based on the asymptotic results associated with each estimator given in Section 3. The ML estimates are obtained from the `gamlss()` function and the GM estimates from the `gmm()` function. The CIs based on the GM estimates are obtained by using the `R` function `confint()`, where the main argument is an object of the `gmm` class. The scenario of this simulation study considers 10 000 MC replications in each case, sample sizes $n \in \{30, 50, 75, 100, 200\}$ and values for $\delta \in \{0.5, 2.0, 8.0, 32.0, 200\}$ (according to different levels of skewness) and $\mu = 1.0$ (without loss of generality). The obtained results are presented in Tables 2, 3, 4 and 5.

To perform the GM estimation of the parameters μ and δ of the BS distribution, we consider the following vector of moment conditions:

$$\mathrm{E}\left[g(Y_j, \boldsymbol{\theta})\right] = \mathrm{E}\begin{bmatrix} \mu - Y_j \\ \frac{\mu^2\{2\delta+5\}}{\{\delta+1\}^2} - \{Y_j - \mu\}^2 \\ \frac{\{\delta+1\}^2}{\mu\delta^2} - \frac{1}{Y_j} \end{bmatrix} = \mathbf{0}\,,$$

where the gradient function of $g_n(\boldsymbol{\theta})$ is given by

$$G = \frac{\partial g_n(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = \mathrm{E}\begin{bmatrix} 1 & 0 \\ \frac{2\mu\{2\delta+5\}}{\{\delta+1\}^2} - 2\mu + 2\bar{Y} & -\frac{2\mu^2\{\delta+4\}}{\{\delta+1\}^3} \\ -\frac{\{\delta+1\}^2}{\{\mu\delta\}^2} & -\frac{2\{\delta+1\}}{\mu\delta^3} \end{bmatrix}.$$

From Tables 2 through 5, note that the ML, MM and moment estimators of the parameter μ present similar statistical properties in relation to the empirical bias and $\sqrt{\text{MSE}}$. However, the GM estimator presents similar properties to the other estimators only when the sample size is large. In the case of the parameter δ, its ML and MM estimators present similar properties for the different sample sizes and true values assumed for this parameter. Table 3 shows that, in general, the GM method underestimates the true value of μ. From Tables 4 and 5, note that the values of the empirical SE and $\sqrt{\text{MSE}}$ increase as δ increases, for all the considered methods, in the case of the parameter δ. Nevertheless, in the case of the parameter μ, we have a reverse behavior, that is, the values of the empirical SE and $\sqrt{\text{MSE}}$ decrease as δ increases, for all the considered methods. In addition, the GM estimator presents the worse behavior in terms of statistical properties, but, as the sample size increases, the estimators obtained by this method turn to be more competitive, with respect to the other estimators considered.

Table 6 provides empirical CPs of 95% CIs for the parameters of the BS(μ, δ) distribution. Note that the CIs based on the GM estimates have CPs smaller than those from the other methods. However, as the sample size increases, the distance between CPs for the fixed confidence levels decreases. Also, when the true value of δ increases, the distance between the confidence level (0.95) and the empirical CP decreases. Thus, such as in the study based on point estimation, for interval estimation, ML and MM estimators present similar statistical properties and better than the other estimators considered.

Table 2: Empirical mean of the estimator of the indicated parameter, method, n and δ, with $\mu = 1.0$.

n	δ	μ				δ			
		ML	Moment	MM	GM	ML	Moment	MM	GM
30	0.5	1.004	1.002	1.002	0.869	0.561	0.772	0.561	0.633
	2.0	1.001	1.001	1.001	0.929	2.232	2.526	2.232	2.508
	8.0	1.000	1.000	1.000	0.978	8.886	9.285	8.886	9.949
	32.0	1.000	1.000	1.000	1.005	35.477	35.920	35.477	40.352
	200.0	1.000	1.000	1.000	1.003	221.734	222.150	221.734	245.663
50	0.5	0.999	0.998	0.998	0.896	0.536	0.668	0.536	0.578
	2.0	0.999	0.999	0.999	0.946	2.137	2.321	2.137	2.319
	8.0	1.000	1.000	1.000	0.981	8.522	8.775	8.522	9.174
	32.0	1.000	1.000	1.000	1.002	34.058	34.339	34.058	37.064
	200.0	1.000	1.000	1.000	1.002	212.782	213.040	212.782	227.794
75	0.5	0.998	0.996	0.996	0.916	0.524	0.610	0.524	0.552
	2.0	0.999	0.998	0.998	0.958	2.092	2.210	2.092	2.220
	8.0	0.999	0.999	0.999	0.985	8.355	8.518	8.355	8.835
	32.0	1.000	1.000	1.000	1.000	33.385	33.559	33.385	35.463
	200.0	1.000	1.000	1.000	1.001	208.676	208.810	208.676	219.441
100	0.5	0.999	0.998	0.998	0.933	0.518	0.581	0.518	0.539
	2.0	0.999	0.999	0.999	0.967	2.068	2.150	2.068	2.163
	8.0	1.000	1.000	1.000	0.988	8.261	8.377	8.261	8.634
	32.0	1.000	1.000	1.000	0.999	33.022	33.148	33.022	34.590
	200.0	1.000	1.000	1.000	1.001	206.366	206.453	206.366	214.828
200	0.5	0.998	0.997	0.997	0.960	0.509	0.541	0.509	0.521
	2.0	0.999	0.999	0.999	0.980	2.036	2.077	2.036	2.085
	8.0	1.000	0.999	0.999	0.993	8.137	8.195	8.137	8.338
	32.0	1.000	1.000	1.000	0.998	32.529	32.600	32.529	33.313
	200.0	1.000	1.000	1.000	1.001	203.274	203.362	203.274	207.927

Table 3: Empirical bias of the estimator of the indicated parameter, method, n and δ, with $\mu = 1.0$.

n	δ	μ				δ			
		ML	Moment	MM	GM	ML	Moment	MM	GM
30	0.5	0.004	0.002	0.002	−0.131	0.061	0.272	0.061	0.133
	2.0	0.001	0.001	0.001	−0.071	0.232	0.526	0.232	0.508
	8.0	0.000	0.000	0.000	−0.022	0.886	1.285	0.886	1.949
	32.0	0.000	0.000	0.000	0.005	3.477	3.920	3.477	8.352
	200.0	0.000	0.000	0.000	0.003	21.734	22.150	21.734	45.663
50	0.5	−0.001	−0.002	−0.002	−0.104	0.036	0.168	0.036	0.078
	2.0	−0.001	−0.001	−0.001	−0.054	0.137	0.321	0.137	0.319
	8.0	0.000	0.000	0.000	−0.019	0.522	0.775	0.522	1.174
	32.0	0.000	0.000	0.000	0.002	2.058	2.339	2.058	5.064
	200.0	0.000	0.000	0.000	0.002	12.782	13.040	12.782	27.794
75	0.5	−0.002	−0.004	−0.004	−0.084	0.024	0.110	0.024	0.052
	2.0	−0.001	−0.002	−0.002	−0.042	0.092	0.210	0.092	0.220
	8.0	−0.001	−0.001	−0.001	−0.015	0.355	0.518	0.355	0.835
	32.0	0.000	0.000	0.000	0.000	1.385	1.559	1.385	3.463
	200.0	0.000	0.000	0.000	0.001	8.676	8.810	8.676	19.441
100	0.5	−0.001	−0.002	−0.002	−0.067	0.018	0.081	0.018	0.039
	2.0	−0.001	−0.001	−0.001	−0.033	0.068	0.150	0.068	0.163
	8.0	0.000	0.000	0.000	−0.012	0.261	0.377	0.261	0.634
	32.0	0.000	0.000	0.000	−0.001	1.022	1.148	1.022	2.590
	200.0	0.000	0.000	0.000	0.001	6.366	6.453	6.366	14.828
200	0.5	−0.002	−0.003	−0.003	−0.040	0.009	0.041	0.009	0.021
	2.0	−0.001	−0.001	−0.001	−0.020	0.036	0.077	0.036	0.085
	8.0	0.000	−0.001	−0.001	−0.007	0.137	0.195	0.137	0.338
	32.0	0.000	0.000	0.000	−0.002	0.529	0.600	0.529	1.313
	200.0	0.000	0.000	0.000	0.001	3.274	3.362	3.274	7.927

Table 4: Empirical SE of the estimator of the indicated parameter, method, n and δ, with $\mu = 1.0$.

n	δ	μ				δ			
		ML	Moment	MM	GM	ML	Moment	MM	GM
30	0.5	0.296	0.298	0.298	0.308	0.162	0.440	0.162	0.257
	2.0	0.182	0.182	0.182	0.195	0.638	0.986	0.638	0.925
	8.0	0.092	0.092	0.092	0.102	2.532	2.993	2.532	3.533
	32.0	0.046	0.046	0.046	0.051	10.100	10.627	10.100	13.359
	200.0	0.018	0.018	0.018	0.020	63.122	63.636	63.122	78.065
50	0.5	0.226	0.228	0.228	0.237	0.113	0.340	0.113	0.153
	2.0	0.139	0.139	0.139	0.148	0.448	0.733	0.448	0.582
	8.0	0.071	0.071	0.071	0.078	1.786	2.186	1.786	2.212
	32.0	0.035	0.035	0.035	0.040	7.134	7.624	7.134	8.809
	200.0	0.014	0.014	0.014	0.015	44.580	45.136	44.580	51.508
75	0.5	0.185	0.187	0.187	0.193	0.089	0.276	0.089	0.104
	2.0	0.114	0.114	0.114	0.121	0.353	0.591	0.353	0.432
	8.0	0.058	0.058	0.058	0.063	1.404	1.744	1.404	1.663
	32.0	0.029	0.029	0.029	0.032	5.609	6.025	5.609	6.693
	200.0	0.012	0.012	0.012	0.013	35.043	35.502	35.043	39.398
100	0.5	0.159	0.160	0.160	0.166	0.075	0.240	0.075	0.084
	2.0	0.099	0.099	0.099	0.104	0.299	0.504	0.299	0.347
	8.0	0.051	0.051	0.051	0.055	1.191	1.484	1.191	1.372
	32.0	0.025	0.025	0.025	0.028	4.764	5.128	4.764	5.535
	200.0	0.010	0.010	0.010	0.011	29.733	30.126	29.733	32.884
200	0.5	0.114	0.115	0.115	0.118	0.051	0.172	0.051	0.055
	2.0	0.070	0.070	0.070	0.073	0.206	0.354	0.206	0.221
	8.0	0.036	0.036	0.036	0.037	0.820	1.028	0.820	0.884
	32.0	0.018	0.018	0.018	0.019	3.283	3.538	3.283	3.563
	200.0	0.007	0.007	0.007	0.008	20.510	20.790	20.510	21.865

Table 5: Empirical $\sqrt{\text{MSE}}$ of the estimator of the indicated parameter, method, n and δ, with $\mu = 1.0$.

n	δ	μ				δ			
		ML	Moment	MM	GM	ML	Moment	MM	GM
30	0.5	0.296	0.298	0.298	0.334	0.173	0.517	0.173	0.290
	2.0	0.182	0.182	0.182	0.208	0.679	1.117	0.679	1.055
	8.0	0.092	0.092	0.092	0.104	2.683	3.257	2.683	4.035
	32.0	0.046	0.046	0.046	0.052	10.682	11.327	10.682	15.755
	200.0	0.018	0.018	0.018	0.020	66.759	67.380	66.759	90.440
50	0.5	0.226	0.228	0.228	0.259	0.119	0.379	0.119	0.172
	2.0	0.139	0.139	0.139	0.158	0.469	0.800	0.469	0.663
	8.0	0.071	0.071	0.071	0.080	1.861	2.320	1.861	2.505
	32.0	0.035	0.035	0.035	0.040	7.425	7.975	7.425	10.161
	200.0	0.014	0.014	0.014	0.016	46.376	46.981	46.376	58.528
75	0.5	0.185	0.187	0.187	0.210	0.092	0.297	0.092	0.116
	2.0	0.114	0.114	0.114	0.128	0.365	0.627	0.365	0.485
	8.0	0.058	0.058	0.058	0.065	1.448	1.819	1.448	1.861
	32.0	0.029	0.029	0.029	0.032	5.777	6.223	5.777	7.536
	200.0	0.012	0.012	0.012	0.013	36.101	36.578	36.101	43.933
100	0.5	0.159	0.161	0.161	0.179	0.077	0.253	0.077	0.093
	2.0	0.099	0.099	0.099	0.109	0.307	0.526	0.307	0.383
	8.0	0.051	0.051	0.051	0.056	1.219	1.531	1.219	1.511
	32.0	0.025	0.025	0.025	0.028	4.873	5.255	4.873	6.111
	200.0	0.010	0.010	0.010	0.011	30.407	30.809	30.407	36.072
200	0.5	0.114	0.115	0.115	0.125	0.052	0.177	0.052	0.059
	2.0	0.070	0.070	0.070	0.075	0.209	0.362	0.209	0.237
	8.0	0.036	0.036	0.036	0.038	0.832	1.046	0.832	0.947
	32.0	0.018	0.018	0.018	0.019	3.325	3.589	3.325	3.797
	200.0	0.007	0.007	0.007	0.008	20.770	21.060	20.770	23.258

Table 6: CP of 95% CIs for the indicated parameter, method, n and δ, with $\mu = 1.0$.

n	δ	μ				δ			
		ML	Moment	MM	GM	ML	Moment	MM	GM
30	0.5	0.899	0.884	0.896	0.622	0.956	0.993	0.957	0.864
	2.0	0.917	0.906	0.916	0.707	0.956	0.983	0.956	0.861
	8.0	0.930	0.924	0.930	0.785	0.956	0.970	0.956	0.858
	32.0	0.937	0.935	0.937	0.826	0.956	0.961	0.956	0.836
	200.0	0.942	0.940	0.942	0.815	0.956	0.958	0.956	0.880
50	0.5	0.999	0.903	0.914	0.703	0.955	0.984	0.955	0.886
	2.0	0.929	0.921	0.930	0.779	0.954	0.978	0.954	0.878
	8.0	0.939	0.934	0.938	0.826	0.954	0.967	0.954	0.886
	32.0	0.943	0.941	0.942	0.857	0.954	0.960	0.953	0.864
	200.0	0.943	0.943	0.943	0.843	0.953	0.954	0.953	0.896
75	0.5	0.928	0.920	0.926	0.757	0.954	0.982	0.953	0.904
	2.0	0.936	0.930	0.936	0.820	0.954	0.973	0.953	0.899
	8.0	0.941	0.938	0.940	0.862	0.954	0.964	0.954	0.901
	32.0	0.943	0.942	0.942	0.880	0.954	0.957	0.953	0.887
	200.0	0.944	0.944	0.944	0.862	0.954	0.953	0.954	0.906
100	0.5	0.935	0.929	0.933	0.794	0.952	0.978	0.952	0.913
	2.0	0.942	0.939	0.942	0.848	0.952	0.972	0.952	0.910
	8.0	0.944	0.942	0.944	0.879	0.953	0.961	0.953	0.911
	32.0	0.944	0.940	0.944	0.888	0.952	0.955	0.952	0.897
	200.0	0.944	0.944	0.943	0.869	0.952	0.953	0.952	0.912
200	0.5	0.940	0.935	0.938	0.851	0.952	0.978	0.952	0.926
	2.0	0.944	0.942	0.943	0.888	0.951	0.969	0.951	0.926
	8.0	0.949	0.947	0.948	0.916	0.950	0.958	0.950	0.926
	32.0	0.948	0.949	0.948	0.916	0.950	0.952	0.950	0.925
	200.0	0.947	0.947	0.947	0.894	0.950	0.950	0.950	0.927

5. APPLICATIONS

In this section, we provide a practical illustration of the proposed estimation methods based on two real-world data sets, with moderate and large sample sizes and from two fields: economics and engineering.

5.1. Data set I (S1): Griffiths *et al.* (1993)

The first data set (S1) is presented in Griffiths *et al.* (1993) and corresponds to household expenditures for food in the United States (US) expressed in thousands of US dollars (M\$). These data are provided in Table 7.

Table 7: Household expenditures for food (in M\$) (Griffiths *et al.*, 1993).

15.998	16.652	21.741	7.431	10.481	13.548	23.256	17.976	14.161	8.825
14.184	19.604	13.728	21.141	17.446	9.629	14.005	9.160	18.831	7.641
13.882	9.670	21.604	10.866	28.980	10.882	18.561	11.629	18.067	14.539
19.192	25.918	28.833	15.869	14.910	9.550	23.066	14.751		

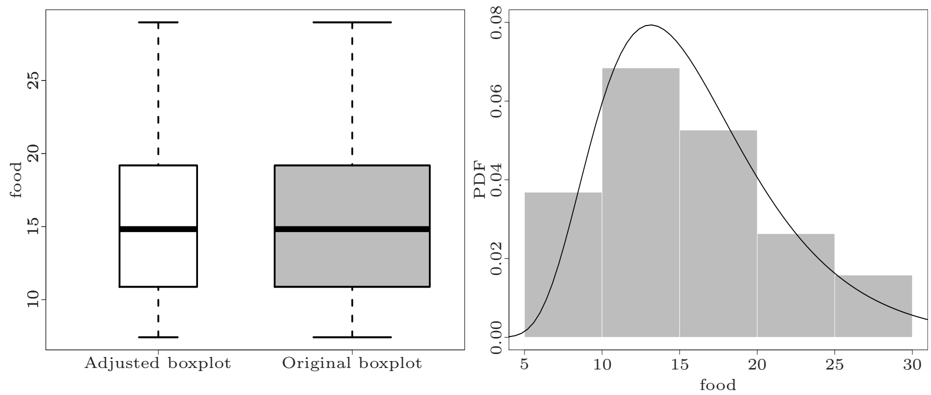
Table 8 presents a descriptive summary of S1 that includes sample mean ($\bar{y}$), median ($\tilde{y}$), standard deviation (SD), CV, coefficients of skewness (CS) and of kurtosis (CK), and minimum ($y_{(1)}$) and maximum ($y_{(n)}$) values. Note that the empirical distribution of the studied RV is slightly positive skewed. Figure 3 presents the boxplots and histogram for S1. From Figure 3(a), note that the adjusted and usual boxplots exhibit the same behavior, which makes sense because the data have little asymmetry. From Figure 3(b), note that the BS distribution fits the data well, whose PDF is estimated with $\widehat{\mu} = 15.95$ and $\widehat{\delta} = 15.57$. Point estimates of the μ and δ parameters of the BS distribution for the proposed methods, and 90% and 95% CIs for these parameters, are displayed in Table 9.

Table 8: Descriptive statistics for S1 (in M\$).

$y_{(1)}$	$\tilde{y}$	$\bar{y}$	$y_{(n)}$	SD	CV	CS	CK
7.431	14.831	15.953	28.980	5.624	0.353	0.525	2.556

Table 9: Estimates and CIs for indicated parameter and method with S1.

Method	μ			δ		
	Estimate	CI(90%)	CI(95%)	Estimate	CI(90%)	CI(95%)
ML	15.95	[14.41;17.50]	[14.11;17.79]	15.57	[9.70;21.45]	[8.57;22.57]
Moment	15.95	[14.47;17.43]	[14.19;17.72]	16.91	[9.51;24.31]	[8.10;25.72]
MM	15.95	[14.41;17.50]	[14.11;17.79]	15.57	[9.70;21.45]	[8.57;22.57]
GM	15.30	[14.31;16.30]	[14.12;16.49]	15.94	[10.96;20.92]	[10.00;21.87]



(a) (b)

Figure 3: Boxplots (a) and histogram with estimated PDF (b) for S1.

Next, we evaluate the fitting of the BS distribution to S1 with goodness-of-fit tests. Consider the null hypothesis H_0: "the data come from a RV $Y \sim \mathrm{BS}(\mu, \delta)$" versus the alternative hypothesis H_1: "the data do not come from this RV". We use the Cramér–von Mises (CM) and Anderson–Darling (AD) statistics to test these hypotheses; see Barros *et al.* (2014). The corresponding p-values of the CM and AD tests obtained for S1, with the BS distribution under H_0, are 0.656 and 0.608, respectively. Thus, we do not have evidence to indicate than the BS distribution does not fit these data well. We check moment conditions of the GM method for S1 with the $\mathcal{J}$ test, by using the `R` function `specTest()`, whose p-value is 0.430. Thus, once again the null hypothesis is not rejected for any usual significance level. Therefore, we do not have evidence to conclude that the moment conditions are incorrect or that the BS distribution does not fit S1 well.

5.2. Data set II (S2): Birnbaum & Saunders (1969b)

The second data set (S2) is a classical one used in the literature on the topic. These data were introduced by Birnbaum & Saunders (1969b) and correspond to lifetimes of 6061-T6 aluminum coupons expressed in cycles ($\times 10^{-3}$) at a maximum stress level of 3.1 psi ($\times 10^4$), until the failure to occur. These coupons were cut parallel to the direction of rolling and oscillating at 18 cycles per seconds. The data are displayed in Table 10.

Table 10: Lifetimes (in cycles $\times 10^{-3}$) (Birnbaum & Saunders, 1969b).

70	90	96	97	99	100	103	104	104	105	107	108	108	108	109
109	112	112	113	114	114	114	116	119	120	120	120	121	121	123
124	124	124	124	124	128	128	129	129	130	130	130	131	131	131
131	131	132	132	132	133	134	134	134	134	134	136	136	137	138
138	138	139	139	141	141	142	142	142	142	142	142	144	144	145
146	148	148	149	151	151	152	155	156	157	157	157	157	158	159
162	163	163	164	166	166	168	170	174	196	212				

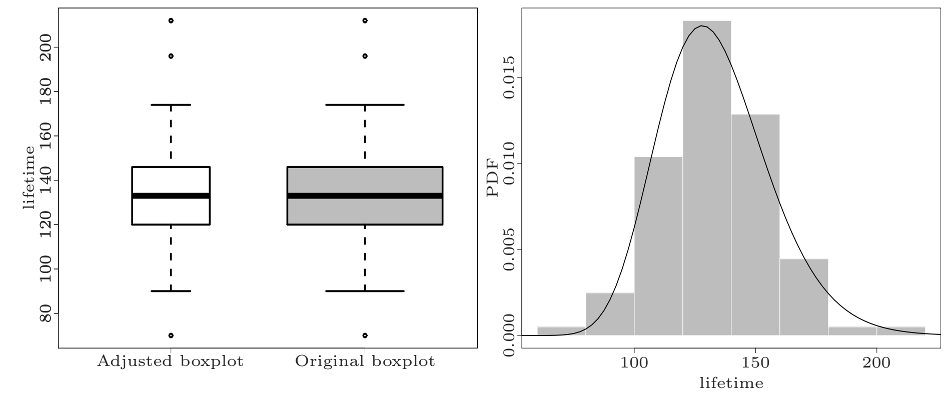
Table 11 presents a descriptive summary of S2 in a similar way to S1. Note that the empirical distribution of the studied RV is relatively symmetric and leptokurtic. Figure 3 presents the boxplots and histogram for S2. From Figure 4(a), note also that the adjusted and usual boxplots exhibit the same behavior, which makes sense because the data have little asymmetry. From Figure 4(b), note that the BS distribution fits the data well, whose PDF is estimated with $\widehat{\mu} = 133.73$ and $\widehat{\delta} = 68.89$. Point estimates of the μ and δ parameters of the BS distribution for the proposed methods, and 90% and 95% CIs for these parameters, for S2, are displayed in Table 12. From this table, we note that less accurate CIs are obtained by the GM method.

Table 11: Descriptive statistics for S2 (in cycles $\times 10^{-3}$).

$y_{(1)}$	$\tilde{y}$	$\bar{y}$	$y_{(n)}$	SD	CV	CS	CK
70.00	133.000	133.733	212.000	22.356	0.167	0.326	3.973

Table 12: Estimates and CIs for indicated parameter and method with S2.

Method	μ			δ		
	Estimate	CI(90%)	CI(95%)	Estimate	CI(90%)	CI(95%)
ML	133.73	[129.99;137.47]	[129.27;138.19]	68.89	[52.95; 84.84]	[49.89;87.89]
Moment	133.73	[130.09;137.37]	[129.39;138.07]	72.76	[55.24; 90.27]	[51.88;93.63]
MM	133.73	[129.99;137.47]	[129.27;138.19]	68.89	[52.95; 84.84]	[49.89;87.89]
GM	137.69	[129.62;145.76]	[128.08;147.31]	75.36	[33.88;116.85]	[25.93;124.80]



(a) (b)

Figure 4: Boxplots (a) and histogram with estimated PDF (b) for S2.

The corresponding p-values of the CM and AD tests obtained for S2 are 0.202 and 0.169, respectively. Thus, we do not have evidence to indicate than the BS distribution does not fit S2 well. The $\mathcal{J}$ test presented a p-value $= 0.720$, so that the null hypothesis is not rejected for any usual significance level. Therefore, we do not have evidence to conclude that the moment conditions are incorrect or that the BS distribution does not fit S2 well.

6. CONCLUSIONS

In this paper, we have provided some novel results on moments and generation of random numbers from a reparameterized version of the Birnbaum–Saunders distribution. In addition, we have studied several estimation methods for this parameterization. We have considered the generalized moment, maximum likelihood, modified moment and moment methods to estimate the corresponding parameters. Furthermore, we have conducted a Monte Carlo study to evaluate the performance of these estimators. From this study, we can conclude that the maximum likelihood and modified moment estimators present similar statistical properties and better that those of the other estimators considered. Therefore, due to the modified moment estimators are easier to compute, we recommend their use for the reparameterized Birnbaum–Saunders distribution. In addition, we have obtained moment estimators in a closed-form, which is not possible with the original parameterization of the Birnbaum–Saunders distribution. However, the parameter estimators obtained by the moment method, as well as those obtained by the generalized moment method, are underperformed with respect to their statistical properties. Nevertheless, for the case of large sample sizes, all the studied estimators have similar statistical properties. We have discussed applications of the BS distribution in different scientific fields and taken advantage of the computational implementation in the `R` software for carrying an application with two real-world data sets, which allowed us to illustrate the obtained results.

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THE k NEAREST NEIGHBORS ESTIMATION OF THE CONDITIONAL HAZARD FUNCTION FOR FUNCTIONAL DATA

Authors: Mohammed Kadi Attouch
– Lab. de Statistique Processus Stochastiques, Univ. Djillali Liabès, Sidi Bel Abbès, BP 89, Sidi Bel Abbès 22000, Algeria
attou_kadi@yahoo.fr

Fatima Zohra Belabed
– Lab. de Statistique Processus Stochastiques, Univ. Djillali Liabès, Sidi Bel Abbès, BP 89, Sidi Bel Abbès 22000, Algeria
zahira_bell@yahoo.fr

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Abstract:

- In this paper, we study the nonparametric estimator of the conditional hazard function using the k nearest neighbors (k-NN) estimation method for a scalar response variable given a random variable taking values in a semi-metric space. We give the almost complete convergence (its corresponding rate) of this estimator and we establish the asymptotic normality. Then the effectiveness of this method is exhibited by a comparison with the kernel method estimation given in Ferraty *et al.* ([12]) and Laksaci and Mechab ([15]) in both cases simulated data and real data.

Key-Words:

- *functional data; nonparametric regression; k-NN estimator; the conditional hazard function; rate of convergence; random bandwidth; asymptotic normality.*

AMS Subject Classification:

- 62G05, 62G08, 62G20, 62G35.

1. INTRODUCTION

The conditional hazard function remains an indispensable tool in survival analysis and many other fields (medicine, reliability or seismology).

The nonparametric estimation of this function in the case of multivariate data is abundant. The first works date back to Waston and Leadbetter ([31]), they introduce the hazard estimate method, since, several results have been developed, see for example, Roussas ([26]) (for previous works), Li and Tran ([18]) (for recent references). The literature has paid quite some attention to nonparametric hazard rate estimation when the data are functional. The first work which deals with this question is Ferraty *et al.* ([12]). They established the almost complete convergence of the kernel estimate of the conditional hazard function in the independent case. This result was extended to the dependent case by Quintela-del-Río ([23]), he treats the almost complete convergence, the mean quadratic convergence and the asymptotic normality of this estimate. The uniform version of the almost complete convergence (with rate) in the i.i.d. case was obtained by Ferraty *et al.* ([10]). Recently, Laksaci and Mechab ([16]) consider the spatial case. The almost complete convergence rate of an adapted estimate of this model are given.

Estimating the conditional hazard function is closely related to the conditional density, and for the last one, the bandwidth selection is very important for the performance of an estimate. The bandwidth must not be too large, so as to prevent over-smoothing, i.e. substantial bias, and must not be too small either, so as prevent detecting the underlying structure. Particularly, in nonparametric curve estimation, the smoothing parameter is critical for the performance.

Starting from this point of view, this work deals with the nonparametric estimation with k nearest neighbors method k-NN, more precisely we consider a kernel estimator of the hazard function constructed from a local window to take into account the exact k nearest neighbors with real response variable Y and functional curves X.

The k nearest neighbor or k-NN estimator is a weighted average of response variables in the neighborhood of x. The existent bibliography of the k-NN method estimation dates back to Royall ([27]) and Stone ([30]) and has received, since, continuous developments (Mack ([20]) derived the rates of convergence for the bias and variance as well as asymptotic normality in the multivariate case, Collomb ([4]) studied different types of convergence (probability, a.s., a.co.) of the estimator of the regression function. Devroye ([6]) obtained the strong consistency and the uniform convergence. For the functional data studies, the k-NN kernel estimate was first introduced in the monograph of Ferraty and Vieu ([13]), Burba *et al.* ([2]) obtained the rate of almost complete convergence of the regression function using the k-NN method for independent data and the asymptotic normality of robust nonparametric regression function was established in Attouch and Benchikh ([1]).

This paper is organized as follows. In Section 2 we present the model and the k-NN estimator. Section 3, is dedicated to fix notations, hypotheses and the presentation of the main results, the almost complete convergence and the asymptotic normality. Section 4 is devoted to some applications in several problems of nonparametric statistics. Some technical auxiliary results are deployed in Section 5, subsequently, in Section 6, we show the proofs of our main result.

2. MODELS AND ESTIMATORS

Let $(X_i, Y_i)_{i=\overline{1,n}}$ be an independent sequence identically distributed (i.i.d.) as (X, Y) which is a random pair valued in $\mathcal{E}\times\mathbb{R}$. Here $(\mathcal{E}, d)$ is a semi-metric space. $\mathcal{E}$ is not necessarily of a finite dimension, and we do not suppose the existence of a density for the functional random variable X.

Our goal, in this article, is to estimate the conditional hazard function defined by:

$$h^X(Y) \;=\; \frac{f^X(Y)}{1 - F^X(Y)}\;, \tag{2.1}$$

where

$f^X(Y)$ is the conditional density function of Y given X,
$F^X(Y)$ is the conditional distribution function of Y given X.

For a fixed $x \in \mathcal{E}$, the k-NN kernel estimator of $h^x(Y{=}y)$ is given by:

$$\widehat{h}^x_{k-NN}(Y{=}y) \;=\; \widehat{h}^x(y) \;=\; \frac{\widehat{f}^x(y)}{1 - \widehat{F}^x(y)}\;, \tag{2.2}$$

with

$$\begin{aligned} F^x(y) \;&=\; \mathbb{P}\big[Y{\leq}\, y \,/\, X{=}\,x\big] \\ &=\; \mathbb{E}\big[\mathbb{1}_{]-\infty,y]} \,/\, X{=}\,x\big] \\ &=\; r\big(\mathbb{1}_{]-\infty,y]}\big)\;, \end{aligned}$$

where $r(\cdot)$ is the regression function defined in Ferraty and Vieu ([13]). Therefore:

$$\widetilde{F}^x(y) \;=\; \widehat{r}\big(\mathbb{1}_{]-\infty,y]}\big) \;=\; \frac{\displaystyle\sum_{i=1}^{n} \mathbb{1}_{]-\infty,y]}\, K\big(H_n^{-1} d(x, X_i)\big)}{\displaystyle\sum_{i=1}^{n} K\big(H_n^{-1} d(x, X_i)\big)}\;.$$

Finally, by Roussas ([25]), Samanta ([28]) and Ferraty and Vieu ([13]), the estimator of the conditional distribution function is given by

$$\widehat{F}^x(y) \;=\; \frac{\displaystyle\sum_{i=1}^{n} K\big(H_n^{-1}d(x,X_i)\big)\, R\big(g_n^{-1}(y-Y_i)\big)}{\displaystyle\sum_{i=1}^{n} K\big(H_n^{-1}d(x,X_i)\big)}\,, \qquad \forall\, y\in\mathbb{R}\,, \tag{2.3}$$

where K is an asymmetrical kernel, H_n is a positive random variable, defined as follows:

$$H_n(x) \;=\; \min\left\{ h\in\mathbb{R}^+ /\ \sum_{i=1}^{n} 1\!\!1_{B(x,h)}(X_i) = k \right\}, \tag{2.4}$$

with

$$B(x,h) \;=\; \Big\{ x'\in\mathcal{E};\ d(x,x') < h \Big\}\,.$$

R is a distribution function and $(g_n)_{n\in\mathbb{N}}$ is a sequence of strictly positive real numbers (depending on n). Under a differentiability assumption of $\widehat{F}^x(y)$, we can obtain the conditional density function by differentiating the conditional distribution function, then we have

$$\widehat{f}^x(y) \;=\; \frac{\partial}{\partial y}\widehat{F}^x(y)$$

and then

$$\widehat{f}^x(y) \;=\; \frac{\displaystyle\sum_{i=1}^{n} K\big(H_n^{-1}d(x,X_i)\big)\, g_n^{-1} R'\big(g_n^{-1}(y-Y_i)\big)}{\displaystyle\sum_{i=1}^{n} K\big(H_n^{-1}d(x,X_i)\big)}\,. \tag{2.5}$$

In parallel, in order to emphasize differences between the k-NN method and the traditional kernel approach, we define the estimator of the conditional hazard function Ferraty *et al.* ([12]) by:

$$\widehat{h}^x_{\text{kernel}}(y) \;=\; \frac{\widehat{f}^x_{\text{kernel}}(y)}{1-\widehat{F}^x_{\text{kernel}}(y)}\,, \tag{2.6}$$

with

$$\widehat{f}^x_{\text{kernel}}(y) \;=\; \frac{\displaystyle\sum_{i=1}^{n} K\big(h_n^{-1}d(x,X_i)\big)\, g_n^{-1} R'\big(g_n^{-1}(y-Y_i)\big)}{\displaystyle\sum_{i=1}^{n} K\big(h_n^{-1}d(x,X_i)\big)} \tag{2.7}$$

and

$$\widehat{F}^x_{\text{kernel}}(y) = \frac{\sum_{i=1}^{n} K\big(h_n^{-1} d(x,X_i)\big)\, R\big(g_n^{-1}(y-Y_i)\big)}{\sum_{i=1}^{n} K\big(h_n^{-1} d(x,X_i)\big)}, \tag{2.8}$$

where K is a kernel, R is a distribution function and $(h_n)_{n\in\mathbb{N}}$, $(g_n)_{n\in\mathbb{N}}$ are sequences of strictly positive numbers.

3. ASYMPTOTIC PROPERTIES OF THE k-NN METHOD

3.1. The almost complete convergence (a.co.)

We focus in the pointwise the almost complete convergence[1] and rate of convergence[2] of the k-NN estimator of the conditional hazard function $\widehat{h}^x(y)$ defined on (2.2).

Before giving the main asymptotic result, we need some assumptions. The first one is about the concentration function $\varphi_x(h)$ and can be interpreted as a small ball probability of the functional variable x given by:

(H1)
$$\begin{aligned}\varphi_x(h) &= \mathbb{P}\big(X \in B(x,h)\big)\\ &= \mathbb{P}\big[X \in \{x'\in\mathcal{E};\ d(x,x')<h\}\big],\end{aligned}$$

with $\varphi_x(h)$ continuous and strictly increasing in a neighborhood of 0 and $\varphi_x(0)=0$.

(H2) We also need a kernel K:
The kernel K is a function from $\mathbb{R}$ into $\mathbb{R}^+$, we say that K is a kernel of type I, so that: there exist two real constants C_1, C_2, $0 < C_1 < C_2 < \infty$, such that

$$C_1 \mathbb{1}_{[0,1]} < K < C_2 \mathbb{1}_{[0,1]} .$$

[1]Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of real random variables. We say that $(X_n)_{n\in\mathbb{N}}$ converges almost completely (a.co.) to some r.r.v. X if and only if:

$$\forall \epsilon > 0, \qquad \sum_{n=1}^{\infty} \mathbb{P}\big[|X_n - X| > \epsilon\big] < \infty .$$

[2]Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of positive real number. We say that $X_n = O_{\text{a.co.}}(u_n)$ if and only if: $\exists\, \epsilon > 0$, so that, $\sum_{n=1}^{\infty} \mathbb{P}\big[|X_n| > \epsilon\, u_n\big] < \infty$. This kind of convergence implies both almost sure convergence and convergence in probability.

K is a kernel of type II, so that: the support of K is $[0,1]$ and if its derivative K' exists on $[0,1]$ and satisfies, for two real constants $-\infty < C_1 < C_2 < 0$,

$$C_1 < K' < C_2 \ .$$

In this case, we also suppose that: $\exists\, C_3 > 0$, $\exists\, \epsilon_0$:

$$\forall\, \epsilon < \epsilon_0 \ , \qquad \int_0^{\epsilon} \varphi_x(u)\, du \ > \ C_3\, \epsilon\, \varphi_x(\epsilon) \ .$$

(**H3**) R is a differentiable function such that:

$$\exists\, C < \infty \, , \quad \forall\, (x_1, x_2) \in \mathbb{R}^2, \qquad |R'(x_1) - R'(x_2)| \le C|x_1 - x_2| \ .$$

$$R' \quad \text{is of the support compact} \ \ [-1, 1] \ .$$

(**H4**) $\exists\, \zeta > 0$:

$$\begin{cases} \forall\, (x_1, x_2) \in \mathbb{R}^2 \, , \quad |R(x_1) - R(x_2)| \ \le \ C|x_1 - x_2| \ , \\ \displaystyle\int |t|^{\zeta} R'(t)\, dt \ < \ \infty \ . \end{cases}$$

(**H5**) $(g_n)_{n \in \mathbb{N}}$ is a strictly positive sequence such that:

$$\begin{cases} \displaystyle\lim_{n \to \infty} g_n = 0 \ , \qquad \exists\, a > 0, \ \ \lim_{n \to \infty} n^a g_n = \infty \ , \\ \displaystyle\lim_{n \to \infty} \frac{\log n}{n\, g_n\, \varphi_x(h)} = 0 \ . \end{cases}$$

The nonparametric model of the function h^x will be determined by regularity conditions of the conditional distribution of Y given X. These conditions are:

(**H6**) N_x will denote a fixed neighborhood of x, S will be a fixed compact subset of $\mathbb{R}$:
We will consider two kinds of nonparametric models. The first one is called the "Lipschitz-type" model that is defined:

$$Lip_{\mathcal{E} \times \mathbb{R}} : \begin{cases} f : \mathcal{E} \times \mathbb{R} \to \mathbb{R} \, , \quad \forall\, (x_1, x_2) \in N_x^2 \, , \quad \forall\, (y_1, y_2) \in S^2, \\ \qquad \exists\, C < \infty \, , \quad \exists\, \alpha, \beta > 0 \, , \\ \big| f(x_1, y_1) - f(x_2, y_2) \big| \ \le \ C\big(d(x_1, x_2)^{\alpha} + |y_1 - y_2|^{\beta} \big) \, . \end{cases}$$

(**H7**) The second one, called the "Continuity type" model, is defined as:

$$C^0_{\mathcal{E} \times \mathbb{R}} = \Big\{ f : \mathcal{E} \times \mathbb{R} \to \mathbb{R} \, , \ \forall\, x' \in N_x, \ \lim_{d(x, x') \to 0} f(x', y) = f(x, y) \Big\} \, .$$

(**H8**) Finally, we will consider the conditional moments of the response random variable Y:

$$\forall\, m \ge 2, \ \ \mathbb{E}\big[|Y|^m / X = x \big] \ = \ \sigma_m(x) < \infty \ ,$$

with $\sigma_m(\cdot)$ continuous on x.

Before studying the k-NN estimator, we remind asymptotic properties of $\widehat{h}^x_{\text{kernel}}$ defined by equation (2.6). Ferraty *et al.* ([12]), showed the almost complete convergence of this estimator.

Theorem 3.1.

- *In the "continuity type" model and under the assumptions (H1), (H2), (H6) and (H8) we have:*

$$\widehat{h}^x_{\text{kernel}}(y) \longrightarrow h^x(y) \qquad \text{a.co.}$$

- *Under the "Lipschitz type" model and the hypotheses (H1), (H2), (H3), (H5), (H8), we have:*

$$\widehat{h}^x_{\text{kernel}}(y) - h^x(y) \;=\; O(h_n^\alpha) + O(g_n^\beta) + O\left(\sqrt{\frac{\log n}{n\,\varphi_x(h)}}\right).$$

Now we state the almost complete convergence for the nonparametric k-NN method estimate, defined in (2.2).

Theorem 3.2. *In the "continuity type" model and under the hypotheses (H1), (H2), (H4), (H5) and (H6), suppose that $k = k_n$ is a sequence of positive real numbers such that $\frac{k_n}{n} \to 0$ and $\frac{\log n}{k_n} \to 0$, then we have:*

$$\lim_{n\to\infty} \widehat{h}^x(y) \;=\; h^x(y) \qquad \text{a.co.}$$

Proof: We consider the following decomposition:

$$\widehat{h}^x(y) - h^x(y) \;=\; \frac{1}{1-\widehat{F}^x(y)}\left[\widehat{f}^x(y)-f^x(y)\right] + h^x(y)\,\frac{1}{1-\widehat{F}^x(y)}\left[\widehat{F}^x(y)-F^x(y)\right]. \tag{3.1}$$

Then the proof of Theorem 3.2 can be deduced from the following intermediate results. □

Lemma 3.1. *Under the hypotheses of Theorem 3.2, we have:*

$$\lim_{n\to\infty} \widehat{f}^x(y) = f^x(y) \qquad \text{a.co.} \tag{3.2}$$

and

$$\lim_{n\to\infty} \widehat{F}^x(y) = F^x(y) \qquad \text{a.co.} \tag{3.3}$$

Lemma 3.2. *Under the hypotheses of Theorem 3.2, we have:*

$$\exists\,\delta>0\,, \quad \sum_{n\in\mathbb{N}} \mathbb{P}\left[\left(1-\widehat{F}^x(y)\right)<\delta\right] < \infty\,. \tag{3.4}$$

Theorem 3.3. *The hypotheses (H1)–(H8) imply*

$$\widehat{h}^x(y) - h^x(y) \;=\; O\left(\varphi_x^{-1}\left(\frac{k}{n}\right)^{\alpha}\right) + O\big(g_n^{\beta}\big) + O\left(\sqrt{\frac{\log n}{k_n\, g_n}}\right) \qquad \text{a.co.}$$

Proof: We consider the decomposition (3.1), and the proof of this Theorem is a consequence of these results. □

Lemma 3.3. *Under the hypotheses of Theorem (3.3), we have:*

$$\widehat{f}^x(y) - f^x(y) \;=\; O\left(\varphi_x^{-1}\left(\frac{k_n}{n}\right)^{\alpha}\right) + O\big(g_n^{\beta}\big) + O\left(\sqrt{\frac{\log n}{k_n\, g_n}}\right) \qquad \text{a.co.} \tag{3.5}$$

Lemma 3.4. *Under the hypotheses of Theorem (3.3), we have:*

$$\widehat{F}^x(y) - F^x(y) \;=\; O\left(\varphi_x^{-1}\left(\frac{k_n}{n}\right)^{\alpha}\right) + O\big(g_n^{\beta}\big) + O\left(\sqrt{\frac{\log n}{k_n}}\right) \qquad \text{a.co.} \tag{3.6}$$

3.2. Asymptotic normality

This section contains results on the asymptotic normality of $\widehat{h}^x(y)$. For this, we have to add the followings assumptions:

- (**H9**) For each sequence $U_n \downarrow 0$ as $n \to \infty$ of positive real numbers, there exists a function $\lambda(\cdot)$ such that:
$$\forall\, t \in [0,1]\,, \quad \lim_{U_n \to \infty} \frac{\varphi_x(tU_n)}{\varphi_x(U_n)} = \lambda(t)\ .$$
- (**H10**) $\lim_{n\to\infty} \left(g_n^2 - \varphi_x^{-1}\left(\frac{k}{n}\right)\right)\sqrt{k_n} = 0$ and $\frac{1}{k_n g_n} = o\big(g_n^{\beta}\big)$.

Theorem 3.4. *Assume that (H1), (H9), (H10) hold, then for any $x \in \mathcal{A}$, we have:*

$$\left(\frac{k_n\, g_n}{\sigma_h^2(x,y)}\right)^{1/2} \Big[\widehat{h}^x(y) - h^x(y)\Big] \xrightarrow{\ \mathcal{D}\ } \mathcal{N}(0,1) \qquad \text{as} \quad n \to \infty\,, \tag{3.7}$$

where

$$\sigma_h^2(x,y) \;=\; \frac{\alpha_2\, h^x(y)}{\alpha_1^2\big(1 - F^x(y)\big)} \tag{3.8}$$

$$\Big(\text{with: } \alpha_j = K^j(1) - \int_0^1 (K^j)'(s)\,\lambda(s)\; ds \quad \text{for } j = 1,2\Big)\,,$$

$$\mathcal{A} = \Big\{x \in \mathcal{E}\,;\; f^x(y)\big[1 - F^x(y)\big] \neq 0\Big\}\,,$$

$\xrightarrow{\ \mathcal{D}\ }$ *means the convergence in distribution.*

Proof: We consider the decomposition (3.1) and we show that the proof of Theorem (3.4) is a consequence of the following results. □

Lemma 3.5. *Under the hypotheses of Theorem (3.4), we have:*

$$\left(\frac{k_n g_n}{\sigma_f^2(x,y)}\right)^{1/2} \left[\widehat{f}^x(y) - f^x(y)\right] \xrightarrow{\mathcal{D}} \mathcal{N}(0,1) \qquad \text{as} \quad n \to \infty ,$$

where

$$\sigma_f^2(x,y) \;=\; f^x(y) \int R'^2(t)\, dt \; . \tag{3.9}$$

Lemma 3.6. *Under the hypotheses of Theorem 3.4, we have:*

$$\left(\frac{k_n g_n}{\sigma_F^2(x,y)}\right)^{1/2} \left[\widehat{F}^x(y) - F^x(y)\right] \xrightarrow{\mathcal{D}} \mathcal{N}(0,1) \qquad \text{as} \quad n \to \infty ,$$

where

$$\sigma_F^2(x,y) \;=\; F^x(y)\left[1 - F^x(y)\right] . \tag{3.10}$$

Lemma 3.7. *Under the hypotheses of Theorem 3.4, we have:*

$$\left(1 - \widehat{F}^x(y)\right) \to \left(1 - F^x(y)\right) \qquad \textit{in probability}\,.$$

4. APPLICATIONS

4.1. Conditional Confidence Interval

The main application of the Theorem (3.4) is the to build confidence interval for the true value of $h^x(y)$ for a given curve $X = x$. A plug-in estimate for the asymptotic standard deviation $\sigma(x,\theta_x)$ can be obtained using the estimators $\widehat{h}^x(y)$ and $\widehat{F}^x(y)$ of $h^x(y)$, $F^x(y)$ respectively. We get $\widehat{\sigma}(x,y) := \left(\frac{\widehat{\alpha_2}\widehat{h}^x(y)}{(\widehat{\alpha_1})^2\,(1 - \widehat{F}^x(y))}\right)^{1/2}$. Then $\widehat{h}^x(y)$ can be used to get the following approximate $(1-\zeta)$ confidence interval for $h^x(y)$

$$\widehat{h}^x(y) \,\pm\, t_{1-\zeta/2} \times \left(\frac{\widehat{\sigma}_n^2(x,y)}{g_n k}\right)^{1/2}$$

where $t_{1-\zeta/2}$ denotes the $1-\zeta/2$ quantile of the standard normal distribution.

We estimate empirically α_1 and α_2 by

$$\widehat{\alpha_1} = \frac{1}{kg(x)} \sum_{i=1}^{n} K_i \qquad \text{and} \qquad \widehat{\alpha_2} = \frac{1}{kg(x)} \sum_{i=1}^{n} K_i^2 \ ,$$

where $K_i = K\left(\dfrac{d(x, X_i)}{\phi^{-1}(k/n)}\right)$.

This last estimation is justified by the fact that, under (H1), (H5) and (H6), we have, (see Ferraty and Vieu ([13]) p. 44)

$$\frac{1}{kg(x)} \mathbb{E}[K_1^j] \to \alpha_j \ , \qquad j = 1, 2 \ .$$

4.2. A Simulation study

In this section we will show the effectiveness of k-NN method compared to the kernel estimation using simulated data. For this we considered a sample of a diffusion process on interval $[0,1]$, $Z_1(t) = 2 - \cos(\pi t W)$ and $Z_2(t) = \cos(\pi t W)$, where W is the standard normal distribution and take $X(t) = AZ_1(t) + (1-A)Z_2(t)$, where A is random variable Bernoulli distributed. We carried out the simulation with a 200-sample of the curve X which is represented by the following graph:

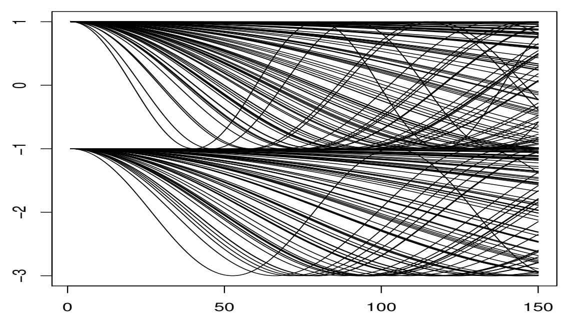

Figure 1: The 200 curves X.

For the scalar response variable, we took $Y = Ar_1(X) + (1-A)r_2(X)$ where r_1 (resp. r_2) is the nonlinear regression model $r_1(X) = 0.25 \times \left(\int_0^1 X'(t)\, dt\right)^2 + \epsilon$, with ϵ is $U([0, 0.5])$ (resp. $r_2(X)$ is the null function). We choose a quadratic kernel K defined by:

$$K(x) = \frac{3}{2}\,(1 - x^2)\, \mathbb{1}_{[0,1]} \ .$$

In practice, the semi-metric choice is based on the regularity of the curves X. For this we use the semi-metric defined by the L_2-distance between the q^{th} derivatives of the curves. In order to evaluate the MSE (Mean Square Error) we proceed by the following algorithm:

Step 1. We split our data into two subsets; the first sample, of size $n=120$ corresponds to the learning sample which will be used, as a sample, to compute our conditional hazard function estimators for the 80 remaining curves (considered as the test sample).

- $(X_j, Y_j)_{j\in J}$ learning sample,
- $(X_i, Y_i)_{i\in I}$ test sample.

Step 2.
- We use the learning sample for computing the hazard function estimator $\widehat{h}_j$, for all $j \in J$.
- We set: $i^* = \arg\min_{j\in J} d(X_i, X_j)$.
- We put: $\forall\, i \in I$,

$$\widehat{T_i} = \widehat{h}^{X_{i^*}}(Y_i) \quad \text{for kernel method}\,,$$
$$\widehat{T_i} = \widehat{h}^{X_{k_{opt}}}(Y_i) \quad \text{for } k\text{-NN method}\,,$$

where

$$X_{i^*}: \ \text{is the nearest curve to } X_j\,,$$
$$k_{opt}: \ \arg\min_a (CV(a))\,,$$

with

$$CV(a) \;=\; \frac{1}{n}\left[\sum_{i\in J}\int \left(\widehat{f}^{-i}_{(a,b)}(X_i,y)\right)^2 dy \;-\; 2\sum_{i\in J}\widehat{f}^{-i}_{(a,b)}(X_i,Y_i)\right]$$

and

$$\widehat{f}^{-k}_{(a,b)}(x,y) \;=\; \frac{b^{-1}\sum_{i\in J, i\neq k} K\left(\dfrac{d(x,X_i)}{a}\right) R\left(\dfrac{y-Y_i}{b}\right)}{\sum_{i\in J} K\left(\dfrac{d(x,X_i)}{a}\right)}\,.$$

Step 3. The error used to evaluate this comparison is the mean of square error (MSE) expressed by

$$\frac{1}{\operatorname{card}(I)}\sum_{i\in I}\left|h(Y_i) - \widehat{T}(X_i,Y_i)\right|^2\,,$$

where $\widehat{T}$ designate the estimator used: kernel or k-NN method estimation and h is the true hazard function.

Consequently, the k-NN method gives slightly better results than the kernel method. This is confirmed by the MSE-k-NN$=0.8227394$ and MSE-Kernel$=1.347982$.

4.3. Real data application

To highlight the efficiency and robustness of the method of k nearest neighbors with respect to the kernel method in estimating the conditional hazard function, we will test these two methods in the presence or not of heterogeneous data.

To do this, based on the study of Burba *et al.* (2009) which emphasizes the effect of the nature of the data (homogeneous or heterogeneous) on the quality of the estimate, especially the superiority of the k-nearest neighbors in the presence of very heterogeneous data.

For this purpose, we apply the described algorithm used in the simulation study to some chemiometrical real data available on the site[3], the original of these data (215 selected pieces of meat) comes from a quality control problem in the food industry that controls grease on a sample of finely chopped meat by chemical processes.

The sample of size 215 was split into learning sample of size 205 (with all data), 178 (without the heterogeneous data, 27 values) and testing sample of size 10. Figure 2 displays the curves of learning sample for all data and the curves of learning sample without the heterogeneous one.

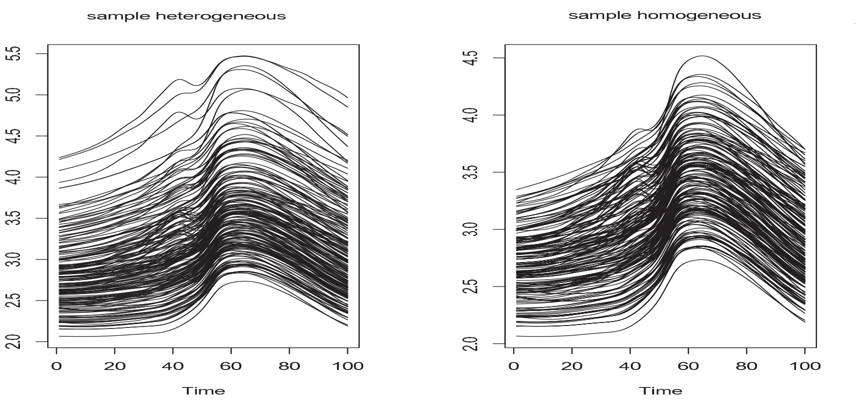

Figure 2: The learning curves.

For our study, we use the standard L^2 semi-metric and a quadratic kernel function K.

We plot the conditional hazard function estimated for the first 3 values of the testing sample, Figure 3 depicts that the k-NN method in presence of hetero-

[3] `http://lib.stat.cmu.edu/datasets/tecator`.

geneous data give a better estimation of the conditional hazard function prediction (regular function) than the kernel method estimation (non-regular function) and when the data are homogeneous the two method give the same result which can be easily seen in Figure 4.

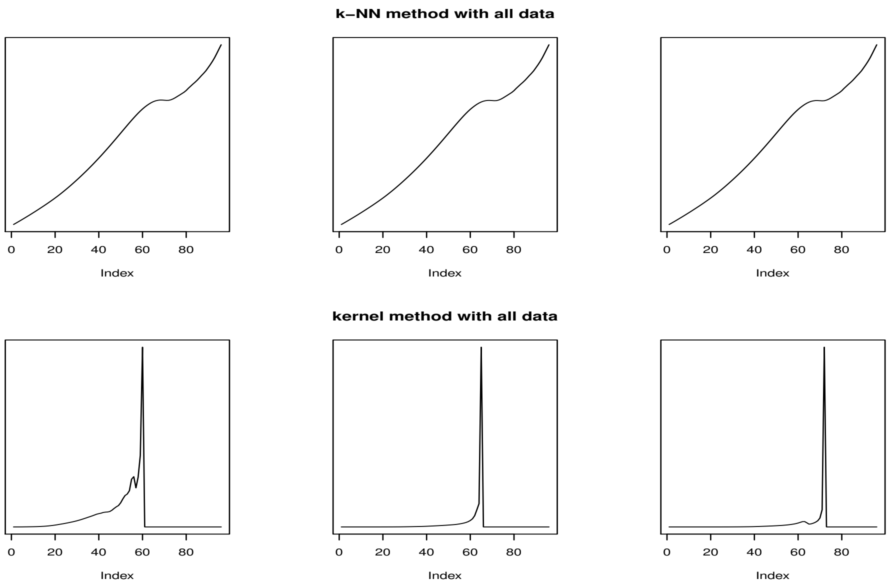

Figure 3: *k*-NN method (upper panels) *vs* kernel method (lower panels) of conditional hazard function far all data.

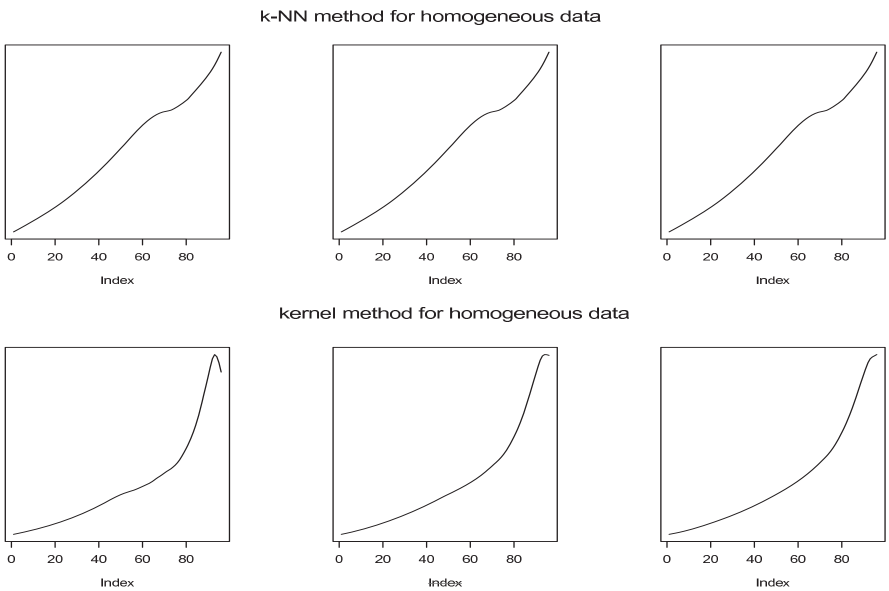

Figure 4: *k*-NN method (upper panels) *vs* kernel method (lower panels) of conditional hazard function for homogeneous data.

5. GENERAL TECHNICAL TOOLS

Let $(A_i, B_i)_{i\in\mathbb{N}}$ be a sequence of random variables with values in $(\Omega\times\mathbb{R}, \mathcal{A}\otimes\mathcal{B})$, independent but not necessarily identically distributed,where $(\Omega, \mathcal{A})$ is a general measurable space, let $G\colon \mathbb{R}\times\Omega \to \mathbb{R}^+$ a measurable function such that: $\forall\, w, w' \in \mathbb{R}$,

$$w \le w' \implies G(w, z) \le G(w', z)\ , \qquad \forall\, z \in \Omega\ .$$

Let c be a not random positive real number and T a real random variable: we define, $\forall\, n \in \mathbb{N}^*$,

$$C_n(T) \;=\; \frac{\displaystyle\sum_{i=1}^{n} B_i\, G(T, A_i)}{\displaystyle\sum_{i=1}^{n} G(T, A_i)}\ .$$

Lemma 5.1 (Burba *et al.* ([3])). *Let $(D_n)_{n\in\mathbb{N}}$ be a sequence of real random variables and $(u_n)_{n\in\mathbb{N}}$ be a decreasing sequence of positive numbers. If $l = \lim u_n \ne 0$, and if, for all increasing sequence $\beta_n \in\,]0, 1[$, there exist two sequences of real random variables $(D_n^+(\beta_n))_{n\in\mathbb{N}}$ and $(D_n^-(\beta_n))_{n\in\mathbb{N}}$:*

(**L1**) $\forall\, n \in \mathbb{N}, \quad D_n^- \le D_n^+$ *and* $\mathbb{1}_{D_n^- \le D_n \le D_n^+} \to 1$ *a.co.*

(**L2**) $$\frac{\displaystyle\sum_{i=1}^{n} G(D_n^-, A_i)}{\displaystyle\sum_{i=1}^{n} G(D_n^+, A_i)} - \beta_n = O(u_n) \qquad \text{a.co.}$$

(**L3**) $C_n(D_n^-) - c = O(u_n)$ *a.co.*
$C_n(D_n^+) - c = O(u_n)$ *a.co.*

Then:

$$C_n(D_n) - c \;=\; O(u_n) \qquad \text{a.co.}$$

If $l = 0$ and if (L1), (L2), (L3) hold for any increasing sequence $\beta_n \in\,]0, 1[$ with limit 1, the same result holds.

Lemma 5.2 (Burba *et al.* ([3])). *Let $(D_n)_{n\in\mathbb{N}}$ be a sequence of real random variables and $(v_n)_{n\in\mathbb{N}}$ be a decreasing positive sequence. If $l' = \lim v_n \ne 0$, and if, for all increasing sequence $\beta_n \in\,]0, 1[$, there exist two sequences of real random variables $(D_n^+(\beta_n))_{n\in\mathbb{N}}$ and $(D_n^-(\beta_n))_{n\in\mathbb{N}}$:*

(**L′1**) $\forall\, n \in \mathbb{N}, \quad D_n^- \le D_n^+$ *and* $\mathbb{1}_{D_n^- \le D_n \le D_n^+} \to 1$ *a.co.*

$$(\mathbf{L'2})\quad \frac{\sum_{i=1}^{n} G(D_n^-, A_i)}{\sum_{i=1}^{n} G(D_n^+, A_i)} - \beta_n = o(v_n) \qquad a.co.$$

$$(\mathbf{L'3})\quad \begin{aligned} C_n(D_n^-) - c = o(v_n) \qquad a.co.\\ C_n(D_n^+) - c = o(v_n) \qquad a.co.\end{aligned}$$

Then:

$$C_n(D_n) - c = o(v_n) \qquad a.co.$$

If $l'=0$ and if (L′1), (L′2), (L′3) hold for any increasing sequence $(\beta_n) \in]0,1[$ with limit 1, the same result holds.

Burba *et al.* ([3]) use in their consistency proof of the k-NN kernel estimate for independent data a Chernoff-type exponential inequality to check conditions (L1) or (L′1).

Lemma 5.3 (Burba *et al.* ([3])). *Let $(X_1, X_2, ..., X_n)$ be independent random variable in $\{0,1\}$. We note $X = \sum_{i=1}^{n} X_i$ and $\mu = \mathbb{E}(X)$: then, $\forall\, \delta > 0$,*

$$\mathbb{P}\big[X > (1+\delta)\,\mu\big] \;<\; \big[e^{\delta}/(1+\delta)^{1+\delta}\big]^{\mu},$$
$$\mathbb{P}\big[X < (1-\delta)\,\mu\big] \;<\; \big[e^{-\delta^2/2\mu}\big].$$

APPENDIX

Proof of Section 3.1

Proof of Lemma 3.1: On one hand, to prove the first result, we apply Lemma 5.2 with:

$$\text{(A.1)}\qquad \begin{cases} v_n = 1 \ , \\ H_n = D_n \ , \\ \widehat{f}^x(y) = C_n(D_n) \ , \\ f^x(y) = c \ . \end{cases}$$

Choose $\beta_n \in]0,1[$, (D_n^-) and (D_n^+) such that:

$$\text{(A.2)}\qquad \begin{cases} \varphi_x(D_n^-) = \sqrt{\beta_n}\,\varphi_x(h) = \sqrt{\beta_n}\dfrac{k_n}{n} \ , \\ \varphi_x(D_n^+) = \dfrac{1}{\sqrt{\beta_n}}\,\varphi_x(h) = \dfrac{1}{\sqrt{\beta_n}}\dfrac{k_n}{n} \ . \end{cases}$$

Define

$$\text{(A.3)}\qquad \begin{cases} h^- = D_n^- = \varphi_x^{-1}\left(\sqrt{\beta_n}\,\dfrac{k_n}{n}\right), \\ h^+ = D_n^+ = \varphi_x^{-1}\left(\dfrac{1}{\sqrt{\beta_n}}\dfrac{k_n}{n}\right). \end{cases}$$

Ferraty and Vieu ([13]) proved under the conditions of Theorem 3.1 that:

$$\frac{1}{n\,\varphi_x(h)}\sum_{i=1}^n K\big(h^{-1}d(x,X_i)\big) \;\to\; 1 \qquad \text{a.co.}$$

Under the conditions (A.2) and (A.3), we have:

$$\begin{cases} \dfrac{1}{n\,\varphi_x(D_n^-)}\displaystyle\sum_{i=1}^n K\left((D_n^-)^{-1}d(x,X_i)\right) \to 1 \qquad \text{a.co.} \\ \dfrac{1}{n\,\varphi_x(D_n^+)}\displaystyle\sum_{i=1}^n K\left((D_n^+)^{-1}d(x,X_i)\right) \to 1 \qquad \text{a.co.} \end{cases}$$

Then:

$$\frac{\displaystyle\sum_{i=1}^n K\left((D_n^-)^{-1}d(x,X_i)\right)}{\displaystyle\sum_{i=1}^n K\left((D_n^+)^{-1}d(x,X_i)\right)} \;\to\; \beta_n \qquad \text{a.co.}\,,$$

so that (L′2) is checked. Now by using Lemma (6.15) in Ferraty and Vieu ([13]) under the conditions of Theorem 3.1 and

$$D_n \longrightarrow 0\,, \quad \frac{\log n}{n\varphi_x(D_n)} \longrightarrow 0 \qquad (n\to\infty)\,, \tag{A.4}$$

we have:

$$C_n(D_n^-) \to c \qquad \text{a.co.}$$
$$C_n(D_n^+) \to c \qquad \text{a.co.}$$

so (L′3) is verified. Finally, we check (L′1). The first part is obvious, and the second one that: $\forall\,\epsilon>0$,

$$\sum_{n\ge 0} \mathbb{P}\Big[\big|\mathbb{1}_{D_n^- < H_n < D_n^+} - 1\big| > \epsilon\Big] < \infty\,.$$

We know that:

$$\mathbb{P}\Big[\big|\mathbb{1}_{D_n^- < H_n < D_n^+} - 1\big| > \epsilon\Big] \le \underbrace{\mathbb{P}\big[H_n < D_n^-\big]}_{A_1} + \underbrace{\mathbb{P}\big[H_n > D_n^+\big]}_{A_2}\,,$$

$$A_1 \le \mathbb{P}\left[\sum_{i=1}^{n} \mathbb{1}_{B(x,D_n^-)} > k_n\right].$$

And by using Lemma 5.3 with

$$\begin{cases} X_i = \mathbb{1}_{B(x,D_n^-)}\,, \\ X = \sum_{i=1}^{n} \mathbb{1}_{B(x,D_n^-)}\,, \\ \mathbb{P}\big(X_i{=}1\big) = \varphi_x(D_n^-)\,, \\ \mu = \mathbb{E}(X) = \sum_{i=1}^{n} \mathbb{E}\big[\mathbb{1}_{B(x,D_n^-)}\big] = n\,\varphi_x(D_n^-)\,, \end{cases} \tag{A.5}$$

we get:

$$\mathbb{P}\big[H_n < D_n^-\big] < \left[e^{\left(\frac{1}{\sqrt{\beta}}-1\right)} \Big/ \left(\frac{1}{\sqrt{\beta}}\right)^{-\frac{1}{\sqrt{\beta}}}\right]^{n\varphi_x(D_n^-)}$$
$$< n^{\left(-\log\sqrt{\beta_n}\, e^{(1-\sqrt{\beta_n})}\right)^{\frac{-k_n}{\log n}}}.$$

Under the hypotheses (A.4) and as $\sqrt{\beta}(e^{(1-\sqrt{\beta})}) < 1$ then:

$$A_1 = \mathbb{P}\big[H_n < D_n^-\big] < \infty\,.$$

Turning now to the study of A_2, we obtain

$$\mathbb{P}\big[H_n > D_n^+\big] = \mathbb{P}\left[\sum_{i=1}^{n} \mathbb{1}_{B(x,D_n^+)} < n\sqrt{\beta}\varphi_x(D_n^+)\right]$$

under the modification (A.5), and by applying the Lemma 5.3 we obtain:

$$\begin{aligned}\mathbb{P}\big[H_n > D_n^+\big] &< e^{\frac{-k_n(1-\sqrt{\beta})^2}{2\sqrt{\beta}}} \\ &< \left(n^{\frac{(1-\sqrt{\beta})^2}{2\sqrt{\beta}}}\right)^{\frac{-k_n}{\log n}} .\end{aligned}$$

Since $\dfrac{(1-\sqrt{\beta})^2}{2\sqrt{\beta}} > 0$ and $\dfrac{n\varphi_x(h)}{\log n} \to \infty$ then:

$$\mathbb{P}\big[H_n > D_n^+\big] < \infty .$$

Finally:

$$\mathbb{P}\Big[\big|\mathbb{1}_{D_n^- < H_n < D_n^+} - 1\big| > \epsilon\Big] < \infty .$$

On the other hand, we prove the second result. For this, we use the preceding steps with:

$$\text{(A.6)} \qquad \begin{cases} v_n = 1 , \\ H_n = D_n , \\ \widehat{F}^x(y) = C_n(D_n) , \\ F^x(y) = c . \end{cases}$$

□

Proof of Lemma 3.2: It is clear that:

$$\big|1 - \widehat{F}^x(y)\big| < \frac{1 - F^x(y)}{2} \implies \big|\widehat{F}^x(y) - F^x(y)\big| > \frac{1 - F^x(y)}{2} .$$

Turning now, to the term of probability, we obtain:

$$\mathbb{P}\left[\big|1 - \widehat{F}^x(y)\big| < \frac{1 - F^x(y)}{2}\right] \leq \mathbb{P}\left[\big|\widehat{F}^x(y) - F^x(y)\big| > \frac{1 - F^x(y)}{2}\right] ,$$

$$\sum_{n\in\mathbb{N}} \mathbb{P}\left[\big|1 - \widehat{F}^x(y)\big| < \frac{1 - F^x(y)}{2}\right] \leq \sum_{n\in\mathbb{N}} \mathbb{P}\left[\big|\widehat{F}^x(y) - F^x(y)\big| > \frac{1 - F^x(y)}{2}\right] .$$

For the second term, by result 3.3, we have:

$$\sum_{n\in\mathbb{N}} \mathbb{P}\left[\big|\widehat{F}^x(y) - F^x(y)\big| > \frac{1 - F^x(y)}{2}\right] < \infty .$$

Then, for $\delta = \dfrac{1 - F^x(y)}{2}$, we obtain:

$$\sum_{n\in\mathbb{N}} \mathbb{P}\left[\big|\widehat{F}^x(y) - F^x(y)\big| > \frac{1 - F^x(y)}{2}\right] < \infty .$$

□

Proof of Lemma 3.3: To prove this lemma, we use Lemma 5.1. Choose β_n as an increasing sequence in $]0,1[$ with limit 1. Furthermore, we choose D_n^- and D_n^+ under (A.2), Ferraty and Vieu ([13]) proved under the conditions of Theorem 3.1 that:

$$\widehat{r}_3(x) - \mathbb{E}\left[\widehat{r}_3(x)\right] \;=\; O\left(\sqrt{\frac{\log n}{n h_n \varphi_x(h)}}\right) ,$$

with

$$\begin{aligned}
\widehat{r}_3(x) \;&=\; \frac{1}{n}\sum_{i=1}^{n} \frac{K\left(h_n^{-1} d(x,X_i)\right) R\left(g_n^{-1}(y-Y_i)\right)}{\mathbb{E}\left[K\left(h_n^{-1} d(x,X_1)\right)\right]} ,\\
\widehat{r}_3(x) \;&=\; \frac{1}{n}\sum_{i=1}^{n} \frac{K\left(h_n^{-1} d(x,X_i)\right) \Gamma_i(y)}{\mathbb{E}\left[K\left(h_n^{-1} d(x,X_1)\right)\right]} ,\\
\Gamma_i(y) \;&=\; R\left(g_n^{-1}(y-Y_i)\right) .
\end{aligned}$$

Then,

$$\begin{aligned}
\widehat{r}_3(x) - \mathbb{E}\left[\widehat{r}_3(x)\right] \;&=\; \frac{1}{n}\sum_{i=1}^{n} \frac{K\left(h_n^{-1} d(x,X_i)\right)}{\mathbb{E}K\left(h_n^{-1} d(x,X_1)\right)}\,\Gamma_i(y)\\
&\quad - \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}\left[\frac{K\left(h_n^{-1} d(x,X_i)\right)}{\mathbb{E}K\left(h_n^{-1} d(x,X_1)\right)}\,\Gamma_i(y)\right]\\
&=\; \frac{1}{n\mathbb{E}K\left(h_n^{-1} d(x,X_1)\right)} \sum_{i=1}^{n} K\left(h_n^{-1} d(x,X_i)\right) \Gamma_i(y)\\
&\quad - \frac{1}{\mathbb{E}K\left(h_n^{-1} d(x,X_1)\right)}\, \mathbb{E}\left[K\left(h_n^{-1} d(x,X_1)\right) \mathbb{E}(\Gamma_1(y)/X_1)\right]\\
&=\; \frac{1}{n\mathbb{E}K\left(h_n^{-1} d(x,X_1)\right)} \sum_{i=1}^{n} K\left(h_n^{-1} d(x,X_i)\right) \Gamma_i(y) - \mathbb{E}(\Gamma_1(y)/X_1) .
\end{aligned}$$

Using the fact that $\mathbb{E}\left[K\left(h_n^{-1} d(x,X_i)\right)\right] = O(\varphi_x(h)$ (see Ferraty and Vieu ([13]) and under the notations (A.2) and (A.3), we have:

$$\begin{cases}
\dfrac{1}{n\varphi_x(D_n^-)} \displaystyle\sum_{i=1}^{n} K\left(\frac{d(x,X_i)}{D_n^-}\right) \Gamma_i(y) \;=\; \mathbb{E}(\Gamma_1(y)/X_1) + O\left(\sqrt{\dfrac{\log n}{g_n k_n}}\right) ,\\[2ex]
\dfrac{1}{n\varphi_x(D_n^+)} \displaystyle\sum_{i=1}^{n} K\left(\frac{d(x,X_i)}{D_n^+}\right) \Gamma_i(y) \;=\; \mathbb{E}(\Gamma_1(y)/X_1) + O\left(\sqrt{\dfrac{\log n}{g_n k_n}}\right) .
\end{cases}$$

By this, we obtain:

$$\frac{\displaystyle\sum_{i=1}^{n} K\left(\frac{d(x,X_i)}{D_n^-}\right)}{\displaystyle\sum_{i=1}^{n} K\left(\frac{d(x,X_i)}{D_n^+}\right)} - \beta_n \;=\; O\left(\sqrt{\frac{\log n}{g_n k_n}}\right) \qquad \text{a.co.}$$

that (L2) is verified. Now, we apply Lemma (6.15) for Ferraty and Vieu ([13]) under (A.2) and (A.1), we get:

$$C_n(D_n^-) - c = O\left(\varphi_x^{-1}\left(\frac{k}{n}\right)^{\alpha}\right) + O(g_n^{\beta}) + O\left(\sqrt{\frac{\log n}{g_n k_n}}\right) \qquad \text{a.co.}$$

$$C_n(D_n^+) - c = O\left(\varphi_x^{-1}\left(\frac{k}{n}\right)^{\alpha}\right) + O(g_n^{\beta}) + O\left(\sqrt{\frac{\log n}{g_n k_n}}\right) \qquad \text{a.co.}$$

that verifies condition (L3). □

Proof of Lemma 3.4: To verify this Lemma, we pass by the same steps as before, such that: Ferraty and Vieu ([13]) showed that

$$\widehat{r}_1(x) - 1 = O\left(\sqrt{\frac{\log n}{n\varphi_x(h)}}\right),$$

with

$$\widehat{r}_1(x) = \frac{1}{n}\sum_{i=1}^{n} \frac{K\left(h_n^{-1} d(x, X_i)\right)}{\mathbb{E}K\left(h_n^{-1} d(x, X_1)\right)}.$$

Then

$$\frac{1}{n}\sum_{i=1}^{n} K\left(h_n^{-1} d(x, X_i)\right) - \varphi_x(h) = O\left(\sqrt{\frac{\log n}{n\varphi_x(h)}}\right)$$

and under the same choice of $h^- = D_n^-$ and $h^+ = D_n^+$ as above, we have:

$$\begin{cases} \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} K\left(\dfrac{d(x, X_i)}{D_n^-}\right) = \sqrt{\beta_n}\dfrac{k_n}{n} + O\left(\sqrt{\dfrac{\log n}{k_n}}\right), \\ \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} K\left(\dfrac{d(x, X_i)}{D_n^+}\right) = \dfrac{1}{\sqrt{\beta_n}}\dfrac{k_n}{n} + O\left(\sqrt{\dfrac{\log n}{k_n}}\right). \end{cases}$$

We get

$$\frac{\displaystyle\sum_{i=1}^{n} K\left(\frac{d(x, X_i)}{D_n^-}\right)}{\displaystyle\sum_{i=1}^{n} K\left(\frac{d(x, X_i)}{D_n^+}\right)} - \beta_n = O\left(\sqrt{\frac{\log n}{k_n}}\right)$$

so that, (L2) is checked. Now we are able to apply Lemma (6.14) in Ferraty and Vieu ([13]) under (A.6), we obtain

$$C_n(D_n^-) - c = O\left(\varphi_x^{-1}\left(\frac{k_n}{n}\right)^{\alpha}\right) + O(g_n^{\beta}) + O\left(\sqrt{\frac{\log n}{k_n}}\right),$$

$$C_n(D_n^+) - c = O\left(\varphi_x^{-1}\left(\frac{k_n}{n}\right)^{\alpha}\right) + O(g_n^{\beta}) + O\left(\sqrt{\frac{\log n}{k_n}}\right),$$

and (L3) is verified. □

Proof of Section 3.2

Proof of Lemma 3.5: We denote:

$$\text{(A.7)}\qquad \begin{cases} C_n(H_n) = \widehat{f}^x(y)\ , \\ c = f^x(y)\ . \end{cases}$$

Under (A.2) and (A.3), we have:

$$\text{(A.8)}\qquad \begin{aligned} &\left(\frac{k_n g_n}{\sigma_f^2(x,y)}\right)^{1/2} \left[C_n(H_n) - c\right] = \\ &\quad = \left(\frac{k_n g_n}{\sigma_f^2(x,y)}\right)^{1/2} \left[C_n(D_n^+) - c\right] + \left(\frac{k_n g_n}{\sigma_f^2(x,y)}\right)^{1/2} \left[C_n(H_n) - C_n(D_n^+)\right] . \end{aligned}$$

Then, to establish the asymptotic normality of the conditional density function, we need to show the asymptotic normality of the first term in equation (A.8) and the second term converges a.co. to 0.

For this, we remind that, under the same assumptions as Lemma 3.5, Quintela-del-Río ([23]) in Theorem 5 proved that

$$\left(\frac{k_n g_n}{\sigma_f^2(x,y)}\right)^{1/2} \left[C_n(D_n^+) - c\right] \xrightarrow{\mathcal{D}} \mathcal{N}(0,1) \qquad \text{as} \quad n \to \infty\ .$$

On the other hand, by hypothesis (H2) and the fact that $\mathbb{1}_{\{D_n^- \le H_n \le D_n^+\}} \to 1$ where $\frac{k_n}{n} \to 0$ (see Burba *et al.* ([3])), we have:

$$C_n(D_n^+) \le C_n(H_n) \le C_n(D_n^-)\ .$$

Using the fact that:

$$\text{(A.9)}\qquad \begin{aligned} |C_n(H_n) - C_n(D_n^+)| &\le |C_n(D_n^-) - C_n(D_n^+)| \\ &\le |C_n(D_n^-) - \mathbb{E}\left[C_n(D_n^-)\right]| + |C_n(D_n^+) - \mathbb{E}[C_n(D_n^+)]| \\ &\quad + |\mathbb{E}[C_n(D_n^-)] - \mathbb{E}[C_n(D_n^+)]|\ . \end{aligned}$$

For the first term, we can write:

$$|C_n(D_n^-) - \mathbb{E}\left[C_n(D_n^-)\right]| \le |C_n(D_n^-) - c| + |\mathbb{E}\left[C_n(D_n^-)\right] - c|$$

by Lemma (3.3), we have:

$$|C_n(D_n^-) - c| = O\left(\varphi_x^{-1}\left(\frac{k_n}{n}\right)^{\alpha}\right) + O(g_n^{\beta}) + O\left(\sqrt{\frac{\log n}{k_n g_n}}\right)$$

and Quintela-del-Río ([23]) proved that:

$$\text{(A.10)} \qquad |\mathbb{E}\left[C_n(D_n^-)\right] - c| \;=\; o(g_n^\beta) + O\left(\frac{1}{k_n}\right).$$

Finally, under hypothesis (H10), we obtain the almost complete convergence of the first term of (A.9). And to establish the almost complete convergence of the second term we apply the same steps as before.

Finally for the third term, we have:

$$|\mathbb{E}[C_n(D_n^-)] - \mathbb{E}[C_n(D_n^+)]| \;\leq\; |\mathbb{E}[C_n(D_n^-)] - c| + |\mathbb{E}[C_n(D_n^+)] - c|$$

the almost complete convergence to 0 of these two terms is verified in (A.10). □

Proof of Lemma 3.6: To prove this Lemma, we apply the same steps as preceding with:

$$\text{(A.11)} \qquad \begin{cases} C_n(H_n) = \widehat{F}^x(y) \;, \\ c = F^x(y) \;. \end{cases}$$

□

Proof of Lemma 3.7: It is clear that, the result (3.3) of Lemma (3.1) permits to conclude that:

$$\widehat{F}^x(y) \to F^x(y) \quad \text{in probability}\,.$$

□

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PORT-ESTIMATION OF A SHAPE SECOND-ORDER PARAMETER

Authors: Lígia Henriques-Rodrigues
– CEAUL, University of Lisbon and Instituto Politécnico de Tomar, Portugal
lcphjr@gmail.com

M. Ivette Gomes
– CEAUL and DEIO, FCUL, University of Lisbon, Portugal
ivette.gomes@fc.ul.pt

M. Isabel Fraga Alves
– CEAUL and DEIO, FCUL, University of Lisbon, Portugal
isabel.alves@fc.ul.pt

Cláudia Neves
– CEAUL and Department of Mathematics, University of Aveiro, Portugal
claudia.neves@ua.pt

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Abstract:

- In this paper we study, under a semi-parametric framework and for heavy right tails, a class of location invariant estimators of a shape second-order parameter, ruling the rate of convergence of the normalised sequence of maximum values to a non-degenerate limit. This class is based on the PORT methodology, with PORT standing for peaks over random thresholds. Asymptotic normality of such estimators is achieved under a third-order condition on the right-tail of the underlying model F and for suitable large intermediate ranks. An illustration of the finite sample behaviour of the estimators is provided through a small-scale Monte-Carlo simulation study.

Key-Words:

- *asymptotic properties; location/scale invariant estimation; Monte-Carlo simulation;* PORT *methodology; sample of excesses; semi-parametric estimation; shape second-order parameters; statistics of extremes; third-order framework.*

AMS Subject Classification:

- 62G32, 62E20; 65C05.

1. INTRODUCTION AND MOTIVATION

Let $\underline{X}_n = (X_1, ..., X_n)$ denote a random sample of n independent, identically distributed (i.i.d.) random variables (r.v.'s) with distribution function (d.f.) F. We are interested in heavy-tailed models, i.e. in d.f.'s with a regularly varying right-tail. This means that, for $\xi > 0$, the right tail-function

$$\overline{F} := 1 - F$$

is such that

$$\lim_{t\to\infty} \overline{F}(tx)/\overline{F}(t) \;=\; x^{-1/\xi}, \quad \text{for all } x > 0. \tag{1.1}$$

We then say that $\overline{F}$ is of regular variation at infinity with an index equal to $-1/\xi$, and define

$$G_\xi(x) := \begin{cases} \exp\left(-(1+\xi\ x)^{-1/\xi}\right), & 1+\xi\ x > 0, \text{ if } \xi \neq 0 \\ \exp(-\exp(-x)), & x \in \mathbb{R}, \qquad\qquad \text{if } \xi = 0, \end{cases} \tag{1.2}$$

the general extreme-value (EV) distribution function. If (1.1) holds, we are in the domain of attraction for maxima of G_ξ, with $\xi > 0$, and we write $F \in \mathcal{D}_{\mathcal{M}}(G_{\xi>0})$, meaning that it is possible to find sequences of real constants $\{a_n > 0\}$ and $\{b_n \in \mathbb{R}\}$ such that the maximum $X_{n:n} := \max(X_1, ..., X_n)$, linearly normalized, i.e. $(X_{n:n} - b_n)/a_n$, converges in distribution to a non-degenerate r.v. with d.f. $G_\xi(x)$, in (1.2), with $\xi > 0$. This type of heavy-tailed models arises often in practice, in fields like telecommunication traffic, finance, insurance, economics, ecology and biometry, among others. The parameter ξ, in (1.2), is the extreme-value index (EVI), one of the primary parameters of extreme events.

Let $F^{\leftarrow}$ denote the generalised inverse function of F, defined by

$$F^{\leftarrow}(t) := \inf\{x : F(x) \geq t\}, \tag{1.3}$$

and let U be the associated (reciprocal) quantile function, defined as

$$U(t) := F^{\leftarrow}(1 - 1/t), \qquad t \geq 1. \tag{1.4}$$

1.1. First and second-order conditions for heavy-tailed models

In a heavy-tailed framework, i.e. if (1.1) holds, with the usual notation RV_a for the class of regularly varying functions at infinity with an index $a \in \mathbb{R}$, and on the basis of the results in Gnedenko (1943), for the right-tail function $\overline{F} = 1 - F$, and in de Haan (1984), for U in (1.4), the following first-order conditions are equivalent,

$$F \in \mathcal{D}_{\mathcal{M}}(G_{\xi>0}) \iff \overline{F} \in RV_{-1/\xi} \iff U \in RV_{\xi}. \tag{1.5}$$

Now we need to say something about the rate of convergence in (1.5), and assume that the following limiting relation holds for every $x > 0$,

$$\lim_{t\to\infty} \frac{\ln U(tx) - \ln U(t) - \xi \ln x}{A(t)} = \begin{cases} \frac{x^\rho - 1}{\rho}, & \text{if } \rho < 0 \\ \ln x, & \text{if } \rho = 0, \end{cases} \tag{1.6}$$

where $|A|$ must then be in RV_ρ (Geluk and de Haan, 1987). The second-order parameter $\rho \le 0$ rules the rate of convergence provided by (1.6), which increases with $|\rho|$. Note further that in the scope of applications, the most common models depend on a location or shift parameter $s \in \mathbb{R}$, not necessarily null, i.e. $F(x) \equiv F_s(x) = F_0(x - s)$. Then, $U(t) \equiv U_s(t) = U_0(t) + s$ and also both A and ρ depend obviously on s, i.e. $A = A_s$ and $\rho = \rho_s$, with

$$\rho_s := \begin{cases} -\xi, & \text{if } \xi + \rho_0 < 0 \ \wedge s \neq 0 \\ \rho_0, & \text{otherwise.} \end{cases} \tag{1.7}$$

Among the literature specifically devoted to the estimation of the second-order parameter ρ, in (1.6), we mention Gomes *et al.* (2002), Fraga Alves *et al.* (2003a), and the more recent articles by Goegebeur *et al.* (2008; 2010), Ciuperca and Mercadier (2010) and Caeiro and Gomes (2012a,b). Indeed, most of the research devised to improve the classical EVI-estimators tries to reduce the dominant component of their asymptotic bias, deals with second-order reduced-bias (SORB) EVI-estimators, and an adequate estimation of ρ is needed, for an adequate reduction of the bias. Some of the pioneering papers in the area of SORB-estimation are the ones by Beirlant *et al.* (1999), Feuerverger and Hall (1999), Gomes *et al.* (2000) and Gomes and Martins (2001; 2002). More recently, the minimum-variance reduced-bias (MVRB) EVI-estimators, studied in Caeiro *et al.* (2005), Gomes *et al.* (2007) and Gomes *et al.* (2008c), among others, also call for an adequate estimation of ρ. An overview of the subject can be found in Chapter 6 of the book by Reiss and Thomas (2007). See also Gomes *et al.* (2008a) and Beirlant *et al.* (2012) in this respect. However, despite of scale-invariant, all these MVRB EVI-estimators are not location-invariant.

1.2. The PORT methodology

Let $X_{i:n}$, $1 \le i \le n$, be the o.s.'s associated with the random sample $\underline{X}_n = (X_1, ..., X_n)$ with common d.f. F_0. The class of estimators suggested here is a function of the sample of excesses over a random threshold $X_{n_q:n}$, with $n_q = \lfloor nq \rfloor + 1$, where $\lfloor x \rfloor$ stands for the integer part of x. Such a sample is denoted by

$$\underline{X}_n^{(q)} := \left(X_{n:n} - X_{n_q:n}, X_{n-1:n} - X_{n_q:n}, ..., X_{n_q+1:n} - X_{n_q:n}\right) , \tag{1.8}$$

where, we can have

- $0 < q < 1$, for any $F_0 \in D_{\mathcal{M}}(G_{\xi>0})$ (the random threshold, $X_{n_q:n}$, is an empirical quantile);
- $q = 0$, for d.f.'s with a finite left endpoint $x_F := \inf\{x : F_0(x) > 0\}$, (the random threshold is the minimum, $X_{1:n}$).

Any statistical inference methodology based on the sample of excesses $\underline{X}_n^{(q)}$, in (1.8), will be called a PORT-methodology, with PORT standing for peaks over random thresholds, a term coined by Araújo Santos *et al.* (2006). This methodology enabled the introduction and study of classical location/scale invariant EVI-estimators, like the PORT-Hill and the PORT-Moment estimators, studied for finite-samples in Gomes *et al.* (2008b). This methodology was also applied in the estimation of high quantiles in Henriques-Rodrigues and Gomes (2009).

Such a methodology leads to location-invariant estimation, where the unshifted model F_0 thus plays a central role. In what follows, we use the notation χ_q for the q-quantile of the d.f. F_0, i.e. the value

$$\chi_q := F_0^{\leftarrow}(q) \tag{1.9}$$

(by convention $\chi_0 := x_F$, the left endpoint of F_0), with $F^{\leftarrow}(\cdot)$ defined in (1.3). Since $n_q/n \to q$, as $n \to \infty$, we then know that the o.s. $X_{n_q:n}$, associated with a sample from F_0, is a consistent estimator for $F_0^{\leftarrow}(q)$ (Mosteller, 1946, under stronger assumptions on F; van der Vaart, 1998, p.308), i.e. we have the following convergence in probability:

$$X_{n_q:n} \underset{n\to\infty}{\overset{p}{\longrightarrow}} \chi_q = F_0^{\leftarrow}(q), \quad \text{for} \quad 0 \le q < 1 \quad \left(\chi_0 = x_F\right). \tag{1.10}$$

1.3. Scope of the paper

We shall make use of the above-mentioned PORT methodology for heavy tails. Henceforth $\xi > 0$ denotes the first-order parameter of the model underlying the available data, $\rho_0 \le 0$ is the second-order parameter of the associated unshifted model, and χ_q has been provided in the limit of (1.10), in order to introduce a class of location-invariant semi-parametric estimators of the so-called PORT-ρ second-order parameter,

$$\rho_q := \begin{cases} -\xi, & \text{if } \xi + \rho_0 < 0 \ \wedge \ \chi_q \ne 0 \\ \rho_0, & \text{otherwise.} \end{cases} \tag{1.11}$$

Note that when applying the PORT-methodology, we are working with the sample of excesses in (1.8), and we can assume that we are dealing with an unshifted d.f. F_0 underlying the r.v. X_0, to which we are inducing a random shift, strictly related to χ_q, in (1.9). More precisely, we have a shift $s = -\chi_q$, i.e. we are working with $X_q := X_0 - \chi_q$, and use the simpler notation ρ_q for $\rho_{-\chi_q}$, with ρ_s defined in (1.7). Hence $\rho_q = -\xi \neq \rho_0$ if and only if $\chi_q \neq 0$ and the underlying model is such that $\xi + \rho_0 < 0$, just as written in (1.11), i.e. $\rho_q \neq \rho_0$ if and only if $s = 0$, $\chi_q \neq 0$ and $\xi + \rho_0 < 0$.

The main motivation for a class of estimators of the shape second-order parameter ρ_q, in (1.11), is related to its possible use, concomitantly with a class of PORT estimators of the functional A, in (1.6), or at least of an adequate location-invariant estimator of the scale parameter of such a A-function, in the building of second-order PORT-MVRB EVI-estimators, invariant for changes in location. The study of the asymptotic behaviour of such EVI-estimators is a challenging theoretical open subject, out of the scope of this paper, but already dealt with by Monte-Carlo simulation, in Gomes *et al.* (2011, 2013).

The building block of our estimators of the shape second-order parameter ρ_q, defined in (1.11) are of the same kind as the statistics used in Dekkers *et al.* (1989), Gomes *et al.* (2002), Fraga Alves *et al.* (2003a) and Caeiro and Gomes (2006), among others, i.e. for $\alpha > 0$ we consider the moment statistics

$$(1.12) \qquad M_{n,k}^{(\alpha)} \equiv M_{n,k}^{(\alpha)}(\underline{X}_n) := \frac{1}{k}\sum_{i=1}^{k} \left(\ln X_{n-i+1:n} - \ln X_{n-k:n}\right)^{\alpha},$$

but now applied to the sample of excesses $\underline{X}_n^{(q)}$, $0 \leq q < 1$, in (1.8). For an intermediate k-sequence, i.e. a sequence $k = k_n$ of positive integers such that

$$(1.13) \qquad k = k_n \to \infty \quad \text{and} \quad k = o(n) \quad \text{as} \quad n \to \infty,$$

we shall thus consider the location and scale-invariant statistics,

$$(1.14) \qquad M_{n,k}^{(\alpha,q)} \equiv M_{n,k}^{(\alpha)}(\underline{X}_n^{(q)}) := \frac{1}{k}\sum_{i=1}^{k} \left(\ln \frac{X_{n-i+1:n} - X_{n_q:n}}{X_{n-k:n} - X_{n_q:n}}\right)^{\alpha},$$

defined for $k < n - n_q$, with $M_{n,k}^{(\alpha)}(\underline{X}_n)$ given in (1.12), $\alpha > 0$.

Regarding the *tuning* parameters $\tau_q \in \mathbb{R}$, $\alpha, \theta_1, \theta_2 \in \mathbb{R}^+$, $\theta_1, \theta_2 \neq 1$ and $\theta_1 < \theta_2$, we shall consider the PORT-versions of the statistics used in Fraga Alves *et al.* (2003a) for the estimation of ρ, in (1.6), i.e.

$$(1.15) \qquad T_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)} := \frac{\left(\frac{M_{n,k}^{(\alpha,q)}}{\Gamma(\alpha+1)}\right)^{\tau_q} - \left(\frac{M_{n,k}^{(\alpha\theta_1,q)}}{\Gamma(\alpha\theta_1+1)}\right)^{\tau_q/\theta_1}}{\left(\frac{M_{n,k}^{(\alpha\theta_1,q)}}{\Gamma(\alpha\theta_1+1)}\right)^{\tau_q/\theta_1} - \left(\frac{M_{n,k}^{(\alpha\theta_2,q)}}{\Gamma(\alpha\theta_2+1)}\right)^{\tau_q/\theta_2}} =: \frac{D_{n,k}^{(\alpha,1,\theta_1,\tau_q,q)}(\xi)}{D_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)}(\xi)},$$

with $\Gamma(t)$ denoting the complete Gamma function. As detailed in Section 3.1, under adequate conditions upon the growth of $k = k_n$, $T_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)}$ converges in probability to

$$t_{\alpha,\theta_1,\theta_2}(\rho_q) := \theta_2 \frac{(\theta_1-1)(1-\rho_q)^{\alpha\theta_2} - \theta_1(1-\rho_q)^{\alpha(\theta_2-1)} + (1-\rho_q)^{\alpha(\theta_2-\theta_1)}}{(\theta_2-\theta_1)(1-\rho_q)^{\alpha\theta_2} - \theta_2(1-\rho_q)^{\alpha(\theta_2-\theta_1)} + \theta_1}. \tag{1.16}$$

Remark 1.1. Note that the function $t_{\alpha,\theta_1,\theta_2}(\rho_q)$, defined for $\rho_q \le 0$, $\alpha > 0$, $\theta_1, \theta_2 \in \mathbb{R}^+ \setminus \{1\}$, $\theta_1 < \theta_2$, is a decreasing function of ρ_q if $\theta_1, \theta_2 > 1$ or $\theta_1, \theta_2 < 1$ and increasing otherwise. Since $t_{\alpha,\theta_1,\theta_2}(\rho_q)$ is always monotone continuous then it is invertible. Moreover,

$$\lim_{\rho_q \to -\infty} t_{\alpha,\theta_1,\theta_2}(\rho_q) = \frac{\theta_2(\theta_1-1)}{\theta_2-\theta_1} \quad \text{and} \quad \lim_{\rho_q \to 0} t_{\alpha,\theta_1,\theta_2}(\rho_q) = \frac{\theta_1-1}{\theta_2-\theta_1}.$$

The general class of consistent ρ_q-estimators, invariant for changes in location, already introduced and validated under a second-order framework in Henriques-Rodrigues and Gomes (2012), and named PORT-ρ class of estimators, it is now written as

$$\widehat{\rho}_{n,k|T}^{(\alpha,\theta_1,\theta_2,\tau_q,q)} := -\left| t_{\alpha,\theta_1,\theta_2}^{\leftarrow}\left(T_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)}\right)\right|. \tag{1.17}$$

with $T_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)}$ given in (1.15).

The simplest choice of tuning control parameters suggested in Fraga Alves *et al.* (2003a) for the classical ρ-estimators, $(\alpha, \theta_1, \theta_2) = (1, 2, 3)$, gives rise to an explicit ρ-estimator, denoted $\widehat{\rho}_k^{(\tau)}$ in the aforementioned paper, and leads us to a simpler class of PORT-ρ estimators of the shape second-order parameter ρ_q, because it only depends on the tuning parameter τ_q. With ρ_q defined in (1.11), we have that

$$t(\rho_q) = t_{1,2,3}(\rho_q) = \frac{3(1-\rho_q)}{3-\rho_q} = \begin{cases} \frac{3(1+\xi)}{3+\xi}, & \text{if } \xi + \rho_0 < 0 \ \wedge \chi_q \neq 0, \\ \frac{3(1-\rho_0)}{3-\rho_0}, & \text{otherwise.} \end{cases}$$

Thus the PORT-ρ estimator associated with $(\alpha, \theta_1, \theta_2) = (1, 2, 3)$ is explicitly given by

$$\widehat{\rho}_k^{(\tau_q,q)} \equiv \widehat{\rho}_{n,k|T}^{(1,2,3,\tau_q,q)} := -\left| \frac{3\left(T_{n,k}^{(1,2,3,\tau_q,q)} - 1\right)}{T_{n,k}^{(1,2,3,\tau_q,q)} - 3} \right|, \tag{1.18}$$

where

$$T_{n,k}^{(1,2,3,\tau_q,q)} = \frac{\left(M_{n,k}^{(1,q)}\right)^{\tau_q} - \left(M_{n,k}^{(2,q)}/2\right)^{\tau_q/2}}{\left(M_{n,k}^{(2,q)}/2\right)^{\tau_q/2} - \left(M_{n,k}^{(3,q)}/6\right)^{\tau_q/3}},$$

for any $\tau_q \in \mathbb{R}$, with $M_{n,k}^{(\alpha,q)}$ given in (1.14). The notation $a^{b\tau_q} = b \ln a$ is used for $\tau_q = 0$.

In Section 2 of this paper we present preliminary asymptotic results related to the PORT-methodology. In Section 3 we justify the class of PORT-ρ estimators of the shape second-order parameter ρ_q, in (1.11), addressing the possibility of shifted heavy-tailed models, and refer the conditions required for their consistency and asymptotic normality. In Section 4, we illustrate the finite sample behaviour of the new estimators through a small-scale Monte-Carlo simulation study. Finally, in Section 5, we present the proofs of the results in Section 3.

2. TECHNICAL RESULTS RELATED TO THE PORT-METHODOLOGY

2.1. The second-order PORT-framework for heavy-tailed models

Under the aforementioned set-up in Section 1.3, the transformed r.v., $X_q = X_0 - \chi_q$, has an associated quantile function given by $U_q(t) = U_0(t) - \chi_q$. The second-order condition in (1.6) translates as

$$\lim_{t\to\infty} \frac{\ln U_q(tx) - \ln U_q(t) - \xi \ln x}{A_q(t)} = \begin{cases} \frac{x^{\rho_q}-1}{\rho_q}, & \text{if } \rho_q < 0 \\ \ln x, & \text{if } \rho_q = 0, \end{cases} \tag{2.1}$$

for all $x > 0$. Moreover, $|A_q| \in RV_{\rho_q}$, $\rho_q \le 0$, and A_q relates to A_0 according to the following lemma.

Lemma 2.1. *Assume $U_0 \in RV_\xi$ satisfies the second order condition in* (1.6) *with $\rho = \rho_0$ and $A = A_0$. Then $U_q(t) = U_0(t) - \chi_q$, with χ_q defined in* (1.9), *is such that $U_q \in RV_\xi$ and* (2.1) *holds with ρ_q given in* (1.11) *and*

$$A_q(t) := \begin{cases} \xi\chi_q/U_0(t), & \text{if } \xi+\rho_0 < 0 \ \wedge\ \chi_q \ne 0 \\ A_0(t), & \text{if } \xi+\rho_0 > 0 \ \vee\ \chi_q = 0 \\ A_0(t) + \xi\chi_q/U_0(t), & \text{if } \xi+\rho_0 = 0 \ \wedge\ \chi_q \ne 0. \end{cases} \tag{2.2}$$

2.2. Third-order framework and asymptotic behaviour of auxiliary statistics

Next, we present the asymptotic behaviour of the statistics $M_{n,k}^{(\alpha,q)}$ defined in (1.14), based on the sample of excesses $\underline{X}_n^{(q)}$, $0 \le q < 1$, defined in (1.8). This requires a third-order framework because we further need to know the rate of convergence in (1.6). It is common to assume a third-order condition that rules such a rate of convergence through the shape third-order parameter $\rho' \le 0$, assuming that for all $x > 0$,

$$\lim_{t\to\infty} \frac{\frac{\ln U(tx)-\ln U(t)-\xi\ln x}{A(t)} - \frac{x^\rho-1}{\rho}}{B(t)} = \frac{x^{\rho+\rho'}-1}{\rho+\rho'}, \tag{2.3}$$

with $|A| \in RV_\rho$ and $|B| \in RV_{\rho'}$. For technical simplicity, we shall assume that $\rho,\ \rho' < 0$, i.e. we assume to be working in a class $\mathcal{H}$ of heavy-tailed models, such that, as $t \to \infty$,

$$U(t) = Ct^\xi \left\{1 + D_1 t^\rho + D_2 t^{\rho+\rho'} + o(t^{\rho+\rho'})\right\}, \tag{2.4}$$

where $C > 0$. Details on the third-order condition in (2.3) can be found in Fraga Alves *et al.* (2003b, 2006) and more generally in Wang and Cheng (2005).

Note that the statistics $M_{n,k}^{(\alpha,q)}$, in (1.14), depend on q through χ_q, in (1.9) (see also (1.10)), but are obviously independent on any shift s imposed to the data. We can thus assume throughout that $s = 0$.

Let $\mathbb{E}$ and $\mathbb{V}ar$ denote the mean value and variance operators, respectively, and let E denote a unit exponential random variable. For any real $\alpha > 0$, with $\xi > 0$ and $\rho < 0$, let us define

$$\mu_\alpha^{(1)}(\xi) := \mathbb{E}\Big(E^\alpha e^{-\xi E}\Big) = \frac{\Gamma(\alpha+1)}{(1+\xi)^{\alpha+1}}, \qquad \mu_\alpha^{(1)} := \mu_\alpha^{(1)}(0) = \Gamma(\alpha+1), \tag{2.5}$$

$$\sigma_\alpha^{(1)} := \sqrt{\mathbb{V}\mathrm{ar}(E^\alpha)} = \sqrt{\Gamma(2\alpha+1) - \Gamma^2(\alpha+1)}, \tag{2.6}$$

$$\mu_\alpha^{(2)}(\xi,\rho) := \mathbb{E}\big(E^{\alpha-1}\, e^{-\xi E}\, (e^{\rho E}-1)/\rho\big) = \frac{\Gamma(\alpha)}{\rho}\left(\frac{(1+\xi)^\alpha - (1+\xi-\rho)^\alpha}{(1+\xi-\rho)^\alpha (1+\xi)^\alpha}\right),$$

$$\mu_\alpha^{(2)}(\rho) := \mu_\alpha^{(2)}(0,\rho) = \frac{\Gamma(\alpha)}{\rho}\left(\frac{1-(1-\rho)^\alpha}{(1-\rho)^\alpha}\right),$$

$$\sigma_\alpha^{(2)}(\rho) := \sqrt{\mathbb{V}\mathrm{ar}\big(E^{\alpha-1}(e^{\rho E}-1)/\rho\big)} = \sqrt{\mu_{2\alpha}^{(3)}(\rho) - \big(\mu_\alpha^{(2)}(\rho)\big)^2},$$

$$\eta_\alpha^{(3)}(\xi,\rho) := \mathbb{E}\Big(E^{\alpha-2}\ \big((e^{-\xi E}-1)/(-\xi)\big)\ \big((e^{\rho E}-1)/\rho\big)\Big)$$
$$= \begin{cases} -\frac{1}{\xi\rho}\ln\frac{(1+\xi)(1-\rho)}{1+\xi-\rho}, & \text{if } \alpha = 1 \\ -\frac{\Gamma(\alpha)}{\xi\rho(\alpha-1)}\left\{\frac{1}{(1+\xi-\rho)^{\alpha-1}} - \frac{1}{(1+\xi)^{\alpha-1}} - \frac{1}{(1-\rho)^{\alpha-1}} + 1\right\}, & \text{if } \alpha \neq 1, \end{cases}$$

and

$$\mu_\alpha^{(3)}(\rho) := \mathbb{E}\Big(E^{\alpha-2}\ \big((e^{\rho E}-1)/\rho\big)^2\Big)$$
$$= \begin{cases} \frac{1}{\rho^2}\ln\frac{(1-\rho)^2}{1-2\rho}, & \text{if } \alpha = 1 \\ \frac{\Gamma(\alpha)}{\rho^2(\alpha-1)}\left\{\frac{1}{(1-2\rho)^{\alpha-1}} - \frac{2}{(1-\rho)^{\alpha-1}} + 1\right\}, & \text{if } \alpha \neq 1. \end{cases}$$

Let us further introduce the notations:

$$\overline{\mu}_\alpha^{(j)}(\rho) := \frac{\mu_\alpha^{(j)}(\rho)}{\mu_\alpha^{(1)}},\ j=2,3, \quad \overline{\mu}_\alpha^{(2)}(\xi,\rho) := \frac{\mu_\alpha^{(2)}(\xi,\rho)}{\mu_\alpha^{(1)}}, \tag{2.7}$$

$$\overline{\eta}_\alpha^{(3)}(\xi,\rho) := \frac{\eta_\alpha^{(3)}(\xi,\rho)}{\mu_\alpha^{(1)}}, \tag{2.8}$$

$$\overline{\sigma}_\alpha^{(1)} := \frac{\sigma_\alpha^{(1)}}{\mu_\alpha^{(1)}}, \qquad \overline{\sigma}_\alpha^{(2)}(\rho) := \frac{\sigma_\alpha^{(2)}(\rho)}{\mu_\alpha^{(1)}}, \tag{2.9}$$

and for any $\theta_1,\ \theta_2 > 0$, define

$$d_{\alpha,\theta_1,\theta_2}(\rho) := \overline{\mu}_{\alpha\theta_1}^{(2)}(\rho) - \overline{\mu}_{\alpha\theta_2}^{(2)}(\rho). \tag{2.10}$$

Recall that $E_i,\ i \geq 1$, are i.i.d. unit exponential r.v.'s, and, with $\sigma_\alpha^{(1)}$ given in (2.6), define the asymptotically standard normal r.v.'s

$$Z_k^{(\alpha)} := \sqrt{k}\Big(\tfrac{1}{k}\sum_{i=1}^k E_i^\alpha - \Gamma(\alpha+1)\Big)/\sigma_\alpha^{(1)}. \tag{2.11}$$

Now, together with (2.9), we can combine these as follows:

$$W_k^{(\alpha,\theta_1,\theta_2)} := \overline{\sigma}_{\alpha\theta_1}^{(1)} Z_k^{(\alpha\theta_1)}/\theta_1 - \overline{\sigma}_{\alpha\theta_2}^{(1)} Z_k^{(\alpha\theta_2)}/\theta_2. \tag{2.12}$$

Finally, for $\tau \in \mathbb{R}$, $\alpha, \theta > 0$, and with $\big(\overline{\mu}_\alpha^{(2)}(\rho), \overline{\mu}_\alpha^{(2)}(\xi,\rho)\big)$ and $\overline{\eta}_\alpha^{(3)}(\xi,\rho)$ defined in (2.7) and (2.8), respectively, we define

$$c_{\alpha,\theta,\tau}(\rho) := (\alpha\theta-1)\overline{\mu}_{\alpha\theta}^{(3)}(\rho) + \alpha(\tau-\theta)\big(\overline{\mu}_{\alpha\theta}^{(2)}(\rho)\big)^2, \tag{2.13}$$

$$g_{\alpha,\theta,\tau}(\xi,\rho) := \overline{\mu}_{\alpha\theta}^{(2)}(\xi,\rho) + (\alpha\theta-1)\overline{\eta}_{\alpha\theta}^{(3)}(\xi,\rho) + \alpha(\tau-\theta)\overline{\mu}_{\alpha\theta}^{(2)}(\rho)\overline{\mu}_{\alpha\theta}^{(2)}(-\xi), \tag{2.14}$$

$$h_{\alpha,\theta,\tau}(\xi) := 2\overline{\mu}_{\alpha\theta}^{(2)}(-2\xi) + (\alpha\theta-1)\overline{\mu}_{\alpha\theta}^{(3)}(-\xi) + \alpha(\tau-\theta)\left(\overline{\mu}_{\alpha\theta}^{(2)}(-\xi)\right)^2. \tag{2.15}$$

We first state Proposition 2.1, related to the behaviour of $M_{n,k}^{(\alpha)}$, in (1.12), now needed only for $s=0$ ($\rho=\rho_0$), proved in Gomes *et al.* (2002), also under a third-order framework.

Proposition 2.1 (Gomes *et al.*, 2002). *Under the third-order condition* (2.3), *with* $\rho_0,\ \rho_0' < 0$, *for intermediate sequences* $k = k_n$, *i.e. sequences of positive integers such that* (1.13) *holds, and with* $M_{n,k}^{(\alpha)}$, $\mu_\alpha^{(1)}$, $\overline{\mu}_\alpha^{(j)}(\rho)$, $j=2,3$, $\overline{\sigma}_\alpha^{(1)}$ *and* $Z_k^{(\alpha)}$ *defined in* (1.12), (2.5), (2.7), (2.9) *and* (2.11), *respectively,*

$$M_{n,k}^{(\alpha)} \overset{d}{=} \xi^\alpha \mu_\alpha^{(1)}\Big\{1 + \overline{\sigma}_\alpha^{(1)}\ \frac{Z_k^{(\alpha)}}{\sqrt{k}} + \tfrac{\alpha}{\xi}\ \overline{\mu}_\alpha^{(2)}(\rho_0)A_0(n/k)$$
$$+ \Big(\tfrac{\alpha(\alpha-1)}{2\xi^2}\ \overline{\mu}_\alpha^{(3)}(\rho_0)A_0^2(n/k) + \tfrac{\alpha}{\xi}\ \overline{\mu}_\alpha^{(2)}(\rho_0+\rho_0')A_0(n/k)B_0(n/k)\Big)(1+o_p(1))\Big\}.$$

We next provide, under the third-order framework in (2.3), the behaviour of $M_{n,k}^{(\alpha,q)}$, in (1.14).

Proposition 2.2. *Let us assume that* (1.13) *holds, as well as the third-order condition in* (2.3), *with* $\rho_0, \rho_0' < 0$. *We then get for* $M_{n,k}^{(\alpha,q)}$, *in* (1.14), $\alpha > 0$, $k < n - n_q$, *with* χ_q *and* $M_{n,k}^{(\alpha)}$ (*for* $s = 0$), *given in* (1.10) *and* (1.12), *respectively,* $\mu_\alpha^{(1)}$ *and* $\big(\overline{\mu}_\alpha^{(2)}(\rho), \overline{\mu}_\alpha^{(2)}(\xi,\rho), \overline{\mu}_\alpha^{(3)}(\rho)\big)$ *and* $\overline{\eta}_\alpha^{(3)}(\xi,\rho)$ *respectively given in* (2.5), (2.7) *and* (2.8), *the distributional representation,*

$$(2.16)\quad M_{n,k}^{(\alpha,q)} \overset{d}{=} M_{n,k}^{(\alpha)} + \frac{\alpha\xi^\alpha\mu_\alpha^{(1)}\chi_q}{U_0(n/k)}\Big\{\overline{\mu}_\alpha^{(2)}(-\xi) + \frac{\overline{\mu}_\alpha^{(2)}(\xi,\rho_0)+(\alpha-1)\,\overline{\eta}_\alpha^{(3)}(\xi,\rho_0)}{\xi}\,A_0(n/k)(1+o_p(1)) + \frac{\chi_q}{U_0(n/k)}\Big(\overline{\mu}_\alpha^{(2)}(-2\xi) + \frac{(\alpha-1)}{2}\overline{\mu}_\alpha^{(3)}(-\xi)\Big)(1+o_p(1))\Big\}.$$

3. ASYMPTOTIC BEHAVIOUR OF THE PORT-ρ ESTIMATORS

3.1. Consistency of the PORT-ρ estimators

For $\alpha > 0$, let us consider the statistics $M_{n,k}^{(\alpha,q)} = M_{n,k}^{(\alpha)}\big(\underline{X}_n^{(q)}\big)$, in (1.14), defined for $k < n - n_q$, with $\underline{X}_n^{(q)}$ the sample of excesses in (1.8). Under the third-order framework in (2.3), if (1.13) holds, on the basis of the results in Propositions 2.1 and 2.2, similarly to the developments in Fraga Alves *et al.* (2003a), and for real tuning parameters $\tau_q \in \mathbb{R}$ and $\theta \neq 0$,

$$(3.1)\quad \left(\frac{M_{n,k}^{(\alpha\theta,q)}}{\mu_{\alpha\theta}^{(1)}}\right)^{\tau_q/\theta} \overset{d}{=} \xi^{\alpha\tau_q}\Bigg(1 + \frac{\tau_q}{\theta}\,\frac{\overline{\sigma}_{\alpha\theta}^{(1)}}{\sqrt{k}}\,Z_k^{(\alpha\theta)} + \frac{\alpha\tau_q\;\overline{\mu}_{\alpha\theta}^{(2)}(\rho_0)A_0(n/k)}{\xi} + \frac{\alpha\tau_q\;\chi_q\;\overline{\mu}_{\alpha\theta}^{(2)}(-\xi)}{U_0(n/k)} + \left\{\frac{\alpha\tau_q\;c_{\alpha,\theta,\tau_q}(\rho_0)}{2\xi^2}A_0^2(n/k) + \frac{\alpha\tau_q\;\overline{\mu}_{\alpha\theta}^{(2)}(\rho_0+\rho_0')}{\xi}A_0(n/k)B_0(n/k)\right\}(1+o_p(1)) + \left\{\frac{\alpha\tau_q\chi_q}{\xi}g_{\alpha,\theta,\tau_q}(\xi,\rho_0)\frac{A_0(n/k)}{U_0(n/k)} + \frac{\alpha\tau_q\chi_q^2}{2}\;h_{\alpha,\theta,\tau_q}(\xi)\frac{1}{U_0^2(n/k)}\right\}(1+o_p(1))\Bigg).$$

i.e.

$$\left(\frac{M_{n,k}^{(\alpha\theta,q)}}{\mu_{\alpha\theta}^{(1)}}\right)^{\tau_q/\theta} \overset{d}{=} \left(\frac{M_{n,k}^{(\alpha\theta)}}{\mu_{\alpha\theta}^{(1)}}\right)^{\tau_q/\theta} + \frac{\alpha\tau_q\xi^{\alpha\tau_q}\chi_q}{U_0(n/k)}\Bigg\{\overline{\mu}_{\alpha\theta}^{(2)}(-\xi) + \frac{g_{\alpha,\theta,\tau_q}(\xi,\rho_0)}{\xi}A_0(n/k)(1+o_p(1)) + \frac{\chi_q\;h_{\alpha,\theta,\tau_q}(\xi)}{2}\,\frac{1}{U_0(n/k)}(1+o_p(1))\Bigg\},$$

with $M_{n,k}^{(\alpha,q)}$, $\mu_\alpha^{(1)}$, $\overline{\mu}_\alpha^{(j)}(\rho)$, $j = 2,3$, $\overline{\sigma}_\alpha^{(1)}$, $Z_k^{(\alpha)}$, $c_{\alpha,\theta,\tau}(\rho)$, $g_{\alpha,\theta,\tau}(\xi,\rho)$ and $h_{\alpha,\theta,\tau}(\xi)$ given in (1.14), (2.5), (2.7), (2.9), (2.11), (2.13), (2.14) and (2.15), respectively.

Let us next introduce the notations,

$$u_{\alpha,\theta_1,\theta_2,\tau}(\rho) := \big\{c_{\alpha,\theta_1,\tau}(\rho) - c_{\alpha,\theta_2,\tau}(\rho)\big\}/(2\xi), \tag{3.2}$$

$$v_{\alpha,\theta_1,\theta_2}(\rho,\rho') := \overline{\mu}^{(2)}_{\alpha\theta_1}(\rho+\rho') - \overline{\mu}^{(2)}_{\alpha\theta_2}(\rho+\rho') \;\equiv\; d_{\alpha,\theta_1,\theta_2}(\rho+\rho'), \tag{3.3}$$

$$w_{\alpha,\theta_1,\theta_2,\tau}(\xi,\rho) := \{g_{\alpha,\theta_1,\tau}(\xi,\rho) - g_{\alpha,\theta_2,\tau}(\xi,\rho)\}/\xi, \tag{3.4}$$

$$y_{\alpha,\theta_1,\theta_2,\tau}(\xi) := \{h_{\alpha,\theta_1,\tau}(\xi) - h_{\alpha,\theta_2,\tau}(\xi)\}/2, \tag{3.5}$$

with $d_{\alpha,\theta_1,\theta_2}(\rho)$, $c_{\alpha,\theta,\tau}(\rho)$, $g_{\alpha,\theta,\tau}(\xi,\rho)$ and $h_{\alpha,\theta,\tau}(\xi)$ defined in (2.10), (2.13), (2.14) and (2.15), respectively. On the basis of (3.1), using the notation $W_k^{(\alpha,\theta_1,\theta_2)}$ in (2.12), and with $D_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)}(\xi)$ defined in (1.15), we can write

$$\begin{aligned} D_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)}(\xi) \overset{d}{=} \xi^{\alpha\tau_q}\Bigg(&\frac{\tau_q}{\sqrt{k}}\, W_k^{(\alpha,\theta_1,\theta_2)} + \frac{\alpha\tau_q\; A_0(n/k)}{\xi}\Big\{d_{\alpha,\theta_1,\theta_2}(\rho_0) \\ &+ u_{\alpha,\theta_1,\theta_2,\tau}(\rho_0)A_0(n/k)(1+o_p(1)) + v_{\alpha,\theta_1,\theta_2}(\rho_0,\rho_0')B_0(n/k)(1+o_p(1))\Big\} \\ &+ \frac{\alpha\tau_q\chi_q}{U_0(n/k)}\Big\{d_{\alpha,\theta_1,\theta_2}(-\xi) + w_{\alpha,\theta_1,\theta_2,\tau}(\xi,\rho_0)A_0(n/k)(1+o_p(1)) \\ &+ \frac{\chi_q\; y_{\alpha,\theta_1,\theta_2,\tau}(\xi)}{U_0(n/k)}(1+o_p(1))\Big\}\Bigg), \end{aligned} \tag{3.6}$$

i.e.

$$\begin{aligned} D_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)}(\xi) \;\overset{d}{=}\; & D_{n,k}^{(\alpha,\theta_1,\theta_2,\tau)}(\xi) + \frac{\alpha\tau_q\chi_q\xi^{\alpha\tau_q}}{U_0(n/k)}\Big\{d_{\alpha,\theta_1,\theta_2}(-\xi) \\ &+ w_{\alpha,\theta_1,\theta_2,\tau}(\xi,\rho_0)A_0(n/k)(1+o_p(1)) + \frac{\chi_q\; y_{\alpha,\theta_1,\theta_2,\tau}(\xi)}{U_0(n/k)}(1+o_p(1))\Big\}. \end{aligned}$$

The dominant component of the right hand-side of (3.6) depends on the relative behaviour of the functions $A_0(t)$ and $1/U_0(t)$. We shall thus consider three different regions related to χ_q, in (1.9), and the vector (ξ,ρ_0) of the unshifted model F_0 associated with the available data:

- $\mathcal{R}_1 := \{F_0 : \xi+\rho_0 < 0 \wedge \chi_q \neq 0\}$,
- $\mathcal{R}_2 := \{F_0 : \xi+\rho_0 > 0 \vee \chi_q = 0\}$,
- $\mathcal{R}_3 := \{F_0 : \xi+\rho_0 = 0 \wedge \chi_q \neq 0\}$.

We now state the following:

Theorem 3.1 (Henriques-Rodrigues and Gomes, 2013, Theorem 1)**.** *Under the validity of the second-order condition in* (1.6)*, with $\rho=\rho_0<0$, ρ_q defined in* (1.11)*, $\widehat{\rho}_{n,k|T}^{(\alpha,\theta_1,\theta_2,\tau_q,q)}$ defined in* (1.17)*, and with an explicit expression given in* (1.18) *for the particular case $(\alpha,\theta_1,\theta_2)=(1,2,3)$, is consistent for the estimation of ρ_q, i.e.*

$$\widehat{\rho}_{n,k|T}^{(\alpha,\theta_1,\theta_2,\tau_q,q)} \;\underset{n\to\infty}{\overset{p}{\longrightarrow}}\; \rho_q,$$

for any real $\alpha > 0$, $\tau_q \in \mathbb{R}$, $\theta_1, \theta_2 \in \mathbb{R}^+ \backslash \{1\}$, $\theta_1 < \theta_2$ *and* $0 < q < 1$ *or* $q = 0$ *if* $\chi_0 = x_F$, *the left endpoint of the underlying parent, is finite, provided that* k *is an intermediate sequence, and moreover, with* A_q *defined in* (2.2),

$$\sqrt{k} A_q(n/k) \to \infty, \text{ as } n \to \infty. \tag{3.7}$$

Remark 3.1. Note that when we consider models $F_0 \in \mathcal{R}_1$, $A_0(t) = o(1/U_0(t))$ and with $A_q(t) = \xi\chi_q/U_0(t)$, by (2.2), condition (3.7) corresponds to $\sqrt{k}/U_0(n/k) \to \infty$, as $n \to \infty$. For models $F_0 \in \mathcal{R}_2$, $1/U_0(t) = o(A_0(t))$ and since $A_q(t) = A_0(t)$, condition (3.7) is equivalent to $\sqrt{k}A_0(n/k) \to \infty$, as $n \to \infty$. Finally, for models $F_0 \in \mathcal{R}_3$, $1/U_0(t) = O(A_0(t))$ and since $A_q(t) = A_0(t) + \xi\chi_q/U_0(t)$, condition (3.7) is equivalent to $\sqrt{k}A_0(n/k) \to \infty$ or $\sqrt{k}/U_0(n/k) \to \infty$, as $n \to \infty$.

3.2. Non-degenerate asymptotic behaviour of the PORT-ρ estimators

In this section, and under a third-order framework, we derive the non-degenerate asymptotic properties of the PORT-ρ classes of estimators introduced with all the generality in (1.17), and particularised in (1.18). We first state the following result:

Proposition 3.1 (Fraga Alves *et al.*, 2003). *Under the validity of the second-order condition in* (1.6), *with* $\rho < 0$, *if* (1.13) *holds and* $\sqrt{k}A(n/k) \to \infty$, *as* $n \to \infty$, *the asymptotic variance of* $W_k^{(\alpha,\theta_1,\theta_2)}$, *in* (2.12), *is*

$$\sigma^2_{W|\alpha,\theta_1,\theta_2} = \frac{2}{\alpha}\left(\frac{\Gamma(2\alpha\theta_1)}{\theta_1^3\Gamma^2(\alpha\theta_1)} + \frac{\Gamma(2\alpha\theta_2)}{\theta_2^3\Gamma^2(\alpha\theta_2)} - \frac{(\theta_1+\theta_2)\Gamma(\alpha(\theta_1+\theta_2))}{\theta_1^2\theta_2^2\Gamma(\alpha\theta_1)\Gamma(\alpha\theta_2)}\right) - \left(\frac{1}{\theta_1} - \frac{1}{\theta_2}\right)^2, \tag{3.8}$$

and the asymptotic covariance of $(W_k^{(\alpha,1,\theta_1)}, W_k^{(\alpha,\theta_1,\theta_2)})$ *is given by*

$$\begin{aligned}\sigma_{W|\alpha,1,\theta_1,\theta_2} = \frac{1}{\alpha}\Bigg(&\frac{(\theta_1+1)\Gamma(\alpha(\theta_1+1))}{\theta_1^2\Gamma(\alpha)\Gamma(\alpha\theta_1)} - \frac{(\theta_2+1)\Gamma(\alpha(\theta_2+1))}{\theta_2^2\Gamma(\alpha)\Gamma(\alpha\theta_2)} - \frac{2\Gamma(2\alpha\theta_1)}{\theta_1^3\Gamma^2(\alpha\theta_1)} \\ &+ \frac{(\theta_1+\theta_2)\Gamma(\alpha(\theta_1+\theta_2))}{\theta_1^2\theta_2^2\Gamma(\alpha\theta_1)\Gamma(\alpha\theta_2)}\Bigg) - \left(1 - \frac{1}{\theta_1}\right)\left(\frac{1}{\theta_1} - \frac{1}{\theta_2}\right).\end{aligned} \tag{3.9}$$

Note that $t'_{\alpha,\theta_1,\theta_2}(\rho) := dt_{\alpha,\theta_1,\theta_2}(\rho)/d\rho$, with $t_{\alpha,\theta_1,\theta_2}(\rho_q)$ defined in (1.16), is given by

$$\begin{aligned} t'_{\alpha,\theta_1,\theta_2}(\rho)(1-\rho)&\left((\theta_2-\theta_1)(1-\rho)^{\alpha\theta_2} - \theta_2(1-\rho)^{\alpha(\theta_2-\theta_1)} + \theta_1\right)^2 \\ &= \alpha\theta_1\theta_2\Big\{\theta_1(\theta_2-1)(1-\rho)^{\alpha(\theta_2-1)}\left(1+(1-\rho)^{\alpha(\theta_2-\theta_1+1)}\right) \\ &\quad - (\theta_2-\theta_1)(1-\rho)^{\alpha(\theta_2-\theta_1)}\left(1+(1-\rho)^{\alpha(\theta_2-\theta_1-1)}\right) \\ &\quad - \theta_2(\theta_1-1)(1-\rho)^{\alpha\theta_2}\left(1+(1-\rho)^{\alpha(\theta_2-\theta_1-1)}\right)\Big\}.\end{aligned} \tag{3.10}$$

Let us further use the notations,

$$(3.11)\quad y_T^{(\alpha,\theta_1,\theta_2,\tau)}(\xi,\rho) := \frac{y_{\alpha,1,\theta_1,\tau}(\xi)-t_{\alpha,\theta_1,\theta_2}(\rho)\,y_{\alpha,\theta_1,\theta_2,\tau}(\xi)}{d_{\alpha,\theta_1,\theta_2}(\rho)},\qquad y_{\rho|T}^{(\alpha,\theta_1,\theta_2,\tau)}(\xi,\rho) := \frac{y_T^{(\alpha,\theta_1,\theta_2,\tau)}(\xi,\rho)}{t'_{\alpha,\theta_1,\theta_2}(\rho)},$$

$$(3.12)\quad z_T^{(\alpha,\theta_1,\theta_2)}(\xi,\rho) := \frac{d_{\alpha,1,\theta_1}(\rho)-t_{\alpha,\theta_1,\theta_2}(-\xi)\,d_{\alpha,\theta_1,\theta_2}(\rho)}{\xi\, d_{\alpha,\theta_1,\theta_2}(-\xi)},\qquad z_{\rho|T}^{(\alpha,\theta_1,\theta_2)}(\xi,\rho) := \frac{z_T^{(\alpha,\theta_1,\theta_2)}(\xi,\rho)}{t'_{\alpha,\theta_1,\theta_2}(-\xi)},$$

$$(3.13)\quad u_T^{(\alpha,\theta_1,\theta_2,\tau)}(\rho) := \frac{u_{\alpha,1,\theta_1,\tau}(\rho)-t_{\alpha,\theta_1,\theta_2}(\rho)\,u_{\alpha,\theta_1,\theta_2,\tau}(\rho)}{d_{\alpha,\theta_1,\theta_2}(\rho)},\qquad u_{\rho|T}^{(\alpha,\theta_1,\theta_2,\tau)}(\rho) := \frac{u_T^{(\alpha,\theta_1,\theta_2,\tau)}(\rho)}{t'_{\alpha,\theta_1,\theta_2}(\rho)},$$

$$(3.14)\quad v_T^{(\alpha,\theta_1,\theta_2)}(\rho,\rho') := \frac{v_{\alpha,1,\theta_1}(\rho,\rho')-t_{\alpha,\theta_1,\theta_2}(\rho)\,v_{\alpha,\theta_1,\theta_2}(\rho,\rho')}{d_{\alpha,\theta_1,\theta_2}(\rho)},\qquad v_{\rho|T}^{(\alpha,\theta_1,\theta_2)}(\rho,\rho') := \frac{v_T^{(\alpha,\theta_1,\theta_2)}(\rho,\rho')}{t'_{\alpha,\theta_1,\theta_2}(\rho)},$$

$$(3.15)\quad f_T^{(\alpha,\theta_1,\theta_2)}(\xi,\rho) := \frac{\xi\left\{d_{\alpha,1,\theta_1}(-\xi)-t_{\alpha,\theta_1,\theta_2}(\rho)\,d_{\alpha,\theta_1,\theta_2}(-\xi)\right\}}{d_{\alpha,\theta_1,\theta_2}(\rho)},\qquad f_{\rho|T}^{(\alpha,\theta_1,\theta_2)}(\xi,\rho) := \frac{f_T^{(\alpha,\theta_1,\theta_2)}(\xi,\rho)}{t'_{\alpha,\theta_1,\theta_2}(\rho)},$$

$$(3.16)\quad g_T^{(\alpha,\theta_1,\theta_2,\tau)}(\xi,\rho) := \frac{w_{\alpha,1,\theta_1,\tau}(\xi,\rho)-t_{\alpha,\theta_1,\theta_2}(\rho)\,w_{\alpha,\theta_1,\theta_2,\tau}(\xi,\rho)}{d_{\alpha,\theta_1,\theta_2}(\rho)},\qquad g_{\rho|T}^{(\alpha,\theta_1,\theta_2,\tau)}(\xi,\rho) := \frac{g_T^{(\alpha,\theta_1,\theta_2,\tau)}(\xi,\rho)}{t'_{\alpha,\theta_1,\theta_2}(\rho)},$$

with $t_{\alpha,\theta_1,\theta_2}(\rho)$, $d_{\alpha,\theta_1,\theta_2}(\rho)$, $u_{\alpha,\theta_1,\theta_2,\tau}(\rho)$, $v_{\alpha,\theta_1,\theta_2,\tau}(\rho,\rho')$, $w_{\alpha,\theta_1,\theta_2,\tau}(\xi,\rho)$, $y_{\alpha,\theta_1,\theta_2,\tau}(\xi)$ and $t'_{\alpha,\theta_1,\theta_2}(\rho)$ given in (1.16), (2.10), (3.2), (3.3), (3.4), (3.5) and (3.10), respectively.

We can finally derive the non-degenerate asymptotic behaviour of the class of PORT-ρ estimators, in (1.17).

Theorem 3.2. *Let us assume that the third-order condition in* (2.3) *holds, with* ρ_0, $\rho_0' < 0$ *and consider the* PORT-ρ *class of estimators,* $\widehat{\rho}_{n,k|T}^{(\alpha,\theta_1,\theta_2,\tau_q,q)}$, *defined in* (1.17), *with* ρ_q *given in* (1.11). *Then, with* $\theta_1 < \theta_2$, *real numbers different from* 1, $\alpha > 0$, $\tau_q \in \mathbb{R}$ *and* $0 < q < 1$ *or* $q = 0$ *provided that* $\chi_0 = x_F$ *is finite, and intermediate sequences of positive integers* $k = k_n$, *as in* (1.13), *such that* (3.7) *holds, we have:*

i) *In $\mathcal{R}_1$, let us consider the regions $\mathcal{R}_{11} := \{\rho_0 < -2\xi \wedge \chi_q \neq 0\}$, $\mathcal{R}_{12} := \{\rho_0 = -2\xi \wedge \chi_q \neq 0\}$ and $\mathcal{R}_{13} := \{-2\xi < \rho_0 < -\xi \wedge \chi_q \neq 0\}$. If we further assume that $\lim_{n\to\infty} \sqrt{k}A_0(n/k) = \lambda$ and $\lim_{n\to\infty} \sqrt{k}/U_0^2(n/k) = \lambda_U$, we get*

$$\frac{\sqrt{k}}{U_0(n/k)} \left(\widehat{\rho}_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)} - \rho_q\right) \underset{n\to\infty}{\overset{d}{\longrightarrow}} \mathcal{N}\left(\overset{\bullet}{\mu}{}_{\rho_0|T}^{(\alpha,\theta_1,\theta_2,\tau_q,q)}, \overset{\bullet}{\sigma}{}^2_{\rho_0|T,\alpha,\theta_1,\theta_2,q}\right),$$

with

$$\overset{\bullet}{\mu}{}_{\rho_0|T}^{(\alpha,\theta_1,\theta_2,\tau_q,q)} = \begin{cases} \chi_q\, \lambda_U\, y_{\rho_0|T}^{(\alpha,\theta_1,\theta_2,\tau_q)}(\xi,-\xi), & \text{in } \mathcal{R}_{11} \\ \frac{\lambda\, z_{\rho_0|T}^{(\alpha,\theta_1,\theta_2)}(\xi,\rho_0) + \chi_q^2\, \lambda_U\, y_{\rho_0|T}^{(\alpha,\theta_1,\theta_2,\tau_q)}(\xi,-\xi)}{\chi_q}, & \text{in } \mathcal{R}_{12} \\ \frac{\lambda\, z_{\rho_0|T}^{(\alpha,\theta_1,\theta_2)}(\xi,\rho_0)}{\chi_q}, & \text{in } \mathcal{R}_{13}, \end{cases}$$

$y_{\rho|T}^{(\alpha,\theta_1,\theta_2,\tau)}(\xi,\rho)$ and $z_{\rho|T}^{(\alpha,\theta_1,\theta_2)}(\xi,\rho)$ defined in (3.11) *and* (3.12), *respectively. Moreover,*

$$\overset{\bullet}{\sigma}{}^2_{\rho_0|T,\alpha,\theta_1,\theta_2,q} \equiv \overset{\bullet}{\sigma}{}^2_{\rho_0|T,\alpha,\theta_1,\theta_2} = \left\{\overset{\bullet}{\sigma}_{T|\alpha,\theta_1,\theta_2} / t'_{\alpha,\theta_1,\theta_2}(-\xi)\right\}^2,$$

where

$$\begin{aligned} \overset{\bullet}{\sigma}{}^2_{T|\alpha,\theta_1,\theta_2} &= \left(\frac{1}{\alpha\chi_q d_{\alpha,\theta_1,\theta_2}(-\xi)}\right)^2 \mathbb{V}\mathrm{ar}\left(W_k^{(\alpha,1,\theta_1)} - t_{\alpha,\theta_1,\theta_2}(-\xi) W_k^{(\alpha,\theta_1,\theta_2)}\right) \\ &= \frac{\sigma^2_{W|\alpha,1,\theta_1} + t^2_{\alpha,\theta_1,\theta_2}(-\xi)\sigma^2_{W|\alpha,\theta_1,\theta_2} - 2t_{\alpha,\theta_1,\theta_2}(-\xi)\sigma_{W|\alpha,1,\theta_1,\theta_2}}{\left(\alpha\chi_q d_{\alpha,\theta_1,\theta_2}(-\xi)\right)^2}, \end{aligned}$$

with $\sigma^2_{W|\alpha,\theta_1,\theta_2}$, $\sigma_{W|\alpha,1,\theta_1,\theta_2}$ and $t'_{\alpha,\theta_1,\theta_2}(\rho)$ given in (3.8), (3.9) *and* (3.10), *respectively.*

ii) *In $\mathcal{R}_2$, let us consider the regions $\mathcal{R}_{21} := \{-\xi < \rho_0 < -\frac{\xi}{2} \wedge \chi_q \neq 0\}$, $\mathcal{R}_{22} := \{\rho_0 = -\frac{\xi}{2} \wedge \chi_q \neq 0\}$ and $\mathcal{R}_{23} := \{\frac{\xi}{2} < \rho_0 < 0 \vee (\xi > -\rho_0 \wedge \chi_q = 0)\}$. If we further assume that $\lim_{n\to\infty} \sqrt{k}A_0^2(n/k) = \lambda_A$, $\lim_{n\to\infty} \sqrt{k}A_0(n/k)B_0(n/k) = \lambda_B$ and $\lim_{n\to\infty} \sqrt{k}/U_0(n/k) = \lambda'$, we get*

$$\sqrt{k}A_0(n/k)\left(\widehat{\rho}_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)} - \rho_q\right) \underset{n\to\infty}{\overset{d}{\longrightarrow}} \mathcal{N}\left(\mu_{\rho_0|T}^{(\alpha,\theta_1,\theta_2,\tau_q,q)}, \sigma^2_{\rho_0|T,\alpha,\theta_1,\theta_2,q}\right),$$

where with $\mu_{\rho_0|T}^{(\alpha,\theta_1,\theta_2,\tau_q)} := u_{\rho_0|T}^{(\alpha,\theta_1,\theta_2,\tau_q)}(\rho_0)\lambda_A + v_{\rho_0|T}^{(\alpha,\theta_1,\theta_2)}(\rho_0,\rho_0')\lambda_B$, and $u_{\rho|T}^{(\alpha,\theta_1,\theta_2,\tau)}(\rho)$, $v_{\rho|T}^{(\alpha,\theta_1,\theta_2)}(\rho,\rho')$ and $f_{\rho|T}^{(\alpha,\theta_1,\theta_2)}(\xi,\rho)$ given in (3.13), (3.14) *and* (3.15), *respectively,*

$$\mu_{\rho_0|T}^{(\alpha,\theta_1,\theta_2,\tau_q,q)} = \begin{cases} \chi_q\lambda' f_{\rho_0|T}^{(\alpha,\theta_1,\theta_2)}(\xi,\rho_0), & \textit{in } \mathcal{R}_{21} \\ \mu_{\rho_0|T}^{(\alpha,\theta_1,\theta_2,\tau_q)} + \chi_q\lambda' f_{\rho_0|T}^{(\alpha,\theta_1,\theta_2)}(\xi,\rho_0), & \textit{in } \mathcal{R}_{22} \\ \mu_{\rho_0|T}^{(\alpha,\theta_1,\theta_2,\tau_q)}, & \textit{in } \mathcal{R}_{23}. \end{cases}$$

Additionally,

$$\sigma^2_{\rho_0|T,\alpha,\theta_1,\theta_2,q} = \sigma^2_{\rho_0|T,\alpha,\theta_1,\theta_2} = \left\{\sigma_{T|\alpha,\theta_1,\theta_2}/t'_{\alpha,\theta_1,\theta_2}(\rho_0)\right\}^2,$$

with $\sigma^2_{T|\alpha,\theta_1,\theta_2}$ *given by*

$$\begin{aligned}\sigma^2_{T|\alpha,\theta_1,\theta_2} &= \left(\frac{\xi}{\alpha d_{\alpha,\theta_1,\theta_2}(\rho_0)}\right)^2 \mathbb{V}\text{ar}\left(W_k^{(\alpha,1,\theta_1)} - t_{\alpha,\theta_1,\theta_2}(\rho_0) W_k^{(\alpha,\theta_1,\theta_2)}\right)\\ &= \frac{\xi^2\left(\sigma^2_{W|\alpha,1,\theta_1} + t^2_{\alpha,\theta_1,\theta_2}(\rho_0)\sigma^2_{W|\alpha,\theta_1,\theta_2} - 2t_{\alpha,\theta_1,\theta_2}(\rho_0)\sigma_{W|\alpha,1,\theta_1,\theta_2}\right)}{\left(\alpha d_{\alpha,\theta_1,\theta_2}(\rho_0)\right)^2},\end{aligned} \tag{3.17}$$

$\sigma^2_{W|\alpha,\theta_1,\theta_2}$ *and* $\sigma_{W|\alpha,1,\theta_1,\theta_2}$ *defined in* (3.8) *and* (3.9), *respectively.*

iii) *In* $\mathcal{R}_3$, *if we further assume that* $\lim_{n\to\infty} \sqrt{k}A_0^2(n/k) = \lambda_A$, $\lim_{n\to\infty} \sqrt{k}A_0(n/k)B_0(n/k) = \lambda_B$ *and* $\lim_{n\to\infty} \sqrt{k}A_0(n/k)/U_0(n/k) = \lambda_{AU}$, *we get*

$$\sqrt{k}A_0(n/k)\left(\widehat{\rho}_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)} - \rho_q\right) \underset{n\to\infty}{\overset{d}{\longrightarrow}} \mathcal{N}\left(\widetilde{\mu}_{\rho_0|T}^{(\alpha,\theta_1,\theta_2,\tau_q,q)}, \widetilde{\sigma}^2_{\rho_0|T,\alpha,\theta_1,\theta_2,q}\right),$$

where, with $\widetilde{\lambda} = \lim_{n\to\infty} 1/\bigl(A_0(n/k)U_0(n/k)\bigr) \neq 0$, $w_{\rho|T}^{(\alpha,\theta_1,\theta_2,\tau)} := g_{\rho|T}^{(\alpha,\theta_1,\theta_2)}(\xi,\rho) + \chi_q\widetilde{\lambda}\; y_{\rho_0|T}^{(\alpha,\theta_1,\theta_2,\tau)}(\xi,\rho)$, $y_{\rho|T}^{(\alpha,\theta_1,\theta_2,\tau)}(\xi,\rho)$, $g_{\rho|T}^{(\alpha,\theta_1,\theta_2)}(\xi,\rho)$ *and* $\sigma^2_{T|\alpha,\theta_1,\theta_2}$ *defined in* (3.11), (3.16) *and* (3.17), *respectively,*

$$\widetilde{\mu}_{\rho_0|T}^{(\alpha,\theta_1,\theta_2,\tau_q,q)} = \frac{u_{\rho_0|T}^{(\alpha,\theta_1,\theta_2,\tau_q)}(\rho_0)\lambda_A + v_{\rho_0|T}^{(\alpha,\theta_1,\theta_2)}(\rho_0,\rho_0')\lambda_B + \xi\chi_q w_{\rho_0|T}^{(\alpha,\theta_1,\theta_2,\tau_q)}\lambda_{AU}}{1+\xi\widetilde{\lambda}\chi_q},$$

$$\widetilde{\sigma}^2_{\rho_0|T,\alpha,\theta_1,\theta_2,q} = \widetilde{\sigma}^2_{\rho_0|T,\alpha,\theta_1,\theta_2} = \frac{\sigma^2_{\rho_0|T,\alpha,\theta_1,\theta_2}}{(1+\xi\widetilde{\lambda}\chi_q)^2} = \left\{\frac{\sigma_{T|\alpha,\theta_1,\theta_2}}{(1+\xi\widetilde{\lambda}\chi_q)\; t'_{\alpha,\theta_1,\theta_2}(\rho_0)}\right\}^2.$$

We finally present the non-degenerate behaviour of the PORT-ρ estimators, in (1.18).

Corollary 3.1. *Under the validity of the third-order condition in* (2.3), *with* $\rho = \rho_0$, $\rho' = \rho_0' < 0$, *and for the particular case* $(\alpha,\theta_1,\theta_2) = (1,2,3)$, *we have the validity of the following asymptotic distributional representation for the* PORT-ρ *estimator,* $\widehat{\rho}_k^{(\tau_q,q)}$, *in* (1.18).

i) *In* $\mathcal{R}_1$, *and with the same notation as before for* $\mathcal{R}_{11}$, $\mathcal{R}_{12}$ *and* $\mathcal{R}_{13}$,

$$\widehat{\rho}_k^{(\tau_q,q)} \overset{d}{=} \rho_q + \frac{\overset{\bullet}{\sigma}_{\rho_0,q}}{\sqrt{k}/U_0(n/k)} W_k^{R_1} + \begin{cases} \frac{\chi_q\; y_{\rho_0|T}(\xi)}{U_0(n/k)}(1+o_p(1)), & \text{in } \mathcal{R}_{11}\\ \left(\frac{z_{\rho_0|T}(\xi,\rho_0)A_0(n/k)U_0(n/k)}{\chi_q} + \frac{\chi_q\; y_{\rho_0|T}(\xi)}{U_0(n/k)}\right)(1+o_p(1)), & \text{in } \mathcal{R}_{12}\\ \frac{z_{\rho_0|T}(\xi,\rho_0)A_0(n/k)U_0(n/k)}{\chi_q}(1+o_p(1)), & \text{in } \mathcal{R}_{13}, \end{cases}$$

where $W_k^{R_1}$ *is asymptotically standard normal,*

$$\begin{aligned} y_{\rho_0|T}(\xi) &= \tfrac{6\xi\left(-4+\xi\left(-13+2\xi\left(-3+2\xi(2+\xi)^2\right)\right)\right)-\xi(3+\xi)(1+2\xi)^3(3+2\xi)\tau}{12(1+\xi)^2(1+2\xi)^3}, \\ z_{\rho_0|T}(\xi,\rho_0) &= -\tfrac{(1+\xi)^3\rho_0(\xi+\rho_0)}{\xi^2(1-\rho_0)^3} \end{aligned}$$

and $\overset{\bullet}{\sigma}{}^2_{\rho_0,q} = (1+\xi)^6\left(2\,\xi^2+2\,\xi+1\right)/(\xi\chi_q)^2$.

ii) *In* $\mathcal{R}_2$*, and again with the same notation as before for* $\mathcal{R}_{21}$*,* $\mathcal{R}_{22}$ *and* $\mathcal{R}_{23}$*,*

$$\begin{aligned} \widehat{\rho}_k^{(\tau_q,q)} &\overset{d}{=} \rho_q + \tfrac{\sigma_{\rho_0,q}}{\sqrt{k}A_0(n/k)}W_k^{R_2} \\ &+ \begin{cases} \left(\tfrac{\chi_q f_{\rho_0|T}(\xi,\rho_0)}{A_0(n/k)U_0(n/k)}\right)(1+o_p(1)), & \text{in } \mathcal{R}_{21} \\ \left(m_{\rho_0,\rho_0'|T} + \tfrac{\chi_q f_{\rho_0|T}(\xi,\rho_0)}{A_0(n/k)U_0(n/k)}\right)(1+o_p(1)), & \text{in } \mathcal{R}_{22} \\ m_{\rho_0,\rho_0'|T}(1+o_p(1)), & \text{in } \mathcal{R}_{23}, \end{cases} \end{aligned}$$

where $m_{\rho,\rho'|T} = u_{\rho|T}(\rho)A_0(n/k) + v_{\rho|T}(\rho,\rho')B_0(n/k)$*, with* $u_{\rho|T}(\rho) \equiv u_\rho(\tau=\tau_q)$ *and* $v_{\rho|T}(\rho,\rho') \equiv v_{\rho,\rho'}$*, given by*

(3.18)
$$u_\rho \equiv u_\rho(\tau) = \tfrac{\rho\left(\rho\,(42-45\,\tau)+\rho^3\,(96-44\,\tau)+8\,\rho^4\,(\tau-3)+9\,\tau+2\,\rho^2\,(37\,\tau-60)\right)}{12\,\xi\,(1-3\,\rho+2\,\rho^2)^2}$$

and

(3.19) $$v_{\rho,\rho'} = (1-\rho)^3\,\rho'\,(\rho+\rho')/\{\rho\,(1-\rho-\rho')^3\},$$

respectively. Moreover, $W_k^{R_2}$ *is asymptotically standard normal,*

$$\begin{aligned} \sigma^2_{\rho_0,q} &\equiv \sigma^2_{\rho_0} = \xi^2(1-\rho_0)^6\left(2\rho_0^2-2\rho_0+1\right)/\rho_0{}^2, \\ f_{\rho_0|T}(\xi,\rho_0) &= \tfrac{\xi^2(1-\rho_0)^3(\xi+\rho_0)}{(1+\xi)^3\rho_0}. \end{aligned}$$

iii) *In* $\mathcal{R}_3$*, and with* $\widetilde{\lambda} = \lim_{n\to\infty} 1/\big(A_0(n/k)U_0(n/k)\big) = (\xi\beta_0 C)^{-1} \neq 0$*, with* C *given in (2.4),*

$$\begin{aligned} \widehat{\rho}_k^{(\tau_q,q)} &\overset{d}{=} \rho_q + \tfrac{\widetilde{\sigma}_{\rho_0,q}}{\sqrt{k}A_0(n/k)}W_k^{R_3} \\ &+\left(\widetilde{u}_{\rho_0|T}A_0(n/k) + \widetilde{v}_{\rho_0,\rho_0'|T}B_0(n/k) + \xi\chi_q\tfrac{\widetilde{g}_{\xi,\rho_0|T}+\chi_q\widetilde{\lambda}\,\widetilde{y}_{\xi,\rho_0|T}}{U_0(n/k)}\right)(1+o_p(1)), \end{aligned}$$

where $W_k^{R_3}$ *is an asymptotically standard normal r.v.,* $u_{\rho|T} \equiv u_\rho(\tau=\tau_q)$ *and* $v_{\rho,\rho'|T} \equiv v_{\rho,\rho'}$*, defined in (3.18) and (3.19), respectively,* $\widetilde{u}_{\rho|T} = u_{\rho|T}/(1+\xi\widetilde{\lambda}\chi_q)$*,* $\widetilde{v}_{\rho,\rho'|T} = v_{\rho,\rho'|T}/(1+\xi\widetilde{\lambda}\chi_q)$*, and* $\widetilde{\bullet}_{\xi,\rho|T} = \bullet_{\xi,\rho|T}/(1+\xi\widetilde{\lambda}\chi_q)$*, with* $\bullet = g, y$*, with*

$$\begin{aligned} g_{\xi,\rho_0|T} = g_{-\rho_0,\rho_0|T} &\equiv g_{\rho_0|T} \\ &= -\tfrac{6\left(4+\rho_0\left(-13+2\rho_0\left(3+2\rho_0(2-\rho_0)^2\right)\right)\right)+(3-\rho_0)(3-2\rho_0)(1-2\rho_0)^3\tau}{6(1-\rho_0)^2(1-2\rho_0)^3}, \end{aligned}$$

$$y_{\xi,\rho_0|T} = y_{-\rho_0,\rho_0|T} \;\equiv\; y_{\rho_0|T} = \tfrac{(3-\rho_0)(1-\rho_0)^3}{2\rho_0} b(\rho_0,\tau),$$

$$b(\rho,\tau) = -\tfrac{(\rho-2)^2(\tau-2)}{4(1-\rho)^4} + \tfrac{\tau-1}{(1-\rho)^2} - \tfrac{2(1-\rho)}{(1-2\rho)^2} + \tfrac{2}{1-2\rho} - \tfrac{1}{1-\rho(3-2\rho)}$$
$$+ \tfrac{(1-\rho)\rho\left\{-(\rho+3)(5\rho(\rho+3)+12)(2\rho+1)^3\tau-6(6+\rho(3+2\rho)(4\rho^5+24\rho^4+42\rho^3+31\rho^2+14\rho+9))\right\}}{12(3-\rho)(1+\rho)^6(1+2\rho)^3}$$

$$\text{and } \widetilde{\sigma}^2_{\rho_0,q} = (1-\rho_0)^6 \left(2\,{\rho_0}^2 - 2\,\rho_0 + 1\right)/\left(1 - \widetilde{\lambda}\chi_q\,\rho_0\right)^2.$$

3.3. A few comments and conclusions

- We consider that the class of PORT-ρ estimators introduced and studied in this article is, from a theoretical point of view, a nice alternative to the classical ρ-estimators whenever, in a real data analysis, we are led to a bad performance of the classical estimators. Such a bad performance is usually due to the existence of a location $s \neq 0$ in the available data, associated with unshifted models with $\xi + \rho_0 < 0$, a quite common situation in practical applications.
- Concomitantly, the development and the theoretical study of a new class of PORT-estimators of the functional A, in (1.6), can lead us to SORB EVI-estimators, invariant for changes in location and MVRB for an adequate choice of q, i.e. EVI-estimators of the type of the ones in Caeiro *et al.* (2005), Gomes *et al.* (2007) and Gomes *et al.* (2008c), but invariant for changes in location, the so-called PORT-MVRB EVI-estimators. Note that these PORT-MVRB EVI-estimators have already been studied for finite samples in Gomes *et al.* (2011, 2012), and exhibit a quite interesting performance.

4. A SMALL-SCALE MONTE-CARLO SIMULATION

We next present in Figures 1 and 2, respectively the mean values (E) and the root mean squared errors (RMSE), of the classical estimator $\widehat{\rho}_k^{(0)}$ and the PORT-ρ estimators $\left\{\widehat{\rho}_k^{(0,q)}\right\}_{q=0,0.1,0.25}$, as defined in Eq. (1.18), as a function of the sample fraction k/n, for sample sizes $n = 5000$ and $n = 10000$. The results are associated with the output of a small-scale simulation, of size 5000, related to underlying Fréchet parents $F_0(x) = \exp(-x^{-1/\xi}), x > 0$, with $\xi = 0.25$, and the shifted model $F_s(x) = \exp\left(-(x-s)^{-1/\xi}\right),\ x > s$, with $s = 1$.

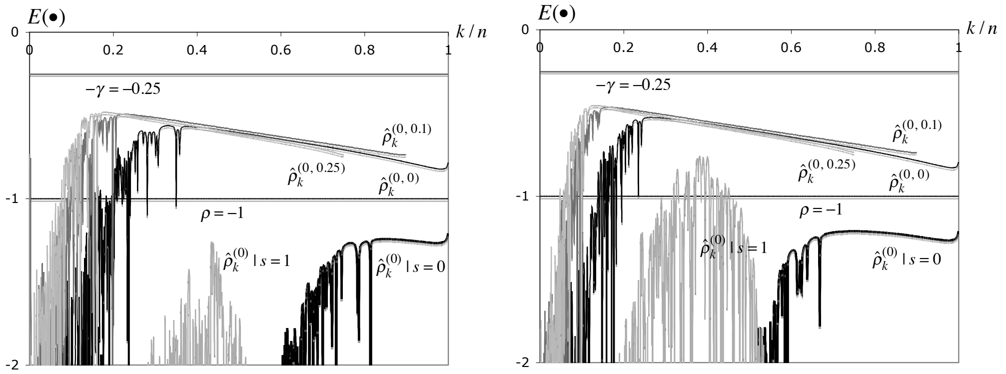

Figure 1: Mean values of the estimators under consideration for Fréchet unshifted ($s = 0$) and shifted ($s = 1$) parents, with $\xi = 0.25$, and sample size $n = 5000$ (left) and $n = 10000$ (right).

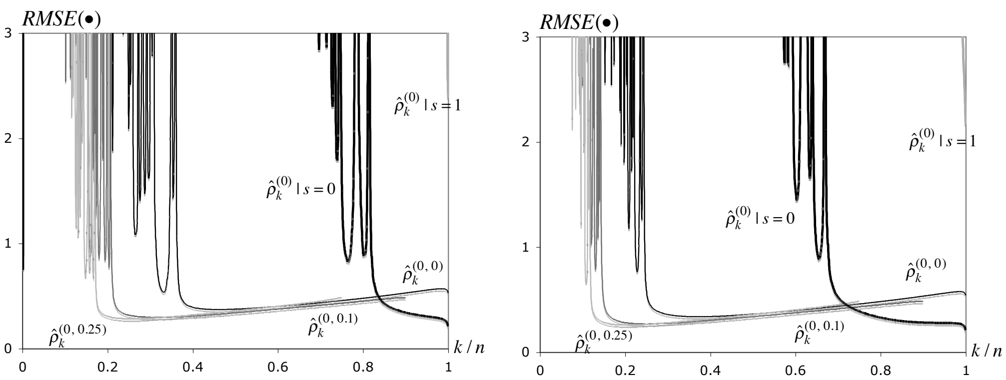

Figure 2: RMSEs of the estimators under consideration for Fréchet unshifted ($s = 0$) and shifted ($s = 1$) parents, with $\xi = 0.25$, and sample size $n = 5000$ (left) and $n = 10000$ (right).

There is indeed only a light improvement in all estimators as the sample size increases, and a high volatility of the classical ρ-estimators for shifted models, as can be seen, in either Figure 1 or in Figure 2, where the RMSE of such estimator is above 2, even for $n = 10000$. For smaller values of n, the sample paths of all estimators are even more volatile, particularly for small sample fractions k/n. But if we consider a much larger sample size, $n = 100000$, there is a clear improvement only in the classical ρ-estimators for shifted models, as can be seen, in Figure 3.

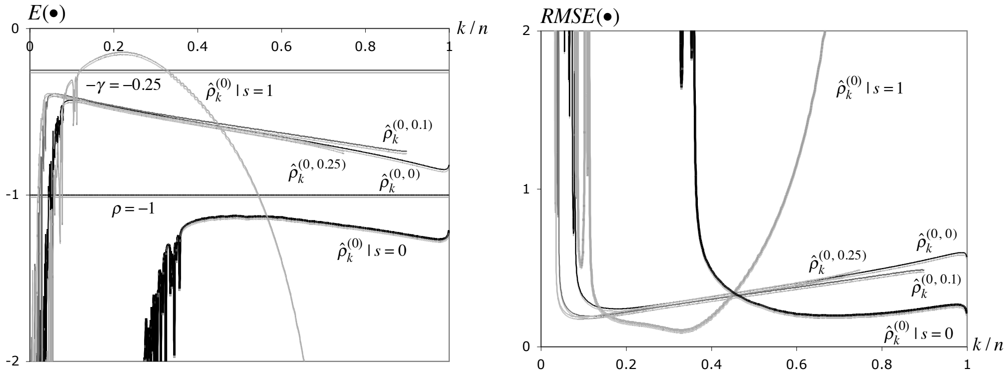

Figure 3: Mean values (left) and RMSEs (right) of the estimators under consideration for Fréchet unshifted ($s = 0$) and shifted ($s = 1$) parents, with $\xi = 0.25$, and sample size $n = 100000$.

We now would like to emphasise the following points:

- The stability of the classical ρ-estimators around the ‘target’ for large k can be fictitious or even non-existent, unless the model is an unshifted model. As can be seen in Figures 1 and 3, left, the classical ρ-estimator associated with the unshifted model, $\widehat{\rho}_k^{(0)}|s = 0$ is close to -1 for large values of k, as expected, but the ρ-estimator associated with the shifted model, $\widehat{\rho}_k^{(0)}|s = 1$, that should converge to -0.25, exhibits no stability in the sample paths.

- We are in the region $\xi + \rho_0 < 0$ ($\xi = 0.25,\ \rho_0 = -1$). Consequently, the PORT-ρ estimator should converge to $-\xi = -0.25$ for $\chi_q \neq 0$ and to $\rho_0 = -1$ for $\chi_q = 0$. Unfortunately, the pattern of the PORT-ρ estimators does not depend strongly on χ_q. If we decide for a large value of k, we obtain a value close to -1 if $\chi_q = 0$, but a value not a long way from -1 when $\chi_q \neq 0$. But if we look at the region of k/n close to 0.2, the PORT-ρ estimators associated with $\chi_q \neq 0$ are reasonably close to $-\xi = -0.25$, with a not too large RMSE. We shall thus be again confronted with an adequate choice of the threshold k.

- This means that for shifted models or PORT-ρ estimators associated with $\chi_q \neq 0$, the optimal level is clearly attained for not very large k, as can be seen in Figures 2 and 3, right, when we look at the minimal RMSE.

- For $\chi_q = 0$, the PORT-ρ estimator is able to beat the classical one regarding minimum RMSE, even for very large sample sizes.

- Similar comments apply to other simulated underlying models.

- The choice of the tuning parameters τ and τ_q is also crucial. We have here used $\tau_q = \tau = 0$. The choice $\tau = 0$ has been heuristically suggested and used before for the ρ-estimation and the region $|\rho| \leq 1$, but it is possibly not the most adequate choice for the PORT-ρ estimation. This is another interesting topic out of the scope of this paper.

5. PROOFS

Proof: [**Lemma 2.1**]. We begin by writing

$$\begin{aligned}\ln U_q(tx) - \ln U_q(t) &= \ln \tfrac{U_0(tx)-\chi_q}{U_0(t)-\chi_q} = \ln\left(\tfrac{U_0(tx)}{U_0(t)}\,\tfrac{1-\frac{\chi_q}{U_0(tx)}}{1-\frac{\chi_q}{U_0(t)}}\right)\\ &= \xi \ln x + \ln\left(x^{-\xi}\tfrac{U_0(tx)}{U_0(t)}\right) + \ln\left(1-\tfrac{\chi_q}{U_0(tx)}\right) - \ln\left(1-\tfrac{\chi_q}{U_0(t)}\right).\end{aligned}$$

Using Taylor's expansion of $\ln(1+x)$, as $x \to 0$, we obtain

$$\begin{aligned}\ln U_q(tx) - \ln U_q(t) &= \xi \ln x + \ln\left(x^{-\xi}\tfrac{U_0(tx)}{U_0(t)}\right) - \tfrac{\chi_q}{U_0(tx)} + \tfrac{\chi_q}{U_0(t)} + o\left(\tfrac{1}{U_0(t)}\right),\\ &= \xi \ln x + \ln\left(x^{-\xi}\tfrac{U_0(tx)}{U_0(t)}\right) + \tfrac{\chi_q}{U_0(t)}\left(1-\tfrac{U_0(t)}{U_0(tx)}\right) + o\left(\tfrac{1}{U_0(t)}\right),\end{aligned}$$

as $t \to \infty$. Since $U_0(tx) \sim x^{\xi} U_0(t),\ t \to \infty$, we thus have that

$$\begin{aligned}&\ln U_q(tx) - \ln U_q(t) - \xi \ln x\\ &= \ln\left(x^{-\xi}\tfrac{U_0(tx)}{U_0(t)}\right) + \tfrac{\chi_q}{U_0(t)}(1-x^{-\xi}) - \tfrac{\chi_q}{U_0(t)}\left(\tfrac{U_0(t)}{U_0(tx)} - x^{-\xi}\right) + o\left(\tfrac{1}{U_0(t)}\right).\end{aligned}$$

Now, condition (1.6) with U, A and ρ replaced with U_0, A_0 and ρ_0, respectively, ascertains

$$\begin{aligned}\ln U_q(tx) - \ln U_q(t) - \xi \ln x = A_0(t)\,\tfrac{x^{\rho_0}-1}{\rho_0} + \tfrac{\chi_q}{U_0(t)}(1-x^{-\xi})\\ - \tfrac{\chi_q}{U_0(t)}\left(\tfrac{U_0(t)}{U_0(tx)} - x^{-\xi}\right) + o\left(\tfrac{1}{U_0(t)}\right) + o\big(A_0(t)\big).\end{aligned}$$

The precise result thus follows by noting that $1/U_0 \in RV_{-\xi}$ (hence χ_q/U_0 is also in $RV_{-\xi}$) and that $x^{\xi}U_0(t)/U_0(tx) - 1$ divided by $A_0(t)$ has the same limit as in (1.6), with the same second order parameter ρ_0 (cf. Proposition 6 and Corollary 7 of Neves, 2009). This result confirms a similar one for the rate of convergence of $U_q(tx)/U_q(t)$ to x^{ξ} as obtained in Araújo Santos *et al.* (2006, Lemma 2.1). □

Proof: [**Proposition 2.2**]. Using the same arguments as in Fraga Alves *et al.* (2009), bearing in mind the unshifted model ($s = 0$), we can write the PORT log-excesses of the observations over the random quantile $X_{n_q:n}$, i.e. $X_{n-i+1:n} - X_{n_q:n}$, for $i = 1, ..., k$, in terms of the POT log-excesses, $X_{n-i+1:n} - \chi_q$, over $\chi_q := F_0^{\leftarrow}(q) = U_0(1/(1-q))$, as follows:

$$\ln\left(X_{n-i+1:n} - X_{n_q:n}\right) = \ln\left(X_{n-i+1:n} - \chi_q\right) + \ln\left(1 - \tfrac{X_{n_q:n}-\chi_q}{X_{n-i+1:n}-\chi_q}\right).$$

Now for the second term holds the inequality

$$\ln\left(1-\tfrac{X_{n_q:n}-\chi_q}{X_{n-i+1:n}-\chi_q}\right) \le \ln\left(1-\tfrac{X_{n_q:n}-\chi_q}{X_{n:n}-\chi_q}\right).$$

Since we are assuming $\xi>0$ we have that $X_{n:n}-\chi_q \underset{n\to\infty}{\overset{p}{\longrightarrow}} \infty$, which in conjunction with the asymptotical normality of the empirical quantile $\sqrt{n}\left(X_{n_q:n}-\chi_q\right) = O_p(1)$ ascertains

$$\begin{aligned}\sqrt{k}\ln\left(1-\tfrac{X_{n_q:n}-\chi_q}{X_{n:n}-\chi_q}\right) &= \sqrt{k}\,\tfrac{X_{n_q:n}-\chi_q}{X_{n:n}-\chi_q}(1+o_p(1)) = \sqrt{k/n}\,o_p\left(\sqrt{n}(X_{n_q:n}-\chi_q)\right)\\ &= o_p\left(\sqrt{k/n}\right) \underset{n\to\infty}{\overset{p}{\longrightarrow}} 0\,.\end{aligned}$$

Then it is easily seen that, for any $\alpha>0$, the PORT-moment statistics $M_{n,k}^{(\alpha,q)}$ provided in (1.14) are asymptotically identically distributed to their POT-moment counterparts

$$\widetilde{M}_{n,k}^{(\alpha,q)} = \tfrac{1}{k}\sum_{i=1}^{k}\left(\ln\tfrac{X_{n-i+1:n}-\chi_q}{X_{n-k:n}-\chi_q}\right)^{\alpha}.$$

In fact, $\widetilde{M}_{n,k}^{(\alpha,q)}$ differs from $M_{n,k}^{(\alpha)} = \frac{1}{k}\sum_{i=1}^{k}\left(\ln\frac{X_{n-i+1:n}}{X_{n-k:n}}\right)^{\alpha}$ by a deterministic shift $-\chi_q = -U_0(1/(1-q))$ in the observations X_i, $1\le i\le n$. Then the asymptotic results for $\widetilde{M}_{n,k}^{(\alpha,q)} \equiv \frac{1}{k}\sum_{i=1}^{k}\left(\ln\frac{\widetilde{X}_{n-i+1:n}}{\widetilde{X}_{n-k:n}}\right)^{\alpha}$ can be obtained in view of the shifted observations from $\widetilde{X} := X_q = X_0-\chi_q$, with associated $U_q(t) = U_0(t)-\chi_q$.

Let us begin with the first moment of the log-excesses. With $\{Y_i\}_{i=1,\dots,n}$ i.i.d. unit Pareto r.v.'s, we have the equality in distribution

$$\{\widetilde{X}_{n-i+1:n}\}_{i=1}^{n} := \{X_{n-i+1:n}-\chi_q\}_{i=1}^{n} \overset{d}{=} \{U_q(Y_{n-i+1:n})\}_{i=1}^{n},$$

and we can write,

$$\begin{aligned}(5.1)\quad \widetilde{M}_{n,k}^{(1,q)} &= \tfrac{1}{k}\sum_{i=1}^{k}\ln\widetilde{X}_{n-i+1:n} - \ln\widetilde{X}_{n-k:n}\\ &\overset{d}{=} \tfrac{1}{k}\sum_{i=1}^{k}\ln U_q\left(Y_{n-i+1:n}\right) - \ln U_q(Y_{n-k:n}).\end{aligned}$$

We note that

$$\begin{aligned}&\ln U_q(tx)-\ln U_q(t)-\big(\ln U_0(tx)-\ln U_0(t)\big)\\ &= \ln\frac{\frac{U_0(tx)}{U_0(t)}-\frac{\chi_q}{U_0(t)}}{1-\frac{\chi_q}{U_0(t)}} - \left(\ln U_0(tx)-\ln U_0(t)\right)\\ &= \ln\Big(\big(x^{-\xi}\tfrac{U_0(tx)}{U_0(t)}-1\big)-x^{-\xi}\tfrac{\chi_q}{U_0(t)}+1\Big) - \ln\Big(\big(x^{-\xi}\tfrac{U_0(tx)}{U_0(t)}-1\big)+1\Big) - \ln\big(1-\tfrac{\chi_q}{U_0(t)}\big).\end{aligned}$$

Next, we deal with the first two terms in the above. Towards this end, we define for each $x>0$,

$$\begin{aligned}y_1(t) &:= \big(x^{-\xi}\tfrac{U_0(tx)}{U_0(t)}-1\big)-x^{-\xi}\tfrac{\chi_q}{U_0(t)},\\ y_2(t) &:= x^{-\xi}\tfrac{U_0(tx)}{U_0(t)}-1,\end{aligned}$$

with $y_1(t)$ and $y_2(t)$ converging to zero as $t \to \infty$ (see text in the end of the proof of lemma 2.1). MacLaurin's expansion of the logarithm, i.e. $\ln(1+y) = y - y^2/2 + o(y^2)$, applied to both $y_1(t)$ and $y_2(t)$ now yields

$$\begin{aligned}&\ln U_q(tx) - \ln U_q(t) - \big(\ln U_0(tx) - \ln U_0(t)\big)\\ &= -x^{-\xi}\tfrac{\chi_q}{U_0(t)} - \frac{1}{2}\Big(x^{-\xi}\tfrac{\chi_q}{U_0(t)}\Big)^2\big(1+o(1)\big) + \big(x^{-\xi}\tfrac{U_0(tx)}{U_0(t)} - 1\big)x^{-\xi}\tfrac{\chi_q}{U_0(t)}\big(1+o(1)\big)\\ &\quad - \ln\big(1 - \tfrac{\chi_q}{U_0(t)}\big).\end{aligned}$$

In order to have a grasp at the remainder $o(1)$-terms, we require the following uniform bounds, which arise in connection with the third-order framework in (2.3) and Remark B.3.12 of de Haan and Ferreira (2006): for any ε, $\delta > 0$, there exists a $t_0 = t_0(\varepsilon, \delta)$ such that for $t \geq t_0$, $x \geq 1$,

$$\left|\frac{\frac{x^{-\xi}\frac{U_0(tx)}{U_0(t)}-1}{A_0(t)} - \frac{x^{\rho_0}-1}{\rho_0}}{B_0(t)} - \frac{x^{\rho_0+\rho_0'}-1}{\rho_0+\rho_0'}\right| \leq \varepsilon x^{\rho_0+\rho_0'+\delta}.$$

Furthermore, since $0 < -\ln(1-v) - v - v^2/2 < v^3/\big(3(1-v)\big)$, $v \in (0,1)$, we can set $v = \chi_q/U_0$ in order to establish the upper bound

$$\begin{aligned}&\ln U_q(tx) - \ln U_q(t) - \big(\ln U_0(tx) - \ln U_0(t)\big)\\ &\qquad - \xi\big(\tfrac{x^{-\xi}-1}{-\xi}\big)\tfrac{\chi_q}{U_0(t)} - \xi\big(\tfrac{x^{-2\xi}-1}{-2\xi}\big)\big(\tfrac{\chi_q}{U_0(t)}\big)^2 - x^{-\xi}\big(\tfrac{x^{\rho_0}-1}{\rho_0}\big)\tfrac{\chi_q A_0(t)}{U_0(t)}\\ &\leq \tfrac{\chi_q^3}{3}\Big(U_0^3(t)\big(1-\tfrac{\chi_q}{U_0(t)}\big)\Big)^{-1} + x^{-\xi}\tfrac{x^{\rho_0+\rho_0'}-1}{\rho_0+\rho_0'}\chi_q\tfrac{A_0(t)}{U_0(t)}B_0(t) + \varepsilon\big|\tfrac{A_0(t)}{U_0(t)}B_0(t)\big|x^{-\xi+\rho_0+\rho_0'+\delta}.\end{aligned}$$

We can also establish a similar lower bound. In this development, and with respect to the right hand-side of (5.1), assuming $k = k_n$ an intermediate sequence of positive integers, i.e. such that (1.13) holds, then taking average across $i = 1, 2, ..., k$, for arbitrary ε, $\delta > 0$, the weak law of large numbers ensures that

$$M_{n,k}^{(1,q)} - M_{n,k}^{(1)} = \tfrac{\chi_q}{U_0(n/k)}\Big(\tfrac{\xi}{1+\xi} + \tfrac{\xi}{1+2\xi}\tfrac{\chi_q}{U_0(n/k)}(1+o_p(1)) + \tfrac{A_0(n/k)}{(1+\xi)(1+\xi-\rho_0)}(1+o_p(1))\Big).$$

We are then led to (2.16) for $\alpha = 1$ where

$$\tfrac{\xi}{1+\xi} = \xi\overline{\mu}_1^{(2)}(-\xi), \quad \tfrac{1}{(1+\xi)(1+\xi-\rho_0)} = \overline{\mu}_1^{(2)}(\xi,\rho_0) \quad \text{and} \quad \tfrac{\xi}{1+2\xi} = \xi\overline{\mu}_1^{(2)}(-2\xi).$$

Let us next consider a general α. Similarly as before, we can write

$$\begin{aligned}&\Big(\ln U_q(tx) - \ln U_q(t)\Big)^\alpha - \Big(\ln U_0(tx) - \ln U_0(t)\Big)^\alpha = \tfrac{\alpha(\xi\ln x)^\alpha\chi_q}{U_0(t)}\Bigg(\tfrac{1}{\ln x}\ \Big(\tfrac{x^{-\xi}-1}{-\xi}\Big)\\ &\qquad + \tfrac{1}{\xi}\Big(\tfrac{x^{-\xi}}{\ln x}\Big(\tfrac{x^{\rho_0}-1}{\rho_0}\Big) + \tfrac{(\alpha-1)}{(\ln x)^2}\Big(\tfrac{x^{\rho_0}-1}{\rho_0}\Big)\Big(\tfrac{x^{-\xi}-1}{-\xi}\Big)\Big)A_0(t)\\ &\qquad + \tfrac{1}{\ln x}\ \tfrac{\chi_q}{U_0(t)}\Bigg(\Big(\tfrac{x^{-2\xi}-1}{-2\xi}\Big) + \tfrac{\alpha-1}{2\ln x}\Big(\tfrac{x^{-\xi}-1}{-\xi}\Big)^2\Bigg)\Bigg) + o(1/U_0^2(t)).\end{aligned}$$

Considering again $k = k_n$ as an intermediate sequence of integers, i.e. (1.13) holds, the same type of arguments of the previous case ($\alpha = 1$), and the weak law of large numbers enable us to write (2.16) for any $\alpha > 0$. □

Proof: [**Theorem 3.2**]. (**i**) In the region $\mathcal{R}_1$, $A_0(t) = o(1/U_0(t))$, as $t \to \infty$, the third and last term of the right-hand side of (3.6) is the dominant one, and the r.v.'s $D_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)}(\xi)/(1/U_0(n/k))$ converge in probability to $\alpha\ \tau_q \xi^{\alpha\tau_q} \chi_q\ d_{\alpha,\theta_1,\theta_2}(-\xi)$ provided that (3.7) holds, i.e. if $\sqrt{k}/U_0(n/k) \to \infty$, as $n \to \infty$ (see Remark 3.1). Moreover, we get

$$\frac{D_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)}(\xi)}{1/U_0(n/k)} \stackrel{d}{=} \xi^{\alpha\tau_q}\Biggl(\alpha\ \tau_q\chi_q\ d_{\alpha,\theta_1,\theta_2}(-\xi) + \frac{\tau_q W_k^{(\alpha,\theta_1,\theta_2)} U_0(n/k)}{\sqrt{k}} + \alpha\tau_q\left\{\frac{d_{\alpha,\theta_1,\theta_2}(\rho_0)A_0(n/k)U_0(n/k)(1+o_p(1))}{\xi} + \frac{\chi_q^2 y_{\alpha,\theta_1,\theta_2,\tau_q}(\xi)(1+o_p(1))}{U_0(n/k)}\right\}\Biggr).$$

For levels k such that (1.13) holds, with $W_k^{(\alpha,\theta_1,\theta_2)}$ given in (2.12), and with $T_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)}$ defined in (1.15), we can say that if (3.7) holds,

$$T_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)} \stackrel{d}{=} t_{\alpha,\theta_1,\theta_2}(-\xi) + \frac{(d_{\alpha,\theta_1,\theta_2}(-\xi))^{-1}\left(W_k^{(\alpha,1,\theta_1)} - t_{\alpha,\theta_1,\theta_2}(-\xi)W_k^{(\alpha,\theta_1,\theta_2)}\right)}{\alpha\chi_q\sqrt{k}/U_0(n/k)} + \frac{z_T^{(\alpha,\theta_1,\theta_2)}(\xi,\rho_0)A_0(n/k)U_0(n/k)(1+o_p(1))}{\chi_q} + \frac{\chi_q y_T^{(\alpha,\theta_1,\theta_2\tau_q)}(\xi,-\xi)(1+o_p(1))}{U_0(n/k)}.$$

For sequences of positive intermediate integers $k = k_n$ such that $k_n = o(n)$, $\sqrt{k}/U_0(n/k) \to \infty$, $\sqrt{k}A_0(n/k) \to \lambda$ and $\sqrt{k}/U_0^2(n/k) \to \lambda_{_U}$, as $n \to \infty$, let us consider the following cases:

- if $\xi + \rho_0 < -\xi$ and $\chi_q \neq 0$, then

$$T_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)} \stackrel{d}{=} t_{\alpha,\theta_1,\theta_2}(-\xi) + \frac{(d_{\alpha,\theta_1,\theta_2}(-\xi))^{-1}\left(W_k^{(\alpha,1,\theta_1)} - t_{\alpha,\theta_1,\theta_2}(-\xi)W_k^{(\alpha,\theta_1,\theta_2)}\right)}{\alpha\chi_q\sqrt{k}/U_0(n/k)} + \frac{\chi_q y_T^{(\alpha,\theta_1,\theta_2,\tau_q)}(\xi,-\xi)(1+o_p(1))}{U_0(n/k)},$$

  and

$$\frac{\sqrt{k}}{U_0(n/k)}\left(T_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)} - t_{\alpha,\theta_1,\theta_2}(-\xi)\right) \xrightarrow[n\to\infty]{d} \mathcal{N}(\overset{\bullet}{\mu}_{T|\alpha,\theta_1,\theta_2,\tau_q,q}, \overset{\bullet}{\sigma}{}^2_{T|\alpha,\theta_1,\theta_2}),$$

  where $\overset{\bullet}{\mu}_{T|\alpha,\theta_1,\theta_2,\tau_q,q} = \lambda_{_U}\chi_q y_T^{(\alpha,\theta_1,\theta_2,\tau_q)}(\xi,-\xi)$, with $y_T^{(\alpha,\theta_1,\theta_2,\tau)}(\xi,\rho)$ defined in (3.11).

- if $\xi + \rho_0 = -\xi$ and $\chi_q \neq 0$, then

$$T_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)} \stackrel{d}{=} t_{\alpha,\theta_1,\theta_2}(-\xi) + \frac{(d_{\alpha,\theta_1,\theta_2}(-\xi))^{-1}\left(W_k^{(\alpha,1,\theta_1)} - t_{\alpha,\theta_1,\theta_2}(-\xi)W_k^{(\alpha,\theta_1,\theta_2)}\right)}{\alpha\chi_q\sqrt{k}/U_0(n/k)} + \frac{z_T^{(\alpha,\theta_1,\theta_2)}(\xi,\rho_0)\ A_0(n/k)U_0(n/k)(1+o_p(1))}{\chi_q} + \frac{\chi_q\ y_T^{(\alpha,\theta_1,\theta_2,\tau_q)}(\xi,-\xi)(1+o_p(1))}{U_0(n/k)},$$

and

$$\frac{\sqrt{k}}{U_0(n/k)}\Big(T_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)}-t_{\alpha,\theta_1,\theta_2}(-\xi)\Big)\underset{n\to\infty}{\overset{d}{\longrightarrow}}\mathcal{N}(\overset{\bullet}{\mu}_{T|\alpha,\theta_1,\theta_2,\tau_q,q},\overset{\bullet}{\sigma}{}^2_{T|\alpha,\theta_1,\theta_2}),$$

where $\overset{\bullet}{\mu}_{T|\alpha,\theta_1,\theta_2,\tau_q,q}=\frac{\lambda z_T^{(\alpha,\theta_1,\theta_2)}(\xi,\rho_0)}{\chi_q}+\lambda_{_U}\chi_q y_T^{(\alpha,\theta_1,\theta_2,\tau_q)}(\xi,-\xi)$, with $y_T^{(\alpha,\theta_1,\theta_2,\tau)}(\xi,\rho)$ and $z_T^{(\alpha,\theta_1,\theta_2)}(\xi,\rho)$ defined in (3.11) and (3.12), respectively.

- if $\xi+\rho_0>-\xi$ and $\chi_q\neq 0$, then

$$\begin{aligned}T_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)}&\overset{d}{=}t_{\alpha,\theta_1,\theta_2}(-\xi)\\&+\frac{(d_{\alpha,\theta_1,\theta_2}(-\xi))^{-1}\Big(W_k^{(\alpha,1,\theta_1)}-t_{\alpha,\theta_1,\theta_2}(-\xi)W_k^{(\alpha,\theta_1,\theta_2)}\Big)}{\alpha\chi_q\sqrt{k}/U_0(n/k)}\\&\qquad+\frac{z_T^{(\alpha,\theta_1,\theta_2)}(\xi,\rho_0)A_0(n/k)U_0(n/k)(1+o_p(1))}{\chi_q},\end{aligned}$$

and

$$\frac{\sqrt{k}}{U_0(n/k)}\Big(T_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)}-t_{\alpha,\theta_1,\theta_2}(-\xi)\Big)\underset{n\to\infty}{\overset{d}{\longrightarrow}}\mathcal{N}(\overset{\bullet}{\mu}_{T|\alpha,\theta_1,\theta_2,\tau_q,q},\overset{\bullet}{\sigma}{}^2_{T|\alpha,\theta_1,\theta_2}),$$

where $\overset{\bullet}{\mu}_{T|\alpha,\theta_1,\theta_2,\tau_q,q}=\frac{\lambda z_T^{(\alpha,\theta_1,\theta_2)}(\xi,\rho_0)}{\chi_q}$, with $z_T^{(\alpha,\theta_1,\theta_2)}(\xi,\rho)$ defined in (3.12).

(**ii**) In the region $\xi+\rho_0>0$, where $1/U_0(t)=o(A_0(t))$, as $t\to\infty$, or more generally in the region $\mathcal{R}_2$, the second term of the right-hand side of (3.6) is the dominant one. In $\mathcal{R}_2$, $A_q(t)=A_0(t)$, so condition (3.7) can be rewritten as $\sqrt{k}A_0(n/k)\to\infty$, as $n\to\infty$ and if we assume that this condition holds,

$$\begin{aligned}\frac{D_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)}(\xi)}{A_0(n/k)}&\overset{d}{=}\xi^{\alpha\tau_q}\Bigg(\frac{\alpha\tau_q\ d_{\alpha,\theta_1,\theta_2}(\rho_0)}{\xi}+\frac{\tau_q W_k^{(\alpha,\theta_1,\theta_2)}}{\sqrt{k}A(n/k)}\\&+u_{\alpha,\theta_1,\theta_2,\tau_q}(\rho_0)A_0(n/k)(1+o_p(1))+v_{\alpha,\theta_1,\theta_2}(\rho_0,\rho_0')B_0(n/k)(1+o_p(1))\\&\qquad+\frac{\alpha\tau_q\chi_q}{A_0(n/k)U_0(n/k)}d_{\alpha,\theta_1,\theta_2}(-\xi)\Bigg).\end{aligned}$$

If $\xi>-\rho_0$ or ($\xi\le-\rho_0$, $\chi_q=0$), and (3.7) holds,

$$\begin{aligned}T_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)}&\overset{d}{=}t_{\alpha,\theta_1,\theta_2}(\rho_0)+\frac{\xi(d_{\alpha,\theta_1,\theta_2}(\rho_0))^{-1}\Big(W_k^{(\alpha,1,\theta_1)}-t_{\alpha,\theta_1,\theta_2}(\rho_0)W_k^{(\alpha,\theta_1,\theta_2)}\Big)}{\alpha\sqrt{k}A_0(n/k)}\\&+\Big(u_T^{(\alpha,\theta_1,\theta_2,\tau_q)}(\rho_0)A_0(n/k)+v_T^{(\alpha,\theta_1,\theta_2)}(\rho_0,\rho_0')B_0(n/k)\Big)(1+o_p(1))\\&\qquad+\frac{\chi_q\ f_T^{(\alpha,\theta_1,\theta_2)}(\xi,\rho_0)}{A_0(n/k)U_0(n/k)}(1+o_p(1)).\end{aligned}$$

For sequences of positive intermediate integers $k=k_n$ such that $k_n=o(n)$, $\sqrt{k}A_0(n/k)\to\infty$, $\sqrt{k}A_0^2(n/k)\to\lambda_{_A}$, $\sqrt{k}A_0(n/k)B_0(n/k)\to\lambda_{_B}$ and $\sqrt{k}/U_0(n/k)$

$\to \lambda'$, as $n \to \infty$, let us consider the following cases:

- if $0 < \xi + \rho_0 < -\rho_0$ and $\chi_q \neq 0$, then

$$T_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)} \stackrel{d}{=} t_{\alpha,\theta_1,\theta_2}(\rho_0) + \frac{\xi(d_{\alpha,\theta_1,\theta_2}(\rho_0))^{-1}\left(W_k^{(\alpha,1,\theta_1)} - t_{\alpha,\theta_1,\theta_2}(\rho_0) W_k^{(\alpha,\theta_1,\theta_2)}\right)}{\alpha\sqrt{k}A_0(n/k)} + \frac{\chi_q \; f_T^{(\alpha,\theta_1,\theta_2)}(\xi,\rho_0)}{A_0(n/k)U_0(n/k)}(1+o_p(1)),$$

and

$$\sqrt{k}A_0(n/k)\left(T_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)} - t_{\alpha,\theta_1,\theta_2}(\rho_0)\right) \xrightarrow[n\to\infty]{d} \mathcal{N}(\mu_{T|\alpha,\theta_1,\theta_2,\tau_q,q}, \sigma^2_{T|\alpha,\theta_1,\theta_2}),$$

where $\mu_{T|\alpha,\theta_1,\theta_2,\tau_q,q} = \chi_q \; f_T^{(\alpha,\theta_1,\theta_2)}(\xi,\rho_0)\lambda'$, with $f_T^{(\alpha,\theta_1,\theta_2)}(\xi,\rho)$ and $\sigma^2_{T|\alpha,\theta_1,\theta_2}$ defined in (3.15) and (3.17), respectively.

- if $\xi + \rho_0 = -\rho_0$ and $\chi_q \neq 0$, then

$$T_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)} \stackrel{d}{=} t_{\alpha,\theta_1,\theta_2}(\rho_0) + \frac{\xi(d_{\alpha,\theta_1,\theta_2}(\rho_0))^{-1}\left(W_k^{(\alpha,1,\theta_1)} - t_{\alpha,\theta_1,\theta_2}(\rho_0) W_k^{(\alpha,\theta_1,\theta_2)}\right)}{\alpha\sqrt{k}A_0(n/k)} + \left(u_T^{(\alpha,\theta_1,\theta_2,\tau_q)}(\rho_0)A_0(n/k) + v_T^{(\alpha,\theta_1,\theta_2)}(\rho_0,\rho_0')B_0(n/k)\right)(1+o_p(1)) + \frac{\chi_q \; f_T^{(\alpha,\theta_1,\theta_2)}(\xi,\rho_0)}{A_0(n/k)U_0(n/k)}(1+o_p(1)),$$

and

$$\sqrt{k}A_0(n/k)\left(T_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)} - t_{\alpha,\theta_1,\theta_2}(\rho_0)\right) \xrightarrow[n\to\infty]{d} \mathcal{N}(\mu_{T|\alpha,\theta_1,\theta_2,\tau_q,q}, \sigma^2_{T|\alpha,\theta_1,\theta_2}),$$

where $\mu_{T|\alpha,\theta_1,\theta_2,\tau_q,q} = u_T^{(\alpha,\theta_1,\theta_2,\tau_q)}(\rho_0)\lambda_A + v_T^{(\alpha,\theta_1,\theta_2)}(\rho_0,\rho_0')\lambda_B + \chi_q \; f_T^{(\alpha,\theta_1,\theta_2)}(\xi,\rho_0)\lambda'$, $u_T^{(\alpha,\theta_1,\theta_2,\tau_q)}(\rho)$, $v_T^{(\alpha,\theta_1,\theta_2)}(\rho,\rho')$, $f_T^{(\alpha,\theta_1,\theta_2)}(\xi,\rho)$ and $\sigma^2_{T|\alpha,\theta_1,\theta_2}$ defined in (3.13), (3.14), (3.15) and (3.17), respectively.

- if $\xi + \rho_0 > -\rho_0$ or $(\xi + \rho_0 > 0 \wedge \chi_q = 0)$, then

$$T_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)} \stackrel{d}{=} t_{\alpha,\theta_1,\theta_2}(\rho_0) + \frac{\xi(d_{\alpha,\theta_1,\theta_2}(\rho_0))^{-1}\left(W_k^{(\alpha,1,\theta_1)} - t_{\alpha,\theta_1,\theta_2}(\rho_0) W_k^{(\alpha,\theta_1,\theta_2)}\right)}{\alpha\sqrt{k}A_0(n/k)} + \left(u_T^{(\alpha,\theta_1,\theta_2,\tau_q)}(\rho_0)A_0(n/k) + v_T^{(\alpha,\theta_1,\theta_2)}(\rho_0,\rho_0')B_0(n/k)\right)(1+o_p(1)),$$

and

$$\sqrt{k}A_0(n/k)\left(T_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)} - t_{\alpha,\theta_1,\theta_2}(\rho_0)\right) \xrightarrow[n\to\infty]{d} \mathcal{N}(\mu_{T|\alpha,\theta_1,\theta_2,\tau_q,q}, \sigma^2_{T|\alpha,\theta_1,\theta_2}),$$

where $\mu_{T|\alpha,\theta_1,\theta_2,\tau_q,q} = \mu_{T|\alpha,\theta_1,\theta_2,\tau_q} = u_T^{(\alpha,\theta_1,\theta_2,\tau_q)}(\rho_0)\lambda_A + v_T^{(\alpha,\theta_1,\theta_2)}(\rho_0,\rho_0')\lambda_B$, with $u_T^{(\alpha,\theta_1,\theta_2,\tau_q)}(\rho)$ and $v_T^{(\alpha,\theta_1,\theta_2)}(\rho,\rho')$ defined in (3.13) and (3.14), respectively, and $\sigma^2_{T|\alpha,\theta_1,\theta_2}$ is defined in (3.17).

(iii) In the region $\mathcal{R}_3$, $A_0(t)$ and $1/U_0(t)$ are of the same order, i.e. the dominant terms of the right-hand side of (3.6) are the second and the third. In $\mathcal{R}_3$, $A_q(t) = A_0(t) + \xi\chi_q/U_0(t)$, so condition (3.7) can be rewritten as $\sqrt{k}A_0(n/k) \to \infty$, as $n \to \infty$. If we assume that this condition holds with $\widetilde{\lambda} = \lim_{n\to\infty} 1/(A_0(n/k)U_0(n/k)) \neq 0$, then

$$\frac{D_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)}(\xi)}{A_0(n/k)} \stackrel{d}{=} \xi^{\alpha\tau_q}\left(\frac{\alpha\tau_q}{\xi}\Big\{ d_{\alpha,\theta_1,\theta_2}(\rho_0) + \xi\,\widetilde{\lambda}\chi_q d_{\alpha,\theta_1,\theta_2}(-\xi)\Big\} + \frac{\tau_q W_k^{(\alpha,\theta_1,\theta_2)}}{\sqrt{k}A_0(n/k)}\right.$$
$$+\frac{\alpha\tau_q}{\xi}\Big\{u_{\alpha,\theta_1,\theta_2,\tau_q}(\rho_0)A_0(n/k)(1+o_p(1)) + v_{\alpha,\theta_1,\theta_2}(\rho_0,\rho_0')B_0(n/k)(1+o_p(1))\Big\}$$
$$\left.+\frac{\alpha\tau_q\chi_q}{U_0(n/k)}\Big\{w_{\alpha,\theta_1,\theta_2,\tau_q}(\xi,\rho_0) + y_{\alpha,\theta_1,\theta_2,\tau_q}(\xi)\widetilde{\lambda}\chi_q(1+o_p(1))\Big\}\right),$$

If $\xi + \rho_0 = 0$ and $\chi_q \neq 0$, if we consider levels k such that (1.13) and (3.7) hold,

$$T_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)} \stackrel{d}{=} t_{\alpha,\theta_1,\theta_2}(\rho_0) + \frac{\xi(d_{\alpha,\theta_1,\theta_2}(\rho_0))^{-1}\Big(W_k^{(\alpha,1,\theta_1)} - t_{\alpha,\theta_1,\theta_2}(\rho_0)W_k^{(\alpha,\theta_1,\theta_2)}\Big)}{\alpha(1+\xi\widetilde{\lambda}\chi_q)\sqrt{k}A_0(n/k)}$$
$$+\frac{u_T^{(\alpha,\theta_1,\theta_2,\tau_q)}(\rho_0)A_0(n/k) + v_T^{(\alpha,\theta_1,\theta_2)}(\rho_0,\rho_0')B_0(n/k)}{1+\xi\widetilde{\lambda}\chi_q}(1+o_p(1))$$
$$+\left\{\frac{\xi\chi_q\; g_T^{(\alpha,\theta_1,\theta_2,\tau_q)}(\xi,\rho_0)}{(1+\xi\widetilde{\lambda}\chi_q)U_0(n/k)} + \frac{\xi\chi_q^2\;\widetilde{\lambda}\; y_T^{(\alpha,\theta_1,\theta_2,\tau_q)}(\xi,\rho_0)}{(1+\xi\widetilde{\lambda}\chi_q)U_0(n/k)}\right\}(1+o_p(1)),$$

with $y_T^{(\alpha,\theta_1,\theta_2,\tau)}(\xi,\rho)$, $u_T^{(\alpha,\theta_1,\theta_2,\tau)}(\xi,\rho)$, $v_T^{(\alpha,\theta_1,\theta_2)}(\xi,\rho)$ and $g_T^{(\alpha,\theta_1,\theta_2,\tau)}(\xi,\rho)$ defined in (3.11), (3.13), (3.14) and (3.16), respectively. The proof of the theorem follows for sequences of positive intermediate integers $k = k_n$ such that $k_n = o(n)$, $\sqrt{k}A_0(n/k) \to \infty$, $\sqrt{k}A_0^2(n/k) \to \lambda_{_A}$, $\sqrt{k}A_0(n/k)B_0(n/k) \to \lambda_{_B}$ and $\sqrt{k}A_0(n/k)/U_0(n/k) \to \lambda_{AU}$, as $n \to \infty$. □

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In 1998 it was decided to publish papers also in English. This step has been taken to achieve a larger diffusion, and to encourage foreign contributors to submit their work.

At the time, the Editorial Board was mainly composed by Portuguese university professors, being now composed by national and international university professors, and this has been the first step aimed at changing the character of *Revista de Estatística* from a national to an international scientific journal.

In 2001, the *Revista de Estatística* published three volumes special issue containing extended abstracts of the invited contributed papers presented at the 23rd European Meeting of Statisticians.

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