REVSTAT – Statistical Journal
Volume 0, Number 0, Month 0000, 000–000
https://doi.org/00.00000/revstat.v00i0.000

Evaluation of Robust Confidence Interval for the Standard Deviation under Non-Normality

Authors: Theophilus Oppong Kyeremeh ✉
– Department of Mathematics and Statistics, Stephen F. Austin State University, USA
oppongkyt@sfasu.edu

Robert K. Henderson
– Department of Mathematics and Statistics, Stephen F. Austin State University, USA
hendersork@sfasu.edu

Received: Month 0000 Revised: Month 0000 Accepted: Month 0000

Abstract:

- This paper evaluates the performance of Abu-Shawiesh et al. (2011)'s robust confidence interval for the population standard deviation which was designed to achieve a coverage probability close to a nominal of $100(1-\alpha)\%$, particularly for skewed distributions. A simulation study was conducted to evaluate the performance of this method, and a discrepancy was discovered between the simulation results and those reported by the original authors. Specifically, for skewed distributions, the procedure exhibited very poor coverage-approaching zero in some cases. Motivated by this finding, this paper introduces a modified version of the robust method that provides coverage probabilities much closer to the nominal level of $100(1-\alpha)\%$. The modification builds upon the approach of Abu-Shawiesh et al. (2011) but incorporates an adjustment to account for skewness. Results from Monte Carlo simulations using both normal and non-normal distributions indicate that the modified method produces confidence intervals with substantially improved coverage probabilities compared to the original procedure, particularly under skewed distributions.

Keywords:

- *population standard deviation; robust confidence interval; coverage probability; non-normal distribution; bootstrap; average width; median width.*

AMS Subject Classification:

- 62F25, 62F40, 62G05, 62G35.

✉ Corresponding author

1. INTRODUCTION

A confidence interval is a range of values with a positive probability of including the unknown parameter. The exact 100(1 - α)% confidence interval for σ^2 is based on the assumption that the sample data are drawn from a normally distributed population. Let $x_1, x_2, \ldots, x_n$ be a random sample from a normal distribution with mean, μ and variance, σ^2, that is, $x_i \sim N(\mu, \sigma^2)$ for all i, then $\sum_{i=1}^{n} \frac{(x_i-\bar{x})^2}{\sigma^2} \sim \chi^2_{n-1}$, where $\bar{x} = \sum_{i=1}^{n} \frac{x_i}{n}$, and $\chi^2_{(n-1)}$ is a chi-square distribution with $n-1$ degrees of freedom. Then the exact 100(1-α)% confidence interval for σ^2 is given as

$$\frac{(n-1)s^2}{U} < \sigma^2 < \frac{(n-1)s^2}{L} \tag{1.1}$$

where $s^2 = \sum_{i=1}^{n} \frac{(x_i-\bar{x})^2}{(n-1)}$, $L = \chi^2_{\alpha/2,n-1}$ and $U = \chi^2_{1-\alpha/2,n-1}$, and $\chi^2_{p,df}$ is the p^{th} percentile of a chi-square distribution with *df* degrees of freedom. To obtain the confidence interval for the population standard deviation σ, take the square root of the endpoints of (1.1).

The sample standard deviation s is not the most efficient scale estimator in leptokurtic and skewed distributions, and is not robust to slight deviations from normality Tukey (1960), making (1.1) hypersensitive to minor violations of the normality assumption Bonett (2006). Therefore, when data are heavy-tailed or skewed, (1.1) performs poorly and often results in inaccurate coverage probabilities Harter (1961).

The sample variance, s^2, is very sensitive to outliers. As demonstrated by Lehmann (1986), (1.1) is also highly sensitive to the presence of outliers. Bonett (2006) through simulation showed that given $\alpha = 0.05$, and $n = 25$, (1.1) yields an asymptotic coverage probability of 0.83, 0.72 and 0.59 for the distributions $t(5)$, $Exp(2)$, and $\chi^2(1)$, respectively. These major setbacks of (1.1) call for the need for an alternative 100(1 - α)% confidence interval for σ that is more robust and consistently provides accurate coverage in a wider range of conditions.

Abu-Shawiesh et al. (2011) proposed a robust method for constructing interval estimates of σ that is not overly sensitive to changes in distributions and is designed to deal with problems associated with skewed distributions and outliers. However, examination of the method proposed by Abu-Shawiesh et al. (2011) suggests that these intervals have very poor coverage probabilities for skewed distributions. Specifically, for highly skewed distributions, the procedure exhibited very poor coverage-approaching zero.

This work proposes a modification to the Abu-Shawiesh et al. (2011) method that has a higher coverage probability for non-normal data. The performance of (1.1), the confidence interval method by Abu-Shawiesh et al. (2011), as well as the proposed modification is evaluated. The paper is organized as follows: Section 2 describes the methodological framework, introducing both the original Abu-Shawiesh et al. (2011) and the proposed modified confidence interval methods. Section 3 presents the findings of the simulation study, while Section 4 offers concluding insights.

2. METHODOLOGY

2.1. Robust method

To address the challenges posed by nonnormal distributions and outliers on (1.1), Abu-Shawiesh et al. (2011) adopted the robust estimator Q_n, originally proposed by Rousseeuw and Croux (1993), and outlined a method for more reliable estimation of the population standard deviation. Suppose $x_1, x_2, \ldots, x_n$ are random observations from a continuous, independent, and identically distributed random variable. The random variable T is defined as

$$T = \frac{d_n Q_n}{\sigma}. \tag{2.1}$$

Here, $d_n Q_n$ is an unbiased estimator for σ, ensuring that $E(T) = 1$ under the normal distribution. The Q_n is defined as

$$Q_n = d\{|x_i - x_j|; i < j\}_{(k)}$$

where d is a correction factor for consistency and $k = \binom{h}{2} \approx \frac{\binom{n}{2}}{4}$, where $h = [\frac{n}{2}] + 1$, representing half of the observations. In the case of Gaussian distributions, Rousseeuw and Croux (1993) showed that $d = 2.2219$. For large sample sizes, the following asymptotic result holds Rousseeuw and Croux (1993):

$$T = \frac{d_n Q_n}{\sigma} \sim N\left(1, \frac{1}{1.65n}\right), \tag{2.2}$$

$$\sigma T = d_n Q_n \sim N\left(\sigma, \frac{1}{1.65n}\sigma^2\right). \tag{2.3}$$

Therefore, (2.3) leads to the following pivotal quantity

$$\frac{d_n Q_n - \sigma}{\frac{1}{1.28\sqrt{n}}\sigma} \sim N(0,1). \tag{2.4}$$

Based on the pivotal quantity, the $100(1-\alpha)\%$ robust confidence interval for σ is derived as follows:

$$P\left(z_{\frac{\alpha}{2}} < \frac{d_n Q_n - \sigma}{\frac{1}{1.28\sqrt{n}}\sigma} < z_{1-\frac{\alpha}{2}}\right) = 1 - \alpha, \tag{2.5}$$

where $z_{\frac{\alpha}{2}}$ and $z_{1-\frac{\alpha}{2}}$ are the $(\frac{\alpha}{2})^{th}$ and $(1-\frac{\alpha}{2})^{th}$ percentiles of the standard normal distribution. Isolating σ gives

$$P\left(\frac{1.28\sqrt{n} \cdot d_n Q_n}{z_{1-\frac{\alpha}{2}} + 1.28\sqrt{n}} < \sigma < \frac{1.28\sqrt{n} \cdot d_n Q_n}{z_{\frac{\alpha}{2}} + 1.28\sqrt{n}}\right) = 1 - \alpha.$$

The proposed $100(1-\alpha)\%$ robust confidence interval by Abu-Shawiesh et al. (2011) is

$$\frac{1.28\sqrt{n}\cdot d_nQ_n}{z_{1-\frac{\alpha}{2}}+1.28\sqrt{n}} < \sigma < \frac{1.28\sqrt{n}\cdot d_nQ_n}{z_{\frac{\alpha}{2}}+1.28\sqrt{n}} \tag{2.6}$$

where $z_{\frac{\alpha}{2}}$ and $z_{1-\frac{\alpha}{2}}$ are the $(\frac{\alpha}{2})^{th}$ and $(1-\frac{\alpha}{2})^{th}$ percentiles of the standard normal distribution, respectively. Rousseeuw and Croux (1993), derived the factor d_n so that d_nQ_n becomes an unbiased estimator of σ when sampling from a normal distribution. The values of d_n for $n < 10$ are provided in Table 1:

n	2	3	4	5	6	7	8	9
d_n	0.399	0.994	0.512	0.844	0.611	0.857	0.669	0.872

Table 1: Unbiasing Factor (d_n) Values.

For $n \geq 10$, d_n can be defined as

$$d_n = \begin{cases} \frac{n}{n+3.8}, & n \text{ even} \\ \frac{n}{n+1.4}, & n \text{ odd.} \end{cases}$$

Abu-Shawiesh et al. (2011) also proposed the following bootstrap procedure for the $100(1-\alpha)\%$ confidence interval for σ

$$LCL = \frac{1.28\sqrt{n}\cdot d_nQ_n}{Z^*_{\alpha/2}+1.28\sqrt{n}} \text{ and } UCL = \frac{1.28\sqrt{n}\cdot d_nQ_n}{Z^*_{1-\alpha/2}+1.28\sqrt{n}} \tag{2.7}$$

where $Z^*_{\alpha/2}$ and $Z^*_{1-\alpha/2}$ are the $(\frac{\alpha}{2})^{th}$ and $(1-\frac{\alpha}{2})^{th}$ sample quantiles of the bootstrap test statistics $Z^*_i = \frac{\bar{x}_i{}^*-\bar{\bar{x}}}{\hat{\sigma}_B}$, and $\hat{\sigma}_B = \sqrt{\frac{1}{B-1}\sum_{i=1}^{B}(\bar{x}_i{}^*-\bar{\bar{x}})^2}$; $\bar{x}_i{}^*$ is the i^{th} bootstrap sample mean, $\bar{\bar{x}}$ is the overall bootstrap mean; hence $\hat{\sigma}_B$ is the overall bootstrap standard deviation of the bootstrap means.

2.2. Simulation results

This section presents a simulation study to assess the performance of Equations (2.6) and (2.7). Coverage probabilities, along with the mean and standard deviation of the interval widths, were estimated using 10,000 Monte Carlo samples for each of the following sample sizes: 5, 10, 20, 30, 50, 70, and 100, consistent with the setup of Abu-Shawiesh et al. (2011). All simulations were carried out in R with a random seed of 1234 and run on a Pentium 4 processor. Abu-Shawiesh et al. (2011) evaluated (2.6) under three distributions

1. Normal distribution with a mean and standard deviation of 3 and 1, respectively.
2. Chi-square distribution with 1 degree of freedom.
3. Lognormal distribution with a mean of 1 and a standard deviation of 0.80.

To estimate coverage probabilities, random samples of the specified sizes were generated from each distribution. For each replication, the statistic Q_n was computed, and the corresponding estimate of d_n was obtained for the given sample size. A confidence interval for σ was constructed using $d_n Q_n$ based on the methods under evaluation. Repeating this process 10,000 times yielded a large collection of simulated intervals from which coverage probability was estimated as the proportion of intervals containing the true value of σ. The mean, median, and standard deviation of the interval widths were also recorded.

The simulated coverage probabilities, the mean and standard deviation of the interval widths, and the results reported in Abu-Shawiesh et al. (2011) were used to generate Figures 1, 2, and 3. These figures present comparative plots of the simulation results against those reported in Abu-Shawiesh et al. (2011) for the normal, chi-square, and lognormal distributions, respectively. Each figure consists of four subplots:

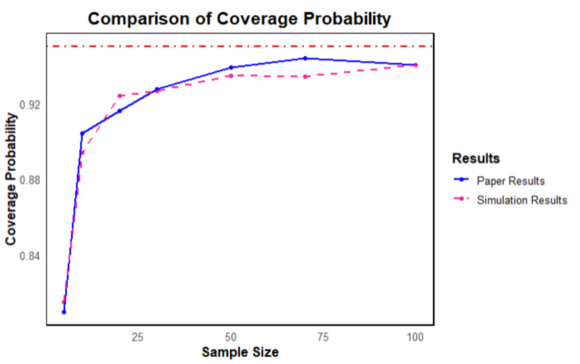
a) coverage probabilities.

b) mean interval width.

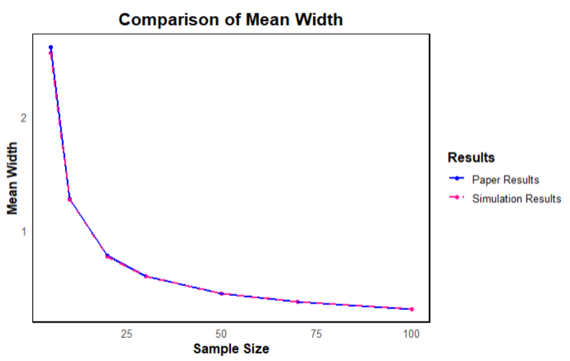
c) standard deviation of the width of the interval.

d) an area plot displaying the proportion of intervals that

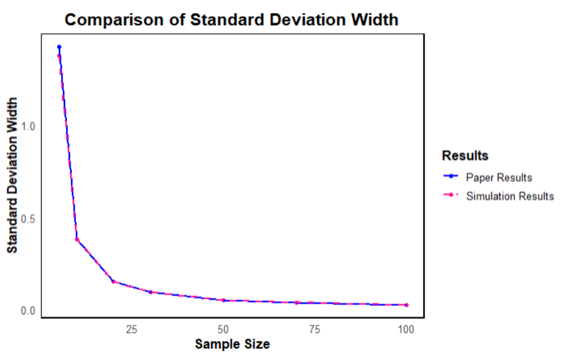
- captures the true value of σ.
- miss high (that is, the lower confidence limit is greater than σ). Labeled as "Over" in plots and tables.
- miss low (that is, the upper confidence limit is less than σ). Labeled as "Under" in plots and tables.



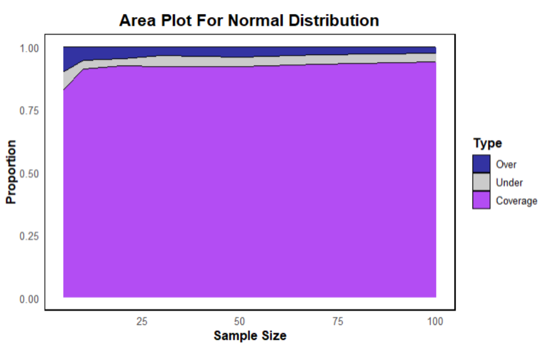
(a) Coverage Probability.



(b) Mean Width.



(c) Standard Deviation Width.



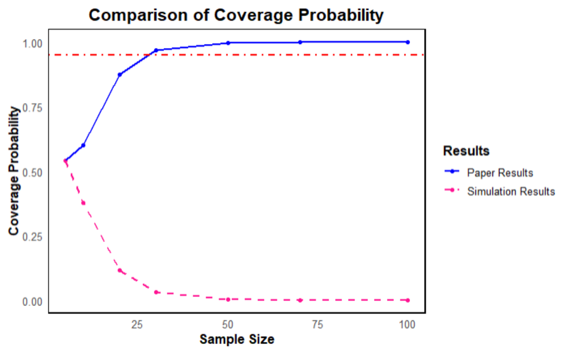
(d) Area Plot.

Figure 1: Normal(3,1).

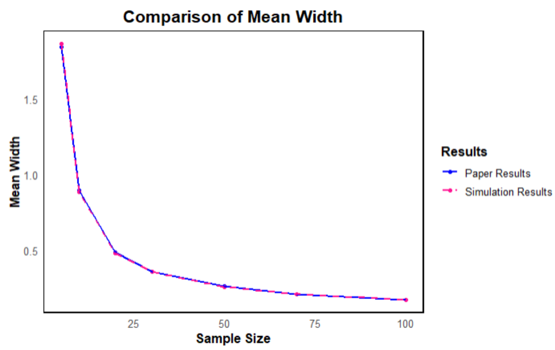
In subplots a), b), and c) of each figure, the solid line represents the results reported in Abu-Shawiesh et al. (2011), while the dashed line represents the results of the simulation. As illustrated in Figure 1a, the simulated coverage probabilities align closely with those reported in Abu-Shawiesh et al. (2011), considering a simulation error of 0.00436.

Figures 1b and 1c provide comparisons of the mean and standard deviation of the interval widths, respectively. Both simulation metrics closely mirror those reported in Abu-Shawiesh et al. (2011), as indicated by the near overlap between the solid and dashed lines. A similar pattern of agreement is observed in Figures 2b and 2c. However, Figure 2a shows a notable discrepancy: as the sample size increases, the simulation's coverage probabilities decrease sharply to 0, whereas the findings reported in Abu-Shawiesh et al. (2011) increase slightly above the nominal level of 0.95.

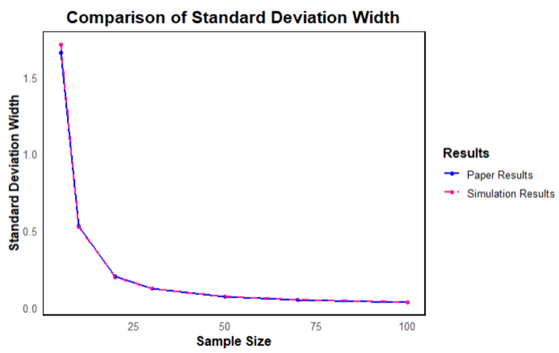
Although the method appears to perform well for the normal distribution with $\mu = 3$ and $\sigma = 1$, it performs poorly for the $\chi^2_{(1)}$ and lognormal distributions with $\mu_{\log} = 1$ and $\sigma_{\log} = 0.8$. The coverage probabilities of these skewed right distributions decrease sharply to 0 as reflected in Figures 2a and 3a and deviate substantially from the values reported in Abu-Shawiesh et al. (2011). Figures 1d, 2d, and 3d further illustrate how the interval captures, underestimates, or overestimates the true parameter for the normal, chi-square, and lognormal distributions, respectively.



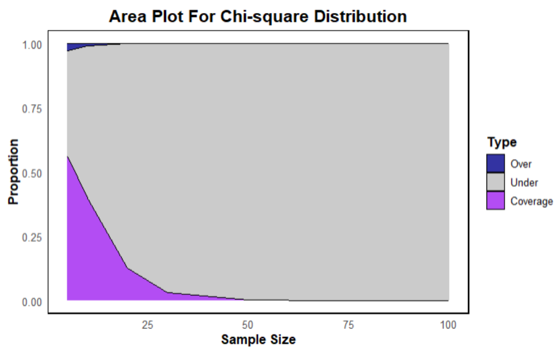
(a) Coverage Probability.



(b) Mean Width.



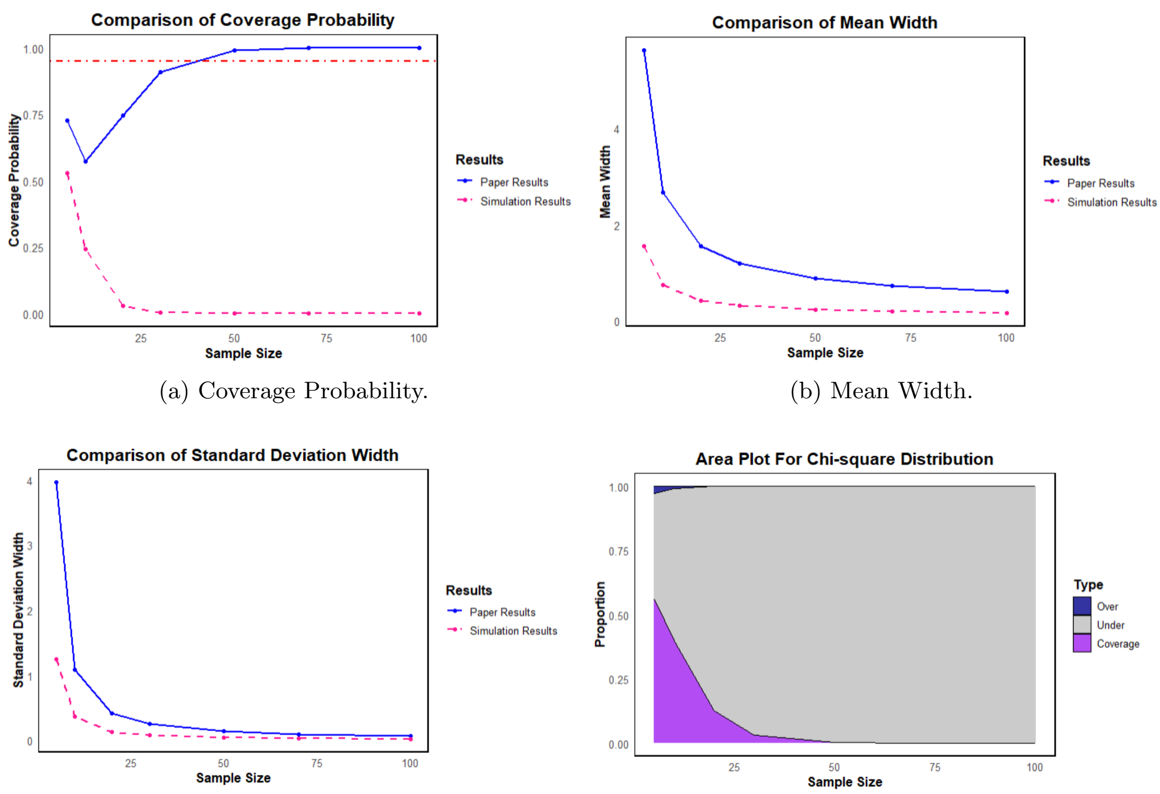
(c) Standard Deviation Width.



(d) Area Plot.

Figure 2: $\chi^2_{(1)}$.

Figures 2d and 3d demonstrate how (2.6) consistently fails to capture the true standard deviation of the $\chi^2_{(1)}$ and lognormal distributions with $\mu_{\log} = 1$ and $\sigma_{\log} = 0.8$. The plots highlight that the upper bounds of the constructed intervals frequently fall below the actual standard deviation values for these skewed distributions. To remedy this, extending the upper bound of (2.6) appears necessary. However, any such modification must preserve the strong performance of equation (2.6) for symmetric and less skewed distributions.



(a) Coverage Probability. (b) Mean Width.

(c) Standard Deviation Width. (d) Area Plot.

Figure 3: Lognormal(1,0.8).

2.3. Proposed confidence interval

The potential modification considered to address the issue observed when using the Abu-Shawiesh et al. (2011) method was to introduce a skew scaling factor into the upper tail of (2.6). The motivation for incorporating skewness to the upper limit of (2.6) was that the method continued to underestimate σ for highly skewed data and to obtain coverage close to the nominal value of 0.95, the upper limit of (2.6) needed to be extended for skewed distributions whilst also making sure that the modification made retains the performance of (2.6) under symmetric distributions. Therefore, the proposed $100(1-\alpha)\%$ confidence interval for σ is:

$$\frac{1.28\sqrt{n}(d_nQ_n)}{z_{1-\frac{\alpha}{2}}+1.28\sqrt{n}} < \sigma < \frac{1.28\sqrt{n}(d_nQ_n)(1+|\hat{\gamma}_3|(\mathbb{I}_{(>1)}(|\hat{\gamma}_3|))}{z_{\frac{\alpha}{2}}+1.28\sqrt{n}}, \tag{2.8}$$

where $\mathbb{I}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$, $z_{\frac{\alpha}{2}}$ and $z_{1-\frac{\alpha}{2}}$ are the $(\frac{\alpha}{2})^{th}$ and $(1-\frac{\alpha}{2})^{th}$ percentiles of the standard normal distribution. Here, $\hat{\gamma}_3$ is the sample skew defined as $\frac{n}{(n-1)(n-2)}\sum_{i=1}^{n}(\frac{x_i-\bar{x}}{s})^3$

with $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$, $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$. In addition, $|x|$ is the absolute value of x. The same adjustment was made for the robust bootstrap method proposed by Abu-Shawiesh et al. (2011).Therefore, the bootstrap procedure proposed for $100\%(1-\alpha)$ confidence interval for σ is:

$$LCL = \frac{1.28\sqrt{n} \cdot d_n Q_n}{Z^*_{\alpha/2} + 1.28\sqrt{n}} \text{ and } UCL = \frac{1.28\sqrt{n} \cdot d_n Q_n (1 + |\hat{\gamma}_3|(\mathbb{I}_{(>1)}(|\hat{\gamma}_3|))}{Z^*_{1-\alpha/2} + 1.28\sqrt{n}} \tag{2.9}$$

where $Z^*_{\alpha/2}$ and $Z^*_{1-\alpha/2}$ are the $(\frac{\alpha}{2})^{th}$ and $(1-\frac{\alpha}{2})^{th}$ quantiles from the distribution of the bootstrap test statistics $Z^*_i = \frac{\bar{x_i}^* - \bar{\bar{x}}}{\hat{\sigma}_B}$, and $\hat{\sigma}_B = \sqrt{\frac{1}{B-1}\sum_{i=1}^{B}(\bar{x_i}^* - \bar{\bar{x}})^2}$; $\bar{x_i}^*$ is the i^{th} bootstrap sample mean, $\bar{\bar{x}}$ is the overall bootstrap mean; hence $\hat{\sigma}_B$ is the overall bootstrap standard deviation of the bootstrap means.

3. SIMULATION STUDIES

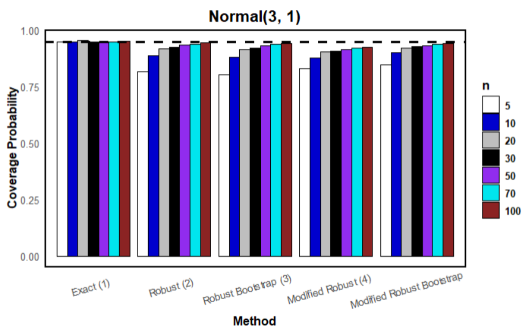
3.1. Method

A simulation study was conducted to investigate the performance of (1.1), (2.6), (2.7), (2.8) and (2.9). For each of the sample sizes-5, 10, 20, 30, 50, 70, and 100-10,000 Monte Carlo samples were generated from each of the following distributions:

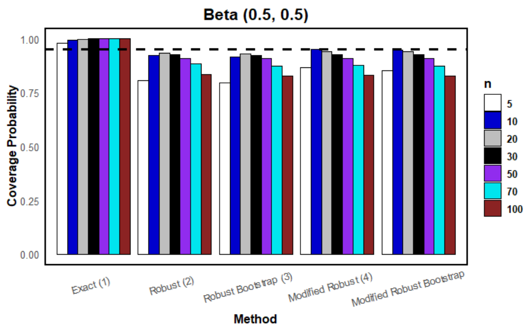
1. Normal distribution with a mean and standard deviation of 3 and 1, respectively.
2. Beta distribution with shape parameters $\alpha = 0.5$ and $\beta = 0.5$
3. Laplace distribution with location $\mu = 0$ and scale parameter 0
4. Lognormal distribution with a mean of 1 and a standard deviation of 0.80.
5. Chi-square distribution with 1 degree of freedom.
6. Beta distribution with shape parameters $\alpha = 10$ and $\beta = 4$
7. Beta distribution with shape parameters $\alpha = 20$ and $\beta = 1$

These simulations were used to estimate the coverage probabilities, average interval widths, median widths, and standard deviations (SD) of the interval widths. The coverage probability was obtained as the proportion of the 10,000 simulated intervals that contained the true value of σ. The mean, median, and SD of the interval widths were computed by first determining the width of each of the 10,000 intervals and then calculating the corresponding summary statistics. Over-coverage and under-coverage were defined as the proportions of simulated intervals that lay above or below the true value of σ, respectively. For the bootstrap-based methods, 1,000 bootstrap samples were generated for each simulation sample. A significance level of $\alpha = 0.05$ was used, corresponding to a 95% confidence interval, which served as the target coverage rate. The primary performance metric was coverage probability, with mean or median width used for further comparison when methods exhibited similar coverage.

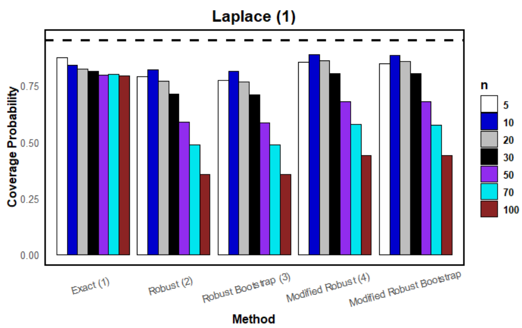
3.2. Simulation results



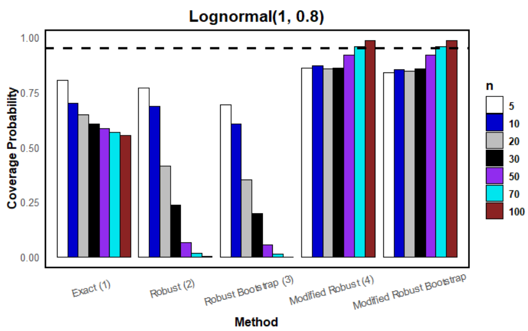
(a) N(3, 1)



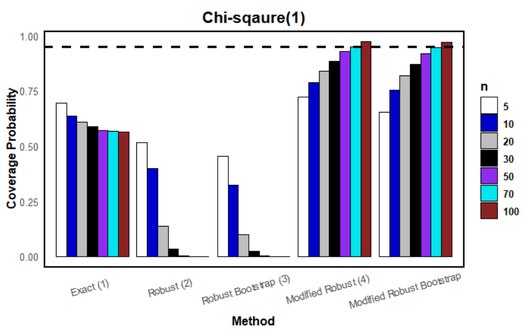
(b) Beta (0.5, 0.5)



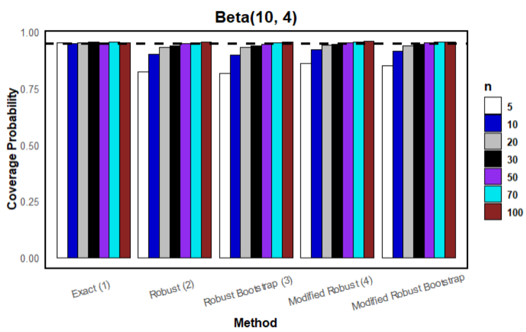
(c) Laplace (0, 4)



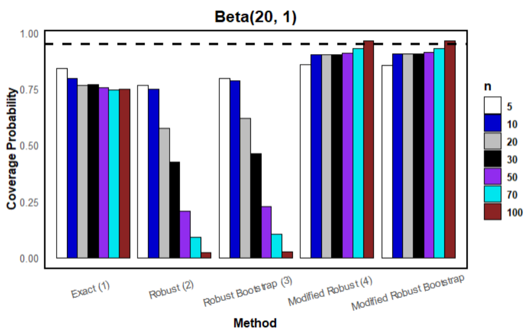
(d) Lognormal (1, 0.8)



(e) $\chi^2_{(1)}$



(f) Beta (10, 4)



(g) Beta (20, 1)

Figure 4: Each confidence interval procedure is represented by seven distinct bars, displaying the coverage probabilities for sample sizes, n, arranged from smallest to largest, read from left to right in this study.

Figure 4 compares the coverage probabilities of methods (1.1), (2.6), (2.7), (2.8) and (2.9). Under normality (Figure 4a), the exact confidence interval (1.1) attains coverage closest to the nominal 0.95 and produces the narrowest interval (Table 3, appendix). Although (2.6) and (2.8) approach nominal coverage for large sample sizes ($n = 70$ and 100), their intervals remain wider than those from (1.1). As shown by Cohen (1972), no other confidence interval for s^2 under normality is shorter than (1.1).

Figures 4b–4g present the results for symmetric heavy-tailed and skewed distributions. In these settings, methods (1.1) and (2.6) generally perform poorly, with coverage more than 10% below the nominal, except for the $Beta(0.5, 0.5)$ and $Beta(10, 4)$ distributions. For $Beta(0.5, 0.5)$, method (1.1) is conservative, producing coverage above 0.95. Under highly skewed distributions, method (2.6) deteriorates sharply, with coverage approaching zero as sample size increases; for large n, its coverage is 0 or near 0 (Tables 6, 7, and 9 in the appendix). The modified method (2.8) performs more reliably under skewness, producing a coverage closer to 0.95, but under unbounded symmetric heavy-tailed distributions, its coverage drops below 0.70 for larger n. The bootstrap methods (2.7) and (2.9) closely mirror the behavior of their corresponding parent methods.

Tables 3 - 9 (see the appendix) show that (2.8) is wider and more variable than the other confidence interval procedures, especially for small sample sizes ($n = 5$ and 10). Although narrower intervals are preferable, (2.1) required a widening under skewness, resulting in (2.8), because its intervals were too narrow to capture σ. This is the trade-off for achieving coverage closer to 0.95. Comparative analysis shows that the modified robust method outperforms the original robust confidence interval method in symmetric, skewed, and heavy-tailed distributions. This improvement is also evident when considering bootstrap techniques.

4. Example

Bluman (2018, p. 404) provides the mean mathematics SAT scores by state. The scores of 12 states are provided below.

$$490, 502, 211, 209, 499, 565, 469, 543, 572, 550, 515, 500.$$

The mean, standard deviation, and skewness of this sample are 468.75, 124.88, and −1.77, respectively. The samples were assumed to have originated from a normal population. The resulting 95% confidence intervals and widths for the different methods are presented in the table below.

Method	Confidence Interval	Width
Exact	(88.46, 212.02)	123.56
Robust	(36.25, 93.69)	57.44
Modified Robust	(8.18, 58.43)	50.25
Robust Bootstrap	(37.72, 105.03)	67
Modified Robust Bootstrap	(37.62, 290.21)	252.59

Table 2: 95% Confidence Intervals for the SAT Data.

Table 2 shows substantial differences in the width of the interval between methods. The

Exact interval is wide, indicating sensitivity to distributional assumptions, while the Robust and Modified Robust methods yield much narrower and more precise intervals. The Robust Bootstrap interval is slightly wider but still consistent with its analytic counterpart. In contrast, the Modified Robust Bootstrap interval is extremely wide, suggesting instability of the modified estimator under resampling and indicating that this method may not perform reliably for the SAT data. From the above Table, we observed that the Modified Robust methods yield much narrower and more precise intervals.

5. CONCLUSION

This paper evaluates the performance of Abu-Shawiesh et al. (2011) (2.6) and introduces a modification that achieves coverage probabilities closer to the nominal level $1 - \alpha$. A simulation study was conducted to compare the performance of several confidence intervals for σ. For normally distributed data, (1.1) yields the most reliable and narrowest intervals. In non-normal settings, (2.8) generally performs well, although it tends to generate wider intervals than the other methods and produces poor coverage for large samples from symmetric unbounded heavy-tailed distributions. Abu-Shawiesh et al. (2011)'s (2.1) performs well for symmetric and mildly skewed distributions. The bootstrap procedures reflected the behavior of their corresponding parent methods. The proposed bootstrap version outperformed the bootstrap method of Abu-Shawiesh et al. (2011), although, like its parent method, (2.9) yields wider intervals than (2.7).

Overall, the modified robust method is recommended for practitioners working with highly skewed distributions, while the original robust method from Abu-Shawiesh et al. (2011) is preferable for symmetric and mildly skewed data. Both robust and modified robust intervals are straightforward to compute and less computationally demanding than their bootstrap counterparts.

ACKNOWLEDGMENTS

The authors gratefully acknowledge the insightful comments and constructive suggestions provided by the reviewers and referees, which significantly improved the quality of this manuscript. The authors also acknowledge all individuals who contributed to the development of this work.

REFERENCES

Abu-Shawiesh, M. O. A., Banik, S., and Kibria, B. M. G. (2011). A simulation study on some confidence intervals for the population standard deviations. *SORT*, 35(2):83–102.

Bluman, A. G. (2018). *Elementary Statistics: A Step by Step Approach*. McGraw-Hill Education.

Bonett, D. G. (2006). Approximate confidence interval for standard deviation of nonnormal distributions. *Computaional Statistics & Data Analysis*, 50(3):775–782.

Cohen, A. (1972). Improved confidence intervals for the variance of a normal distribution. *Journal of the American Statistical Association*, 67(338):382–387.

Croux, C. and Rousseuw, P. J. (1992). Time-efficient algorithms for two highly robust estimators of scale. In Dodge, Y. and Whittaker, J., editors, *Computational Statistics*, pages 411–428. Heidelberg: Physica-Verlag.

Harter, H. L. (1961). Expected values of normal order statistics. *Biometrika*, 48(1 and 2):151–165.

Lehmann, E. L. (1986). *Testing Statistical Hypothesis*. Springer-Verlag.

Niwitpong, S. and Kirdwichai, P. (2008). Adjusted bonett confidence interval for standard deviation of non-normal distributions. *Journal of the Thai Statistical Association*, 6(1):1–6.

Rousseeuw, P. J. and Croux, C. (1993). Alternatives to the median absolute deviation. *Journal of the American Statistical Association*, 88(424):1273–1283.

Tukey, J. W. (1960). A survey of sampling from contaminated distributions. In Olkin, I., editor, *Contributions to Probability and Statistics, Essays in Honor of Harold Hotelling*, pages 448–485. Heidelberg: Physica-Verlag.

Vesna, C. and Andrija, T. (2007). Nonparametric confidence intervals for population variance of one sample and the difference of variances of two samples. *Computational Statistics & Data Analysis*, 51(12):5562–5578.

A. APPENDIX

		Sample Sizes						
Methods	Measuring Criteria	5	10	20	30	50	70	100
Exact	Cover	0.9499	0.9486	0.9544	0.9478	0.9493	0.9500	0.9514
	Under	0.0263	0.0258	0.0213	0.0248	0.0269	0.0243	0.0242
	Over	0.0238	0.0256	0.0243	0.0274	0.0238	0.0257	0.0244
	Mean Width	2.1189	1.1114	0.6903	0.5441	0.4087	0.3410	0.2827
	Median Width	2.0701	1.0977	0.6877	0.5420	0.4082	0.3405	0.2826
	SD Width	0.7689	0.2687	0.1114	0.0722	0.0416	0.0293	0.0200
Robust	Cover	0.8154	0.8868	0.9169	0.9255	0.9337	0.9374	0.9448
	Under	0.0789	0.0422	0.0362	0.0314	0.0300	0.0294	0.035
	Over	0.1057	0.0710	0.0469	0.0431	0.0363	0.0332	0.0295
	Mean Width	2.5638	1.2790	0.7739	0.6080	0.4542	0.3785	0.3135
	Median Width	2.3936	1.2549	0.7704	0.6049	0.4535	0.3779	0.3130
	SD Width	1.3754	0.3921	0.1533	0.0952	0.0536	0.0369	0.0251
Modified Robust	Cover	0.8558	0.9042	0.9242	0.9304	0.9353	0.9377	0.9448
	Under	0.0385	0.0248	0.0289	0.0265	0.0284	0.0291	0.0257
	Over	0.1057	0.0710	0.0469	0.0431	0.0363	0.0332	0.0295
	Mean Width	3.4651	1.5688	0.8584	0.6405	0.4614	0.3802	0.3137
	Median Width	3.0424	1.3749	0.7879	0.6104	0.4542	0.3781	0.3131
	SD Width	2.0742	0.7837	0.3709	0.2215	0.1057	0.0570	0.0279
Robust Bootstrap	Cover	0.8037	0.8818	0.9148	0.9215	0.9309	0.9369	0.9420
	Under	0.0851	0.0464	0.0379	0.0330	0.0317	0.0297	0.0262
	Over	0.1112	0.0718	0.0473	0.0455	0.0374	0.0334	0.0318
	Mean Width	2.5027	1.2648	0.7689	0.6048	0.4518	0.3767	0.3120
	Median Width	2.3153	1.2430	0.7640	0.6021	0.4501	0.3757	0.3118
	SD Width	1.4010	0.3974	0.1548	0.0963	0.0546	0.0378	0.0259
Modified Robust Bootstrap	Cover	0.8470	0.9012	0.9226	0.9226	0.9325	0.9372	0.9420
	Under	0.0418	0.0270	0.0301	0.0279	0.0301	0.0294	0.0262
	Over	0.1112	0.0718	0.0473	0.0455	0.0374	0.0334	0.0318
	Mean Width	3.4151	1.5526	0.8533	0.6372	0.4590	0.3783	0.3122
	Median Width	2.8908	1.3594	0.7812	0.6071	0.4508	0.3759	0.3118
	SD Width	2.2759	0.7889	0.3723	0.2219	0.1062	0.0573	0.0287

Table 3: Coverage Properties for N(3, 1).

		Sample Sizes						
Methods	Measuring Criteria	5	10	20	30	50	70	100
Exact	Cover	0.9784	0.9937	0.9983	0.9995	0.9998	0.9998	0.9999
	Under	0.0216	0.0063	0.0017	0.0005	0.0002	0.0002	0.0001
	Over	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	Mean Width	0.7747	0.3986	0.2464	0.1935	0.1450	0.1210	0.1002
	Median Width	0.8085	0.4034	0.2476	0.1939	0.1451	0.1211	0.1002
	SD Width	0.2030	0.0571	0.0221	0.0133	0.0075	0.0052	0.0036
Robust	Cover	0.8048	0.9230	0.9346	0.9269	0.9075	0.8837	0.8329
	Under	0.1346	0.0658	0.0652	0.0730	0.0925	0.1163	0.1671
	Over	0.0606	0.0112	0.0002	0.0001	0.0000	0.0000	0.0000
	Mean Width	0.8091	0.4186	0.2486	0.1926	0.1438	0.1198	0.0991
	Median Width	0.7755	0.4282	0.2533	0.1954	0.1455	0.1208	0.0998
	SD Width	0.4434	0.1150	0.0417	0.0243	0.0129	0.0086	0.0057
Modified Robust	Cover	0.8650	0.9508	0.9419	0.9268	0.9088	0.8772	0.8323
	Under	0.0787	0.0359	0.0578	0.0732	0.0912	0.1228	0.1677
	Over	0.0563	0.0133	0.0003	0.0000	0.0000	0.0000	0.0000
	Mean Width	1.0612	0.4739	0.2536	0.1934	0.1439	0.1198	0.0991
	Median Width	0.9923	0.4528	0.2541	0.1960	0.1456	0.1209	0.0998
	SD Width	0.6096	0.1673	0.0521	0.0266	0.0129	0.0087	0.0057
Robust Bootstrap	Cover	0.7955	0.9165	0.9313	0.9228	0.9071	0.8736	0.8278
	Under	0.1454	0.0688	0.0683	0.0772	0.0929	0.1264	0.1722
	Over	0.0591	0.0147	0.0004	0.0000	0.0000	0.0000	0.0000
	Mean Width	0.7842	0.4135	0.2466	0.1915	0.1431	0.1191	0.0986
	Median Width	0.7492	0.4207	0.2506	0.1943	0.1445	0.1200	0.0991
	SD Width	0.4500	0.1176	0.0418	0.0251	0.0132	0.0091	0.0061
Modified Robust Bootstrap	Cover	0.8514	0.9479	0.9409	0.9247	0.9071	0.8736	0.8278
	Under	0.0895	0.0374	0.0587	0.0753	0.0929	0.1264	0.1722
	Over	0.0591	0.0147	0.0004	0.0000	0.0000	0.0000	0.0000
	Mean Width	1.0554	0.4680	0.2516	0.1921	0.1431	0.1191	0.0987
	Median Width	0.9446	0.4459	0.2520	0.1944	0.1445	0.1201	0.0991
	SD Width	0.6875	0.1707	0.0525	0.0271	0.0133	0.0091	0.0062

Table 4: Coverage Properties for Beta(0.5, 0.5).

		Sample Sizes						
Methods	Measuring Criteria	5	10	20	30	50	70	100
Exact	Cover	0.8748	0.8396	0.8245	0.8121	0.7973	0.7986	0.7925
	Under	0.0684	0.0819	0.0960	0.0991	0.1088	0.1077	0.1104
	Over	0.0568	0.0785	0.0795	0.0888	0.0939	0.0937	0.0971
	Mean Width	11.4875	6.1339	3.8507	3.0414	2.2932	1.9193	1.5939
	Median Width	10.5406	5.8575	3.7720	2.9929	2.2706	1.9075	1.5861
	SD Width	5.7038	2.1396	0.9271	0.6102	0.3593	0.2534	0.1765
Robust	Cover	0.7897	0.8193	0.7701	0.7109	0.5862	0.4862	0.3571
	Under	0.1256	0.1336	0.2104	0.2775	0.4101	0.5129	0.6424
	Over	0.0847	0.0471	0.0195	0.0116	0.0037	0.0009	0.0005
	Mean Width	12.6783	6.2207	3.6674	2.8462	2.1118	1.7521	1.4502
	Median Width	11.0767	5.9415	3.6050	2.8081	2.0935	1.7422	1.4422
	SD Width	8.0310	2.3580	0.9487	0.5923	0.3411	0.2339	0.1636
Modified Robust	Cover	0.8521	0.8873	0.8591	0.8021	0.6789	0.5770	0.4402
	Under	0.0632	0.0656	0.1214	0.1863	0.3174	0.4221	0.5593
	Over	0.0847	0.0471	0.0195	0.0116	0.0037	0.0009	0.0005
	Mean Width	19.4204	10.0773	6.3107	4.9429	3.5727	2.8619	2.2521
	Median Width	15.6448	7.6747	4.2001	3.1024	2.2135	1.8056	1.4753
	SD Width	14.0576	6.7581	4.6981	4.0529	3.2775	2.8506	2.3613
Robust Bootstrap	Cover	0.7711	0.8123	0.7662	0.7094	0.5841	0.4852	0.3569
	Under	0.1421	0.1399	0.2134	0.2791	0.4122	0.5139	0.6427
	Over	0.0868	0.0478	0.0204	0.0115	0.0037	0.0009	0.0004
	Mean Width	12.5269	6.1646	3.6481	2.8316	2.1029	1.7456	1.4440
	Median Width	10.8695	5.8563	3.5788	2.7948	2.0833	1.7356	1.4366
	SD Width	8.4389	2.4232	0.9570	0.5986	0.3439	0.2366	0.1661
Modified Robust Bootstrap	Cover	0.8450	0.8837	0.8556	0.8016	0.6777	0.5752	0.4400
	Under	0.0682	0.0685	0.1240	0.1869	0.3186	0.4239	0.5596
	Over	0.0868	0.0478	0.0204	0.0115	0.0037	0.0009	0.0004
	Mean Width	19.4076	9.9927	6.2888	4.9219	3.5620	2.8551	2.2456
	Median Width	14.8695	7.6113	4.1799	3.0861	2.2083	1.7998	1.4691
	SD Width	15.8803	6.8243	4.7086	4.0430	3.2733	2.8515	2.3621

Table 5: Coverage Properties for Laplace(0, 4).

		Sample Sizes						
Methods	Measuring Criteria	5	10	20	30	50	70	100
Exact	Cover	0.6948	0.6366	0.6103	0.5864	0.5718	0.5689	0.5645
	Under	0.2215	0.2526	0.2620	0.2584	0.2627	0.2593	0.2578
	Over	0.0837	0.1108	0.1277	0.1552	0.1655	0.1718	0.1777
	Mean Width	2.5653	1.4180	0.9164	0.7421	0.5634	0.4728	0.3941
	Median Width	2.0968	1.2680	0.8629	0.7073	0.5465	0.4622	0.3874
	SD Width	1.8758	0.7558	0.3494	0.2372	0.1419	0.1017	0.0708
Robust	Cover	0.5454	0.3875	0.1217	0.0293	0.0024	0.0000	0.0000
	Under	0.4279	0.6047	0.8782	0.9707	0.9976	1.0000	1.0000
	Over	0.0267	0.0078	0.0001	0.0000	0.0000	0.0000	0.0000
	Mean Width	1.8578	0.8909	0.4835	0.3603	0.2595	0.2131	0.1748
	Median Width	1.3724	0.7849	0.4559	0.3469	0.2539	0.2100	0.1728
	SD Width	1.7012	0.5230	0.1979	0.1212	0.0693	0.0483	0.0332
Modified Robust	Cover	0.7219	0.7894	0.8397	0.8849	0.9288	0.9507	0.9729
	Under	0.2514	0.2028	0.1602	0.1151	0.0712	0.0493	0.0271
	Over	0.0267	0.0078	0.0001	0.0000	0.0000	0.0000	0.0000
	Mean Width	3.5578	2.5229	2.0786	1.9824	1.8735	1.8251	1.7864
	Median Width	2.4824	2.1101	1.9721	1.8741	1.7708	1.7229	1.6976
	SD Width	3.4795	1.7443	1.0375	0.8437	0.6603	0.5916	0.5239
Robust Bootstrap	Cover	0.4535	0.3232	0.0974	0.0236	0.0016	0.0000	0.0000
	Under	0.5210	0.6696	0.9025	0.9764	0.9984	1.0000	1.0000
	Over	0.0255	0.0072	0.0001	0.0000	0.0000	0.0000	0.0000
	Mean Width	1.4839	0.7717	0.4442	0.3383	0.2488	0.2063	0.1705
	Median Width	1.0193	0.6703	0.4173	0.3244	0.2429	0.2032	0.1685
	SD Width	1.5057	0.4794	0.1882	0.1167	0.0676	0.0475	0.0328
Modified Robust Bootstrap	Cover	0.6545	0.7516	0.8195	0.8720	0.9199	0.9453	0.9702
	Under	0.3200	0.2412	0.1804	0.1280	0.0801	0.0547	0.0298
	Over	0.0255	0.0072	0.0001	0.0000	0.0000	0.0000	0.0000
	Mean Width	2.6662	2.1889	1.9409	1.8934	1.8229	1.7898	1.7621
	Median Width	1.9532	1.8492	1.8512	1.8004	1.7297	1.6935	1.6755
	SD Width	2.5018	1.4805	0.9499	0.7885	0.6312	0.5716	0.5113

Table 6: Coverage Properties for Chi-square df = 1.

		Sample Sizes						
Methods	Measuring Criteria	5	10	20	30	50	70	100
Exact	Cover	0.8036	0.6968	0.6458	0.6058	0.5844	0.5655	0.5530
	Under	0.1327	0.1997	0.2425	0.2558	0.2689	0.2763	0.2774
	Over	0.0637	0.1035	0.1117	0.1384	0.1467	0.1582	0.1696
	Mean Width	1.4795	0.8223	0.5187	0.4172	0.3177	0.2667	0.2222
	Median Width	1.2330	0.7215	0.4793	0.3914	0.3038	0.2558	0.2158
	SD Width	0.9877	0.4253	0.1975	0.1355	0.0855	0.0641	0.0428
Robust	Cover	0.7678	0.6847	0.4146	0.2370	0.0658	0.0158	0.0010
	Under	0.1985	0.3059	0.5846	0.7630	0.9342	0.9842	0.9990
	Over	0.0337	0.0094	0.0008	0.0000	0.0000	0.0000	0.0000
	Mean Width	1.3938	0.6781	0.3964	0.3084	0.2288	0.1899	0.1570
	Median Width	1.1982	0.6383	0.3876	0.3031	0.2267	0.1886	0.1564
	SD Width	0.9359	0.2684	0.1015	0.0629	0.0353	0.0244	0.0165
Modified Robust	Cover	0.8597	0.8678	0.8532	0.8598	0.9180	0.9570	0.9828
	Under	0.1066	0.1228	0.1460	0.1402	0.0820	0.0430	0.0172
	Over	0.0337	0.0094	0.0008	0.0000	0.0000	0.0000	0.0000
	Mean Width	2.4284	1.6230	1.3940	1.3902	1.4169	1.4267	1.4458
	Median Width	1.8219	1.3144	1.3949	1.3815	1.3511	1.3348	1.3354
	SD Width	2.0713	1.1327	0.8199	0.7415	0.6526	0.5989	0.5559
Robust Bootstrap	Cover	0.6927	0.6056	0.3516	0.1959	0.0532	0.0115	0.0007
	Under	0.2750	0.3858	0.6477	0.8041	0.9468	0.9885	0.9993
	Over	0.0323	0.0086	0.0007	0.0000	0.0000	0.0000	0.0000
	Mean Width	1.1637	0.6040	0.3694	0.2925	0.2205	0.1845	0.1535
	Median Width	0.9712	0.5661	0.3603	0.2873	0.2182	0.1832	0.1529
	SD Width	0.8264	0.2503	0.0982	0.0618	0.0350	0.0244	0.0166
Modified Robust Bootstrap	Cover	0.8366	0.8523	0.8435	0.8548	0.9173	0.9567	0.9828
	Under	0.1311	0.1391	0.1558	0.1452	0.0827	0.0433	0.0172
	Over	0.0323	0.0086	0.0007	0.0000	0.0000	0.0000	0.0000
	Mean Width	1.9144	1.4294	1.3081	1.3319	1.3803	1.4002	1.4268
	Median Width	1.5438	1.1786	1.3251	1.3373	1.3239	1.3158	1.3211
	SD Width	1.4578	0.9420	0.7453	0.6934	0.6219	0.5766	0.5406

Table 7: Coverage Properties for Lognormal (1, 0.8).

		Sample Sizes						
Methods	Measuring Criteria	5	10	20	30	50	70	100
Exact	Cover	0.9528	0.9473	0.9515	0.9540	0.9522	0.9552	0.9535
	Under	0.0232	0.0280	0.0238	0.0219	0.0238	0.0231	0.0234
	Over	0.0240	0.0247	0.0247	0.0241	0.0240	0.0217	0.0231
	Mean Width	0.2491	0.1295	0.0806	0.0634	0.0476	0.0398	0.0329
	Median Width	0.2439	0.1283	0.0802	0.0632	0.0475	0.0397	0.0329
	SD Width	0.0897	0.0307	0.0129	0.0082	0.0048	0.0033	0.0023
Robust	Cover	0.8238	0.9009	0.9311	0.9386	0.9474	0.9520	0.9568
	Under	0.0750	0.0415	0.0339	0.0295	0.0270	0.0253	0.0236
	Over	0.1012	0.0576	0.0350	0.0319	0.0256	0.0227	0.0196
	Mean Width	0.2971	0.1474	0.0897	0.0704	0.0527	0.0439	0.0363
	Median Width	0.2779	0.1458	0.0895	0.0701	0.0527	0.0439	0.0363
	SD Width	0.1555	0.0434	0.0171	0.0104	0.0059	0.0040	0.0027
Modified Robust	Cover	0.8611	0.9207	0.9413	0.9459	0.9517	0.9551	0.9578
	Under	0.0377	0.0217	0.0237	0.0222	0.0.0227	0.0222	0.0226
	Over	0.1012	0.0576	0.0350	0.0319	0.0256	0.0227	0.0196
	Mean Width	0.4027	0.1870	0.1087	0.0820	0.0579	0.0465	0.0375
	Median Width	0.3549	0.1607	0.0931	0.0717	0.0531	0.0441	0.0364
	SD Width	0.2413	0.0992	0.0585	0.0433	0.0275	0.0190	0.0122
Robust Bootstrap	Cover	0.8165	0.8971	0.9314	0.9394	0.9465	0.9509	0.9558
	Under	0.0762	0.0398	0.0319	0.0266	0.0252	0.0248	0.0228
	Over	0.1073	0.0631	0.0367	0.0340	0.0283	0.0243	0.0214
	Mean Width	0.3032	0.1495	0.0904	0.0707	0.0528	0.0439	0.0363
	Median Width	0.2856	0.1477	0.0901	0.0705	0.0527	0.0439	0.0363
	SD Width	0.1636	0.0448	0.0173	0.0106	0.0060	0.0041	0.0028
Modified Robust Bootstrap	Cover	0.8510	0.9145	0.9398	0.9456	0.9505	0.9537	0.9567
	Under	0.0417	0.0224	0.0235	0.0204	0.0212	0.0220	0.0219
	Over	0.1073	0.0631	0.0367	0.0340	0.0283	0.0243	0.0214
	Mean Width	0.4251	0.1918	0.1099	0.0826	0.0579	0.0465	0.0374
	Median Width	0.3465	0.1614	0.0935	0.0719	0.0531	0.0441	0.0364
	SD Width	0.3063	0.1105	0.0614	0.0445	0.0278	0.0192	0.0123

Table 8: Coverage Properties for Beta (10, 4).

		Sample Sizes						
Methods	Measuring Criteria	5	10	20	30	50	70	100
Exact	Cover	0.8417	0.7956	0.7647	0.7696	0.7558	0.7464	0.7498
	Under	0.0890	0.1217	0.1334	0.1263	0.1324	0.1383	0.1338
	Over	0.0693	0.0827	0.1019	0.1041	0.1118	0.1153	0.1164
	Mean Width	0.0912	0.0482	0.0306	0.0243	0.0184	0.0153	0.0127
	Median Width	0.0823	0.0457	0.0297	0.0238	0.0181	0.0152	0.0127
	SD Width	0.0489	0.0181	0.0083	0.0052	0.0031	0.0022	0.0015
Robust	Cover	0.7662	0.7484	0.5757	0.4234	0.2067	0.0927	0.0235
	Under	0.1825	0.2332	0.4216	0.5761	0.7932	0.9073	0.9765
	Over	0.0513	0.0184	0.0027	0.0005	0.0001	0.0000	0.0000
	Mean Width	0.0877	0.0426	0.0250	0.0192	0.0143	0.0118	0.0097
	Median Width	0.0755	0.0406	0.0244	0.0190	0.0141	0.0118	0.0097
	SD Width	0.0584	0.0168	0.0066	0.0041	0.0023	0.0016	0.0011
Modified Robust	Cover	0.8558	0.9010	0.9004	0.9011	0.9095	0.9288	0.9620
	Under	0.0929	0.0806	0.0969	0.0984	0.0904	0.0712	0.0380
	Over	0.0513	0.0184	0.0027	0.0005	0.0001	0.0000	0.0000
	Mean Width	0.1477	0.0927	0.0756	0.0724	0.0698	0.0679	0.0673
	Median Width	0.1158	0.0736	0.0781	0.0767	0.0719	0.0687	0.0668
	SD Width	0.1163	0.0596	0.0427	0.0361	0.0286	0.0238	0.0189
Robust Bootstrap	Cover	0.7950	0.7852	0.6196	0.4617	0.2259	0.1037	0.0267
	Under	0.1466	0.1934	0.3771	0.5377	0.7740	0.8963	0.9733
	Over	0.0584	0.0214	0.0033	0.0006	0.0001	0.0000	0.0000
	Mean Width	0.0992	0.0456	0.0260	0.0198	0.0145	0.0119	0.0098
	Median Width	0.0851	0.0434	0.0254	0.0195	0.0143	0.0119	0.0097
	SD Width	0.0660	0.0178	0.0068	0.0042	0.0023	0.0016	0.0011
Modified Robust Bootstrap	Cover	0.8552	0.9036	0.9054	0.9058	0.9117	0.9300	0.9621
	Under	0.0864	0.0750	0.0913	0.0936	0.0882	0.0700	0.0379
	Over	0.0584	0.0214	0.0033	0.0006	0.0001	0.0000	0.0000
	Mean Width	0.1779	0.1003	0.0788	0.0743	0.0709	0.0686	0.0677
	Median Width	0.1255	0.0783	0.0809	0.0785	0.0728	0.0692	0.0672
	SD Width	0.1595	0.0676	0.0455	0.0375	0.0293	0.0242	0.0191

Table 9: Coverage Properties for Beta (20, 1).