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ON ESTIMATION FOLLOWING SUBSET SELECTION FROM TRUNCATED POISSON DISTRIBUTIONS UNDER STEIN LOSS FUNCTION

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Abstract:

- In this paper, we consider the problem of estimating the parameters of a subset selected from $p\,(p \geq 2)$ left-truncated Poisson distributions under Stein loss function. Two problems of estimations are considered; average worth and simultaneous estimation. For the average worth, the natural estimator is shown to be positively biased with respect to Stein loss function and the Unique Minimum Risk Unbiased Estimator *UMRUE* is obtained. For the simultaneous estimation problem, we have shown that the natural estimator is positively biased with respect to Stein loss function and the *UMRUE* is obtained. The inadmissibility of the natural estimator of the simultaneous estimation is also proved and a class of dominating estimators is obtained. Monte Carlo simulation is undertaken to compute the biases and risks of the two problems of estimation.

Key-Words:

- *simultaneous estimation after subset selection; average worth estimation; Stein loss function; difference inequalities; truncated Poisson distributions.*

AMS Subject Classification:

- 62F10, 62F07.

1. INTRODUCTION

Estimating the parameter of the selected population is an important practical problem which arises in various disciplines such as agriculture, medicine and industry. Say, we wish to select the most productive machine from p different types of machines and then estimate the mean of the production of the selected machine. The problem of estimation after selection has received considerable attention from many researchers. Some references in this area include, Sackrowitz and Samuel-Cahn (1984), Kumar and Gangopadhyay (2005), Misra, van der Meulen and Branden (2006a, 2006b), Sill and Sampson (2007) and Vellaisamy and Jain (2008). All these studies considered the problem of estimation when the selection rule selects only one population. However sometimes we are interested to select a subset of good populations (including the best) rather than only one population (the best) and then estimating the parameters of the selected subset. The problem of estimation after subset selection was initially formulated and studied by Jayaratnam and Panchapakesan (1984) for two normal populations. They proposed three classes of estimators for the average worth of the selected subset and compared numerically their biases and mean squared errors. Jayaratnam and Panchapakesan (1986) considered the case of two independent exponential populations and they proved that the natural estimator of the average worth of the selected subset is positively biased. They suggested an adjusted estimator by adjusting the bias of the natural estimator and compared the bias and mean squared error of the natural estimator with the adjusted estimator. Vellaisamy (1992) considered the average worth estimation and simultaneous estimation of the subset selected from independent gamma population with unknown scale parameters and common known shape parameter. He proved that the natural estimator of the average worth is positively biased and inadmissible and also, he obtained the *UMVUE* of it using the *UV* method of Robbins (1988). Also, he observed similar results for the simultaneous estimation of the selected subset. Misra (1994) derived the *UMVUE* of the average worth of the selected subset from p independent gamma populations with common known shape parameter and unknown scale parameters. He also proved the inadmissibility of the natural estimator of the average worth under squared error loss function by constructing improved estimators. Vellaisamy (1996) considered the case of subset selection from uniform populations. He proved for the simultaneous estimation of the parameters associated with the selected populations, the natural estimators as well as the unbiased estimator are inadmissible under the squared error loss function and the dominating ones were obtained. The problem of estimating the average worth of the selected subset from exponential populations with a common unknown location parameter and unknown scale parameters has been investigated by Gangopadhyay and Kumar (2005). They derived the *UMVUE* of the average worth of the selected subset and they also, compared, numerically, the bias and the mean squared error of the *UMVUE*, *BAEE* and *MLE* of the average worth.

They observed that the natural estimator dominates the unbiased estimator and the natural estimator itself is inadmissible. The literatures, so far, deal with the problem of estimating the parameters (average worth or simultaneous estimation) of a selected subset containing the best population when the distributions of populations are continuous. In this paper, we take up the problem of estimating the parameters of the selected subset under the asymmetric loss function when the distributions of populations are discrete. The loss function considered here is Stein loss function defined as

$$L(\theta, d) = \frac{d}{h(\theta)} - \log\left(\frac{d}{h(\theta)}\right) - 1 \,, \tag{1.1}$$

where d is an estimate of $h(\theta)$ and log denotes the natural logarithm. The loss function (1.1) was first introduced in James and Stein (1961) for estimation of the multinormal covariance matrix. Also, it was considered by Dey and Srinivasan (1985) and Dey and Chung (1991) for simultaneous estimation. In Section 2, we introduce some notations, definitions and lemmas and formulate the problem. In Section 3, the natural estimators of the average worth and simultaneous estimation of the selected subset are shown to be positively biased with respect to Stein loss function. In Section 4, the *UMRUE*'s of the average worth and simultaneous estimation are derived. In Section 5, the inadmissibility of the natural estimator of the simultaneous estimation is proved by solving certain difference inequality and a class of improved estimators is constructed. In Section 6, Monte Carlo simulation is undertaken to compute the biases and risks of the estimators under the two problems of estimation.

2. NOTATIONS, DEFINITIONS AND FORMULATION OF THE PROBLEM

Let $\Pi_1, ..., \Pi_p$ be p $(p \geq 2)$ independent populations such that the random variable X_i represents the population Π_i has left-truncated Poisson p.d.f.

$$P(X_i = x_i) = \frac{\theta_i^{x_i}}{x_i!\, f(\theta_i, t)} \,, \qquad x_i = t, t+1, ...; \quad t > 0; \quad \theta_i > 0; \quad i = 1, ..., p \,,$$

where $f(\theta_i, t) = e^{\theta_i} - \sum_{k=o}^{t-1} \theta_i^k / k!$. We assume that $\theta_1, ..., \theta_p$ are unknown parameters. Suppose from each population Π_i we have a random sample $X_{i1}, ..., X_{in}$ and let $Z_i = \sum_{j=1}^{n} X_{ij}$. It is well-known (see for example, Jani (1977)) that the distribution of Z_i is given by

$$P(Z_i = z_i) = \frac{n!\, S(z_i, n, t)\, \theta_i^{z_i}}{z_i!\, f^n(\theta_i, t)} \,, \qquad z_i = nt, nt+1, ... \,,$$

where $S(z_i, n, t)$ is the Stirling number of the second kind and the *UMVUE* of θ_i is given by

$$\delta(Z_i) = \begin{cases} Z_i\, S(Z_i - 1, n, t)/S(Z_i, n, t), & \text{if } Z_i \geq nt + 1\,; \\ 0, & \text{otherwise}\,. \end{cases} \tag{2.1}$$

Without loss of generality, we consider the case $n = 1$. So that the *UMVUE*, defined in (2.1), reduces to

$$\delta(X_i) = \begin{cases} X_i, & \text{if } X_i \geq t + 1\,; \\ 0, & \text{otherwise}\,. \end{cases} \tag{2.2}$$

Let $\chi = \{x \colon x = (x_1, ..., x_p),\ x_i \geq t,\ i = 1, ..., p\}$ and $\Omega = \{\theta \colon \theta = (\theta_1, ..., \theta_p),\ \theta_i > 0,\ i = 1, ..., p\}$ denote the sample space and the parameter space, respectively, and let $\theta_{[1]} \geq \theta_{[2]} \geq \cdots \geq \theta_{[p]}$ represent the ordered parameters and $X_{(1)} \geq X_{(2)} \geq \cdots \geq X_{(p)}$ represent the ordered values of $X_1, ..., X_p$ (use arbitrary ordering if some of the θ_i's (X_i's) are equal). The population associated with $\theta_{[1]}$ is called the best population. In the subset selection approach, we want to select a non-empty subset from the p populations so that the best population is included in the selected subset with a minimum pre-assigned probability $P^*(1/p < P^* < 1)$ (Gupta (1965)). To select such a subset, we consider, in this paper, the following modified selection rule which was suggested by Gupta and Huang (1975).

$$R\colon \text{ Choose } \Pi_i \text{ in the subset iff } X_i + 1 \geq c X_{(1)}\ , \tag{2.3}$$

where $c = c(p, P^*)$ $(0 < c < 1)$ is some suitable constant satisfying the basic probability requirement

$$\inf_{\theta \in \Omega} P_\theta\big(CS|R\big) = P^*\ ,$$

and CS stands for "Correct Selection" (i.e. the selection contains the best population). Let $X_{(1)i} \geq X_{(2)i} \geq \cdots \geq X_{(p-1)i}$ denote the ordered values of $X_1, ..., X_{i-1}$, $X_{i+1}, ..., X_p$. Note that

$$\{X_i + 1 \geq c X_{(1)}\} = \{X_i + 1 \geq c X_{(1)i}\}\,.$$

Suppose a subset (of random size) is selected using the rule R. The problems that we are interested here are the estimation of the average worth M and the simultaneous estimation of Q, defined by

$$M = \frac{\sum_{i=1}^{p} \theta_i I_i(X)}{\sum_{i=1}^{p} I_i(X)}$$

and

$$Q = \big(\theta_1 I_1(X), ..., \theta_p I_p(X)\big)\,,$$

where $I_i(X) = I\big(X_i + 1 \geq c X_{(1)i}\big)$ and $I(A)$ denotes the indicator function of an event A. It can be seen that the dimension of the estimand M is random, as it varies with X, unlike in the case of classical estimation problem. The natural analogues of M and Q for the selection problem are as follows

$$\hat{M}_1(X) = \frac{\sum_{i=1}^{p} \delta(X_i)\, I_i(X)}{\sum_{i=1}^{p} I_i(X)} \tag{2.4}$$

and

$$\hat{Q}_1(X) = \big(\delta(X_1)\, I_1(X), ..., \delta(X_p)\, I_p(X)\big) , \tag{2.5}$$

and we will call them, the natural estimators of M and Q, respectively, where δ is as in (2.2). The loss function (1.1) can be written for the case of estimating M and Q as in the following

$$L(M, \hat{M}) = \frac{\hat{M}}{M} - \log\left(\frac{\hat{M}}{M}\right) - 1$$

and

$$L(Q, \hat{Q}) = \sum_{j=1}^{p} \left[\frac{d_j}{\theta_j} - \log\left(\frac{d_j}{\theta_j}\right) - 1\right] I_j(X) ,$$

where $\hat{M}$ is an estimate of M, d_j is an estimate of θ_j and $\hat{Q} = (d_1, ..., d_p)$. The loss function is well defined in our problem, since we considered distributions truncated at zero. Now, we introduce the following lemmas which will be used in the next sections. The following lemma is from Chou (1991).

Lemma 2.1. *Let f_1 be a real-valued function defined on p-fold Cartesian product of $\mathbb{I}^+$, the set of positive integers, such that $E_\theta|f_1(X)| < \infty$ and $f_1(x) = 0$ if $x_i \le t$. Then*

$$E_\theta\, f_1(X)/\theta_i = E_\theta\big(f_1(X + e_i)/\delta(X_i + 1)\big) ,$$

where e_i is the p-dimensional vector whose i-th coordinate is 1 and the rest are zeros and δ is as in (2.2).

Lemma 2.2. *Let f_2 be a real-valued function defined on p-fold Cartesian product of $\mathbb{I}^+$ such that $E_\theta|f_2(X)| < \infty$. Then*

$$E_\theta\, f_2(X)\, \theta_i = E_\theta\, f_2(X - e_i)\, \delta(X_i) ,$$

where δ is as in (2.2)

Lemma 2.3. *If $|w| \le 1/2$, then $\log(1 + w) \ge w - w^2$.*

Proof: Similar to the proof of Lemma 2.2 of Dey and Srinivasan (1985), we observe that

$$\begin{aligned}
\log(1+w) &= w - \frac{w^2}{2} + \frac{w^3}{3} - \frac{w^4}{4} + \cdots \\
&\geq w - \frac{w^2}{2} - \frac{|w|^3}{2} - \frac{|w|^4}{2} - \cdots \\
&= w - \frac{w^2}{2}\big(1 + |w| + |w|^2 + \cdots\big) \\
&= w - \frac{w^2}{2\big(1-|w|\big)} \\
&\geq w - w^2 \ ,
\end{aligned}$$

since $|w| \leq 1/2$. □

3. ESTIMATION OF M AND Q

In this section, the natural estimators of M and Q are shown to be positively biased with respect to Stein loss function. First of all we need to impose a condition on the estimator δ to be unbiased under Stein loss function using the definition of the risk-unbiasedness of Lehmann (1951).

Definition 3.1. An estimator $\eta(Y)$ of $g(\theta)$ is said to be risk-unbiased if it satisfies

$$E_\theta\, L\big(\theta, \eta(Y)\big) \leq E_\theta\, L\big(\theta', \eta(Y)\big) \qquad \forall\, \theta' \neq \theta \ . \tag{3.1}$$

Following Nematollahi and Motamed-Shariati (2009), the estimator η of θ is said to be unbiased under Stein loss function if $E_\theta\, \eta(Y) = \theta$, $\forall\, \theta \in \Omega$, otherwise, it is biased and its bias is $B_\theta(\eta) = E_\theta\big(\eta(Y) - \theta\big)$. Clearly, this is the same definition of the usual unbiasedness (unbiasedness under the squared error loss function). Consider first the estimation problem of the average worth M. The estimator $\hat{M}$ of M is said to be unbiased under Stein loss function if $E_\theta \hat{M} = E_\theta M$, otherwise, it is biased and its bias is $B_\theta(\hat{M}, M) = E_\theta(\hat{M} - M)$. Without loss of generality, consider $p = 2$. Then, the average worth M can be written as

$$M = \begin{cases} \theta_1, & \text{if } \ X_1 > c^{-1}(X_2+1)\,, \\ \theta_2, & \text{if } \ X_1 < c X_2 - 1\,, \\ \frac{1}{2}(\theta_1+\theta_2), & \text{if } \ cX_2 - 1 \leq X_1 \leq c^{-1}(X_2+1)\ , \end{cases} \tag{3.2}$$

and hence the natural estimator of M is

$$\hat{M}_1 = \begin{cases} \delta(X_1), & \text{if } \ X_1 > c^{-1}(X_2+1)\,, \\ \delta(X_2), & \text{if } \ X_1 < c X_2 - 1\,, \\ \frac{1}{2}\big(\delta(X_1)+\delta(X_2)\big) & \text{if } \ cX_2 - 1 \leq X_1 \leq c^{-1}(X_2+1)\ , \end{cases} \tag{3.3}$$

where δ is as in (2.2).

Theorem 3.1. *The natural estimator $\hat{M}_1$ of M is positively biased under Stein loss function.*

Proof: Without loss of generality consider $p=2$. From (3.2) and (3.3), it follows

$$\begin{aligned} B_\theta(\hat{M}_1, M) &= E_\theta(\hat{M}_1 - M) \\ &= E_\theta\big(\delta(X_1)-\theta_1\big)\, I\big(X_1 > c^{-1}(X_2+1)\big) + E_\theta\big(\delta(X_2)-\theta_2\big)\, I\big(X_1 < cX_2 - 1\big) \\ &\quad + 0.5\, E_\theta\big(\delta(X_1)-\theta_1+\delta(X_2)-\theta_2\big)\, I\big(cX_2 - 1 \le X_1 \le c^{-1}(X_2+1)\big)\ . \end{aligned}$$

Since $\delta(X_i)$ is unbiased estimator for θ_i, $i=1,2$, and

$$I\big(cX_2 - 1 \le X_1 \le c^{-1}(X_2+1)\big) = 1 - I\big(X_1 > c^{-1}(X_2+1) - I(X_1 < cX_2 - 1)\big)\ .$$

Then

$$\begin{aligned} B_\theta(\hat{M}_1, M) &= 0.5\, E_\theta\Big[\big(\delta(X_1)-\theta_1\big) - \big(\delta(X_2)-\theta_2\big)\Big]\, I\big(X_1 > c^{-1}(X_2+1)\big) \\ &\quad + 0.5\, E_\theta\Big[\big(\delta(X_2)-\theta_2\big) - \big(\delta(X_1)-\theta_1\big)\Big]\, I\big(X_2 > c^{-1}(X_1+1)\big) \\ &= 0.5\, A(\theta_1,\theta_2) + 0.5\, A(\theta_2,\theta_1)\ , \end{aligned}$$

where

$$A(\theta_1,\theta_2) = E_\theta\big(\delta(X_1)-\theta_1\big)\, I\big(X_1 > c^{-1}(X_2+1)\big) - E_\theta\big(\delta(X_2)-\theta_2\big)\, I\big(X_1 > c^{-1}(X_2+1)\big),$$

and $A(\theta_2,\theta_1)$ follows by interchanging the role of (X_1,θ_1) and (X_2,θ_2) in $A(\theta_1,\theta_2)$. Consider the term $A(\theta_1,\theta_2)$.

$$\begin{aligned} A(\theta_1,\theta_2) &= E_\theta\, \delta(X_1)\, I\big(X_1 > c^{-1}(X_2+1)\big) - \theta_1\, E_\theta\, I\big(X_1 > c^{-1}(X_2+1)\big) \\ &\quad - E_\theta\, \delta(X_2)\, I(X_2 < cX_1 - 1) + \theta_2\, E_\theta\, I(X_2 < cX_1 - 1) \end{aligned}$$

and by using Lemma 2.2 we get

$$\begin{aligned} A(\theta_1,\theta_2) &= E_\theta\, \delta(X_1)\, I\big(X_1 > c^{-1}(X_2+1)\big) - E_\theta\, \delta(X_1)\, I\big(X_1 > c^{-1}(X_2+1)+1\big) \\ &\quad - E_\theta\, \delta(X_2)\, I(X_2 < cX_1 - 1) + E_\theta\, \delta(X_2)\, I(X_2 < cX_1) \\ &= E_\theta\, \delta(X_1)\, I\big(c^{-1}(X_2+1) < X_1 \le c^{-1}(X_2+1)+1\big) \\ &\quad + E_\theta\, \delta(X_2)\, I\big(cX_1 - 1 \le X_2 < cX_1\big) \\ &> 0\ . \end{aligned}$$

This completes the proof. □

Consider next the problem of estimating Q. The estimator $\hat{Q} = (q_1 I_1(X), ..., q_p I_p(X))$ of Q is said to be unbiased under Stein loss function if $E_\theta\, q_i I_i(X) = E_\theta\, \theta_i I_i(X)\ \forall i$, otherwise, it is biased and its bias is

$$B_\theta(\hat{Q}, Q) = E_\theta \sum_{i=1}^{p} (q_i - \theta_i)\, I_i(X)\ .$$

Theorem 3.2. *The natural estimator $\hat{Q}_1$ of Q is positively biased under Stein loss function.*

Proof: Observe that

$$\begin{aligned}
B_\theta(\hat{Q}, Q) &= \sum_{i=1}^{p} E_\theta\, \delta(X_i)\, I\big(X_i+1 \ge cX_{(1)i}\big) - \sum_{i=1}^{p} E_\theta\, \theta_i I\big(X_i+1 \ge cX_{(1)i}\big) \\
&= \sum_{i=1}^{p} E_\theta\, \delta(X_i)\, I\big(X_i+1 \ge cX_{(1)i}\big) - \sum_{i=1}^{p} E_\theta\, \delta(X_i)\, I\big(X_i \ge cX_{(1)i}\big) \\
&\quad \text{(using Lemma 2.2)} \\
&= \sum_{i=1}^{p} E_\theta\, \delta(X_i)\, I_i\big(cX_{(1)i} - 1 \le X_i < cX_{(1)i}\big) \\
&> 0
\end{aligned}$$

since $P\big(cX_{(1)i} - 1 \le X_i < cX_{(1)i}\big) > 0$ for some i. This completes the proof. □

4. THE UMRUE's OF M AND Q

In this section we derive the *UMRUE's* of M and Q using the *UV* method of estimation of Robbins (1988) and the generalization of Misra (1994) of the Lehmann–Scheffe theorem. These estimators are also the *UMVUE's* since the definition of risk-unbiasedness under Stein loss function coincides with the definition of usual unbiasedness. First, consider the *UMRUE* of the average worth M. Let

$$U_i(X) = \frac{I\big(X_i+1 \ge cX_{(1)i}\big)}{\sum\limits_{j=1}^{p} I\big(X_j+1 \ge cX_{(1)j}\big)}\,, \qquad i=1,...,p\,,$$

then $M = \sum_{i=1}^{p} \theta_i U_i(X)$. From Lemma 2.2, we have $E_\theta\, \theta_i U_i(X) = E_\theta\, \delta(X_i)\, U_i(X - e_i)$. Let $\hat{M}_2 = \sum_{i=1}^{p} V_i(X)$ and $V_i(X) = \delta(X_i)\, U_i(X - e_i)$. Then, $E_\theta M = E_\theta \hat{M}_2$ and hence $\hat{M}_2$ is an unbiased estimator of M. Let $Y_1 \ge Y_2 \ge \cdots \ge Y_p$ denote the ordered values of $X_1, ..., X_p$ and $Y = (Y_1, ..., Y_p)$. It is easy to see that

$$\sum_{i=1}^{p} V_i(X) = \sum_{i=1}^{p} V_i^*(Y)\,.$$

Now, since

$$U_1^*(Y) = \frac{I(Y_1+1 \ge cY_2)}{I(Y_1+1 \ge cY_2) + \sum\limits_{j=2}^{p} I(Y_j+1 \ge cY_1)}$$

and

$$U_i^*(Y) = \frac{I(Y_i+1 \ge cY_1)}{I(Y_1+1 \ge cY_2) + I(Y_i+1 \ge cY_1) + \sum\limits_{j=2, j\ne i}^{p} I(Y_j+1 \ge cY_1)}\,, \qquad i=2,...,p\,,$$

we get

$$\begin{aligned} V_1^*(Y) &= \delta(Y_1)\, U_1^*(Y-e_1) \\ &= \frac{\delta(Y_1)\, I(Y_1 \geq cY_2)}{I(Y_1 \geq cY_2) + \sum_{j=2}^{p} I\big(Y_j + 1 \geq c\max(Y_1-1, Y_2)\big)} \\ &= \frac{\delta(Y_1)}{1 + I(Y_1 = Y_2) \sum_{j=2}^{p} I\big(Y_j + 1 \geq cY_1\big) + I(Y_1 > Y_2) \sum_{j=2}^{p} I\big(Y_j + 1 \geq c(Y_1 - 1)\big)} \end{aligned}$$

and

$$\begin{aligned} V_i^*(Y) &= \delta(Y_i)\, U_i^*(Y-e_i) \\ &= \frac{\delta(Y_i)\, I(Y_i \geq cY_1)}{1 + I(Y_i \geq cY_1) + \sum_{j=2, j\neq i}^{p} I(Y_j + 1 \geq cY_1)}\ , \qquad i = 2, ..., p\ . \end{aligned}$$

To find an explicit form of $V^*(Y)$, we need the following definitions. Let

$$S_0 = \text{empty set}\ , \qquad S_i = \big\{Y_j,\ j = 1, ..., p : Y_j = Y_{1+m_i}\big\}\ ,$$

and $m_i = \sum_{j=1}^{i-1} \#(S_j)$ where $i = 1, ..., r$, and $1 \leq r \leq p$. Note that $\sum_{i=1}^{r} \#(S_i) = p$ and the subsets $\{S_i\}_{i=1}^{r}$ represent a partition of the set of variables $Y = (Y_1, ..., Y_p)$. Let $W_1, ..., W_{r+1}$ be random variables such that $W_i \in S_i$, $i = 1, ..., r$ and $W_{r+1} = -1$. It is obvious that $W_{i+1} < W_i$ and $W_i = Y_{1+m_i} = Y_{2+m_i} = \cdots = Y_{m_{i+1}}$ for $i = 1, ..., r$. Define the following partition of χ

$$\chi = \left(\bigcup_{l=1}^{r-2} \bigcup_{k=l}^{l+1} \chi_{1,l,k} \right) \bigcup \chi_{1,r-1,r-1} \bigcup \left(\bigcup_{l=1}^{r-1} \bigcup_{k=l-1}^{l} \chi_{2,l,k} \right) \bigcup \chi_{2,r,r-1}$$

where

$$\chi_{1,l,k} = \Big\{ X \in \chi :\ W_{l+1} < cW_1 \leq W_{l+1} + 1,\ W_{k+2} + 1 < c(W_1 - 1) \leq W_{k+1} + 1 \Big\}$$
$$\text{for} \quad l = 1, ..., r-2;\ k = l, l+1 \quad \text{and} \quad l = r-1;\ k = l\ ,$$

and

$$\chi_{2,l,k} = \Big\{ X \in \chi :\ W_{l+1} + 1 < cW_1 \leq W_l\,,\ W_{k+2} + 1 < c(W_1 - 1) \leq W_{k+1} + 1 \Big\}$$
$$\text{for} \quad l = 1, ..., r-1;\ k = l-1, l \quad \text{and} \quad l = r;\ k = l-1\ .$$

Note that if $W_l = W_{l+1} + 1$ for some $l = 1, ..., r$, then $\chi_{2,l,k} =$ empty set for all $k = l-1, l$.

Case I: When $X \in \chi_{1,l,k}$.

In this case we have

$$\begin{aligned} &I\big(W_j \geq cW_1\big) = 1 \qquad \text{for}\ \ j = 1, ..., l\,, \\ &I\big(W_j + 1 \geq cW_1\big) = 1 \qquad \text{for}\ \ j = 1, ..., l+1\,, \\ &I\big(W_j + 1 \geq c(W_1 - 1)\big) = 1 \qquad \text{for}\ \ j = 1, ..., k+1\ , \end{aligned}$$

then

$$
\begin{aligned}
&I\big(Y_j \geq cY_1\big) = 1 \qquad \text{for} \;\; j = 1, ..., m_{l+1}\,, \\
&I\big(Y_j + 1 \geq cY_1\big) = 1 \qquad \text{for} \;\; j = 1, ..., m_{l+2}\,, \\
&I\big(Y_j + 1 \geq c(Y_1 - 1)\big) = 1 \qquad \text{for} \;\; j = 1, ..., m_{k+2}\,.
\end{aligned}
$$

So that

$$V_1^*(Y) \;=\; \frac{\delta(Y_1)}{m_{l+2}\, I(Y_1 = Y_2) + m_{k+2}\, I(Y_1 > Y_2)}$$

and

$$V_i^*(Y) = \frac{\delta(Y_i)}{m_{l+2}}\,, \qquad i = 2, ..., m_{l+1}\,.$$

Case II: When $X \in \chi_{2,l,k}$.

Similar to the Case I, we obtain

$$
\begin{aligned}
&I\big(W_j \geq cW_1\big) = 1 \qquad \text{for} \;\; j = 1, ..., l\,, \\
&I\big(W_j + 1 \geq cW_1\big) = 1 \qquad \text{for} \;\; j = 1, ..., l\,, \\
&I\big(W_j + 1 \geq c(W_1 - 1)\big) = 1 \qquad \text{for} \;\; j = 1, ..., k+1\,,
\end{aligned}
$$

then

$$V_1^*(Y) \;=\; \frac{\delta(Y_1)}{m_{l+1}\, I(Y_1 = Y_2) + m_{k+2}\, I(Y_1 > Y_2)}$$

and

$$V_i^*(Y) = \frac{\delta(Y_i)}{m_{l+1}}\,, \qquad i = 2, ..., m_{l+1}\,.$$

Since $(X_1, ..., X_p)$ is sufficient and complete statistic for $(\theta_1, ..., \theta_p)$, the following theorem is now established.

Theorem 4.1. *The UMRUE of M is $\hat{M}_2 = \sum_{i=1}^{m_{l+1}} V_i^*(Y)$ where*

$$V_1^*(Y) \;=\; \frac{\delta(Y_1)}{m_u\, I(Y_1 = Y_2) + m_{k+2}\, I(Y_1 > Y_2)}\,,$$

$$V_i^*(Y) = \frac{\delta(Y_i)}{m_u}\,, \qquad i = 2, 3, ..., m_{l+1}\,,$$

and

$$u = \begin{cases} l+2, & \text{if } X \in \chi_{1,l,k}\,, \\ l+1, & \text{if } X \in \chi_{2,l,k}\,. \end{cases}$$

Consider next the *UMRUE* of the simultaneous estimation Q. Observe that

$$\begin{aligned}
E_\theta Q &= E_\theta \sum_{i=1}^{p} \theta_i \, I\big(X_i + 1 \geq c X_{(1)i}\big) \\
&= \sum_{i=1}^{p} E_\theta \, \theta_i \, I\big(X_i + 1 \geq c X_{(1)i}\big) \\
&= \sum_{i=1}^{p} E_\theta \, \delta(X_i) \, I\big(X_i \geq c X_{(1)i}\big) \qquad \text{(using Lemma 2.2)} \\
&= E_\theta \sum_{i=1}^{p} \delta(X_i) \, I\big(X_i \geq c X_{(1)i}\big) \\
&= E_\theta \hat{Q}_2 \quad (\text{say})\,.
\end{aligned}$$

Hence the following theorem.

Theorem 4.2. *The UMRUE of Q is $\hat{Q}_2$ whose the i-th component equals to $\delta(X_i)\, I(X_i \geq c X_{(1)i})$.*

5. INADMISSIBILITY OF THE NATURAL ESTIMATOR OF Q

In this section, we prove the inadmissibility of the natural estimator Q_1 of the simultaneous estimation Q using the technique of Stein (1973). The basic idea of Stein is to find an unbiased estimator $\Delta(X)$ of the risk difference $\Delta(\theta) = R(\theta, \delta + \psi) - R(\theta, \delta)$ and then finding a function ψ such that $\Delta(X) \leq 0 \ \forall x$ and $\Delta(x) < 0$ for some x. This technique has been used extensively in the simultaneous estimation problem when no selection involved (see for example Hudson (1978), Hwang (1982) and Chou (1991)). Consider the following rival estimator of $\hat{Q}_1$,

$$\begin{aligned}
\hat{Q}_3 &= \hat{Q}_1 + \big(\phi_1(X)\, I_1(X), \, ..., \, \phi_p(X)\, I_p(X)\big) \\
&= \Big(\big(\delta(X_1) + \phi_1(X)\big)\, I_1(X), \, ..., \, \big(\delta(X_p) + \phi_p(X)\big)\, I_p(X)\Big)\,,
\end{aligned} \tag{5.1}$$

where ϕ_i is any real-valued functions satisfying the conditions of Lemma 2.1. First, we find an unbiased estimator of the risk difference of estimators $\hat{Q}_3$ and $\hat{Q}_1$.

An unbiased estimator of the risk difference of $\hat{Q}_3$ and $\hat{Q}_1$ is

$$\begin{aligned}\Delta(\theta) &= R(Q,\hat{Q}_3) - R(Q,\hat{Q}_1)\\ &= E_\theta \sum_{i=1}^{p}\left(\frac{\delta(X_i)+\phi_i(X)}{\theta_i} - \log\left(\frac{\delta(X_i)+\phi_i(X)}{\theta_i}\right) - 1\right) I_i(X)\, I(X_i \geq t+1)\\ &\quad - E_\theta \sum_{i=1}^{p}\left(\frac{\delta(X_i)}{\theta_i} - \log\left(\frac{\delta(X_i)}{\theta_i}\right) - 1\right) I_i(X)\, I(X_i \geq t+1)\\ &= E_\theta \sum_{i=1}^{p}\left(\frac{\phi_i(X)}{\theta_i} - \log\left(1+\frac{\phi_i(X)}{\delta(X_i)}\right)\right) I_i(X)\, I(X_i \geq t+1)\,.\end{aligned}$$

Applying Lemma 2.1, we get

$$\Delta(\theta) = E_\theta \sum_{i=1}^{p}\left(\frac{\phi_i(X+e_i)}{\delta(X_i+1)}\, I_i(X+e_i)\, I(X_i \geq t) - \log\left(1+\frac{\phi_i(X)}{\delta(X_i)}\right) I_i(X)\, I(X_i \geq t+1)\right).$$

So that the following lemma is now established.

Lemma 5.1. *An unbiased estimator of the risk difference of the estimators $\hat{Q}_3$ and $\hat{Q}_1$ is given by*

(5.2)
$$D(X) = \sum_{i=1}^{p}\left(\frac{\phi_i(X+e_i)}{\delta(X_i+1)}\, I_i(X+e_i)\, I(X_i \geq t) - \log\left(1+\frac{\phi_i(X)}{\delta(X_i)}\right) I_i(X)\, I(X_i \geq t+1)\right).$$

Following Peng (1975) and Hudson (1978), we introduce the following notations. Let

$$\begin{aligned}&l = X_{(1)} \quad \text{(the largest observation)}\,,\\ &m = X_{(p)} \quad \text{(the smallest observation)}\,,\\ &N_i = \#\{j : X_j = i\}, \quad i = m, \ldots, l \ \text{ and } \ j = 1, \ldots, p\,,\\ &N = (N_m, \ldots, N_l)\,.\end{aligned}$$

If $X_i = r$, let

$$\begin{aligned}&\psi_i(X) = \zeta_r(N)\,,\\ &\delta(X_i) = \begin{cases} r, & r \geq t+1\,,\\ 0, & r < t+1\,,\end{cases}\\ &I(X_i \geq t+1) = I(r \geq t+1)\,,\\ &I_i(X) = I(r+1 \geq c\,l) = J_r \quad \text{(say)}\,.\end{aligned}$$

Then, we have

$$\begin{aligned}
\phi_i(X+e_i) &= \zeta_{r+1}(N-e_r+e_{r+1})\,,\\
\delta(X_i+1) &= \begin{cases} r+1, & r\ge t\,,\\ 0, & r<t\,,\end{cases}\\
I(X_i\ge t) &= I(r\ge t)\,,\\
I_i(X+e_i) &= J_{r+1}\,.
\end{aligned}$$

Define

$$\begin{aligned}
\lfloor x\rfloor_{\text{ge}} &= \text{smallest integer greater than or equal to } x\,,\\
[a]^+ &= \max(0,a)\,.
\end{aligned}$$

Now, using the above notations, the unbiased estimator (5.2) becomes

(5.3)
$$D(X)=\sum_{r=m}^{l} N_r\left(\frac{\zeta_{r+1}(N+e_{r+1}-e_r)}{r+1}\,I(r\ge t)\,J_{r+1}-\log\left(1+\frac{\zeta_r(N)}{r}\right)I(r\ge t+1)\,J_r\right).$$

Next, we consider a solution of a general difference inequality that will be useful for constructing a class of improved estimators of the natural estimator $\hat{Q}_1$. Consider the following general difference inequality

(5.4)
$$\eta(X)=\sum_{r=m}^{l} N_r\left(\frac{\zeta_{r+1}(N+e_{r+1}-e_r)}{r+1}\,I(r\ge t)\,J_{r+1}-\left(\frac{\zeta_r(N)}{r}-\frac{\zeta_r^2(N)}{r^2}\right)I(r\ge t+1)\,J_r\right)$$
$$\le 0\,.$$

In the following theorem, we solve the general difference inequality (5.4) borrowing some ideas from Dey and Chung (1991).

Theorem 5.1. *Consider the difference inequality (5.4). The function*

$$\zeta_r(N)=-\frac{b\,r^2}{d+w}$$

represents a solution of the inequality where

(1) $w=\sum_{s=\lfloor cl-1\rfloor}^{l} N_s s^2\,,$

(2) $d\ge 9\sum_{s=\lfloor cl-1\rfloor}^{l} N_s\,,$

(3) $0<b\le\left[\sum_{r=\lfloor cl-1\rfloor}^{l} N_r\,I(r\ge t+1)-\frac{47}{18}\right]^+.$

Proof: It is clear that

$$\begin{aligned}
\eta(X) &= \sum_{r=m}^{l} N_r\left(-\frac{b(r+1)}{d+w+2r+1}\, I(r\geq t)\, J_{r+1} + \left(\frac{b\,r}{d+w}+\frac{b^2 r^2}{(d+w)^2}\right) I(r\geq t+1)\, J_r\right)\\
&\leq b \sum_{r=\lfloor cl-1\rfloor_{\rm ge}}^{l} N_r\left(\frac{r}{d+w} - \frac{(r+1)}{d+w+2r+1} + \frac{b\,r^2}{(d+w)^2}\right) I(r\geq t+1)\\
&\quad \big(\text{since } J_{r+1}\, I(r\geq t) \,\geq\, J_r\, I(r\geq t+1)\big)\\
&\leq b \sum_{r=\lfloor cl-1\rfloor_{\rm ge}}^{l} \frac{N_r(2r^2+r-d-w)}{(d+w)\,(d+w+2r+1)}\, I(r\geq 2) + \frac{b^2 w}{(d+w)^2}\\
&\leq \frac{2\,b\,w}{(d+w)^2} + \frac{3b \sum\limits_{r=\lfloor cl-1\rfloor_{\rm ge}}^{l} N_r\, r}{(d+w)^2} + \frac{b \sum\limits_{r=\lfloor cl-1\rfloor_{\rm ge}}^{l} N_r}{(d+w)^2} - \frac{b \sum\limits_{r=\lfloor cl-1\rfloor_{\rm ge}}^{l} N_r\, I(r\geq t+1)}{(d+w)} + \frac{b^2 w}{(d+w)^2}\ .
\end{aligned}$$

Since

$$\frac{w}{d+w} \leq 1\,,$$

$$\frac{\sum\limits_{r=\lfloor cl-1\rfloor_{\rm ge}}^{l} N_r}{d+w} \leq \frac{\sum\limits_{r=\lfloor cl-1\rfloor_{\rm ge}}^{l} N_r}{d} \leq \frac{1}{9} \qquad \text{(from assumption (2))}$$

and

$$\frac{\sum\limits_{r=\lfloor cl-1\rfloor_{\rm ge}}^{l} N_r\, r}{d+w} \leq \frac{\sqrt{\sum\limits_{r=\lfloor cl-1\rfloor_{\rm ge}}^{l} N_r}\,\sqrt{\sum\limits_{r=\lfloor cl-1\rfloor_{\rm ge}}^{l} N_r\, r^2}}{d+w} \leq \frac{\sqrt{\sum\limits_{r=\lfloor cl-1\rfloor_{\rm ge}}^{l} N_r}\ \sqrt{w}}{2\sqrt{d}\,\sqrt{w}} \leq \frac{1}{6}$$

it follows that

$$\eta(X) \leq \frac{b\left(47/18 \;-\; \sum\limits_{r=\lfloor cl-1\rfloor_{\rm ge}}^{l} N_r\, I(r\geq t+1) + b\right)}{d+w}\ .$$

Clearly, $\eta(X)\leq 0$ if $b \leq \sum_{r=\lfloor cl-1\rfloor_{\rm ge}}^{l} N_r\, I(r\geq t+1) - 47/18$. This completes the proof of the theorem. □

Now, we are in a position to construct classes of dominating estimators of $\hat{Q}_1$ by solving the inequality $D(x)\leq 0\ \forall x\in\chi$ using Theorem 5.1, where D is as in (5.3). In the following theorem, we construct a class of improved estimators of the natural estimator $\hat{Q}_1$.

Theorem 5.2. *Consider the rival estimator $\hat{Q}_3$ given in (5.1) where*

$$\phi_i(X) = \zeta_r(N) = -\frac{b\,r^2}{d + \sum\limits_{s=\lfloor cl-1\rfloor_{\rm ge}}^{l} N_s\, s^2}$$

if $X_i = r$. Assume further,

$$(\mathbf{1}) \quad d \geq 9 \sum_{s=\lfloor cl-1 \rfloor_{\text{ge}}}^{l} N_s \,,$$

$$(\mathbf{2}) \quad 0 < b \leq \left[\min\left(\sqrt{d}\,, \sum_{s=\lfloor cl-1 \rfloor_{\text{ge}}}^{l} N_s\, I(s \geq t+1) - 47/18\right)\right]^{+} .$$

Then, $\hat{Q}_3$ dominates $\hat{Q}_1$ in terms of risk where $\hat{Q}_1$ is as in (2.5).

Proof: Clearly, the ζ_r's satisfy the conditions of Lemma 2.1. It is easily seen that

$$\left|\frac{\zeta_r(N)}{r}\right| = \left|\frac{-b\,r}{d + \sum_{s=\lfloor cm-1 \rfloor_{\text{ge}}}^{l} N_s\, s^2}\right| \leq \frac{b\,r}{d + r^2} \leq \frac{b\,r}{2\,r\,\sqrt{d}} \leq \frac{1}{2}\,,$$

since $b \leq \sqrt{d}$. Then using Lemma 2.3 in (5.3) gives us

$$\begin{aligned} D(X) &\leq \sum_{r=m}^{l} N_r \left(\frac{\zeta_{r+1}(N+e_{r+1}-e_r)}{r+1}\, I(r \geq t)\, J_{r+1} - \left(\frac{\zeta_r(N)}{r} - \frac{\zeta_r^2(N)}{r^2}\right) I(r \geq t+1)\, J_r\right) \\ &= \eta(X)\,. \end{aligned}$$

Applying Theorem 5.1 in the above inequality completes the proof the theorem. □

The class of estimators in Theorem 5.2 dominates the natural estimator $\hat{Q}_1$, so that the natural estimator (2.5) is inadmissible.

6. SIMULATION RESULTS

In this section, we compute the biases and risks of the estimators $\hat{M}_1$, $\hat{M}_2$, $\hat{Q}_1$, $\hat{Q}_2$ and $\hat{Q}_3$ using the Monte Carlo simulation technique. Also, we compute the percentages of the risk improvement of the estimator $\hat{Q}_3$ over the estimators $\hat{Q}_i$, $i = 1, 2$. We follow the simulation procedure used by Tsue and Press (1982). First the value of p is chosen and then a set of $\{\theta_1, ..., \theta_p\}$ of parameter values are chosen at random within the range (c, d). In the second step, an observation X_i is randomly chosen from zero-truncated Poisson distribution $TP(\theta_i)$, $1 \leq i \leq p$. In step 3, the selection rule R, defined in (2.3), is used to select the subset. To estimate the parameters associated with the selected populations, we compute the biases of the estimators $\hat{M}_1$, $\hat{Q}_1$ and $\hat{Q}_3$ and the risks of the estimators

$\hat{M}_1$, $\hat{M}_2$, $\hat{Q}_1$, $\hat{Q}_2$ and $\hat{Q}_3$. The above procedure is repeated 4 000 times and the averages of the biases and risks are calculated. Then the percentage improvements of the estimator $\hat{Q}_3$ over $\hat{Q}_i$, $i=1,2$, namely

$$RPI(\hat{Q}_i,\hat{Q}_3) = \frac{R(Q,\hat{Q}_i)-R(Q,\hat{Q}_3)}{R(Q,\hat{Q}_i)} \times 100 , \qquad i=1,2 ,$$

are obtained. The above procedure is repeated a number of times with different sets of parameters in (c,d) and then the percentages of risk improvement and averages of biases are calculated and presented in Table 1 and Table 2, respectively. The simulation is carried out using Matlab version 7.4 and considering the values $d=9\sum_{s=\lfloor cl-1\rfloor_{\rm ge}}^{l} N_s$ and $b=\left[\min\left(\sqrt{d}\,,\,\sum_{s=\lfloor cl-1\rfloor_{\rm ge}}^{l} N_s\, I(s\geq 2)-47/18\right)\right]^+$ with the number of populations $p=3,5,7,10$.

We observe the following facts from the simulation results. From Table 1, the risks of the estimators decrease as the range of θ_i's increases. For the percentages of the risk improvement, the two percentages increase for small values of θ_i's while decrease for large values and the highest values appear when all the θ_i's are in the interval $(1.5,3)$. From Table 2, clearly, the biases of M_1, Q_1 and Q_3 are positive and gradually increasing as the range of θ_i's increases. Also, Q_1 has bias less than Q_3 and the bias of both of theme gradually increase as p increases.

Table 1: The risks of the estimators M_1 and M_2 and the percentage improvement of Q_3 over Q_1 and Q_2.

		Range of θ_i's				
		$(0.0,0.5)$	$(0.5,1.5)$	$(1.5,3.0)$	$(3.0,6.0)$	$(6.0,15.0)$
$p=3$	M_1	1.23	0.60	0.42	0.34	0.32
	M_2	3.44	0.61	0.40	0.33	0.32
	$RPI(\hat{Q}_1,\hat{Q}_3)$	0.28	2.65	4.46	3.42	1.90
	$RPI(\hat{Q}_2,\hat{Q}_3)$	0.27	2.08	4.28	3.15	1.83
$p=5$	M_1	1.02	0.60	0.42	0.34	0.32
	M_2	2.81	0.60	0.37	0.32	0.31
	$RPI(\hat{Q}_1,\hat{Q}_3)$	1.07	9.72	16.89	12.34	6.43
	$RPI(\hat{Q}_2,\hat{Q}_3)$	1.07	8.97	16.53	12.05	6.30
$p=7$	M_1	0.93	0.57	0.42	0.34	0.32
	M_2	2.68	0.53	0.36	0.31	0.31
	$RPI(\hat{Q}_1,\hat{Q}_3)$	2.03	15.85	22.77	16.95	8.29
	$RPI(\hat{Q}_2,\hat{Q}_3)$	2.02	14.50	22.27	16.68	8.15
$p=10$	M_1	0.88	0.53	0.37	0.32	0.31
	M_2	3.04	0.62	0.36	0.31	0.30
	$RPI(\hat{Q}_1,\hat{Q}_3)$	2.79	19.33	27.36	20.75	10.08
	$RPI(\hat{Q}_2,\hat{Q}_3)$	2.79	19.32	27.32	20.52	10.07

Table 2: The biases of the estimators M_1, Q_1 and Q_3.

		Range of θ_i's				
		$(0.0, 0.5)$	$(0.5, 1.5)$	$(1.5, 3.0)$	$(3.0, 6.0)$	$(6.0, 15.0)$
$p = 3$	M_1	0.44	1.42	2.56	4.67	9.95
	Q_1	1.31	4.17	7.56	13.84	29.68
	Q_3	1.30	4.08	7.31	13.50	29.31
$p = 5$	M_1	0.43	1.33	2.59	4.80	10.62
	Q_1	2.15	6.54	12.70	23.61	52.68
	Q_3	2.12	5.96	11.09	21.52	50.39
$p = 7$	M_1	0.44	1.43	2.60	4.64	10.85
	Q_1	3.10	9.78	17.85	32.12	75.44
	Q_3	3.00	8.39	14.80	28.24	71.21
$p = 10$	M_1	0.41	1.31	2.55	4.42	10.56
	Q_1	4.07	13.11	25.48	44.17	105.55
	Q_3	3.89	10.79	20.17	37.52	98.31

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