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Some Robust Bayesian Estimators in the Burr XII Distribution under Reflected Gamma Loss Function

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Abstract:

- In this paper, we derive two robust Bayesian estimators, namely the conditional gamma-minimax and posterior regret gamma-minimax estimators, for the shape parameter of the Burr XII distribution. These estimators are developed using the reflected gamma loss function and three classes of prior distributions.
  To evaluate the effectiveness of these estimators, a simulation study is conducted. The study also explores the impact of varying levels of censoring on the inferences. Furthermore, a real data example is provided to illustrate the paratical application of the obtained results.

Keywords:

- *Burr XII Distribution; Censored Data; Reflected Gamma Loss Function; Robust Bayes Estimator.*

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- 62C10, 62F15.

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1. INTRODUCTION

Within the Bayesian framework, it is typically assumed that non-sample information is accessible prior to observing a random sample. This information is utilized to define the prior distribution. However, in practical scenarios, such information is often unavailable. Moreover, any chosen prior distribution can only serve as an approximation of the true prior distribution. Consequently, selecting an appropriate prior distribution stands as one of the major challenges in the Bayesian framework. This challenge becomes particularly pronounced when one attempt to solve a statistical problem using a Bayesian approach, but there is disagreement regarding the selection of the same prior distribution or the prior parameters (hyperparameters). An often employed strategy to tackle the issue of uncertainty in prior distributions within Bayesian statistics is to select a set of prior distributions and compute a range of Bayesian estimators based on these choices. This approach is known as the Robust Bayesian approach in the Bayesian literature (Hu and Xiao, 2023). The purpose of this approach is to examine the robustness of the results obtained through a Bayesian analysis when faced with uncertainty regarding the specific details of the analysis. In a robust Bayesian approach, a standard Bayesian analysis is employed for all conceivable combinations of prior distributions and likelihood functions chosen from the designated classes of priors. The robust Bayesian approach involves pairing a class of the prior distributions with a class of likelihood functions through Bayes' theorem to obtain a class of posterior distributions. In this approach, changing the prior within a given class of priors has an impact various Bayesian quantities of interest such as posterior risk, Bayesian risk, and others. When applying the robust Bayesian analysis, in practice, obtain a range of Bayes estimators. However, despite having this range of Bayesian estimators, it remains unclear which specific value within the range is the most suitable choice. Therefore, the question arises as to how to obtain an optimal estimator based on the available range of Bayesian estimator values.To this end, there are some methods such as Gamma-minimax (GM), Conditional Gamma Minimax (CGM) and Posterior Regret Gamma Minimax (PRGM) (Hu and Xiao, 2023).

When calculating Bayesian quantities such as posterior risk, Bayesian risk, and Bayes estimator, the choice of the prior distributions is not the only factor to consider. The selection of loss function also plays a crucial role. Among the simplest and most commonly used loss functions is the squared error loss function. This function is convex and symmetric, penalizing overestimation and underestimation equally.

However, there are situations where the cost of overestimation may differ from that of underestimation. In such cases, asymmetric loss functions such as Linex and Entropy are more appropriate choices. It is important to note that none of these mentioned loss functions are bounded. Nevertheless, certain scenarios may be necessitate the use of bounded loss functions, such as inverted gamma, inverted normal, or other bounded alternatives. These bounded loss functions are valuable when the nature of the problem under study requires them. In this paper, we specifically consider the Reflected Gamma(RG) loss function denoted as $L(\lambda, d)$ which is expressed as follows:

$$L(\lambda, d) = k \left\{ 1 - \left(\frac{\lambda}{d}\right)^{\gamma} e^{-\gamma\left(\frac{\lambda}{d} - 1\right)} \right\}; \qquad \lambda > 0, d > 0 \tag{1.1}$$

where k represents the maximum loss, which should be a positive value. The shape parameter γ is also positive and influences the penalty for deviations between the true scale parameter λ and the estimator d chosen by the statistician. The RG loss function is designed to balance the trade-off between underestimation and overestimation of the scale parameter λ. The loss function (1.1) is referred to as Reflected Gamma(RG) since it is derived by inverting the gamma density function. This function is a bounded function with lower and upper bounds of 0 and k, respectively. It exhibits distinct properties depending on the range of the input values. On the interval $(0, 1)$, the RG loss function is decreasing, while on the interval $(1, \infty)$, it is increasing. One important characteristic of the RG loss function is its asymmetry and non-convexity. Unlike symmetric loss functions, the RG loss function penalizes underestimation more severely than overestimation. This feature captures the preference for avoiding underestimation in certain applications.

Let λ represent the parameter of interest, and let d denote an estimate of this parameter. We denote the class of estimators as Δ, indicating that d belongs the set of the estimators Δ. Additionally, we assumed that PR(d,λ) represents the Posterior Risk(PR) function associated with the estimator d.

Definition 1.1. The estimator $\hat{\lambda}_{CGM} \in \Delta$ is defined as the CGM estimator if the following condition holds:

$$\inf_{d\in\Delta} \sup_{\pi\in\Gamma} PR(d, \lambda) = \sup_{\pi\in\Gamma} PR(\hat{\lambda}_{CGM}, \lambda) \tag{1.2}$$

Definition 1.2. The estimator $\hat{\lambda}_{PRGM} \in \Delta$ referred to as the PRGM estimator if the following condition is satisfied:

$$\inf_{d\in\Delta} \sup_{\pi\in\Gamma} \rho(d, \lambda) = \sup_{\pi\in\Gamma} \rho(\hat{\lambda}_{PRGM}, \lambda) \tag{1.3}$$

where $\rho(d, \lambda) = PR(d, \lambda) - PR(\hat{\lambda}, \lambda)$ represents the posterior regret, and $\hat{\lambda}$ is the Bayesian estimator of λ.

In this paper, we focus on deriving the CGM and PRGM estimators. To achieve this, we present a formal definition of these estimators, along with a detailed discussion of their specific characteristics and properties.

In the field of robust Bayesian methods, several researchers have made significant contributions. Betrò and Ruggeri (1992) investigated the CGM estimator under convex loss functions. Insua et al. (1995) derived the PRGM estimator using squared error loss function. Additionally, Boratyńska (2002) obtained the PRGM estimation for the mean of the normal distribution under the Linex loss function. Boratynska (2005) focused on the GM prediction for a random variable from distributions within the quadratic exponential family, employing the squared error loss function. Other researchers have also contributed to robust Bayesian prediction. Some notable examples are included Boratyńska (2006), Jozani and Parsian (2008), Kamińska and Porosiński (2009), Kiapour and Nematollahi (2011), Karimnezhad et al. (2014), Karimnezhad and Moradi (2016), Boratyńska (2017), and Hu and Xiao (2023).

The rest of this paper is organized as follows. Section 2 computes the posterior distribution and subsequently derive the Bayesian estimator using the RG loss function. Sections

3 and 4 focus on obtaining the CGM and PRGM estimators for the parameter λ, considering the RG loss function and three different classes of prior distributions, respectively. Section 5 presents a simulation study, while Section 6 provides an analysis of real data. Finally, concluding remarks are discussed in Section 7.

2. POSTERIOR DISTRIBUTION

In recent years, the Burr XII distribution has gained significant attention and has been applied in various fields, including survival analysis and insurance statistics (Singh and Maddala, 1976). This distribution is characterized by the following survival and density functions:

$$\begin{aligned} S(t|\alpha,\lambda) &= (1+t^{\alpha})^{-\lambda}, \qquad t>0 \\ f(t|\alpha,\lambda) &= \alpha\lambda t^{\alpha-1}(1+t^{\alpha})^{-(\lambda+1)}, \qquad t>0, \end{aligned} \tag{2.1}$$

respectively, where α and λ are positive shape parameters. For each lifetime T_i, consider an indicator function $\delta_i = 1$, if T_i is observed lifetime and $\delta_i = 0$ if T_i is right-censored lifetime at censoring time C. Therefore, the data set is $\{(T_i^*, \delta_i); i = 1, 2, \cdots, n\}$, where $\mathrm{T_i^*} = \min\{\mathrm{T_i}, \mathrm{C}\}$ and $\delta_i = I\{T_i \le C\}$. The likelihood function for $\{(T_i^*, \delta_i); i = 1, 2, \cdots, n\}$ can be expressed as follows:

$$L(\alpha,\lambda|\boldsymbol{t}^*) = \prod_{i=1}^{n} \{f(t_i^*|\alpha,\lambda)\}^{\delta_i} \{S(t_i^*|\alpha,\lambda)\}^{1-\delta_i}. \tag{2.2}$$

where $f(t_i^*|\alpha,\lambda)$ represents the density function of the Burr XII distribution with shape parameters α and λ, and $S(t_i^*|\alpha,\lambda)$ represents the survival function of the Burr XII distribution with the same shape parameters. In this likelihood function, the term $\{f(t_i^*|\alpha,\lambda)\}^{\delta_i}$ corresponds to the contribution of observed lifetimes to the Likelihood, while $\{S(t_i^*|\alpha,\lambda)\}^{1-\delta_i}$ represents the contribution of right-censored lifetimes. By substituting the density function f and survival function S with their respective expressions from (2.1), the likelihood function can be rewritten as:

$$L(\boldsymbol{t}^*|\alpha,\lambda) = \alpha^{\sum_{i=1}^n \delta_i} \lambda^{\sum_{i=1}^n \delta_i} \Big\{ \prod_{i=1}^{n} \big(\frac{{t_i^*}^{\alpha-1}}{1+{t_i^*}^{\alpha}}\big)^{\delta_i} \Big\} e^{-\lambda W}, \tag{2.3}$$

where $\boldsymbol{t}^* = (t_1^*, t_2^* \cdots t_n^*)'$ is the vector of observed and censored lifetimes, and $W = \sum_{i=1}^{n} \log(1+{t_i^*}^{\alpha})$. We assume that the shape parameter α is known. Additionally, the parameter λ is assigned a Gamma prior distribution with shape parameter $a > 0$ and rate parameter $b > 0$. This prior distribution for λ is specified as:

$$\pi(\lambda|a,b) \propto \lambda^{a-1} e^{-b\lambda}, \qquad \lambda > 0, a > 0, b > 0. \tag{2.4}$$

Using Eqs. (2.3) and (2.4), the posterior distribution of parameter λ can be obtained as follows:

$$\pi(\lambda|\boldsymbol{t}^*, W, \alpha, a, b) \propto L(\boldsymbol{t}^*|\alpha,\lambda)\pi(\lambda|a,b) \propto \lambda^{a+\sum_{i=1}^n \delta_i - 1} e^{-(b+W)\lambda}, \ \lambda > 0, a > 0, b > 0, \tag{2.5}$$

which is a gamma distribution with shape and rate parameters $a + \sum_{i=1}^{n} \delta_i$ and $b + W$, respectively. The Posterior Risk (PR) of the estimator d under the RG loss function can be given by

$$
\begin{aligned}
PR(d,\lambda) &= \int_0^\infty L(d,\lambda)\pi(\lambda|\mathbf{t})d\lambda \\
&= \int_0^\infty k\Big\{1-\Big(\frac{\lambda}{d}\Big)^{\gamma} e^{-\gamma(\frac{\lambda}{d}-1)}\Big\}\frac{\lambda^{a+\sum_{i=1}^{n}\delta_i-1}e^{-(b+W)\lambda}(b+W)^{a+\sum_{i=1}^{n}\delta_i}}{\Gamma(a+\sum_{i=1}^{n}\delta_i)}d\lambda \\
&= k - k\frac{\Gamma(a+\sum_{i=1}^{n}\delta_i+\gamma)}{\Gamma(a+\sum_{i=1}^{n}\delta_i)}\frac{(b+W)^{a+\sum_{i=1}^{n}\delta_i}}{\Big(b+W+\dfrac{\gamma}{d}\Big)^{a+\gamma+\sum_{i=1}^{n}\delta_i}}d^{-\gamma}e^{\gamma}.
\end{aligned}
\tag{2.6}
$$

By evaluating this integral, we can obtain the posterior risk of the estimators d under RG loss function, which provides a measure of expected risk associated with the estimator considering the uncertainty in λ and the observed data. Assuming that γ is a positive integer, it can be shown that

$$
\Gamma(a+\sum_{i=1}^{n}\delta_i+\gamma) = \Gamma(a+\sum_{i=1}^{n}\delta_i)\prod_{j=0}^{\gamma-1}(a+\sum_{i=1}^{n}\delta_i+j). \tag{2.7}
$$

Using Eq. (2.7), the Eq. (2.6) can be written as follows:

$$
\mathrm{PR(d,\lambda)} \;=\; k - k\prod_{j=0}^{\gamma-1}\left(a+\sum_{i=1}^{n}\delta_i+j\right)\frac{(b+W)^{a+\sum_{i=1}^{n}\delta_i}}{\Big(b+W+\dfrac{\gamma}{d}\Big)^{a+\gamma+\sum_{i=1}^{n}\delta_i}}d^{-\gamma}e^{\gamma} \tag{2.8}
$$

On differentiating Eq. (2.8) with respect to d, we have

$$
\begin{aligned}
\frac{\partial \mathrm{PR(d,\lambda)}}{\partial d} =& k\gamma e^{\gamma}d^{-\gamma-2}\prod_{j=0}^{\gamma-1}(a+\sum_{i=1}^{n}\delta_i+j)(b+W)^{a+\sum_{i=1}^{n}\delta_i+1} \\
&\times\Big(b+W+\frac{\gamma}{d}\Big)^{-a-\gamma-\sum_{i=1}^{n}\delta_i-1}\Big\{d-\frac{a+\sum_{i=1}^{n}\delta_i}{b+W}\Big\}.
\end{aligned}
\tag{2.9}
$$

This equation represents the derivative of the Posterior Risk (PR) with respect to d, which provides information about the rate of change of the PR as d varies. By analyzing last Eq., we can gain insights into behavior of the PR and identify critical points or values of d that minimize or maximize the PR. By solving the last Eq. with respect to d, the Bayesian estimator of parameter λ under RG loss function is given by

$$
\hat{\lambda}_\pi = \frac{a+\sum_{i=1}^{n}\delta_i}{b+W} \tag{2.10}
$$

which is increasing and decreasing function with respect to a and b, respectively. To obtain the CGM and PRGM estimators for λ, three classes of prior distributions are considered:

$$
\begin{aligned}
\Gamma_a &= \{\mathrm{Gamma(a,b)};\ a=a_0\ ,\ b_1 \le b \le b_2\}, \\
\Gamma_b &= \{\mathrm{Gamma(a,b)};\ b=b_0\ ,\ a_1 \le a \le a_2\}, \\
\Gamma_{a,b} &= \{\mathrm{Gamma(a,b)};\ a_1 \le a \le a_2\ ,\ b_1 \le b \le b_2\},
\end{aligned}
$$

where a_0, a_1, a_2, b_0, b_1, b_2, γ and k are known. It is also assumed that $a_0 \in (a_1, a_2)$ and $b_0 \in (b_1, b_2)$. These classes of prior distributions provide flexibility in modeling the uncertainty in the shape parameters of the Gamma distribution prior to incorporating the observed data. In the $\Gamma_a(\Gamma_b)$ class, there is an agreement between two or more decision makers on the value of the shape (rate) hyperparameter, but there is no agreement on the rate (shape) hyperparameter. While in the third class $\Gamma_{a,b}$, there is no any agreement on the value of any of the shape and rate hyperparameters.

3. CGM ESTIMATION

In this section, we discuss the CGM estimator of λ under the class $\Gamma_{a,b}$ and the RG loss function. Given that Eq. (2.8) is a function of the parameters a, b, and d, for simplicity, we assume it is equal to $PR(d,\lambda) = h(a,b,d)$. Then, the partial derivatives of h with respect to a, b and d are computed as follows:

$$\frac{\partial h(a,b,d)}{\partial a} = \frac{d^2(b+W)+d\gamma}{\gamma} C_1(a,b,d)\Big\{ \log(1+\frac{\gamma}{d(b+W)}) - \sum_{j=0}^{\gamma-1} \frac{1}{a+\sum_{i=1}^n \delta_i + j}\Big\},$$
$$\frac{\partial h(a,b,d)}{\partial b} = dC_1(a,b,d)\left\{ d - \frac{a+\sum_{i=1}^n \delta_i}{b+W}\right\},$$
$$\frac{\partial h(a,b,d)}{\partial d} = (b+W)C_1(a,b,d)\left\{ d - \frac{a+\sum_{i=1}^n \delta_i}{b+W}\right\},$$

where $C_1(a,b,d)$ is

$$C_1(a,b,d) = k\gamma e^{\gamma} d^{-\gamma-2}\left\{\prod_{j=0}^{\gamma-1}(a+\sum_{i=1}^n \delta_i + j)\right\}(b+W)^{a+\sum_{i=1}^n \delta_i} \times \left(b+W+\frac{\gamma}{d}\right)^{-a-\sum_{i=1}^n \delta_i-\gamma-1}$$

The root of the equation $\frac{\partial h(a,b,d)}{\partial a} = 0$ is

$$d_1 = \gamma\Big\{(b+W)\Big(-1+\exp\Big\{\sum_{j=0}^{\gamma-1}\frac{1}{a+\sum_{i=1}^n \delta_i + j}\Big\}\Big)\Big\}^{-1}.$$

Since d_1 is an increasing (decreasing) function with respect to a (b), we have $\inf_{a,b} d_1 = \underline{d}_1$ and $\sup_{a,b} d_1 = \bar{d}_1$, where

$$\underline{d}_1 = \gamma\left\{(b_2+W)\left(-1+\exp\left\{\sum_{j=0}^{\gamma-1}\frac{1}{a_1+\sum_{i=1}^n \delta_i + j}\right\}\right)\right\}^{-1},$$
$$\bar{d}_1 = \gamma\left\{(b_1+W)\left(-1+\exp\left\{\sum_{j=0}^{\gamma-1}\frac{1}{a_2+\sum_{i=1}^n \delta_i + j}\right\}\right)\right\}^{-1}.$$

It can be easily observed that

$$\frac{\partial h(a,b,d)}{\partial a} \begin{cases} > 0, & d < d_1 \\ = 0, & d = d_1 \\ < 0, & d > d_1. \end{cases} \tag{3.1}$$

The root of the Eqs. $\dfrac{\partial h(a,b,d)}{\partial d} = 0$ and $\dfrac{\partial h(a,b,d)}{\partial b} = 0$ is as follows:

$$d = \frac{a + \sum_{i=1}^{n} \delta_i}{b + W} = d_2 \qquad (say). \tag{3.2}$$

It is evident that:

$$\frac{\partial h(a,b,d)}{\partial b}, \frac{\partial h(a,b,d)}{\partial d} \begin{cases} > 0, & d > d_2 \\ = 0, & d = d_2 \\ < 0, & d < d_2. \end{cases} \tag{3.3}$$

Since d_2 is an increasing function of a and a decreasing function of b, we have:

$$\inf_{a,b} \frac{a + \sum_{i=1}^{n} \delta_i}{b + W} = \frac{a_1 + \sum_{i=1}^{n} \delta_i}{b_2 + W} = \underline{d}_2, \tag{3.4}$$

$$\sup_{a,b} \frac{a + \sum_{i=1}^{n} \delta_i}{b + W} = \frac{a_2 + \sum_{i=1}^{n} \delta_i}{b_1 + W} = \bar{d}_2. \tag{3.5}$$

Since for all a and b, d_1 has an upper bound, i.e., $d_1 < d_2$, we conclude that $\underline{d}_1 < \underline{d}_2$ and $\bar{d}_1 < \bar{d}_2$.

Theorem 3.1. *The CGM estimator of λ, denoted as $\hat{\lambda}$, under the class of prior $\Gamma_{a,b}$ and RG loss function does not have a closed form expression. It needs to be obtained numerically by solving the following equality:*

$$\frac{\left(b_1 + W + \frac{\gamma}{\hat{\lambda}}\right)^{a_2 + \sum_{i=1}^{n} \delta_i + \gamma}}{\left(b_2 + W + \frac{\gamma}{\hat{\lambda}}\right)^{a_1 + \sum_{i=1}^{n} \delta_i + \gamma}} = \frac{(b_1 + W)^{a_2 + \sum_{i=1}^{n} \delta_i}}{(b_2 + W)^{a_1 + \sum_{i=1}^{n} \delta_i}} \prod_{j=0}^{\gamma-1} \frac{a_2 + \sum_{i=1}^{n} \delta_i + j}{a_1 + \sum_{i=1}^{n} \delta_i + j}. \tag{3.6}$$

Proof: The CGM estimation of λ can be obtained by considering the following three cases:

Case (I) : $0 < d < \underline{d}_1$.

If $d < \underline{d}_1$, then $h(a,b,d)$ is a increasing (decreasing) function of a (b and d) and therefore

$$\inf_{d < \underline{d}_1} \sup_{\pi \in \Gamma_{a,b}} h(a,b,d) = h(a_2, b_1, \underline{d}_1) = k - k\Big\{ \prod_{j=0}^{\gamma-1} (a_2 + \sum_{i=1}^{n} \delta_i + j) \Big\} \times \frac{(b_1 + W)^{a_2 + \sum_{i=1}^{n} \delta_i}}{(b_1 + W + \frac{\gamma}{\underline{d}_1})^{a_2 + \sum_{i=1}^{n} \delta_i + \gamma}}. \tag{3.7}$$

Case (II) : $d > \bar{d}_2$.

For $d > \bar{d}_2$, $h(a,b,d)$ is a decreasing (an increasing) function of a (b and d) and so

$$\begin{aligned} \inf_{d > \bar{d}_2} \sup_{\pi \in \Gamma_{a,b}} h(a,b,d) &= h(a_1, b_2, \bar{d}_2) \\ &= k - k\Big\{ \prod_{j=0}^{\gamma-1} (a_1 + \sum_{i=1}^{n} \delta_i + j) \Big\} (ez)^{\gamma} (1 + \gamma z)^{-a_1 - \sum_{i=1}^{n} \delta_i - \gamma}, \end{aligned} \tag{3.8}$$

where $z = \dfrac{b_1 + W}{(b_2 + W)(a_2 + \sum_{i=1}^{n} \delta_i)} = \dfrac{1}{(b_2 + W)\bar{d}_2}$.

Case (III) : $\underline{d}_1 < d < \bar{d}_2$.

When $\underline{d}_1 < d < \bar{d}_2$, then according to $d_1 < d_2$, there are three situations to consider:

(1) $\underline{d}_1 < d < d_1 < d_2 < \bar{d}_2$,

(2) $\underline{d}_1 < d_1 < d < d_2 < \bar{d}_2$,

(3) $\underline{d}_1 < d_1 < d_2 < d < \bar{d}_2$.

In each case, $h(a,b,d)$ behaves as follows: Increasing (decreasing), decreasing (decreasing) and decreasing (increasing) function of a (and b). Therefore, $\sup_{\pi\in\Gamma_{a,b}} h(a,b,d) = \max\{h(a_2,b_1,d), h(a_1,b_1,d), h(a_1,b_2,d)\}$. Let $l_1(d) = h(a_2,b_1,d) - h(a_1,b_1,d)$. Then,

$$l_1(d) = kd^{-\gamma}e^{\gamma}\Big\{\prod_{j=0}^{\gamma-1}(a_1+\sum_{i=1}^{n}\delta_i+j)\frac{(b_1+W)^{a_1+\sum_{i=1}^{n}\delta_i}}{\left(b_1+W+\dfrac{\gamma}{d}\right)^{a_1+\sum_{i=1}^{n}\delta_i+\gamma}} - \prod_{j=0}^{\gamma-1}(a_2+\sum_{i=1}^{n}\delta_i+j)\frac{(b_1+W)^{a_2+\sum_{i=1}^{n}\delta_i}}{\left(b_1+W+\dfrac{\gamma}{d}\right)^{a_2+\sum_{i=1}^{n}\delta_i+\gamma}}\Big\}.$$

After differentiating $l_1(d)$ with respect to d, we get:

$$\frac{\partial l_1(d)}{\partial d} = -g_{a_1,b_1}(d)\left(d - \frac{a_1+\sum_{i=1}^{n}\delta_i}{b_1+W}\right) + g_{a_2,b_1}(d)\left(d - \frac{a_2+\sum_{i=1}^{n}\delta_i}{b_1+W}\right),$$

where

$$g_{a,b}(d) = k\gamma e^{\gamma}d^{-\gamma-2}\prod_{j=0}^{\gamma-1}(a+\sum_{i=1}^{n}\delta_i+j)\frac{(b+W)^{a+\sum_{i=1}^{n}\delta_i+1}}{\left(b+W+\dfrac{\gamma}{d}\right)^{a+\sum_{i=1}^{n}\delta_i+\gamma+1}}. \tag{3.9}$$

It is easy to verify that $\dfrac{\partial l_1(d)}{\partial d} < 0$, $l_1(\underline{d}_1) > 0$, $l_1(\bar{d}_2) < 0$ and hence $l_1(\underline{d}_1)l_1(\bar{d}_2) < 0$. Therefore, there exists $d^* \in (\underline{d}_1, \bar{d}_2)$ such that $l_1(d^*) = 0$, or equivalently, $h(a_2,b_1,d^*) = h(a_1,b_1,d^*)$. By solving this equation with respect to d^*, we obtain

$$d^* = \gamma\left\{(b_1+W)\left\{-1+\left[\prod_{j=0}^{\gamma-1}\frac{a_2+\sum_{i=1}^{n}\delta_i+j}{a_1+\sum_{i=1}^{n}\delta_i+j}\right]^{\dfrac{1}{(a_2-a_1)}}\right\}\right\}^{-1}. \tag{3.10}$$

Now, we can observe the following:
$\forall d \in (\underline{d}_1, d^*)$, $l_1(d) > 0 \implies h(a_2,b_1,d) > h(a_1,b_1,d) \implies \sup_{\pi\in\Gamma_{a,b}} h(a,b,d) = \max\{h(a_2,b_1,d), h(a_1,b_2,d)\}$,
$\forall d \in (d^*, \bar{d}_2)$, $l_1(d) < 0 \implies h(a_2,b_1,d) < h(a_1,b_1,d) \implies \sup_{\pi\in\Gamma_{a,b}} h(a,b,d) = \max\{h(a_1,b_1,d), h(a_1,b_2,d)\}$.

Since $h(a_1,b_1,d) < h(a_1,b_2,d)$,we have $\max\{h(a_1,b_1,d), h(a_1,b_2,d)\} = h(a_1,b_2,d)$. Combining these cases, we conclude:

$$\sup_{\pi\in\Gamma_{a,b}} h(a,b,d) = \begin{cases}\max\{h(a_2,b_1,d), h(a_1,b_2,d)\}, & d \in (\underline{d}_1, d^*)\\ h(a_1,b_2,d), & d \in (d^*, \bar{d}_2).\end{cases} \tag{3.11}$$

For $d \in (\underline{d}_1, d^*)$, we set $l_2(d) = h(a_2, b_1, d) - h(a_1, b_2, d)$. Then,

$$l_2(d) = kd^{-\gamma} e^{\gamma} \Big\{ \prod_{j=0}^{\gamma-1} \Big(a_1 + \sum_{i=1}^{n} \delta_i + j \Big) \frac{(b_2 + W)^{a_1 + \sum_{i=1}^n \delta_i}}{\Big(b_2 + W + \dfrac{\gamma}{d} \Big)^{a_1 + \sum_{i=1}^n \delta_i + \gamma}} - \prod_{j=0}^{\gamma-1} \Big(a_2 + \sum_{i=1}^{n} \delta_i + j \Big) \frac{(b_1 + W)^{a_2 + \sum_{i=1}^n \delta_i}}{\Big(b_1 + W + \dfrac{\gamma}{d} \Big)^{a_2 + \sum_{i=1}^n \delta_i + \gamma}} \Big\}.$$

Taking derivative $l_2(d)$ with respect to d, we have

$$\frac{\partial l_2(d)}{\partial d} = -g_{a_1,b_2}(d) \left(d - \frac{a_1 + \sum_{i=1}^n \delta_i}{b_2 + W} \right) + g_{a_2,b_1}(d) \left(d - \frac{a_2 + \sum_{i=1}^n \delta_i}{b_1 + W} \right),$$

where $g_{a,b}(d)$ is defined in (3.9). Similarly, it can be shown that $\dfrac{\partial l_2(d)}{\partial d} < 0$, $l_2(\underline{d}_1) > 0$, $l_2(d^*) < 0$ and hence $l_2(\underline{d}_1) l_2(d^*) < 0$. So, there exists $d_1^* \in (\underline{d}_1, d^*)$ such that $l_2(d_1^*) = 0$, or equivalently, $h(a_2, b_1, d_1^*) = h(a_1, b_2, d_1^*)$. Therefore, d_1^* is obtained by solving the following equation:

$$\frac{\Big(b_1 + W + \dfrac{\gamma}{d_1^*} \Big)^{a_2 + \sum_{i=1}^n \delta_i + \gamma}}{\Big(b_2 + W + \dfrac{\gamma}{d_1^*} \Big)^{a_1 + \sum_{i=1}^n \delta_i + \gamma}} = \frac{(b_1 + W)^{a_2 + \sum_{i=1}^n \delta_i}}{(b_2 + W)^{a_1 + \sum_{i=1}^n \delta_i}} \prod_{j=0}^{\gamma-1} \frac{a_2 + \sum_{i=1}^n \delta_i + j}{a_1 + \sum_{i=1}^n \delta_i + j},$$

which does not have a closed form and its value can be found using numerical methods. It can be easily shown that

$\forall d \in (\underline{d}_1, d_1^*)$, $l_2(d) > 0 \Longrightarrow h(a_2, b_1, d) > h(a_1, b_2, d) \Longrightarrow \max\{h(a_2, b_1, d),\ h(a_1, b_2, d)\} = h(a_2, b_1, d)$,

$\forall d \in (d_1^*, d^*)$, $l_2(d) < 0 \Longrightarrow h(a_2, b_1, d) < h(a_1, b_2, d) \Longrightarrow \max\{h(a_1, b_1, d),\ h(a_1, b_2, d)\} = h(a_1, b_2, d)$.

Now, Eq. (3.11) can be written as follows:

$$\sup_{\pi \in \Gamma_{a,b}} h(a, b, d) = \begin{cases} h(a_2, b_1, d), & d \in (\underline{d}_1, d_1^*), \\ h(a_1, b_2, d), & d \in (d_1^*, d^*), \\ h(a_1, b_2, d), & d \in (d^*, \bar{d}_2), \end{cases}$$

or

$$\sup_{\pi \in \Gamma_{a,b}} h(a, b, d) = \begin{cases} h(a_2, b_1, d), & d \in (\underline{d}_1, d_1^*), \\ h(a_1, b_2, d), & d \in (d_1^*, \bar{d}_2). \end{cases}$$

According to the relation $\inf_{\underline{d}_1 < d < \bar{d}_2} \sup_{\pi \in \Gamma_{a,b}} h(a, b, d) = \min\Big\{ \inf_{\underline{d}_1 < d < d_1^*} h(a_2, b_1, d), \inf_{d_1^* < d < \bar{d}_2} h(a_1, b_2, d) \Big\}$, two situations should be discussed:

$$\forall d \in (\underline{d}_1, d_1^*), \frac{\partial h(a_2, b_1, d)}{\partial d} < 0 \Longrightarrow \inf_{\underline{d}_1 < d < d_1^*} h(a_2, b_1, d) = h(a_2, b_1, d_1^*) < h(a_2, b_1, \underline{d}_1),$$

$$\forall d \in (d_1^*, \bar{d}_2), \frac{\partial h(a_1, b_2, d)}{\partial d} > 0 \Longrightarrow \inf_{d_1^* < d < \bar{d}_2} h(a_1, b_2, d) = h(a_1, b_2, d_1^*) < h(a_1, b_2, \bar{d}_2).$$

Since $h(a_2, b_1, d_1^*) = h(a_1, b_2, d_1^*)$, we have

$$\inf_{\underline{d}_1 < d < \bar{d}_2} \sup_{\pi \in \Gamma_{a,b}} h(a, b, d) = \min\{h(a_2, b_1, d_1^*), h(a_1, b_2, d_1^*)\} = h(a_2, b_1, d_1^*) = h(a_1, b_2, d_1^*).$$

Combining the three cases mentioned earlier, the process for obtaining the CGM estimation of λ can be outlined as follows:

$$\begin{aligned}\inf_{d\in D}\sup_{\pi\in\Gamma_{a,b}} h(a,b,d) &= \min\Big\{\inf_{\underline{d}_1<d<\bar{d}_2}\sup_{\pi\in\Gamma_{a,b}} h(a,b,d), \inf_{d<\underline{d}_1}\sup_{\pi\in\Gamma_{a,b}} h(a,b,d)\\ &\quad , \inf_{d>\bar{d}_2}\sup_{\pi\in\Gamma_{a,b}} h(a,b,d)\Big\}\\ &= \min\left\{h(a_2,b_1,d_1^*), h(a_2,b_1,\underline{d}_1), h(a_1,b_2,\bar{d}_2)\right\}\\ &= h(a_2,b_1,d_1^*) = h(a_1,b_2,d_1^*).\end{aligned}$$

which shows that d_1^* is the CGM estimator of λ. □

CGM estimator of λ under the classes of priors Γ_a and Γ_b are given in the following two propositions. These estimators can be obtained as special cases of Theorem 3.1.

Corollary 3.1. *The CGM estimator of λ under the class of prior Γ_a and the RG loss function is given by:*

$$\hat{\lambda} = \frac{\gamma(A-1)}{(b_2+W) - A(b_1+W)}, \tag{3.12}$$

where A is defined as:

$$A = \left(\frac{b_2+W}{b_1+W}\right)^{\frac{a_0+\sum_{i=1}^n \delta_i}{a_0+\sum_{i=1}^n \delta_i+\gamma}}.$$

Corollary 3.2. *The CGM estimator of λ under the class of prior Γ_b and the RG loss function is given by:*

$$\hat{\lambda} = \frac{\gamma}{(b_0+W)\Big\{-1+\Big[\prod_{j=0}^{\gamma-1}\frac{a_2+\sum_{i=1}^n\delta_i+j}{a_1+\sum_{i=1}^n\delta_i+j}\Big]^{\frac{1}{(a_2-a_1)}}\Big\}}. \tag{3.13}$$

4. PRGM ESTIMATION

In this section, we derive the PRGM estimator of λ under the class $\Gamma_{a,b}$ and the RG loss function. By substituting (2.10) into (2.8), we obtain

$$\mathrm{PR}(\hat{\lambda},\lambda) = k - ke^{\gamma}\prod_{j=0}^{\gamma-1}(a+\sum_{i=1}^n\delta_i+j)\frac{(a+\sum_{i=1}^n\delta_i)^{a+\sum_{i=1}^n\delta_i}}{(a+\sum_{i=1}^n\delta_i+\gamma)^{a+\sum_{i=1}^n\delta_i+\gamma}}. \tag{4.1}$$

Now, by replacing (2.8) and (4.1) into the Eq. $\rho(d,\lambda) = \mathrm{PR}(\mathrm{d},\lambda) - \mathrm{PR}(\hat{\lambda},\lambda)$, we have:

$$\begin{aligned}\rho(d,\lambda) &= ke^{\gamma}\prod_{j=0}^{\gamma-1}(a+\sum_{i=1}^{n}\delta_i+j)\\ &\times\Big\{\frac{(a+\sum_{i=1}^{n}\delta_i)^{a+\sum_{i=1}^{n}\delta_i}}{(a+\sum_{i=1}^{n}\delta_i+\gamma)^{a+\sum_{i=1}^{n}\delta_i+\gamma}} - \frac{(b+W)^{a+\sum_{i=1}^{n}\delta_i}}{(b+W+\frac{\gamma}{d})^{a+\sum_{i=1}^{n}\delta_i+\gamma}}d^{-\gamma}\Big\}\\ &= h_1(a,b,d), \qquad\qquad (say).\end{aligned} \tag{4.2}$$

The partial derivatives of $h_1(a,b,d)$ with respect to a, b and d as follows:

$$\begin{aligned}\frac{\partial h_1(a,b,d)}{\partial a} =& -C_2(a,b) + \Bigg\{\prod_{j=0}^{\gamma-1}(a+\sum_{i=1}^{n}\delta_i+j)\frac{(d(b+W))^{a+\sum_{i=1}^{n}\delta_i}}{(\gamma+d(b+W))^{(a+\sum_{i=1}^{n}\delta_i+\gamma)}}\\ &\times\Big(\log(1+\frac{\gamma}{d(b+W)}) - \sum_{j=0}^{\gamma-1}\frac{1}{a+\sum_{i=1}^{n}\delta_i+j}\Big)\Bigg\},\end{aligned} \tag{4.3}$$

$$\frac{\partial h_1(a,b,d)}{\partial b} = C_3(a,b,d)\left\{d - \frac{a+\sum_{i=1}^{n}\delta_i}{b+W}\right\}, \tag{4.4}$$

$$\frac{\partial h_1(a,b,d)}{\partial d} = C_3(a,b,d)d^{-1}(b+W)\left\{d - \frac{a+\sum_{i=1}^{n}\delta_i}{b+W}\right\}. \tag{4.5}$$

where $C_2(a,b)$ and $C_3(a,b,d)$ are as follows

$$\begin{aligned}C_2(a,b) =& \frac{(a+\sum_{i=1}^{n}\delta_i)^{a+\sum_{i=1}^{n}\delta_i}}{(a+\sum_{i=1}^{n}\delta_i+\gamma)^{(a+\sum_{i=1}^{n}\delta_i+\gamma)}}\prod_{j=0}^{\gamma-1}(a+\sum_{i=1}^{n}\delta_i+j)\\ &\times\Big\{\log\Big(1+\frac{\gamma}{a+\sum_{i=1}^{n}\delta_i}\Big) - \sum_{j=0}^{\gamma-1}\frac{1}{a+\sum_{i=1}^{n}\delta_i+j}\Big\},\end{aligned}$$

$$C_3(a,b,d) = k\gamma e^{\gamma}d^{-\gamma-1}\prod_{j=0}^{\gamma-1}(a+\sum_{i=1}^{n}\delta_i+j)(b+W)^{a+\sum_{i=1}^{n}\delta_i}\Big(b+W+\frac{\gamma}{d}\Big)^{-a-\gamma-\sum_{i=1}^{n}\delta_i-1},$$

respectively. In the process of deriving the function $\prod_{j=0}^{\gamma-1}(a+\sum_{i=1}^{n}\delta_i+j)$ with respect to a, the following relationship has been used;

$$\frac{d}{da}\{\prod_{j=0}^{\gamma-1}(a+\sum_{i=1}^{n}\delta_i+j)\} = \prod_{j=0}^{\gamma-1}(a+\sum_{i=1}^{n}\delta_i+j)\sum_{j=0}^{\gamma-1}\frac{1}{a+\sum_{i=1}^{n}\delta_i+j}$$

By solving the Eq. $\frac{\partial h_1(a,b,d)}{\partial a} = 0$ with respect to d, we can get the following expression;

$$\begin{aligned}&\frac{\log(1+\frac{\gamma}{d(b+W)}) - \sum_{j=0}^{\gamma-1}\frac{1}{a+\sum_{i=1}^{n}\delta_i+j}}{\log(1+\frac{\gamma}{a+\sum_{i=1}^{n}\delta_i}) - \sum_{j=0}^{\gamma-1}\frac{1}{a+\sum_{i=1}^{n}\delta_i+j}} =\\ &(\frac{a+\sum_{i=1}^{n}\delta_i}{d(b+W)})^{a+\sum_{i=1}^{n}\delta_i}(\frac{d(b+W)+\gamma}{a+\sum_{i=1}^{n}\delta_i+\gamma})^{a+\sum_{i=1}^{n}\delta_i+\gamma}\end{aligned} \tag{4.6}$$

It can be seen that the value of d_2 in (3.2) satisfies the recent Eq., and therefore it is one of the roots of the Eq. $\frac{\partial h_1(a,b,d)}{\partial a} = 0$. Also, we can find an interval of the form $(10d_2, 35d_2)$ such that the function $\frac{\partial h_1(a,b,d)}{\partial a}$ changes sign within this interval. By the Intermediate Value Theorem, there exist a $d_3 \in (10d_2, 35d_2)$ such that $\frac{\partial h_1(a,b,d)}{\partial a}|_{d=d_3} = 0$, that is, d_3 is the root of the Eq. $\frac{\partial h_1(a,b,d)}{\partial a} = 0$. The mentioned root does not have a closed-form expression and its value must be found using numerical methods. Thus, the Eq. $\frac{\partial h_1(a,b,d)}{\partial a} = 0$ has two roots d_2 and d_3 where $d_2 < d_3$. Since $\frac{\partial h_1(a,b,d)}{\partial b}|_{d=d_2} = \frac{\partial h_1(a,b,d)}{\partial d}|_{d=d_2} = 0$, it follows that d_2 is also the root of two equations $\frac{\partial h_1(a,b,d)}{\partial b} = 0$ and $\frac{\partial h_1(a,b,d)}{\partial d} = 0$. According to (3.4) and (3.5), we have relations $\inf_{a,b} d_2 = \underline{d}_2$ and $\sup_{a,b} d_2 = \bar{d}_2$. Since for each a and b, $d_3 > d_2$, we can conclude that $\underline{d}_3 = \inf_{a,b} d_3 \geq \inf_{a,b} d_2 = \underline{d}_2$ and $\bar{d}_3 = \sup_{a,b} d_3 \geq \sup_{a,b} d_2 = \bar{d}_2$. Then, the behavior of the partial derivative of $h_1(a,b,d)$ with respect to a, b, and d is as follows:

$$\frac{\partial h_1(a,b,d)}{\partial a} \begin{cases} > 0, & 0 < d < d_2, \\ = 0, & d = d_2, \\ < 0, & d_2 < d < d_3, \\ = 0, & d = d_3, \\ > 0, & d > d_3, \end{cases} \tag{4.7}$$

and

$$\frac{\partial h_1(a,b,d)}{\partial b}, \frac{\partial h_1(a,b,d)}{\partial d} \begin{cases} > 0, & d > d_2, \\ = 0, & d = d_2, \\ < 0, & d < d_2. \end{cases} \tag{4.8}$$

Theorem 4.1. *The PRGM estimator of λ, denoted as $\hat{\lambda}$, under the class of prior $\Gamma_{a,b}$ and the RG loss function does not have a closed form expression. It must to be obtained numerically by solving the following equality:*

$$\{A_2 - A_1\}\,\hat{\lambda}^{\gamma} = A_2^* \frac{(b_1+W)^{a_2+\sum_{i=1}^n \delta_i}}{\left(b_1 + W + \frac{\gamma}{\hat{\lambda}}\right)^{a_2+\sum_{i=1}^n \delta_i+\gamma}} - A_1^* \frac{(b_2+W)^{a_1+\sum_{i=1}^n \delta_i}}{\left(b_2 + W + \frac{\gamma}{\hat{\lambda}}\right)^{a_1+\sum_{i=1}^n \delta_i+\gamma}}, \tag{4.9}$$

where

$$A_l = \left\{1 + \frac{\gamma}{a_l + \sum_{i=1}^n \delta_i}\right\}^{-(a_l+\sum_{i=1}^n \delta_i)} \prod_{j=0}^{\gamma-1} \frac{a_l + \sum_{i=1}^n \delta_i + j}{a_l + \sum_{i=1}^n \delta_i + \gamma}; \qquad l = 1, 2$$

and

$$A_l^* = \prod_{j=0}^{\gamma-1} (a_l + \sum_{i=1}^n \delta_i + j); \qquad l = 1, 2.$$

Proof: The PRGM estimation of λ can be obtained by considering the following three cases:

Case (I) : $0 < d < \underline{d}_2$.

If $d < \underline{d}_2$, then $h_1(a,b,d)$ is an increasing function of a and a decreasing function of b and d. Therefore

$$\inf_{d<\underline{d}_2} \sup_{\pi\in\Gamma_{a,b}} h_1(a,b,d) = h_1(a_2,b_1,\underline{d}_2). \tag{4.10}$$

Case (II) : $d > \bar{d}_3$.

When $d > \bar{d}_3$, then $h_1(a,b,d)$ is a increasing function of a, b and d. Therefore

$$\inf_{d>\bar{d}_3} \sup_{\pi\in\Gamma_{a,b}} h_1(a,b,d) = h_1(a_2,b_2,\bar{d}_3). \tag{4.11}$$

Case (III) : $\underline{d}_2 < d < \bar{d}_3$.

When $\underline{d}_2 < d < \bar{d}_3$, there are three situations to consider:

(1) $\underline{d}_2 < d < d_2 < d_3 < \bar{d}_3$,

(2) $\underline{d}_2 < d_2 < d < d_3 < \bar{d}_3$,

(3) $\underline{d}_2 < d_2 < d_3 < d < \bar{d}_3$.

According to Eqs. (4.3) and (4.4), it can be observed that for the three situations (1, 2, and 3), $h_1(a,b,d)$ exhibits different monotonic behaviors with respect to a and b:

case 1: $h_1(a,b,d)$ is an increasing function of a, and a decreasing function of b.

case 2: $h_1(a,b,d)$ is a decreasing function of a, and an increasing function of b.

case 3: $h_1(a,b,d)$ is an increasing function of both a and b.

Therefore, the $\sup_{\pi\in\Gamma_{a,b}} h_1(a,b,d)$ for the last three situations will be $h_1(a_2,b_1,d)$, $h_1(a_1,b_2,d)$ and $h_1(a_2,b_2,d)$, respectively. So, we have

$$\sup_{\pi\in\Gamma_{a,b}} h_1(a,b,d) = \max\left\{h_1(a_2,b_1,d), h_1(a_1,b_2,d), h_1(a_2,b_2,d)\right\}. \tag{4.12}$$

Let us set $l_3(d) = h_1(a_2,b_2,d) - h_1(a_1,b_2,d)$. Then, $l_3(d)$ is as follows

$$\begin{aligned} l_3(d) =& ke^{\gamma}\prod_{j=0}^{\gamma-1}(a_2+\sum_{i=1}^{n}\delta_i+j) \\ &\times\left\{\frac{(a_2+\sum_{i=1}^{n}\delta_i)^{a_2+\sum_{i=1}^{n}\delta_i}}{(a_2+\sum_{i=1}^{n}\delta_i+\gamma)^{a_2+\sum_{i=1}^{n}\delta_i+\gamma}} - \frac{(b_2+W)^{a_2+\sum_{i=1}^{n}\delta_i}d^{-\gamma}}{\left(b_2+W+\dfrac{\gamma}{d}\right)^{a_2+\sum_{i=1}^{n}\delta_i+\gamma}}\right\} \\ &- ke^{\gamma}\prod_{j=0}^{\gamma-1}(a_1+\sum_{i=1}^{n}\delta_i+j) \\ &\times\left\{\frac{(a_1+\sum_{i=1}^{n}\delta_i)^{a_1+\sum_{i=1}^{n}\delta_i}}{(a_1+\sum_{i=1}^{n}\delta_i+\gamma)^{a_1+\sum_{i=1}^{n}\delta_i+\gamma}} - \frac{(b_2+W)^{a_1+\sum_{i=1}^{n}\delta_i}d^{-\gamma}}{\left(b_2+W+\dfrac{\gamma}{d}\right)^{a_1+\sum_{i=1}^{n}\delta_i+\gamma}}\right\}. \end{aligned}$$

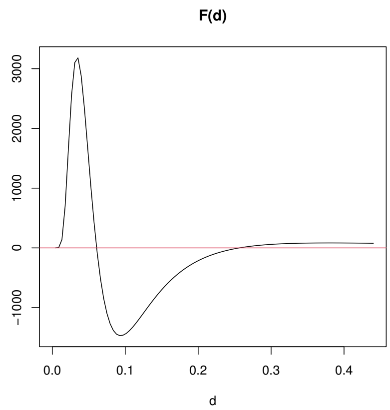

Figure 1: The graph of the function $F(d)$ for the values $\alpha = 2$ and $\lambda = 2$ based on the Burr XII Distribution.

After differentiating $l_3(d)$ with respect to d, we obtain

$$\begin{aligned} \frac{\partial l_3(d)}{\partial d} &= g_{a_2,b_2}(d)\,(d - d_{22}) - g_{a_1,b_2}(d)\,(d - \underline{d}_2) \\ &= F(d), \qquad\qquad (say). \end{aligned} \tag{4.13}$$

where $g_{a,b}(d)$ and $\underline{d}_2$ are defined in Eqs. (3.9) and (3.4), respectively, and $d_{22} = \dfrac{a_2 + \sum_{i=1}^{n}\delta_i}{b_2 + W}$. After some algebraic manipulation, it can be shown that $F(\dfrac{\underline{d}_2}{2}) > 0$, $F(\underline{d}_2) < 0$, $F(d_{22}) < 0$, and $F(2d_{22}) > 0$. Thus, by the Intermediate Value Theorem, there exist $d_1^l \in (\dfrac{\underline{d}_2}{2}, \underline{d}_2)$ and $d_2^l \in (d_{22}, 2d_{22})$ such that $F(d_1^l) = 0$ and $F(d_2^l) = 0$. Therefore, the Eq. $\dfrac{\partial l_3(d)}{\partial d} = 0$ has two roots, d_1^l and d_2^l, which are obtained by solving the following Eq. with respect to d:

$$\begin{aligned} &\left\{\left(1 + \frac{\gamma}{d(b_2 + W)}\right)^{-(a_2 - a_1)} \prod_{j=0}^{\gamma-1} \frac{a_2 + \sum_{i=1}^{n}\delta_i + j}{a_1 + \sum_{i=1}^{n}\delta_i + j}\right\} \\ &\qquad\qquad - \left(1 + \frac{a_2 - a_1}{d(b_2 + W) - (a_2 + \sum_{i=1}^{n}\delta_i)}\right) = 0. \end{aligned}$$

However, these roots do not have a closed form expression and must be approximated using numerical methods. From the plot of $F(d)$ shown in Figure 1, it can be clearly observed and verified that

$$\frac{\partial l_3(d)}{\partial d} \begin{cases} > 0, & 0 < d < d_1^l, \\ = 0, & d = d_1^l, \\ < 0, & d_1^l < d < d_2^l, \\ = 0, & d = d_2^l, \\ > 0, & d_2^l < d < \bar{d}_3 \end{cases} \tag{4.14}$$

The mathematical proof of $\bar{d}_2 < d_2^l$ and $d_2^l < \bar{d}_3$ is not straightforward; in fact, we have verified the validity of these relationships through numerical calculations. According to (4.14), it is evident that for $d \in (d_1^l, d_2^l)$, we have $\frac{\partial l_3(d)}{\partial d} < 0$. Now, since $(\underline{d}_2, \bar{d}_2) \subset (d_1^l, d_2^l)$, it follows that for $d \in (\underline{d}_2, \bar{d}_2)$, we have $\dfrac{\partial l_3(d)}{\partial d} < 0$. It can be easily demonstrated that $l_3(\underline{d}_2) > 0$, $l_3(\bar{d}_2) < 0$

and so $l_3(\bar{d}_2)l_3(\underline{d}_2) < 0$. Therefore there exists $d_1^* \in (\underline{d}_2, \bar{d}_2)$ such that $l_3(d_1^*) = 0$. Similarly, it can be shown that there exists $d_2^* \in (d_2^l, \bar{d}_3)$ such that $l_3(d_2^*) = 0$. Both d_1^* and d_2^* do not have closed-form expressions and therefore must be approximated using numerical methods. It can also be easily demonstrated that

$$l_3(d) \begin{cases} > 0, & \underline{d}_2 < d < d_1^*, \\ = 0, & d = d_1^*, \\ < 0, & d_1^* < d < d_2^*, \\ = 0, & d = d_2^*, \\ > 0, & d_2^* < d < \bar{d}_3. \end{cases} \tag{4.15}$$

This information allows us to divide the range $d \in (\underline{d}_2, \bar{d}_3)$ into three situations:
1: For $d \in (\underline{d}_2, d_1^*)$, $l_3(d) > 0 \implies h_1(a_2, b_2, d) > h_1(a_1, b_2, d) \implies \max\{h_1(a_2, b_2, d), h_1(a_1, b_2, d)\} = h_1(a_2, b_2, d)$,
2: For $d \in (d_1^*, d_2^*)$, $l_3(d) < 0 \implies h_1(a_2, b_2, d) < h_1(a_1, b_2, d) \implies \max\{h_1(a_2, b_2, d), h_1h(a_1, b_2, d)\} = h_1(a_1, b_2, d)$,
3: For $d \in (d_2^*, \bar{d}_3)$, $l_3(d) > 0 \implies h_1(a_2, b_2, d) > h_1(a_1, b_2, d) \implies \max\{h_1(a_2, b_2, d), h_1(a_1, b_2, d)\} = h_1(a_2, b_2, d)$.

This analysis help us understand the behaviour of the function $h_1(a, b, d)$ and the relationships among $h_1(a_2, b_1, d)$, $h_1(a_2, b_2, d)$ and $h_1(a_1, b_2, d)$ in different intervals of d within the range $(\underline{d}_2, \bar{d}_3)$.

Now, we can express Eq. (4.12) as follows:

$$\sup_{\pi \in \Gamma_{a,b}} h_1(a, b, d) = \begin{cases} \max\{h_1(a_2, b_1, d), h_1(a_2, b_2, d)\}, & d \in (\underline{d}_2, d_1^*), \\ \max\{h_1(a_2, b_1, d), h_1(a_1, b_2, d)\}, & d \in (d_1^*, d_2^*), \\ \max\{h_1(a_2, b_1, d), h_1(a_2, b_2, d)\}, & d \in (d_2^*, \bar{d}_3). \end{cases} \tag{4.16}$$

For $d \in (d_1^*, d_2^*)$, let us define $l_4(d) = h_1(a_2, b_1, d) - h_1(a_1, b_2, d)$. Therefore,

$$\begin{aligned} l_4(d) =& k e^{\gamma} \prod_{j=0}^{\gamma-1} (a_2 + \sum_{i=1}^{n} \delta_i + j) \\ & \times \Big\{ \frac{(a_2 + \sum_{i=1}^{n} \delta_i)^{a_2 + \sum_{i=1}^{n} \delta_i}}{(a_2 + \sum_{i=1}^{n} \delta_i + \gamma)^{a_2 + \sum_{i=1}^{n} \delta_i + \gamma}} - \frac{(b_1 + W)^{a_2 + \sum_{i=1}^{n} \delta_i} d^{-\gamma}}{\left(b_1 + W + \frac{\gamma}{d}\right)^{a_2 + \sum_{i=1}^{n} \delta_i + \gamma}} \Big\} \\ & - k e^{\gamma} \prod_{j=0}^{\gamma-1} (a_1 + \sum_{i=1}^{n} \delta_i + j) \\ & \times \Big\{ \frac{(a_1 + \sum_{i=1}^{n} \delta_i)^{a_1 + \sum_{i=1}^{n} \delta_i}}{(a_1 + \sum_{i=1}^{n} \delta_i + \gamma)^{a_1 + \sum_{i=1}^{n} \delta_i + \gamma}} - \frac{(b_2 + W)^{a_1 + \sum_{i=1}^{n} \delta_i} d^{-\gamma}}{\left(b_2 + W + \frac{\gamma}{d}\right)^{a_1 + \sum_{i=1}^{n} \delta_i + \gamma}} \Big\}. \end{aligned}$$

After differentiating $l_4(d)$ with respect to d and simplifying, we obtain:

$$\frac{\partial l_4(d)}{\partial d} = g_{a_2,b_1}(d) \left(d - \bar{d}_2\right) - g_{a_1,b_2}(d) \left(d - \underline{d}_2\right),$$

where $\underline{d}_2$, $\bar{d}_2$, and $g_{a,b}(d)$ are defined in Eqs. (3.4), (3.5), and (3.9), respectively. Similar to $l_3(d) = 0$, it can be shown that Eq. $l_4(d) = 0$ also has two roots, as d_3^* and d_4^* where $d_3^* < d_4^*$.

These roots are obtained by solving the Eq. (4.9). Based on numerical calculations, it can be shown that $d_1^* < d_3^* < d_2^* < d_4^*$. Further, we have

$$l_4(d) = \begin{cases} > 0, & d \in (d_1^*, d_3^*), \\ < 0, & d \in (d_3^*, d_2^*). \end{cases} \tag{4.17}$$

Thus

$$\max\{h_1(a_2, b_1, d), h_1(a_1, b_2, d)\} = \begin{cases} h_1(a_2, b_1, d), & d \in (d_1^*, d_3^*), \\ h_1(a_1, b_2, d), & d \in (d_3^*, d_2^*). \end{cases}$$

If we define $l_5(d) = h_1(a_2, b_2, d) - h_1(a_2, b_1, d)$, then $l_5(\bar{d}_2) l_5(d_1^*) < 0$. Consequently, there exists $d^* \in (d_1^*, \bar{d}_2)$ such that $l_5(d^*) = 0$, where d^* is given in Eq. (3.12) with $a_0 = a_2$. It can be easily shown that for all $d \in (\underline{d}_2, d_1^*)$, $l_5(d) < 0$ and for all $d \in (d_2^*, \bar{d}_2)$, $l_5(d) > 0$, and therefore

$$\max\{h_1(a_2, b_2, d), h_1(a_2, b_1, d)\} = \begin{cases} h_1(a_2, b_1, d), & d \in (\underline{d}_2, d_1^*), \\ h_1(a_2, b_2, d), & d \in (d_2^*, \bar{d}_3). \end{cases}$$

Now, $\sup_{\pi \in \Gamma_{a,b}} h_1(a, b, d)$ can be written as follows:

$$\sup_{\pi \in \Gamma_{a,b}} h_1(a, b, d) = \begin{cases} h_1(a_2, b_1, d), & d \in (\underline{d}_2, d_1^*), \\ h_1(a_2, b_1, d), & d \in (d_2^*, d_3^*), \\ h_1(a_1, b_2, d), & d \in (d_3^*, d_2^*), \\ h_1(a_2, b_2, d), & d \in (d_2^*, \bar{d}_3), \end{cases}$$

or

$$\sup_{\pi \in \Gamma_{a,b}} h_1(a, b, d) = \begin{cases} h_1(a_2, b_1, d), & d \in (\underline{d}_2, d_3^*), \\ h_1(a_1, b_2, d), & d \in (d_3^*, d_2^*), \\ h_1(a_2, b_2, d), & d \in (d_2^*, \bar{d}_3). \end{cases}$$

It is easy to show that

$$\begin{aligned} &For \;\; d \in (\underline{d}_2, d_3^*), \frac{\partial h_1(a_2, b_1, d)}{\partial d} < 0 \Longrightarrow \\ &\qquad \inf_{\underline{d}_2 < d < d_3^*} h_1(a_2, b_1, d) = h_1(a_2, b_1, d_3^*) < h_1(a_2, b_1, \underline{d}_2), \end{aligned}$$

$$\begin{aligned} &For \;\; d \in (d_3^*, d_2^*), \frac{\partial h_1(a_1, b_2, d)}{\partial d} > 0 \Longrightarrow \\ &\qquad \inf_{d_3^* < d < d_2^*} h_1(a_1, b_2, d) = h_1(a_1, b_2, d_3^*) < h_1(a_1, b_2, d_2^*), \end{aligned}$$

$$\begin{aligned} &For \;\; d \in (d_2^*, \bar{d}_3), \frac{\partial h_1(a_2, b_2, d)}{\partial d} > 0 \Longrightarrow \\ &\qquad \inf_{d_2^* < d < \bar{d}_3} h_1(a_2, b_2, d) = h_1(a_2, b_2, d_2^*) < h_1(a_2, b_2, \bar{d}_3). \end{aligned}$$

Now, $\inf_{\underline{d}_2 < d < \bar{d}_3} \sup_{\pi \in \Gamma_{a,b}} h_1(a, b, d)$ can be written as follows:

$$\begin{aligned} &\inf_{\underline{d}_2 < d < \bar{d}_3} \sup_{\pi \in \Gamma_{a,b}} h_1(a, b, d) \\ &\qquad = \min\Big\{ \inf_{\underline{d}_2 < d < d_3^*} h_1(a_2, b_1, d), \inf_{d_3^* < d < d_2^*} h_1(a_1, b_2, d), \inf_{d_2^* < d < \bar{d}_3} h_1(a_2, b_2, d) \Big\} \\ &\qquad = \min\{h_1(a_2, b_1, d_3^*), h_1(a_1, b_2, d_3^*), h_1(a_2, b_2, d_2^*)\} \end{aligned}$$

Since $h_1(a_2, b_1, d_3^*) = h_1(a_1, b_2, d_3^*)$, $h_1(a_2, b_2, d_2^*) = h_1(a_1, b_2, d_2^*)$ and $h_1(a_1, b_2, d)$ is increasing with respect to d on the interval (d_3^*, d_2^*) and so, we have

$$\inf_{\underline{d}_2<d<\bar{d}_3} \sup_{\pi\in\Gamma_{a,b}} h_1(a,b,d) = h_1(a_1, b_2, d_3^*) = h_1(a_2, b_1, d_3^*).$$

By combining the results of three cases I, II and III, we have

$$\begin{aligned}
&\inf_{d\in D} \sup_{\pi\in\Gamma_{a,b}} h_1(a,b,d)\\
&\quad= \min\left\{\inf_{d<\underline{d}_2} \sup_{\pi\in\Gamma_{a,b}} h_1(a,b,d),\ \inf_{\underline{d}_2<d<\bar{d}_3} \sup_{\pi\in\Gamma_{a,b}} h_1(a,b,d),\ \inf_{d>\bar{d}_3} \sup_{\pi\in\Gamma_{a,b}} h_1(a,b,d)\right\}\\
&\quad= \min\Big\{h_1(a_2, b_1, \underline{d}_2), h_1(a_1, b_2, d_3^*), h_1(a_2, b_2, \bar{d}_3)\Big\}
\end{aligned}$$

Since $h_1(a_2, b_1, d)$ is decreasing with respect to d on $(\underline{d}_2, d_3^*)$, so $h_1(a_2, b_1, \underline{d}_2) > h_1(a_2, b_1, d_3^*) = h_1(a_1, b_2, d_3^*)$. It can also be easily shown that $h_1(a_2, b_2, \bar{d}_3) > h_1(a_1, b_2, d_3^*)$, and so

$$\inf_{d\in D} \sup_{\pi\in\Gamma_{a,b}} h_1(a,b,d) = h_1(a_2, b_1, d_3^*) = h_1(a_1, b_2, d_3^*),$$

which shows that d_3^* is a PRGM estimator of λ. □

Corollary 4.1. *The PRGM estimator of λ under the class of prior Γ_a and RG loss function is as follows*

$$\hat{\lambda} = \frac{\gamma(A-1)}{(b_2+W) - A(b_1+W)}, \tag{4.18}$$

where A is

$$A = \left(\frac{b_2+W}{b_1+W}\right)^{\dfrac{a_0+\sum_{i=1}^n \delta_i}{a_0+\sum_{i=1}^n \delta_i+\gamma}}.$$

By referring to Eqs. (3.12) and (4.18), it becomes apparent that when considering the class Γ_a, CGM and PRGM estimators of λ demonstrate equivalence.

Corollary 4.2. *The estimator PRGM of the parameter λ, denoted as $\hat{\lambda}$, under the class of prior Γ_b and the loss function RG does not have a closed form and it is obtained numerically by solving the following Eq.*

$$\begin{aligned}
\Big\{A_2 - A_1\Big\}\{\hat{\lambda}(b_0+W)\Big\}^{\gamma} =& A_2^*\Big\{1+\frac{\gamma}{\hat{\lambda}(b_0+W)}\Big\}^{-a_2-\sum_{i=1}^n \delta_i-\gamma}\\
&- A_1^*\Big\{1+\frac{\gamma}{\hat{\lambda}(b_0+W)}\Big\}^{-a_1-\sum_{i=1}^n \delta_i-\gamma},
\end{aligned} \tag{4.19}$$

where

$$A_l = \left\{1+\frac{\gamma}{a_l+\sum_{i=1}^n \delta_i}\right\}^{-(a_l+\sum_{i=1}^n \delta_i)} \prod_{j=0}^{\gamma-1} \frac{a_l+\sum_{i=1}^n \delta_i+j}{a_l+\sum_{i=1}^n \delta_i+\gamma}; \qquad l=1,2,$$

and

$$A_l^* = \prod_{j=0}^{\gamma-1}(a_l+\sum_{i=1}^n \delta_i+j); \qquad l=1,2.$$

5. SIMULATION STUDY

In this section, we present a simulation study to evaluate the performance of the proposed estimators. We generate $N = 10000$ random samples from the Burr XII distribution using two sets of parameter values: $\alpha = 1, \lambda = 0.4$ and $\alpha = 2, \lambda = 2$. The sample sizes considered are $n = 20$, 50, and 100. To introduce censoring, we set the censoring time such that it corresponds to specific percentiles of the Burr XII distribution. The censoring percentages considered are %20, %35, %50, and %70. Specifically, we set the censoring time equal to the 80th, 65th, 50th, and 30th percentiles of the Burr XII distribution, respectively. For the simulation study, we fix values of $k = 1$, $\gamma = 2$, and consider different values for the shape parameter $a = 2, 5, 7, 9$ and its range $(a_1, a_2) = (1, 10)$. Similarly, we consider different values for the rate parameter $b = 3, 7, 11, 15$ and its range $(b_1, b_2) = (2, 16)$. By systematically varying the parameters a and b within the specified ranges, we assess the performance of the estimators across senarios. This approach enables us to evaluate robustness and accuracy of the estimators under varying conditions.

In tables 1-6, we present the results of the simulation study, including the average biases and risks of the Bayesian, CGM, and PRGM estimations of λ. The tables provide insights into the performance of the estimators under different sample sizes, censoring percentages and the classes Γ_a, Γ_b and $\Gamma_{a,b}$. The values related to risk calculations are indicated in parentheses. Based on the data in these tables, we can draw the following conclusions:

1. The simulation study results show that as the sample size increases, the biases and risks of both the robust (CGM and PRGM) and Bayesian estimators tend to decrease. This observation suggests that larger sample sizes lead to more accurate and precise estimations of the parameter λ.
2. Across all classes (Γ_a, Γ_b and $\Gamma_{a,b}$), the simulation results show, the average biases and risk of Bayesian, CGM, and PRGM estimates tend to increase as the censoring percentages increase.
3. Under classes Γ_a and Γ_b, the robust estimators, i.e. CGM and PRGM, do not uniformly domainate the Bayesian estimator.
4. In Tables 1-3, the bias values for both the robust and Bayesian estimators indicate overestimation. However, the results in Tables 4-6 show underestimation. Under the classes of priors Γ_b and $\Gamma_{a,b}$, the CGM estimator outperforms in cases of overestimation. However, for underestimation the PRGM estimator demonstrates better performance.
5. For large samples sizes, the risk values of CGM and PRGM estimators are nearly equivalent.

6. REAL DATA ANALYSIS

In this section, we analyze a real dataset. The data represent the times to breakdown of an insulating fluid between electrodes at a voltage of 34 kilovolt (see Nelson (2005)). The observed breakdown times are as follows:

$$0.19, 0.78, 0.96, 1.31, 2.78, 3.16, 4.15, 4.67, 4.85, 6.5, 7.35,$$
$$8.01, 8.27, 12.06, 31.75, 32.52, 33.91, 36.71, 72.89$$

Table 1: The average biases and risks (in parentheses) of the Bayes, CGM and PRGM estimators for λ are presented under different censoring percentages, in three class of priors with $\alpha = 1$, $\lambda = 0.4$ and $n = 20$.

		Γ_a			Γ_b			$\Gamma_{a,b}$	
% censor	b(a)	bayesian	CGM	PRGM	bayesian	CGM	PRGM	CGM	PRGM
20	3(2)	0.1034(0.0723)	0.0454(0.0435)	0.0454(0.0435)	−0.0233(0.0529)	0.0326(0.0405)	0.0435(0.0420)	0.0477(0.0452)	0.0549(0.0467)
	7(5)	0.0585(0.0481)	0.0454(0.0435)	0.0454(0.0435)	0.0389(0.0409)	0.0326(0.0405)	0.0435(0.0420)	0.0477(0.0452)	0.0549(0.0467)
	11(7)	0.0210(0.0368)	0.0454(0.0435)	0.0454(0.0435)	0.0804(0.0541)	0.0326(0.0405)	0.0435(0.0420)	0.0477(0.0452)	0.0549(0.0467)
	15(9)	−0.0107(0.0369)	0.0454(0.0435)	0.0454(0.0435)	0.1219(0.0779)	0.0326(0.0405)	0.0435(0.0420)	0.0477(0.0452)	0.0549(0.0467)
35	3(2)	0.1197(0.0847)	0.0509(0.0487)	0.0509(0.0487)	−0.0309(0.0669)	0.0334(0.0455)	0.0464(0.0466)	0.0532(0.0511)	0.0621(0.0524)
	7(5)	0.0652(0539)	0.0509(0.0487)	0.0509(0.0487)	0.0421(0.0452)	0.0334(0.0455)	0.0464(0.0466)	0.0532(0.0511)	0.0621(0.0524)
	11(7)	0.0212(0.0408)	0.0509(0.0487)	0.0509(0.0487)	0.0908(0.0611)	0.0334(0.0455)	0.0464(0.0466)	0.0532(0.0511)	0.0621(0.0524)
	15(9)	−0.0152(0.0431)	0.0509(0.0487)	0.0509(0.0487)	0.1394(0.0909)	0.0334(0.0455)	0.0464(0.0466)	0.0532(0.0511)	0.0621(0.0524)
50	3(2)	0.1457(0.1037)	0.0602(0.0547)	0.0602(0.0547)	−0.0416(0.860)	0.0343(0.0507)	0.0507(0.0513)	0.0622(0.0583)	0.0741(0.0601)
	7(5)	0.0758(0.0608)	0.0602(0.0547)	0.0602(0.0547)	0.0471(0.0495)	0.0343(0.0507)	0.0507(0.0513)	0.0622(0.0583)	0.0741(0.0601)
	11(7)	0.0218(0.0442)	0.0602(0.0547)	0.0602(0.0547)	0.1063(0.0709)	0.0343(0.0507)	0.0507(0.0513)	0.0622(0.0583)	0.0741(0.0601)
	15(9)	−0.0211(0.0496)	0.0602(0.0547)	0.0602(0.0547)	0.1655(0.1109)	0.0343(0.0507)	0.0507(0.0513)	0.0622(0.0583)	0.0741(0.0601)
70	3(2)	0.2196(0.1604)	0.0896(0.0704)	0.0896(0.0704)	−0.0626(0.1313)	0.0376(0.0615)	0.0633(0.0599)	0.0904(0.0771)	0.1128(0.0819)
	7(5)	0.1053(0.0781)	0.0896(0.0704)	0.0896(0.0704)	0.0627(0.0574)	0.0376(0.0615)	0.0633(0.0599)	0.0904(0.0771)	0.1128(0.0819)
	11(7)	0.0267(0.0485)	0.0896(0.0704)	0.0896(0.0704)	0.1462(0.0976)	0.0376(0.0615)	0.0633(0.0599)	0.0904(0.0771)	0.1128(0.0819)
	15(9)	−0.0307(0.0599)	0.0896(0.0704)	0.0896(0.0704)	0.2297(0.1645)	0.0376(0.0615)	0.0633(0.0599)	0.0904(0.0771)	0.1128(0.0819)

Table 2: The average biases and risks (in parentheses) of the Bayes, CGM and PRGM estimators for λ are presented under different censoring percentages, in three class of priors with $\alpha = 1$, $\lambda = 0.4$ and $n = 50$.

		Γ_a			Γ_b			$\Gamma_{a,b}$	
% censor	b(a)	bayesian	CGM	PRGM	bayesian	CGM	PRGM	CGM	PRGM
20	3(2)	0.0427(0.0272)	0.0189(0.0208)	0.0189(0.0208)	−0.0099(0.0232)	0.0164(0.0205)	0.0211(0.0208)	0.0203(0.0211)	0.0233(0.0213)
	7(5)	0.0257(0.0221)	0.0189(0.0208)	0.0189(0.0208)	0.0178(0.0205)	0.0164(0.0205)	0.0211(0.0208)	0.0203(0.0211)	0.0233(0.0213)
	11(7)	0.0101(0.0196)	0.0189(0.0208)	0.0189(0.0208)	0.0363(0.0235)	0.0164(0.0205)	0.0211(0.0208)	0.0203(0.0211)	0.0233(0.0213)
	15(9)	−0.0045(0.0196)	0.0189(0.0208)	0.0189(0.0208)	0.0548(0.0296)	0.0164(0.0205)	0.0211(0.0208)	0.0203(0.0211)	0.0233(0.0213)
35	3(2)	0.0504(0.0328)	0.0216(0.0244)	0.0216(0.0244)	−0.0135(0.0292)	0.0179(0.0241)	0.0236(0.0245)	0.0232(0.0249)	0.0268(0.0251)
	7(5)	0.0296(0.0260)	0.0216(0.0244)	0.0216(0.0244)	0.0199(0.0241)	0.0179(0.0241)	0.0236(0.0245)	0.0232(0.0249)	0.0268(0.0251)
	11(7)	0.0107(0.0231)	0.0216(0.0244)	0.0216(0.0244)	0.0422(0.0277)	0.0179(0.0241)	0.0236(0.0245)	0.0232(0.0249)	0.0268(0.0251)
	15(9)	−0.0067(0.0237)	0.0216(0.0244)	0.0216(0.0244)	0.0645(0.0358)	0.0179(0.0241)	0.0236(0.0245)	0.0232(0.0249)	0.0268(0.0251)
50	3(2)	0.0626(0.0413)	0.0259(0.0293)	0.0259(0.0293)	−0.0189(0.0382)	0.0201(0.0289)	0.0273(0.0292)	0.0277(0.0299)	0.0323(0.0302)
	7(5)	0.0355(0.0314)	0.0259(0.0293)	0.0259(0.0293)	0.0232(0.0287)	0.0201(0.0289)	0.0273(0.0292)	0.0277(0.0299)	0.0323(0.0302)
	11(7)	0.0116(0.0275)	0.0259(0.0293)	0.0259(0.0293)	0.0513(0.0337)	0.0201(0.0289)	0.0273(0.0292)	0.0277(0.0299)	0.0323(0.0302)
	15(9)	−0.0099(0.0291)	0.0259(0.0293)	0.0259(0.0293)	0.0793(0.0454)	0.0201(0.0289)	0.0273(0.0292)	0.0277(0.0299)	0.0323(0.0302)
70	3(2)	0.0981(0.0661)	0.0398(0.0403)	0.0398(0.0403)	−0.0314(0.0626)	0.0262(0.0396)	0.0376(0.0397)	0.0416(0.0421)	0.0493(0.0426)
	7(5)	0.0529(0.0441)	0.0398(0.0403)	0.0398(0.0403)	0.0333(0.0386)	0.0262(0.0396)	0.0376(0.0397)	0.0416(0.0421)	0.0493(0.0426)
	11(7)	0.0152(0.0363)	0.0398(0.0403)	0.0398(0.0403)	0.0763(0.0493)	0.0262(0.0396)	0.0376(0.0397)	0.0416(0.0421)	0.0493(0.0426)
	15(9)	−0.0166(0.0408)	0.0398(0.0403)	0.0398(0.0403)	0.1194(0.0733)	0.0262(0.0396)	0.0376(0.0397)	0.0416(0.0421)	0.0493(0.0426)

Table 3: The average biases and risks (in parentheses) of the Bayes, CGM and PRGM estimators for λ are presented under different censoring percentages, in three class of priors with $\alpha = 1$, $\lambda = 0.4$ and $n = 100$.

		Γ_a			Γ_b			$\Gamma_{a,b}$	
% censor	b(a)	bayesian	CGM	PRGM	bayesian	CGM	PRGM	CGM	PRGM
20	3(2)	0.0210(0.0131)	0.0091(0.0114)	0.0091(0.0114)	−0.0056(0.0122)	0.0084(0.0114)	0.0108(0.0114)	0.0099(0.0114)	0.0114(0.0115)
	7(5)	0.0128(0.0117)	0.0091(0.0114)	0.0091(0.0114)	0.0088(0.0113)	0.0084(0.0114)	0.0108(0.0114)	0.0099(0.0114)	0.0114(0.0115)
	11(7)	0.0049(0.0111)	0.0091(0.0114)	0.0091(0.0114)	0.0184(0.0121)	0.0084(0.0114)	0.0108(0.0114)	0.0099(0.0114)	0.0114(0.0115)
	15(9)	−0.0027(0.0112)	0.0091(0.0114)	0.0091(0.0114)	0.0280(0.0139)	0.0084(0.0114)	0.0108(0.0114)	0.0099(0.0114)	0.0114(0.0115)
35	3(2)	0.0249(0.0158)	0.0104(0.0135)	0.0104(0.0135)	−0.0076(0.0150)	0.0094(0.0134)	0.0124(0.0135)	0.0114(0.0136)	0.0133(0.0136)
	7(5)	0.0149(0.0139)	0.0104(0.0135)	0.0104(0.0135)	0.0099(0.0134)	0.0094(0.0134)	0.0124(0.0135)	0.0114(0.0136)	0.0133(0.0136)
	11(7)	0.0052(0.0132)	0.0104(0.0135)	0.0104(0.0135)	0.0217(0.0144)	0.0094(0.0134)	0.0124(0.0135)	0.0114(0.0136)	0.0133(0.0136)
	15(9)	−0.0039(0.0134)	0.0104(0.0135)	0.0104(0.0135)	0.0334(0.0168)	0.0094(0.0134)	0.0124(0.0135)	0.0114(0.0136)	0.0133(0.0136)
50	3(2)	0.0313(0.019)	0.0126(0.0167)	0.0126(0.0167)	−0.0106(0.0197)	0.0109(0.0167)	0.0148(0.0168)	0.0137(0.0169)	0.0161(0.0169)
	7(5)	0.0181(0.0174)	0.0126(0.0167)	0.0126(0.0167)	0.0118(0.0166)	0.0109(0.0167)	0.0148(0.0168)	0.0137(0.0169)	0.0161(0.0169)
	11(7)	0.0057(0.0163)	0.0126(0.0167)	0.0126(0.0167)	0.0268(0.0179)	0.0109(0.0167)	0.0148(0.0168)	0.0137(0.0169)	0.0161(0.0169)
	15(9)	−0.0059(0.0169)	0.0126(0.0167)	0.0126(0.0167)	0.0417(0.0216)	0.0109(0.0167)	0.0148(0.0168)	0.0137(0.0169)	0.0161(0.0169)
70	3(2)	0.0499(0.0330)	0.0196(0.0252)	0.0196(0.0252)	−0.0181(0.0337)	0.0154(0.0252)	0.0215(0.0252)	0.0211(0.0257)	0.0250(0.0256)
	7(5)	0.0279(0.0265)	0.0196(0.0252)	0.0196(0.0252)	0.0177(0.0250)	0.0154(0.0252)	0.0215(0.0252)	0.0211(0.0257)	0.0250(0.0256)
	11(7)	0.0079(0.0244)	0.0196(0.0252)	0.0196(0.0252)	0.0415(0.0278)	0.0154(0.0252)	0.0215(0.0252)	0.0211(0.0257)	0.0250(0.0256)
	15(9)	−0.0103(0.0264)	0.0196(0.0252)	0.0196(0.0252)	0.0653(0.0359)	0.0154(0.0252)	0.0215(0.0252)	0.0211(0.0257)	0.0250(0.0256)

This data sets were analyzed by Lio et al. (2010), Rao et al. (2015), Soliman (2005), Wu et al. (2014), and Zimmer et al. (1998). Lio et al. (2010) and Zimmer et al. (1998) concluded that the Burr XII distribution provides a good fit for this data set. The hazard function of Burr XII distribution can exhibit either a decreasing or unimodal shape. A useful graphical method to understand the shape of the hazard function of the data is the total time on test (TTT) plot (Aarset, 1987). The TTT plot for this dataset is shown in Figure 2. Based to this plot, there is an indication that the hazard function follows a decreasing pattern. Since

Table 4: The average biases and risks (in parentheses) of the Bayes, CGM and PRGM estimators for λ are presented under different censoring percentages, in three class of priors with $\alpha = 2$, $\lambda = 2$ and $n = 20$.

		Γ_a			Γ_b			$\Gamma_{a,b}$	
% censor	b(a)	bayesian	CGM	PRGM	bayesian	CGM	PRGM	CGM	PRGM
20	3(2)	−0.0589(0.0366)	−0.5912(0.1625)	−0.5912(0.1625)	−0.9329(0.4052)	−0.7739(0.2695)	−0.7429(0.2467)	−0.5470(0.1423)	−0.5302(0.1341)
	7(5)	−0.5870(0.1568)	−0.5912(0.1625)	−0.5912(0.1625)	−0.7557(0.2557)	−0.7739(0.2695)	−0.7429(0.2467)	−0.5470(0.1423)	−0.5302(0.1341)
	11(7)	−0.8883(0.3605)	−0.5912(0.1625)	−0.5912(0.1625)	−0.6376(0.1782)	−0.7739(0.2695)	−0.7429(0.2467)	−0.5470(0.1423)	−0.5302(0.1341)
	15(9)	−1.0833(0.5555)	−0.5912(0.1625)	−0.5912(0.1625)	−0.5194(0.1181)	−0.7739(0.2695)	−0.7429(0.2467)	−0.5407(0.1423)	−0.5302(0.1341)
35	3(2)	−0.0803(0.0420)	−0.6435(0.1958)	−0.6435(0.1958)	−1.0262(0.4996)	−0.8556(0.3386)	−0.8209(0.3095)	−0.5938(0.1701)	−0.5717(0.1579)
	7(5)	−0.6572(0.1998)	−0.6435(0.1958)	−0.6435(0.1958)	−0.8322(0.3184)	−0.8556(0.3386)	−0.8209(0.3095)	−0.5938(0.1701)	−0.5717(0.1579)
	11(7)	−0.9669(0.4371)	−0.6435(0.1958)	−0.6435(0.1958)	−0.7029(0.2213)	−0.8556(0.3386)	−0.8209(0.3095)	−0.5938(0.1701)	−0.5717(0.1579)
	15(9)	−1.1603(0.6409)	−0.6435(0.1958)	−0.6435(0.1958)	−0.5736(0.1454)	−0.8556(0.3386)	−0.8209(0.3095)	−0.5938(0.1701)	−0.5717(0.1579)
50	3(2)	−0.1065(0.0493)	−0.9556(0.2384)	−0.9159(0.2384)	−1.1389(0.6167)	−0.9556(0.4317)	−0.9159(0.3942)	−0.6484(0.2058)	−0.6166(0.1857)
	7(5)	−0.7437(0.2598)	−0.9556(0.2384)	−0.9159(0.2384)	−0.9245(0.4015)	−0.9556(0.4317)	−0.9159(0.3942)	−0.6484(0.2058)	−0.6166(0.1857)
	11(7)	−1.0597(0.5331)	−0.9556(0.2384)	−0.7049(0.2384)	−0.7815(0.2793)	−0.9556(0.4317)	−0.9159(0.3942)	−0.6484(0.2058)	−0.6166(0.1857)
	15(9)	−1.2486(0.7358)	−0.9556(0.2384)	−0.7049(0.2384)	−0.6385(0.1819)	−0.9556(0.4317)	−0.9159(0.3942)	−0.6484(0.2058)	−0.6166(0.1857)
70	3(2)	−0.1589(0.0608)	−0.8086(0.3169)	−0.8086(0.3169)	−1.3329(0.8045)	−1.1333(0.6112)	−1.0819(0.5586)	−0.7423(0.2726)	−0.6747(0.2211)
	7(5)	−0.8987(0.3852)	−0.8086(0.3169)	−0.8086(0.3169)	−1.0829(0.5595)	−1.1333(0.6112)	−1.0819(0.5586)	−0.7423(0.2726)	−0.6747(0.2211)
	11(7)	−1.2143(0.6975)	−0.8086(0.3169)	−0.8086(0.3169)	−0.9162(0.3932)	−1.1333(0.6112)	−1.0819(0.5586)	−0.7423(0.2726)	−0.6747(0.2211)
	15(9)	−1.3892(0.8680)	−0.8086(0.3169)	−0.8086(0.3169)	−0.7496(0.2533)	−1.1333(0.6112)	−1.0819(0.5586)	−0.7423(0.2726)	−0.6747(0.2211)

Table 5: The average biases and risks (in parentheses) of the Bayes, CGM and PRGM estimators for λ are presented under different censoring percentages, in three class of priors with $\alpha = 2$, $\lambda = 2$ and $n = 50$.

		Γ_a			Γ_b			$\Gamma_{a,b}$	
% censor	b(a)	bayesian	CGM	PRGM	bayesian	CGM	PRGM	CGM	PRGM
20	3(2)	−0.0219(0.0206)	−0.3722(0.0657)	−0.3722(0.0657)	−0.5407(0.1257)	−0.4418(0.0855)	−0.4244(0.0795)	−0.3515(0.0604)	−0.3456(0.0587)
	7(5)	−0.3194(0.0524)	−0.3722(0.0657)	−0.3722(0.0.0657)	−0.4368(0.0837)	−0.4418(0.0855)	−0.4244(0.0795)	−0.3515(0.0604)	−0.3456(0.0587)
	11(7)	−0.5388(0.1223)	−0.3722(0.0657)	−0.3722(0.0657)	−0.3676(0.0619)	−0.4418(0.0855)	−0.4244(0.0795)	−0.3515(0.0604)	−0.3456(0.0587)
	15(9)	−0.7074(0.2134)	−0.3722(0.0657)	−0.3722(0.0657)	−0.2983(0.0448)	−0.4418(0.0855)	−0.4244(0.0795)	−0.3515(0.0604)	−0.3456(0.0587)
35	3(2)	−0.0364(0.0245)	−0.4259(0.0862)	−0.4259(0.0862)	−0.6268(0.1741)	−0.5147(0.1175)	−0.4945(0.1089)	−0.4011(0.0785)	−0.3938(0.0760)
	7(5)	−0.3780(0.0713)	−0.4259(0.0862)	−0.4259(0.0862)	−0.5077(0.1144)	−0.5147(0.1175)	−0.4945(0.1089)	−0.4011(0.0785)	−0.3938(0.0760)
	11(7)	−0.6181(0.1657)	−0.4259(0.0862)	−0.4259(0.0862)	−0.4283(0.0836)	−0.5147(0.1175)	−0.4945(0.1089)	−0.4011(0.0785)	−0.3938(0.0760)
	15(9)	−0.7962(0.2812)	−0.4259(0.0862)	−0.4259(0.0862)	−0.3489(0.0593)	−0.5147(0.1175)	−0.4945(0.1089)	−0.4011(0.0785)	−0.3938(0.0760)
50	3(2)	−0.0514(0.0299)	−0.4919(0.1158)	−0.4919(0.1158)	−0.7391(0.2503)	−0.6095(0.1681)	−0.5857(0.1552)	−0.4609(0.1041)	−0.4510(0.0999)
	7(5)	−0.4545(01011)	−0.4919(0.1158)	−0.4919(0.1158)	−0.5993(0.1622)	−0.6095(0.1681)	−0.5857(0.1552)	−0.4609(0.1041)	−0.4510(0.0999)
	11(7)	−0.7192(0.2318)	−0.4919(0.1158)	−0.4919(0.1158)	−0.5061(0.1167)	−0.6095(0.1681)	−0.5857(0.1552)	−0.4609(0.1041)	−0.4510(0.0999)
	15(9)	−0.9064(0.3784)	−0.4919(0.1158)	−0.4919(0.1158)	−0.4129(0.0808)	−0.6095(0.1681)	−0.5857(0.1552)	−0.4609(0.1041)	−0.4510(0.0999)
70	3(2)	−0.0799(0.0412)	−0.6142(0.1829)	−0.6142(0.1829)	−0.9632(0.4428)	−0.8012(0.3011)	−0.7691(0.2761)	−0.5696(0.1605)	−0.5511(0.1502)
	7(5)	−0.6123(0.1804)	−0.6142(0.1829)	−0.6142(0.1829)	−0.7812(0.2847)	−0.8012(0.3011)	−0.7691(0.2761)	−0.5696(0.1605)	−0.5511(0.1502)
	11(7)	−0.9135(0.3916)	−0.6142(0.1829)	−0.6142(0.1829)	−0.6599(0.2001)	−0.8012(0.3011)	−0.7691(0.2761)	−0.5696(0.1605)	−0.5511(0.1502)
	15(9)	−1.1072(0.5836)	−0.6142(0.1829)	−0.6142(0.1829)	−0.5386(0.1334)	−0.8012(0.3011)	−0.7691(0.2761)	−0.5696(0.1605)	−0.5511(0.1502)

Table 6: The average biases and risks (in parentheses) of the Bayes, CGM and PRGM estimators for λ are presented under different censoring percentages, in three class of priors with $\alpha = 2$, $\lambda = 2$ and $n = 100$.

		Γ_a			Γ_b			$\Gamma_{a,b}$	
% censor	b(a)	bayesian	CGM	PRGM	bayesian	CGM	PRGM	CGM	PRGM
20	3(2)	−0.0119(0.0110)	−0.2331(0.0275)	−0.2331(0.0275)	−0.3188(0.0436)	−0.2590(0.0314)	−0.2487(0.0296)	−0.2227(0.0259)	−0.2199(0.0255)
	7(5)	−0.1826(0.0208)	−0.2331(0.0275)	−0.2331(0.0275)	−0.2574(0.0311)	−0.2590(0.0314)	−0.2487(0.0296)	−0.2227(0.0259)	−0.2199(0.0255)
	11(7)	−0.3263(0.0444)	−0.2331(0.0275)	−0.2331(0.0275)	−0.2165(0.0245)	−0.2590(0.0314)	−0.2487(0.0296)	−0.2227(0.0259)	−0.2199(0.0255)
	15(9)	−0.4488(0.0787)	−0.2331(0.0275)	−0.2331(0.0275)	−0.1756(0.0191)	−0.2590(0.0314)	−0.2487(0.0296)	−0.2227(0.0259)	−0.2199(0.0255)
35	3(2)	−0.0187(0.0134)	−0.2740(0.0371)	−0.2740(0.0371)	−0.3794(0.0619)	−0.3093(0.0439)	−0.2972(0.0412)	−0.2612(0.0348)	−0.2577(0.0341)
	7(5)	−0.2207(0.0284)	−0.2740(0.0371)	−0.2740(0.0371)	−0.3070(0.0433)	−0.3093(0.0439)	−0.2972(0.0412)	−0.2612(0.0348)	−0.2577(0.0341)
	11(7)	−0.3853(0.0623)	−0.2740(0.0371)	−0.2740(0.0371)	−0.2587(0.0335)	−0.3093(0.0439)	−0.2972(0.0412)	−0.2612(0.0348)	−0.2577(0.0341)
	15(9)	−0.5219(0.1099)	−0.2740(0.0371)	−0.2740(0.0371)	−0.2105(0.0255)	−0.3093(0.0439)	−0.2972(0.0412)	−0.2612(0.0348)	−0.2577(0.0341)
50	3(2)	−0.0268(0.0172)	−0.3294(0.0529)	−0.3294(0.0529)	−0.4651(0.0945)	−0.3803(0.0656)	−0.3855(0.0613)	−0.3129(0.0491)	−0.3083(0.0479)
	7(5)	−0.2748(0.0414)	−0.3294(0.0529)	−0.3294(0.0529)	−0.3768(0.0645)	−0.3803(0.0656)	−0.3855(0.0613)	−0.3129(0.0491)	−0.3083(0.0479)
	11(7)	−0.4674(0.9313)	−0.3294(0.0529)	−0.3294(0.0529)	−0.3179(0.0488)	−0.3803(0.0656)	−0.3855(0.0613)	−0.3129(0.0491)	−0.3083(0.0479)
	15(9)	−0.6212(0.1624)	−0.3294(0.0529)	−0.3294(0.0529)	−0.2590(0.0362)	−0.3803(0.0656)	−0.3855(0.0613)	−0.3129(0.0491)	−0.3083(0.0479)
70	3(2)	−0.0486(0.0269)	−0.4504(0.0987)	−0.4504(0.0987)	−0.6632(0.2014)	−0.5460(0.1365)	−0.5248(0.1265)	−0.4241(0.0896)	−0.4160(0.0865)
	7(5)	−0.4048(0.0835)	−0.4504(0.0987)	−0.4504(0.0987)	−0.5382(0.1325)	−0.5460(0.1365)	−0.5248(0.1265)	−0.4241(0.0896)	−0.4160(0.0865)
	11(7)	−0.6509(0.1894)	−0.4504(0.0987)	−0.4504(0.0987)	−0.4548(0.0967)	−0.5460(0.1365)	−0.5248(0.1265)	−0.4241(0.0896)	−0.4160(0.0865)
	15(9)	−0.8313(0.3139)	−0.4504(0.0987)	−0.4504(0.0987)	−0.3714(0.0683)	−0.5460(0.1365)	−0.5248(0.1265)	−0.4241(0.0896)	−0.4160(0.0865)

the hazard function of the Burr XII distribution is decreasing, for $0 < \alpha \leq 1$, α parameter can be resonably be set to 1.

The original data is not censored, however we chose to right-censored any data larger than 12. With this approach, approximately %30 of the data is right-censored. In Table 7, the results related to the Bayesian, CGM and PRGM estimates of λ under the classes Γ_a, Γ_b, and $\Gamma_{a,b}$ are given. For the class $\Gamma_a(\Gamma_b)$, it can be seen that with the increase in the value of b(a), the values of Bayesian estimates decrease (increase). In general, in the class $\Gamma_a(\Gamma_b)$, for $b = 3, 7 (a = 2, 5)$ the values of the Bayesian estimations of λ are larger (smaller) compared to the CGM and PRGM values, but for $b = 11, 15 (a = 7, 9)$ this result is reversed. Also, in

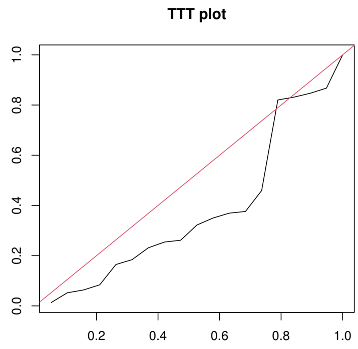

Figure 2: The TTT plot for real dataset.

the classes Γ_b and $\Gamma_{a,b}$, the values of PRGM are greater than the values of CGM.

Table 7: Bayesian, CGM and PRGM estimations for λ under the classes Γ_a, Γ_b and $\Gamma_{a,b}$ in the real dataset.

class			Bayes	CGM	PRGM
Γ_a		3	0.4850	0.4246	0.4246
	b	7	0.4378	0.4246	0.4246
		11	0.3990	0.4246	0.4246
		15	0.3665	0.4246	0.4246
Γ_b		2	0.3479	0.4092	0.4216
	a	5	0.4175	0.4092	0.4216
		7	0.4639	0.4092	0.4216
		9	0.5103	0.4092	0.4216
$\Gamma_{a,b}$	-		-	0.4263	0.4349
	-		-	0.4263	0.4349
	-		-	0.4263	0.4349
	-		-	0.4263	0.4349

7. CONCLUDING REMARKS

In this paper, we developed the Posterior Regret Gamma-Minimax and Conditional Gamma-Minimax estimators for the parameter λ in the Burr XII distribution. These estimators are obtained under the reflected gamma loss function and were evaluated across different classes of prior distributions (Γ_a, Γ_b and $\Gamma_{a,b}$). To assess the performance of the robust estimators, we conducted a simulation study and analyzed a real dataset. In the simulation study, we examined the biases and risks of robust Bayesian estimates of λ. The results revealed that the robust estimators exhibited both overestimations and underestimations in estimating λ, depending on the specific scenario and prior class. Furthermore, we investigated the impact of the censoring on the bias and risk values of the robust Bayesian estimators. The simulation study, demonstrated that as the volume of censorship increased, the biases and risks of the estimators also increased. This observation suggests that higher levels of censoring can introduce additional challenges and uncertainties in accurately estimating the parameter λ.

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