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LIMIT MODEL FOR THE RELIABILITY OF A REGULAR AND HOMOGENEOUS SERIES-PARALLEL SYSTEM

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Abstract:

- In large-scale systems the study of the exact reliability function can be an intricate problem. In these cases it is better to admit that the number of system components goes to infinity so as to find asymptotic models that give a good interpretation of the reliability. In this paper we will use some results of extreme value theory to obtain the asymptotic distribution of the reliability of a regular and homogeneous series-parallel system.

Key-Words:

- *reliability; series-parallel systems; extreme value theory; domains of atraction.*

AMS Subject Classification:

- 60K10, 60G70.

1. INTRODUCTION

When we study the reliability of some technological systems, we frequently find very complex structures due to large numbers of system components and the way the operating process uses such components. Indeed, there are situations that cannot be modelled as a simple parallel (or series) system and are best described as a series-parallel or parallel-series system. Examples of large systems with complex structures arise in transport networks of gas, oil, water and other fluids; also on telecommunication and electrical energy distribution networks and on charge and discharge networks.

The asymptotic theory of extremes, established by Gnedenko ([5]) in 1943, immediately leads to the identification of limit models for the reliability of systems with a large number of components in series or in parallel. Posterior results, such as those by Smirnov ([8]), Chernoff and Teicher ([3]) and Kolowrocki ([6] and [7]), have dealt with the same problem for homogeneous series-parallel (or parallel-series) systems. In turn, our approach will use the characterization of domains of attraction for minima for the known generalized extreme value distributions, as developed by Balkema and de Haan ([1]). In this initial work, we will restrict ourselves identifying limit laws in regular and homogeneous series-parallel systems, whenever the lifetime distribution function of each component belongs to some domain of attraction for minima.

1.1. Some basic notions of extreme value theory

Given a sequence of independent and identically distributed (i.i.d.) random variables, $\{X_i\}$, $i \geq 1$, with distribution function F, the random variable $M_n = \max(X_1, X_2, ..., X_n)$, with $n \geq 1$, has a known distribution function, given by

$$F_{M_n}(x) = \big[F(x)\big]^n .$$

If there exists a pair of sequences (a_n, b_n) where $a_n > 0$ and $b_n \in \mathbb{R}, \forall n \in \mathbb{N}$, and a nondegenerate distribution function G, such that, for all x where G is continuous,

$$P\big[M_n \leq a_n x + b_n\big] = \big[F(a_n x + b_n)\big]^n \underset{n\to\infty}{\longrightarrow} G(x) , \tag{1.1}$$

then G must be a Gumbel, a Fréchet or a Weibull distribution, whose standard forms are

$$\begin{aligned} &\text{Gumbel:} && \Lambda(x) = \exp(-e^{-x}), \quad x \in \mathbb{R} \\ &\text{Fréchet:} && \Phi_\alpha(x) = \exp(-x^{-\alpha}), \quad \alpha > 0, \ \ x \geq 0 \\ &\text{Weibull:} && \Psi_\alpha(x) = \exp\big(-(-x)^{-\alpha}\big), \quad \alpha < 0, \ \ x \leq 0 . \end{aligned}$$

These distributions can be represented uniquely in a parametric form, called the *von Mises–Jenkinson form* or generalized extreme value distribution (GEV),

$$(1.2)\qquad G_\gamma(x) = \begin{cases} \exp\big(-(1+\gamma x)^{-1/\gamma}\big), & 1+\gamma x \ge 0, \quad \gamma \ne 0 \\ \exp(-e^{-x}), & x \in \mathbb{R}, \qquad \gamma = 0~. \end{cases}$$

It is easy to see that

$$G_\gamma(x) = \begin{cases} \Lambda(x), & \gamma = 0 \\ \Phi_{1/\gamma}(1+\gamma x), & \gamma > 0 \\ \Psi_{-1/\gamma}\big(-(1+\gamma x)\big), & \gamma < 0~. \end{cases}$$

Whenever the sequences a_n and b_n exist on the above described conditions, or in other words, verifying (1.1), we will say that the distribution function F belongs to or is in the domain of attraction of G (for maxima) and we write $F \in \mathcal{D}(G)$.

The characterization of domains of attraction is closely related to the study of regular variation. Our approach about the asymptotic behaviour of the distribution function or of the reliability function for a series-parallel system, lies in known results which involve regular varying functions. We say that a real valued function, R, is regularly varying, with index ρ, at infinity and we write, $R \in \mathcal{R}_\rho$, if it is positive and measurable in $[a, +\infty[$, for some $a > 0$ and if $\forall x > 0$,

$$(1.3)\qquad \lim_{t\to\infty} \frac{R(tx)}{R(t)} = x^\rho~,$$

for some $\rho \in \mathbb{R}$. When $\rho = 0$, R is called a slowly varying function.

Gnedenko (1943), Balkema and de Haan (1972) established a relation between regular variation and the characterization of domains of attraction for Weibull and Fréchet laws, described in the following Theorem:

Theorem 1.1.

(1) *A distribution function F is in the domain of attraction of a Weibull law, Ψ_α, iff the right end point*[1] $x^F < \infty$ *and* $1-F\big(x^F - \frac{1}{x}\big) \in R_{-\alpha}$, *when $x \to \infty$. In this case, taking δ_n such that* $n\big(1-F(\delta_n)\big) \underset{n\to\infty}{\longrightarrow} 1$, *we will have*

$$F^n\Big(x^F + (x^F - \delta_n)\,x\Big) \longrightarrow \Psi_\alpha(x)\,, \qquad x < 0~.$$

(2) *A distribution function F is in the domain of attraction of a Fréchet law, Φ_α, iff $1-F \in R_{-\alpha}$. In this case*

$$F^n(a_n x) \longrightarrow \Phi_\alpha(x)\,, \qquad x > 0~,$$

with a_n such that $n\big(1-F(a_n)\big) \underset{n\to\infty}{\longrightarrow} 1$.

[1] Given a distribution function F, absolutely continuous, the right end point of its support is $x^F \equiv \sup\{x\colon F(x) < 1\}$.

It must be noted that only distribution functions with infinite right end point can be in $\mathcal{D}(\Phi_\alpha)$.

For the domain of attraction of a Gumbel law, we will use the characterization established by Balkema and de Haan ([1]):

Theorem 1.2. *A distribution function F belongs to $\mathcal{D}(\Lambda)$ iff there exist a positive function w satisfying $\lim\limits_{x\to x^F} w(x)=1$ and a differentiable, positive function g such that*

$$-\ln F(x) \;=\; w(x)\,\exp\left\{-\int_{z_0}^{x}\frac{1}{g(u)}\,du\right\},$$

for some z_0 and where we have $\lim\limits_{x\to x^F} g'(x)=0$.

The results developed for asymptotic extreme value theory for maxima are readily adapted for minima, since $m_n=\min\limits_{1\le i\le n}(X_i)=-\max\limits_{1\le i\le n}(-X_i)=-\max\limits_{1\le i\le n}(Y_i)$. If the sequence $\max\limits_{1\le i\le n}(-X_i)=\max\limits_{1\le i\le n}(Y_i)$ can be normalized, so as to admit a non degenerate limit Z, then the distribution function Z will be of the same type as G_γ, for some $\gamma\in\mathbb{R}$. Hence the limit law for minima, conveniently normalized, will verify

$$F_{-Z}(x) \;=\; P\big[-Z\le x\big] \;=\; P\big[Z\ge -x\big] \;=\; 1-G_\gamma(-x) \;=:\; H_\gamma(x)\;.$$

Therefore, we say that the distribution function F of a random variable X is in the domain of attraction for minima of H_γ, if the distribution function of $-X$ is in the domain of attraction (for maxima) of G_γ. In this case, there exists a pair of sequences (a_n,b_n) where $a_n>0$ and $b_n\in\mathbb{R}$, $\forall n\in\mathbb{N}$, such that

$$1-\big(1-F(a_nx+b_n)\big)^n \underset{n\to\infty}{\longrightarrow} H_\gamma(x)\;. \tag{1.4}$$

Remark 1.1. In most applications involving lifetimes the limit laws in (1.4) are restricted to the case $\gamma\le 0$. In fact, a lifetime T is always nonnegative, thus $-T$ is a random variable with finite right end point and can only be in the max-domain of attraction of a Weibull ($\gamma<0$) or a Gumbel ($\gamma=0$) (see Theorem 1.1 and Theorem 1.2). However, because there are systems with large durability, we will also study the case $\gamma>0$.

1.2. Regular and homogeneous series-parallel system

In reliability studies, we classify a system as being *series-parallel* if it is composed by subsystems with components in series and if those subsystems are organized in parallel (see Figure 1).

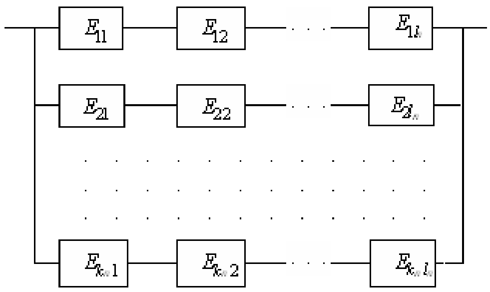

Figure 1: Scheme of a regular homogeneous series-parallel system.

Let E_{ij}, with $i=1,2,...,k$ and $j=1,2,...,l_i$, be the components of a *Series-Parallel System* S, formed by k subsystems in parallel of l_i components in series. Let X_{ij} be the lifetime of E_{ij}, i.e., X_{ij} represents the lifetime of the j-th component of the i-th subsystem. We will assume that all X_{ij}'s are independent. The lifetime T of the whole system is given by

$$T = \max_{1\le i\le k}\left(\min_{1\le j\le l_i} X_{ij}\right) .$$

The system S is called *regular* whenever

$$l_1 = l_2 = ... = l_k = l \ ,$$

and it is *homogeneous* whenever the components E_{ij} have the same *reliability function* $R(x) = P(X_{ij} > x) = 1 - F(x)$, with $x \in]-\infty, +\infty[$, i.e., if the random variables X_{ij} have the same distribution function $F(x) = P(X_{ij} \le x)$.

Suppose now that $k = k_n$ and $l = l_n$, i.e., (k_n) and (l_n) are sequences of real numbers such that at least one of them has a limit equal to infinity when n goes to infinity. Then we obtain sequences of regular systems whose families of reliability functions, in the case of an homogeneous system, are defined by (see in [6])

$$R_n(x) = 1 - \Big[1 - \big(R(x)\big)^{l_n}\Big]^{k_n}, \qquad \text{for} \ \ x \in]-\infty, +\infty[\quad \text{and each} \quad n \in \mathbb{N} \ ,$$

or in terms of the sequence of distribution functions,

$$\begin{aligned} F_n(x) &= 1 - \left(1 - \Big[1 - \big(R(x)\big)^{l_n}\Big]^{k_n}\right) \\ &= \Big[1 - \big(1 - F(x)\big)^{l_n}\Big]^{k_n}, \qquad \text{for} \ \ x \in]-\infty, +\infty[\quad \text{and each} \quad n \in \mathbb{N} \ . \end{aligned} \tag{1.5}$$

Assuming that F is in the domain of attraction of a law for minima, H_γ, our purpose will be to analyse the asymptotic behaviour of the functions $R_n(x)$ and $F_n(x)$ defined above. Although our goal is to treat this problem in its maximum generality, in this paper we will only treat the case where k_n goes to infinity. More precisely, we will suppose that $k_n = n$ and investigate which should be the asymptotic behaviour of l_n, so that, using a suitable normalization, we can find a nondegenerate limit for F_n.

2. CHARACTERIZATION OF THE DOMAINS OF ATTRACTION

From (1.5), it follows that $F_n(x)$ is the distribution function of the maxima of k_n i.i.d. random variables, each one with distribution function given by $H_n(x) = \left[1-\left(1-F(x)\right)^{l_n}\right]$, and we want to determine in which domain of attraction for maxima belongs $H_n(x)$. Now, assuming that F is in the domain of attraction for minima of a law $H_\gamma(x) = 1-G_\gamma(-x)$, then the asymptotic behaviour of the right tail of H_n must be similar to the right tail of the minima law H_γ. In the next paragraphs we will analyse the right tail behaviour of H_γ, for $\gamma<0$, $\gamma=0$ and $\gamma>0$, in order to identify the max-stable law to which it is attracted.

2.1. Case $\gamma < 0$ (Weibull for minima)

The function $H_\gamma(x)$ is defined, for all $x \in \mathbb{R}$, by

$$\begin{aligned}
H_\gamma(x) &= 1 - G_\gamma(-x) \\
&= \begin{cases} 1-\exp\Big(-(1-\gamma x)^{-1/\gamma}\Big), & 1-\gamma x \geq 0 \\ 0, & 1-\gamma x < 0 \end{cases} \qquad (2.1) \\
&= \begin{cases} 1-\exp\Big(-(1-\gamma x)^{-1/\gamma}\Big), & x \geq \frac{1}{\gamma} \\ 0, & x < \frac{1}{\gamma} \;. \end{cases}
\end{aligned}$$

Since the right end point is infinite, we will first check whether or not H_γ is in the Fréchet max-domain of attraction. By Theorem 1, and using (1.3) and (2.1), we have

$$\begin{aligned}
\lim_{t\to+\infty} \frac{1-H_\gamma(tx)}{1-H_\gamma(t)} &= \lim_{t\to+\infty} \frac{e^{-(1-\gamma tx)^{-1/\gamma}}}{e^{-(1-\gamma t)^{-1/\gamma}}} \\
&= \lim_{t\to+\infty} \exp\left\{\Big(-(1-\gamma\, t\, x)^{-1/\gamma}\Big)\left[1-\left(\frac{1-\gamma\, t}{1-\gamma\, t\, x}\right)^{-1/\gamma}\right]\right\} \\
&= \lim_{t\to+\infty} \exp\left\{\Big(-(1-\gamma\, t\, x)^{-1/\gamma}\Big)\left[1-\left(\frac{\frac{1}{\gamma t}-1}{\frac{1}{\gamma t}-x}\right)^{-1/\gamma}\right]\right\} \\
&= \begin{cases} 0, & x^{1/\gamma} < 1 \\ +\infty, & x^{1/\gamma} > 1 \;. \end{cases}
\end{aligned}$$

It follows that $1-H_\gamma$ is not a regularly varying function at infinity and therefore cannot belong to the Fréchet max-domain of attraction. We claim, however, that $H_\gamma \in \mathcal{D}(\Lambda)$. In fact, since

$$\ln H_\gamma(x) = \ln\Big(1-\big(1-H_\gamma(x)\big)\Big) \sim -\big(1-H_\gamma(x)\big) \;,$$

when $x \to x^{H_\gamma}$, we get $-\frac{\ln H_\gamma(x)}{1-H_\gamma(x)} \underset{x\to x^{H_\gamma}}{\longrightarrow} 1$. Without loss of generality, let us suppose $w(x) = -\frac{\ln H_\gamma(x)}{1-H_\gamma(x)} > 0$, for all x, to get

$$\begin{aligned} -\ln H_\gamma(x) &= \left(-\frac{\ln H_\gamma(x)}{1-H_\gamma(x)}\right)\left(1-H_\gamma(x)\right) \\ &= w(x)\exp\Big(-(1-\gamma x)^{-1/\gamma}\Big) \\ &= w(x)\exp\left\{-\int_{1/\gamma}^{x}\frac{1}{g(u)}\,du\right\}, \end{aligned}$$

for $x > \frac{1}{\gamma}$ and where $g(x) = (1-\gamma x)^{1/\gamma+1} > 0$ is such that

$$\lim_{x\to+\infty} g'(x) = \lim_{x\to+\infty}\Big[-(\gamma+1)\,(1-\gamma x)^{1/\gamma}\Big] = 0\ .$$

Consequently, by Theorem 1.2, $H_\gamma(x)$ is in the Gumbel max-domain of attraction. In this case, the attraction constants can be defined by (see [4])

$$\begin{cases} b_n\colon\quad H_\gamma(b_n) = \exp\left(-\dfrac{1}{n}\right) \\ a_n\colon\quad a_n = \dfrac{1}{k(b_n)}\ , \end{cases}$$

where

$$k(b_n) = -\frac{H'_\gamma(b_n)}{H_\gamma(b_n)\,\ln H_\gamma(b_n)}\ .$$

Now,

$$\begin{aligned} H_\gamma(b_n) = \exp\left(-\frac{1}{n}\right) &\iff (1-\gamma b_n)^{-1/\gamma} = -\ln\left[1-\exp\left(-\frac{1}{n}\right)\right] \\ &\iff 1-\gamma b_n = \left(-\ln\left[1-\exp\left(-\frac{1}{n}\right)\right]\right)^{-\gamma} \\ &\iff b_n = \frac{1-\Big(-\ln\Big[1-\exp\big(-\frac{1}{n}\big)\Big]\Big)^{-\gamma}}{\gamma}\ . \end{aligned}$$

On the other hand,

$$\begin{aligned} a_n &= \frac{1}{k(b_n)} \\ &= -\frac{\exp\big(-\frac{1}{n}\big)\,\ln\Big(\exp\big(-\frac{1}{n}\big)\Big)}{(1-\gamma b_n)^{-1/\gamma-1}\Big(1-\exp\big(-\frac{1}{n}\big)\Big)} \\ &= \frac{1}{n\Big(\exp\big(\frac{1}{n}\big)-1\Big)\,(1-\gamma b_n)^{-1/\gamma-1}} \\ &= \frac{1}{n\Big(\exp\big(\frac{1}{n}\big)-1\Big)\Big(-\ln\Big[1-\exp\big(-\frac{1}{n}\big)\Big]\Big)^{\gamma+1}}\ . \end{aligned}$$

It follows that

$$(2.2)\qquad \begin{cases} a_n = \dfrac{1}{n\Big(\exp\big(\frac{1}{n}\big)-1\Big)\Big(-\ln\Big[1-\exp\big(-\frac{1}{n}\big)\Big]\Big)^{\gamma+1}}\,, \\[2ex] b_n = \dfrac{1-\Big(-\ln\Big[1-\exp\big(-\frac{1}{n}\big)\Big]\Big)^{-\gamma}}{\gamma}\,, \qquad\qquad \gamma<0\,. \end{cases}$$

2.2. Case $\gamma > 0$ (Fréchet for minima)

Let us suppose that the distribution function of the lifetime of each component belongs to the domain of attraction of a Fréchet for minima. We have, for $x \in \mathbb{R}$,

$$(2.3)\qquad \begin{aligned} H_\gamma(x) &= 1 - G_\gamma(-x) \\ &= \begin{cases} 1-\exp\Big(-(1-\gamma x)^{-1/\gamma}\Big)\,, & x \le \frac{1}{\gamma} \\ 1\,, & x > \frac{1}{\gamma}\,. \end{cases} \end{aligned}$$

In this case, since $x^{H_\gamma} = \frac{1}{\gamma}$, $H_\gamma(x)$ cannot be in the Fréchet domain of attraction for maxima, but it can, however, be in the max-domain of attraction of a Weibull or a Gumbel. Now,

$$\begin{aligned} 1 - H_\gamma\Big(x^{H_\gamma} - \frac{1}{x}\Big) &= \exp\bigg(-\Big(1-\gamma\Big(\frac{1}{\gamma}-\frac{1}{x}\Big)\Big)^{-1/\gamma}\bigg) \\ &= \exp\bigg(-\Big(\frac{\gamma}{x}\Big)^{-1/\gamma}\bigg) \\ &= e^{-\gamma^{-1/\gamma}x^{1/\gamma}}\,, \qquad \gamma>0, \quad x>0\,, \end{aligned}$$

so applying Theorem 1.2 and (2.3) we get

$$\begin{aligned} \lim_{t\to\infty} \frac{e^{-\gamma^{-1/\gamma}(tx)^{1/\gamma}}}{e^{-\gamma^{-1/\gamma}t^{1/\gamma}}} &= \lim_{t\to\infty} e^{-\gamma^{-1/\gamma}t^{1/\gamma}(x^{1/\gamma}-1)} \\ &= \begin{cases} 0\,, & x^{1/\gamma} > 1 \\ +\infty\,, & x^{1/\gamma} < 1\,. \end{cases} \end{aligned}$$

We conclude therefore that the function H_γ is not in the domain of attraction for maxima of a Weibull. Following the same reasoning as in the previous case, we now prove that H_γ verifies the representation given by Theorem 1.2, and so H_γ

is in the Gumbel max-domain of attraction. In fact, with $w(x) = -\frac{\ln H_\gamma(x)}{1-H_\gamma(x)}$, we have

$$\begin{aligned} -\ln H_\gamma(x) &= w(x) \exp\Big(-(1-\gamma x)^{-1/\gamma}\Big) \\ &= w(x) \exp\left\{-\int_{-\infty}^{x} \frac{1}{g(u)}\, du\right\}, \end{aligned}$$

for $x < \frac{1}{\gamma}$ and where $g(x) = (1-\gamma x)^{1/\gamma+1} > 0$ is such that

$$\lim_{x\to\frac{1}{\gamma}} g'(x) = \lim_{x\to\frac{1}{\gamma}} \Big[-(\gamma+1)\,(1-\gamma x)^{1/\gamma}\Big] = 0 .$$

The possible attraction constants are defined by (see [4])

$$\begin{cases} b_n\colon \quad H_\gamma(b_n) = \exp\Big(-\dfrac{1}{n}\Big) \\[2mm] a_n\colon \quad a_n = \dfrac{1}{k(b_n)} . \end{cases}$$

Calculations similar to the case $\gamma < 0$ yield

$$\text{(2.4)} \qquad \begin{cases} a_n = \dfrac{1}{n\Big(\exp(\frac{1}{n})-1\Big)\Big(-\ln\Big[1-\exp(-\frac{1}{n})\Big]\Big)^{\gamma+1}}, \\[4mm] b_n = \dfrac{1-\Big(-\ln\Big[1-\exp(-\frac{1}{n})\Big]\Big)^{-\gamma}}{\gamma}, \qquad \gamma > 0 . \end{cases}$$

2.3. Case $\gamma = 0$ (Gumbel for minima)

Finally we analyse the case where the lifetime of the components is in the domain of attraction of a Gumbel for minima. The function $H_0(x)$ is defined, for all $x \in \mathbb{R}$, by

$$H_0(x) = 1 - G_0(-x) = 1 - \exp\big(-\exp(x)\big),$$

so that

$$-\ln H_0(x) = w(x) \exp\left\{-\int_{-\infty}^{x} e^u\, du\right\},$$

with $w(x)$ defined as in the previous cases. Once again, the conditions of Theorem 1.2 are verified, considering $g(x) = e^{-x} > 0$, $\forall x \in \mathbb{R}$, and therefore the distribution function $H_0(x)$ is in the max-domain of attraction of a Gumbel law. The sequences (a_n) and (b_n) are now given by

$$\text{(2.5)} \qquad \begin{cases} a_n = \dfrac{1}{n\Big(\exp(\frac{1}{n})-1\Big)\Big(-\ln\Big[1-\exp(-\frac{1}{n})\Big]\Big)}, \\[4mm] b_n = \ln\Big(-\ln\Big[1-\exp(-\frac{1}{n})\Big]\Big), \end{cases}$$

since

$$H_0(b_n) = \exp\Big(-\frac{1}{n}\Big) \iff \exp(-e^{b_n}) = 1 - \exp\Big(-\frac{1}{n}\Big)$$
$$\iff e^{b_n} = -\ln\Big[1 - \exp\Big(-\frac{1}{n}\Big)\Big]$$
$$\iff b_n = \ln\Big(-\ln\Big[1 - \exp\Big(-\frac{1}{n}\Big)\Big]\Big)$$

and

$$\begin{aligned} a_n &= \frac{1}{k(b_n)} \\ &= -\frac{\Big(1-\exp\big(-\exp(b_n)\big)\Big)\ \ln\Big(1-\exp(-\exp\big(b_n)\big)\Big)}{\exp(b_n)\ \exp\big(-\exp(b_n)\big)} \\ &= \frac{\exp\big(-\frac{1}{n}\big)\ \ln\Big(\exp\big(-\frac{1}{n}\big)\Big)}{\ln\Big(1-\exp\big(-\frac{1}{n}\big)\Big)\ \Big(1-\exp\big(-\frac{1}{n}\big)\Big)} \\ &= \frac{1}{n\Big(\exp\big(\frac{1}{n}\big)-1\Big)\ \Big(-\ln\Big[1-\exp\big(-\frac{1}{n}\big)\Big]\Big)}\ . \end{aligned}$$

We can sum up the results derived in the last three paragraphs by saying that for all $\gamma \in \mathbb{R}$ there are sequences (a_n) and (b_n), with $a_n > 0$ and $b_n \in \mathbb{R}$, such that

$$H^n_\gamma(a_n x + b_n) \underset{n\to\infty}{\longrightarrow} \Lambda(x)\ , \tag{2.6}$$

i.e., all stable laws for minima are in the Gumbel max-domain of attraction.

3. LIMIT MODEL FOR THE RELIABILITY OF A REGULAR AND HOMOGENEOUS SERIES-PARALLEL SYSTEM

Using the above results it is possible to obtain the limit behaviour for the reliability of a regular series-parallel system, with a large number of components, whose lifetimes are i.i.d. and belong to the domain of attraction of a stable law for minima, which in turn allows us to establish the following result:

Theorem 3.1. *Let F be a distribution function in the domain of attraction of $H_\gamma(x)$, i.e., assume that there are sequences (a_n) and (b_n), with $a_n > 0$ and $b_n \in \mathbb{R}$, $\forall n \in \mathbb{N}$, such that*

$$1 - \big(1 - F(a_n x + b_n)\big)^n = H_\gamma(x) + \varepsilon_n(x) = 1 - G_\gamma(-x) + \varepsilon_n(x)\ , \tag{3.1}$$

with $\varepsilon_n(x) \to 0$, $\forall x \in \mathbb{R}$ and where $G_\gamma(x)$ is defined in (1.2). Given a sequence of integers such that $\frac{l_n}{n} n^{\frac{n}{l_n}} e_n = o(1)$, with $e_n = \sup\limits_{x\in\mathbb{R}} |\varepsilon_n(x)|$, then for all $\gamma \in \mathbb{R}$, there exist $\alpha_n > 0$ and $\beta_n \in \mathbb{R}$, $\forall n \in \mathbb{N}$ such that, for the sequence of distribution functions, conveniently normalized, the following holds

$$F_n(\alpha_n x + \beta_n) = \Big[1 - \big(1 - F(\alpha_n x + \beta_n)\big)^{l_n}\Big]^n \underset{n\to\infty}{\longrightarrow} \Lambda(x) \;, \tag{3.2}$$

for all $x \in \mathbb{R}$, i.e., for a regular homogeneous series-parallel system, constituted by n parallel subsystems of l_n components in series, the sequence of reliability functions,conveniently normalized, verifies

$$R_n(\alpha_n x + \beta_n) = 1 - \Big[1 - \big(1 - F(\alpha_n x + \beta_n)\big)^{l_n}\Big]^n \underset{n\to\infty}{\longrightarrow} 1 - \Lambda(x) \;,$$

for all $x \in \mathbb{R}$. Further we can consider $\alpha_n = a_n a_n^*$, $\beta_n = a_n b_n^* + b_n$ with

$$a_n^* = \frac{1 - \gamma b_n^*}{n\Big(\exp(\frac{1}{n}) - 1\Big)\Big(-\ln\Big[1 - \exp(-\frac{1}{n})\Big]\Big)} \tag{3.3}$$

and

$$b_n^* = \begin{cases} -\dfrac{1}{\gamma}\left[\left(\dfrac{l_n}{n\Big(-\ln\Big[1-\exp(-\frac{1}{n})\Big]\Big)}\right)^{\gamma} - 1\right], & \gamma \neq 0 \\[2ex] -\ln\left(\dfrac{l_n}{n\Big(-\ln\Big[1-\exp(-\frac{1}{n})\Big]\Big)}\right), & \gamma = 0 \;. \end{cases} \tag{3.4}$$

Proof: Given the sequences (a_n) and (b_n) for which (3.1) is valid and taking (α_n) and (β_n) such that $\alpha_n = a_n a_n^*$ and $\beta_n = a_n b_n^* + b_n$, we have

$$\begin{aligned} \big(1 - F(\alpha_n x + \beta_n)\big)^{l_n} &= \big(1 - F(a_n a_n^* x + a_n b_n^* + b_n)\big)^{l_n} \\ &= \Big[\Big(1 - F\big(a_n(a_n^* x + b_n^*) + b_n\big)\Big)^n\Big]^{\frac{l_n}{n}} \\ &= \Big(1 - H_\gamma(a_n^* x + b_n^*) + \varepsilon_n(a_n^* x + b_n^*)\Big)^{\frac{l_n}{n}} \\ &= \big(1 - H_\gamma(a_n^* x + b_n^*)\big)^{\frac{l_n}{n}} + \rho_n(x) \;. \end{aligned} \tag{3.5}$$

First, we will analyse the component $\big(1 - H_\gamma(a_n^* x + b_n^*)\big)^{\frac{l_n}{n}}$ and later we will prove that $n\rho_n(x) \to 0$, $\forall x \in \mathbb{R}$, where (l_n), (a_n^*) and (b_n^*) satisfy the previously mentioned conditions. Now, for $\gamma \neq 0$ we have, successively,

$$\begin{aligned} \big(1 - H_\gamma(a_n^* x + b_n^*)\big)^{\frac{l_n}{n}} &= \Big[G_\gamma\big(-(a_n^* x + b_n^*)\big)\Big]^{\frac{l_n}{n}} \\ &= \exp\left\{-\left(\left(\frac{l_n}{n}\right)^{-\gamma} - \gamma\left(\left(\frac{l_n}{n}\right)^{-\gamma} a_n^* x + \left(\frac{l_n}{n}\right)^{-\gamma} b_n^*\right)\right)^{-1/\gamma}\right\} \\ &= \exp\Big\{-\big(1 - \gamma(\alpha_n^* x + \beta_n^*)\big)^{-1/\gamma}\Big\} = 1 - H_\gamma(\alpha_n^* x + \beta_n^*) \;, \end{aligned} \tag{3.6}$$

with

$$(3.7) \qquad \begin{cases} \alpha_n^* = \left(\dfrac{l_n}{n}\right)^{-\gamma} a_n^* \\ \beta_n^* = \left(\dfrac{l_n}{n}\right)^{-\gamma} \left(\dfrac{\gamma b_n^* - 1}{\gamma}\right) + \dfrac{1}{\gamma} \,. \end{cases}$$

Hence for $\alpha_n = a_n a_n^*$, $\beta_n = a_n b_n^* + b_n$ and for α_n^* and β_n^* given by (3.7), we can write

$$(3.8) \qquad \big(1 - F(\alpha_n x + \beta_n)\big)^{l_n} = 1 - H_\gamma(\alpha_n^* x + \beta_n^*) + \rho_n(x) \,.$$

Using (3.3), (3.4) and (3.7) the sequence α_n^* verifies

$$\begin{aligned} \alpha_n^* &= \left(\frac{l_n}{n}\right)^{-\gamma} a_n^* \\ &= \left(\frac{l_n}{n}\right)^{-\gamma} \left(\frac{1 + \left(\frac{l_n}{n\left(-\ln\left[1-\exp\left(-\frac{1}{n}\right)\right]\right)}\right)^{\gamma} - 1}{n\Big(\exp\big(\frac{1}{n}\big) - 1\Big)\Big(-\ln\Big[1 - \exp\big(-\frac{1}{n}\big)\Big]\Big)} \right) \\ &= \left(\frac{l_n}{n}\right)^{-\gamma} \left(\frac{\big(\frac{l_n}{n}\big)^{\gamma}}{n\Big(\exp\big(\frac{1}{n}\big) - 1\Big)\Big(-\ln\Big[1 - \exp\big(-\frac{1}{n}\big)\Big]\Big)\Big(-\ln\Big[1 - \exp\big(-\frac{1}{n}\big)\Big]\Big)^{\gamma}} \right) \\ &= \frac{1}{n\Big(\exp\big(\frac{1}{n}\big) - 1\Big)\Big(-\ln\Big[1 - \exp\big(-\frac{1}{n}\big)\Big]\Big)^{\gamma+1}} \,. \end{aligned}$$

Moreover, given (3.4) for $\gamma \neq 0$, $l_n = n\big(-\ln\big[1 - \exp\big(-\frac{1}{n}\big)\big]\big)\,(1 - \gamma b_n^*)^{1/\gamma}$ and it follows that

$$\begin{aligned} \beta_n^* &= \left(\frac{l_n}{n}\right)^{-\gamma} \left(\frac{\gamma b_n^* - 1}{\gamma}\right) + \frac{1}{\gamma} \\ &= \frac{n^{-\gamma}\Big(-\ln\Big[1 - \exp\big(-\frac{1}{n}\big)\Big]\Big)^{-\gamma}(1 - \gamma b_n^*)^{-1}(\gamma b_n^* - 1)}{\gamma\, n^{-\gamma}} + \frac{1}{\gamma} \\ &= -\frac{\Big(-\ln\Big[1 - \exp\big(-\frac{1}{n}\big)\Big]\Big)^{-\gamma}(1 - \gamma b_n^*)^{-1}(1 - \gamma b_n^*)}{\gamma} + \frac{1}{\gamma} \\ &= \frac{1 - \Big(-\ln\Big[1 - \exp\big(-\frac{1}{n}\big)\Big]\Big)^{-\gamma}}{\gamma} \,. \end{aligned}$$

This means that (α_n^*) and (β_n^*) verify (2.2) and (2.4) and consequently are a suitable choice of sequences for the convergence of H_γ^n to the Gumbel law. To prove that $\rho(x)$ in (3.8) is such that $n\,\rho(x)$ goes to zero, we start by observing that since $n\big(\exp\big(\frac{1}{n}\big) - 1\big) \sim 1$ and $-\ln\big(1 - \exp\big(-\frac{1}{n}\big)\big) \sim \ln n$, as $n \to \infty$, the constants (α_n^*) and (β_n^*) are asymptotically given by

$$\alpha_n^* \sim \frac{1}{(\ln n)^{\gamma+1}} \qquad \text{and} \qquad \beta_n^* \sim \frac{1 - (\ln n)^{-\gamma}}{\gamma} \,,$$

and so using (3.7) we get,

$$a_n^* x + b_n^* = \left(\frac{l_n}{n}\right)^{\gamma} \alpha_n^* x + \left(\frac{l_n}{n}\right)^{\gamma} \left(\frac{\gamma \beta_n^* - 1}{\gamma}\right) + \frac{1}{\gamma}$$

$$\sim \left(\frac{l_n}{n \ln n}\right)^{\gamma} \left(\frac{x}{\ln n} - \frac{1}{\gamma}\right) + \frac{1}{\gamma} \; . \tag{3.9}$$

Moreover, from (3.6), (3.7) and (3.9) we obtain

$$\left(1 - H_\gamma(a_n^* x + b_n^*)\right)^{\frac{l_n}{n} - 1} \sim \frac{\exp\left\{-\left(1 - \gamma\left(\frac{1}{(\ln n)^{\gamma+1}}\, x + \frac{1-(\ln n)^{-\gamma}}{\gamma}\right)\right)^{-1/\gamma}\right\}}{\exp\left\{-\left(1 - \gamma\left(\left(\frac{l_n}{n \ln n}\right)^{\gamma} \left(\frac{x}{\ln n} - \frac{1}{\gamma}\right) + \frac{1}{\gamma}\right)\right)^{-1/\gamma}\right\}}$$

$$\sim \frac{\exp\left\{-\ln n \left(1 - \gamma \frac{x}{\ln n}\right)^{-1/\gamma}\right\}}{\exp\left\{-\frac{n \ln n}{l_n} \left(1 - \gamma \frac{x}{\ln n}\right)^{-1/\gamma}\right\}} \tag{3.10}$$

$$\sim \exp\left\{\left(\frac{n}{l_n} - 1\right) \ln n \left(1 - \frac{\gamma}{\ln n}\, x\right)^{-1/\gamma}\right\}$$

$$\sim n^{\frac{n}{l_n} - 1} \; ,$$

when $n \to \infty$ and $\forall x \in \mathbb{R}$. Now, since H_γ is a continuous distribution function on $\mathbb{R}$, the convergence of $\varepsilon_n(x)$ in (3.1) is naturally the uniform convergence and we can write $\lim\limits_{n\to\infty} e_n = \lim\limits_{n\to\infty} \left[\sup\limits_{x\in\mathbb{R}} |\varepsilon_n(x)|\right] = 0$. Furthermore, taking into account that $a_n^* x + b_n^*$ converges to x^{H_γ}, we also have $\varepsilon_n(a_n^* x + b_n^*) \to 0$, uniformly in $\mathbb{R}$, when $n \to \infty$. These results, together with (3.10) and $\frac{l_n}{n}\, n^{\frac{n}{l_n}} e_n \to 0$, when $n \to \infty$, allow us to obtain the following approximation for $\rho_n(x)$ in (3.5),

$$\rho_n(x) = \frac{l_n}{n}\, \varepsilon_n(a_n^* x + b_n^*) \left(1 - H_\gamma(a_n^* x + b_n^*)\right)^{\frac{l_n}{n} - 1} + o(\xi_n)$$

$$\sim \frac{l_n}{n^2}\, \varepsilon_n(a_n^* x + b_n^*)\, n^{\frac{n}{l_n}} + o(\xi_n) \; ,$$

with $\xi_n = \frac{l_n}{n^2}\, n^{\frac{n}{l_n}} e_n$, so that $n\, \rho_n(x) \to 0$. To derive the main result in (3.2) for $\gamma \neq 0$, observe that, using (3.8), we have

$$\left[1 - \left(1 - F(\alpha_n x + \beta_n)\right)^{l_n}\right]^n =$$

$$= \left[H_\gamma(\alpha_n^* x + \beta_n^*) + \rho_n(x)\right]^n$$

$$= \left[H_\gamma(\alpha_n^* x + \beta_n^*)\right]^n \left[1 + \frac{\rho_n(x)}{H_\gamma(\alpha_n^* x + \beta_n^*)}\right]^n$$

$$= \left[H_\gamma(\alpha_n^* x + \beta_n^*)\right]^n \left[1 + \frac{n\, \rho_n(x)}{H_\gamma(\alpha_n^* x + \beta_n^*)} + o\left(\frac{n\, \rho_n(x)}{H_\gamma(\alpha_n^* x + \beta_n^*)}\right)\right],$$

where (α_n^*) and (β_n^*) are normalizing sequences for the convergence of $\big[H_\gamma(\alpha_n^* x + \beta_n^*)\big]^n$ to $\Lambda(x)$. So since $H_\gamma(\alpha_n^* x + \beta_n^*) \to 1$, when n goes to infinity, we finally obtain

$$F_n(\alpha_n x + \beta_n) = \Big[1 - \big(1 - F(\alpha_n x + \beta_n)\big)^{l_n}\Big]^n \underset{n\to\infty}{\longrightarrow} \Lambda(x) \ ,$$

or in other words,

$$R_n(\alpha_n x + \beta_n) = 1 - \Big[1 - \big(1 - F(\alpha_n x + \beta_n)\big)^{l_n}\Big]^n \underset{n\to\infty}{\longrightarrow} 1 - \Lambda(x) \ .$$

In the case $\gamma = 0$, we can also write

$$\big[1 - H_0(a_n^* x + b_n^*)\big]^{\frac{l_n}{n}} = 1 - H_0(\alpha_n^* x + \beta_n^*) \ , \tag{3.11}$$

where

$$\begin{cases} \alpha_n^* = a_n^* \\ \beta_n^* = b_n^* + \ln\Big(\dfrac{l_n}{n}\Big) \ . \end{cases} \tag{3.12}$$

Using (3.3), (3.4) and (3.12) it now follows that

$$\alpha_n^* = \frac{1}{n\Big(\exp\big(\frac{1}{n}\big) - 1\Big)\Big(-\ln\Big[1 - \exp\big(-\frac{1}{n}\big)\Big]\Big)} \ ,$$

and moreover

$$\begin{aligned} \beta_n^* &= b_n^* + \ln\Big(\frac{l_n}{n}\Big) \\ &= -\ln\left(\frac{l_n}{n\Big(-\ln\Big[1 - \exp\big(-\frac{1}{n}\big)\Big]\Big)}\right) + \ln\Big(\frac{l_n}{n}\Big) \\ &= \ln\left(\frac{\frac{l_n}{n}}{\frac{l_n}{n\left(-\ln\left[1-\exp\left(-\frac{1}{n}\right)\right]\right)}}\right) = \ln\Big(-\ln\Big[1 - \exp\Big(-\frac{1}{n}\Big)\Big]\Big) \ . \end{aligned}$$

This means that the sequences (α_n^*) and (β_n^*) verify (2.5) and therefore are a suitable choice of sequences for the convergence of H_0^n to the Gumbel law. Given that $\alpha_n^* \sim \frac{1}{\ln n}$ and $\beta_n^* \sim \ln(\ln n)$ and given (3.12), we have the approximation

$$\begin{aligned} a_n^* x + b_n^* &= \alpha_n^* x + \left(\beta_n^* - \ln\Big(\frac{l_n}{n}\Big)\right) \\ &\sim \frac{x}{\ln n} + \ln\Big(\frac{n \ln n}{l_n}\Big) \ . \end{aligned}$$

Once again we can show that $\big(1 - H_0(a_n^* x + b_n^*)\big)^{\frac{l_n}{n}} \sim n^{\frac{n}{l_n}}$ and $n\,\rho_n(x) \to 0$, when $n \to \infty$, $\forall x \in \mathbb{R}$, yielding

$$F_n(\alpha_n x + \beta_n) = \Big[1 - \big(1 - F(\alpha_n x + \beta_n)\big)^{l_n}\Big]^n \underset{n\to\infty}{\longrightarrow} \Lambda(x) \ ,$$

i.e.,

$$R_n(\alpha_n x + \beta_n) \underset{n\to\infty}{\longrightarrow} 1 - \Lambda(x) \ ,$$

which proves the result. $\square$

Example 3.1. Let be $X \curvearrowright \mathrm{Exp}(1)$.

Observe that

$$\left(1-F\left(\frac{x}{n}\right)\right)^n = \left(e^{-\frac{x}{n}}\right)^n = e^{-x} = \Psi_1(-x) = 1-H_{-1}(x) \;.$$

The conditions of theorem 3 are satisfied, setting $a_n = \frac{1}{n}$, $b_n = 0$ and $\varepsilon_n(x) = 0$, $\forall x \in \mathbb{R}$. For any sequence l_n, by now considering

$$\begin{cases} a_n^* = \dfrac{1+b_n^*}{n\left(\exp\left(\frac{1}{n}\right)-1\right)\left(-\ln\left[1-\exp\left(-\frac{1}{n}\right)\right]\right)} \\[2ex] b_n^* = \left(\dfrac{l_n}{n\left(-\ln\left[1-\exp\left(-\frac{1}{n}\right)\right]\right)}\right)^{-1} - 1 \end{cases}$$

and

$$\begin{cases} \alpha_n = \dfrac{1+b_n^*}{n^2\left(\exp\left(\frac{1}{n}\right)-1\right)\left(-\ln\left[1-\exp\left(-\frac{1}{n}\right)\right]\right)} \\[2ex] \beta_n = \dfrac{\left(-\ln\left[1-\exp\left(-\frac{1}{n}\right)\right]\right)}{l_n} - \dfrac{1}{n} \;, \end{cases}$$

we obtain

$$F_n(x) = \left[1-\left(1-F(\alpha_n x+\beta_n)\right)^{l_n}\right]^n \underset{n\to\infty}{\longrightarrow} \Lambda(x) \;,$$

i.e.,

$$R_n(x) = 1-\left[1-\left(1-F(\alpha_n x+\beta_n)\right)^{l_n}\right]^n \underset{n\to\infty}{\longrightarrow} 1-\Lambda(x) \;.$$

Remark 3.1. Note that if the sequence (l_n) is constant and (k_n) goes to infinity then the limit models for the reliability of the system are the usual models for maxima.

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