REVSTAT – Statistical Journal
Volume 0, Number 0, Month 0000, 000–000
https://doi.org/00.00000/revstat.v00i0.000

Testing uniformity based on maximum extropy

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Received: Month 0000 Revised: Month 0000 Accepted: Month 0000

Abstract:

- In this paper, seven tests for uniformity based on the maximum extropy of uniform distribution are proposed. The properties of the tests are investigated and the percentage points are obtained. Through a Monte Carlo simulation, powers of the tests against various alternatives and for different sample sizes are computed and reported. The results show the tests have a good performance in terms of power. Finally, the tests are applied to a real data set for illustration.

Keywords:

- *Extropy estimator, Testing uniformity, Monte Carlo simulation, Percentage points, Test power.*

AMS Subject Classification:

- 62G10, 62B10.

1. INTRODUCTION

Recently, an alternative measure of uncertainty, termed by extropy, was proposed by Lad et al. (2015). For an absolutely continuous non-negative random variable X with probability density function $f(x)$, the extropy of X is defined as

$$J(f) = -\frac{1}{2}\int_0^\infty f^2(x)\,dx\,.$$

The properties of this measure such as the maximum extropy distribution and statistical applications were presented in Lad et al. (2015). Also, fruitful results can be found in Qiu (2017) and Qiu and Jia (2018b) related with extropy and residual extropy properties of order statistics and record values. Furthermore, Qiu and Wang (2019) obtained some results on extropy properties of mixed systems. The problem of estimation of extropy has been considered by Alizadeh and Jarrahiferiz (2019).

For more recent works on extropy and its applications, one can also refer to Das (2017), Jose and Sathar (2019), Kayal and Moharana (2017), Kelbert et al. (2017), Raqab and Qiu (2019), Jahanshahi et al. (2020), Krishnan et al. (2020), Abdul Sathar and Dhanya Nair (2021), Tahmasebi and Toomaj (2022), Jose and Sathar (2022), Irshad and Maya (2023), Toomaj et al. (2023), Gupta and Chaudhary (2024), Tahmasebi et al. (2023) and Dhanya Nair and Abdul Sathar (2023) and the references therein.

Let F and G be two continuous cdf's with corresponding probability density functions (pdf's) f and g (with respect to Lebesgue measure). Then the relative extropy in a density f relative to g defined over S is:

$$\begin{aligned} d = d(f,g) &= \tfrac{1}{2}\int_S (f(x)-g(x))^2 dx \\ &= -J(f) - J(g) - \int_S f(x)g(x)dx\,, \end{aligned}$$

where $J(f)$ and $J(g)$ are the extropy with respect to f and g. Clearly, $d(f,g) \geq 0$ and the equality holds if and only if $f = g$. Also, $d(f,g) = d(g,f)$. That is, relative extropy is symmetric, though it does not satisfy the triangle inequality.

Given a random sample $X_1,...,X_n$ from a population with absolutely continuous density function $f(x)$ concentrated on the interval [0,1], consider the problem of testing the hypothesis H_0 that the X_i's are uniformly distributed. Several tests for H_0 have been proposed in the statistical literature. For example, Stephens (1974) used tests based on the empirical distribution function and proposed tests for uniformity.

The present paper begins with some test statistics for testing a hypothesis that the sample comes from a uniform distribution, denoted by $U(0,1)$, based on the maximum extropy. The percentage points of the proposed test statistics are obtained for different sample sizes based on 100,000 sample values generated by a Monte Carlo experiment. Also, power values of the proposed tests are computed and power comparisons are performed and then results of our simulation studies are described. Finally, the proposed tests are applied to a real data set for illustration.

2. The proposed tests

Let $X_1, ..., X_n$ be a random sample from a continuous distribution function $F(x)$ with density $f(x)$ concentrated on the interval [0,1]. Let $X_{(1)} \leq X_{(2)} \leq ... \leq X_{(n)}$ denote the order statistics of the sample. Then, the hypothesis of interest is

$$H_0: \ f(x) = 1, \quad 0 < x < 1,$$

against

$$H_1: \ f(x) \neq 1, \quad 0 < x < 1.$$

An important property of uniform distribution is that it obtains the maximum extropy among all distributions that possess a pdf f and have a given support on (0,1). Based on this property, we construct a test for uniformity.

Theorem 2.1. *In the class of continuous distributions f, concentrated on [0,1], it holds*

$$J(f) \leq J(U),$$

and the value of $J(f) = -0.5$, being uniquely attained by the $U(0,1)$ density.

Proof: See Qiu and Jia (2018a). □

A consistent test of the hypothesis of uniformity is then given by

$$T_n = \hat{J}(X),$$

where $\hat{J}(X)$ is the sample estimate of $J(X)$. Here, we consider different estimators for extropy and construct seven test statistics as follows. Clearly, small values of the test statistic will reject the null hypothesis.

1. **The first test statistic**

   Qiu and Jia (2018a) suggested an estimate of $J(f)$ as

$$JQ_{mn} = -\frac{1}{2n} \sum_{i=1}^{n} \frac{2m/n}{X_{(i+m)} - X_{(i-m)}},$$

   where the window size m is a positive integer smaller than $n/2$, $X_{(i)} = X_{(1)}$ if $i < 1$, $X_{(i)} = X_{(n)}$ if $i > n$. Therefore, based on the Qiu and Jia (2018a)'s estimator we can construct the following test statistic.

$$TQ_{mn} = -\frac{1}{2n} \sum_{i=1}^{n} \frac{2m/n}{X_{(i+m)} - X_{(i-m)}}.$$

   Qiu and Jia (2018a) proved that $JQ_{mn} \to J(X)$ as $n \to \infty$, $m \to \infty$, $m/n \to 0$, consequently, under the null hypothesis H_0, TQ_{mn} converges in probability to -0.5 as $n \to \infty$ and under an alternative distribution on [0,1] with absolutely continuous density f, TQ_{mn} converges in probability to a number smaller than -0.5 as $n \to \infty$.

Guided by these properties, given any significance level α, and any finite sample size n, our test procedure is then defined by the critical region

$$TQ_{mn} \leq C^*_\alpha,$$

where C^*_α is set so that the test has the desired level α for given n. For specific α and n, the C^*_α can be obtained by Monte Carlo methods. We determine the C^*_α in follow.

Theorem 2.2. *Let F be a completely unknown continuous distribution and G be the null distribution with unspecified parameters. Then under H_1, the test based on TQ_{mn} is consistent.*

Proof: From Qiu and Jia (2018a), we have

$$JQ_{mn} \to J(f) \qquad as\ n \to \infty,\ m \to \infty,\ m/n \to 0\,.$$

Consequently, TQ_{mn} converges in probability to $J(f)$ as $n \to \infty$ and this completes the proof of the theorem. □

2. **The second test statistic**

Qiu and Jia (2018a) adjusted the weights of the estimator JQ_{mn}, in order to take into account the fact that the differences are truncated around the smallest and the largest data points. (i.e., $X_{(i+m)} - X_{(i-m)}$ is replaced by $X_{(i+m)} - X_{(1)}$ when $i \leq m$ and $X_{(i+m)} - X_{(i-m)}$ is replaced by $X_{(n)} - X_{(i-m)}$ when $i \geq n - m + 1$). Their estimator is given by

$$JQ2_{mn} = -\frac{1}{2n}\sum_{i=1}^{n}\frac{c_i m/n}{X_{(i+m)} - X_{(i-m)}},$$

where

$$c_i = \begin{cases} 1 + \frac{i-1}{m}, & 1 \leq i \leq m, \\ 2, & m+1 \leq i \leq n-m, \\ 1 + \frac{n-i}{m}, & n-m+1 \leq i \leq n. \end{cases}$$

Therefore, we can propose the following test statistic.

$$TE_{mn} = -\frac{1}{2n}\sum_{i=1}^{n}\frac{c_i m/n}{X_{(i+m)} - X_{(i-m)}}\,.$$

Since $JQ2_{mn} \to J(X)$ as $n \to \infty$, $m \to \infty$, $m/n \to 0$, the test based on TE_{mn} is consistent.

3. **The third test statistic**

The third estimator proposed by Qiu and Jia (2018a) is

$$JD = -\frac{1}{2}\int_{-\infty}^{\infty} \hat{f}^2(x)\,dx\,,$$

where $\hat{f}$ is the kernel density function estimation of f and is defined by

$$\hat{f}(x) = \frac{1}{nh}\sum_{j=1}^{n} k\left(\frac{x - X_j}{h}\right),$$

where h is a bandwidth and k is a kernel function which satisfies

$$\int_{-\infty}^{\infty} k(x)dx = 1\,.$$

Usually, k will be a symmetric probability density function. Therefore, we have the following test statistic.

$$TD_n = -\frac{1}{2}\int_0^1 \hat{f}^2(x)\,dx\,.$$

4. **The fourth test statistic**
Alizadeh and Jarrahiferiz (2019), based on a local linear model, proposed an extropy estimator as follows.

$$JC_{mn} = -\frac{1}{2n}\sum_{i=1}^{n}\left(\frac{\sum_{j=i-m}^{i+m}(X_{(j)}-\bar{X}_{(i)})(j-i)}{n\sum_{j=i-m}^{i+m}(X_{(j)}-\bar{X}_{(i)})^2}\right),$$

where

$$\bar{X}_{(i)} = \frac{1}{2m+1}\sum_{j=i-m}^{i+m} X_{(j)}\,.$$

Therefore, we can construct the following test statistic.

$$TC_{mn} = -\frac{1}{2n}\sum_{i=1}^{n}\left(\frac{\sum_{j=i-m}^{i+m}(X_{(j)}-\bar{X}_{(i)})(j-i)}{n\sum_{j=i-m}^{i+m}(X_{(j)}-\bar{X}_{(i)})^2}\right).$$

5. **The fifth test statistic**
The second estimator proposed by Alizadeh and Jarrahiferiz (2019) is as

$$JB_n = -\frac{1}{2n}\sum_{i=1}^{n}\hat{f}(X_i),$$

where

$$\hat{f}(X_i) = \frac{1}{nh}\sum_{j=1}^{n} k\left(\frac{X_i - X_j}{h}\right),$$

and the kernel function is chosen to be the standard normal density function and the bandwidth is chosen to be the normal optimal smoothing formula, $h = 1.06sn^{-\frac{1}{5}}$, where s is the sample standard deviation.
Therefore, we propose the following test statistic for uniformity test.

$$TB_n = -\frac{1}{2n}\sum_{i=1}^{n}\hat{f}(X_i)\,.$$

6. **The sixth test statistic**
Alizadeh and Jarrahiferiz (2019), in a way different from that of Qiu and Jia (2018a), modified the extropy estimator JQ_{mn} as

$$JN_{mn} = -\frac{1}{2n}\sum_{i=1}^{n}\{s_i(n,m)\},$$

where

$$s_i(m,n) = \begin{cases} \hat{f}(X_{(i)})\,, & 1 \le i \le m, \\ \frac{2m/n}{X_{(i+m)} - X_{(i-m)}}, & m+1 \le i \le n-m, \\ \hat{f}(X_{(i)}), & n-m+1 \le i \le n\,. \end{cases}$$

and

$$\hat{f}(X_i) = \frac{1}{nh} \sum_{j=1}^{n} k(\frac{X_i - X_j}{h})\,.$$

Therefore, the following test statistic is proposed.

$$TN_{mn} = -\frac{1}{2n} \sum_{i=1}^{n} \{s_i(n,m)\}.$$

7. **The seventh test statistic**
   Alizadeh and Jarrahiferiz (2019) proposed a sample extropy estimator as

$$JA_{mn} = -\frac{1}{2n} \sum_{i=1}^{n} \left\{ \frac{\hat{f}(X_{(i+m)}) + \hat{f}(X_{(i-m)})}{2} \right\},$$

where

$$\hat{f}(X_i) = \frac{1}{nh} \sum_{j=1}^{n} k(\frac{X_i - X_j}{h})\,,$$

and therefore we can propose the following test statistic.

$$TA_{mn} = -\frac{1}{2n} \sum_{i=1}^{n} \left\{ \frac{\hat{f}(X_{(i+m)}) + \hat{f}(X_{(i-m)})}{2} \right\}.$$

All of the above estimators are consistent and therefore the proposed test statistics are consistent. Clearly, small values of the test statistics will reject the null hypothesis. In the following, we perform a simulation study and obtain the critical values and powers the proposed tests.

3. Simulation study

Under the null hypothesis H_0, the proposed test statistics converge in probability to -0.5 as $n \to \infty$ and under an alternative distribution on [0,1] with absolutely continuous density f, they converge in probability to a number smaller than -0.5 as $n \to \infty$.
Guided by these properties, given any significance level α, and any finite sample size n, test procedures are then defined by the critical region

$$T_{mn} \le C_\alpha,$$

where C_α is set so that the test has the desired level α for given n. For specific α and n, the C_α can be obtained by Monte Carlo methods.
Clearly, the above test statistics are depended on value of m. Therefore, we propose the test statistic

$$T_n = \underset{1 \le m \le n/2}{median}\, T_{mn}\,,$$

eliminating the dependency on the unknown integer parameter m. Consequently, seven proposed test statistics for testing uniformity are as follows.

$$TQ_n = \underset{1 \le m \le n/2}{median}\, TQ_{mn}\,,$$

$$TE_n = \underset{1 \le m \le n/2}{median}\, TE_{mn}\,,$$

$$TD_n = TD_n\,,$$

$$C_n = \underset{1 \le m \le n/2}{median}\, TC_{mn}\,,$$

$$TB_n = TB_n\,,$$

$$TN_n = \underset{1 \le m \le n/2}{median}\, TN_{mn}\,,$$

$$TA_n = \underset{1 \le m \le n/2}{median}\, TA_{mn}\,.$$

Clearly, we reject H_0 for small values of the test statistics.

3.1. Percentage points

At the significance level α, we reject H_0 if the value of the test statistic is smaller than C_α, where C_α the critical value is obtained by the $\alpha-$quantile of the distribution of the test statistic under the null hypothesis H_0.
Distribution of the test statistics under the null hypothesis cannot be evaluated analytically. Therefore, the critical values of the test statistics are computed by the Monte Carlo method. For selected values of the sample size n, 100,000 samples of size n from uniform distribution are generated. For each sample, the test statistics are computed. For level α, the lower-tail percentage points C_α of the distribution of the test statistics are estimated by the α percentiles of the empirical distribution function of the statistics based on the observed 100,000 samples. These estimates are presented in Table 1.

3.2. Power study

The Monte Carlo study of the proposed tests is carried out under seven alternative distributions. The distribution function of the considered alternatives are as follows.

$$A_k : F(x) = 1 - (1-x)^k, \qquad 0 \le x \le 1 \;\; (for\ k = 1.5, 2)\,;$$

$$B_k : F(x) = \begin{cases} 2^{k-1}x^k, & 0 \le x \le 0.5 \\ 1 - 2^{k-1}(1-x)^k, & 0.5 \le x \le 1 \end{cases} \quad (for\ k = 1.5,\ 2,\ 3);$$

$$C_k : F(x) = \begin{cases} 0.5 - 2^{k-1}(0.5-x)^k, & 0 \le x \le 0.5 \\ 0.5 + 2^{k-1}(x-0.5)^k, & 0.5 \le x \le 1 \end{cases} \quad (for\ k = 1.5,\ 2).$$

n	TD_n	TQ_n	TB_n	TE_n	TC_n	TN_n	TA_n
10	-0.5949	-1.3534	-0.6883	-0.9445	-1.0857	-0.7066	-0.6392
20	-0.5182	-0.8995	-0.5790	-0.6815	-0.7618	-0.5843	-0.5310
30	-0.4957	-0.7887	-0.5453	-0.6148	-0.6840	-0.5456	-0.4981
40	-0.4840	-0.7393	-0.5277	-0.5842	-0.6489	-0.5261	-0.4805
50	-0.4773	-0.7109	-0.5175	-0.5670	-0.6288	-0.5152	-0.4705
60	-0.4729	-0.6917	-0.5109	-0.5558	-0.6157	-0.5083	-0.4634
70	-0.4701	-0.6782	-0.5056	-0.5480	-0.6063	-0.5028	-0.4585
80	-0.4679	-0.6691	-0.5024	-0.5424	-0.5996	-0.4997	-0.4548
90	-0.4657	-0.6608	-0.4988	-0.5376	-0.5938	-0.4964	-0.4517
100	-0.4645	-0.6547	-0.4965	-0.5341	-0.5898	-0.4940	-0.4493

Table 1: Critical values of the proposed test statistics at level 5%.

These alternatives were used by Stephens (1974) in his study of power comparisons of some uniformity tests. According to Stephens, alternative A gives points closer to zero than expected under the hypothesis of uniformity. Alternative B gives points near 0.5 and alternative C gives two points close to 0 and 1. The densities of the alternatives A_k, B_k and C_k are depicted in Figure 1.
For each sample size n, 100,000 samples of size n are generated from alternative distribution and the proposed statistics are calculated. For level α, the power values of the tests are estimated by the proportion of the 100,000 samples falling into the critical region. The power estimates resulting based on a Monte Carlo study are given in Tables 2-5 for $\alpha = 0.05$ and $n = 10, 20, 30, 50$.
For each alternative, the bold type in these tables indicates the tests achieving the maximal power.

Alternative	TD_n	TQ_n	TB_n	TE_n	TC_n	TN_n	TA_n
$A_{1.5}$	0.1212	0.1079	0.1288	0.1168	0.1123	0.1289	**0.1338**
A_2	0.2471	0.2353	0.2854	0.2580	0.2477	0.2869	**0.3002**
$B_{1.5}$	**0.2370**	0.1128	0.2138	0.1267	0.1201	0.2074	0.2127
B_2	**0.5208**	0.2471	0.4737	0.2867	0.2702	0.4585	0.4727
B_3	**0.9049**	0.6085	0.8740	0.6707	0.6441	0.8618	0.8701
$C_{1.5}$	0.0112	**0.0788**	0.0191	0.0715	0.0743	0.0264	0.0200
C_2	0.0072	**0.1423**	0.0184	0.1266	0.1302	0.0318	0.0189

Table 2: Power comparisons of the tests for $n = 10$ at the significance level 0.05.

In Tables 2-5, it is evident that for small sample sizes the test TA_n, against alternative

Alternative	TD_n	TQ_n	TB_n	TE_n	TC_n	TN_n	TA_n
$A_{1.5}$	0.1950	0.2233	0.2261	0.2399	0.2316	0.2295	**0.2429**
A_2	0.4452	0.5870	0.5595	**0.6177**	0.6012	0.5714	0.6048
$B_{1.5}$	**0.4397**	0.2188	0.4009	0.2614	0.2392	0.3970	0.3881
B_2	**0.8604**	0.5726	0.8239	0.6545	0.611	0.8207	0.8104
B_3	**0.9989**	0.9618	0.9976	0.9840	0.9737	0.9975	0.9973
$C_{1.5}$	0.0036	**0.1235**	0.0102	0.0970	0.1101	0.0147	0.0172
C_2	0.0015	**0.3095**	0.0100	0.2353	0.2657	0.0198	0.0187

Table 3: Power comparisons of the tests for $n = 20$ at the significance level 0.05.

Alternative	TD_n	TQ_n	TB_n	TE_n	TC_n	TN_n	TA_n
$A_{1.5}$	0.2582	0.3505	0.3149	**0.3704**	0.3589	0.3242	0.3433
A_2	0.6196	0.8281	0.7619	**0.8493**	0.8388	0.7759	0.8084
$B_{1.5}$	**0.6086**	0.3277	0.5656	0.4073	0.3605	0.5685	0.5428
B_2	**0.9664**	0.7871	0.9539	0.8785	0.8296	0.9551	0.9448
B_3	**1.0000**	0.9981	0.9999	0.9998	0.9991	0.9999	0.9999
$C_{1.5}$	0.0017	**0.1703**	0.0081	0.1228	0.1555	0.0098	0.0164
C_2	0.0002	**0.4860**	0.0066	0.3503	0.4243	0.0116	0.0186

Table 4: Power comparisons of the tests for $n = 30$ at the significance level 0.05.

Alternative	TD_n	TQ_n	TB_n	TE_n	TC_n	TN_n	TA_n
$A_{1.5}$	0.3920	0.5698	0.4948	**0.6144**	0.5927	0.5143	0.5406
A_2	0.8410	0.9780	0.9436	**0.9845**	0.9820	0.9527	0.9669
$B_{1.5}$	**0.8304**	0.4947	0.7996	0.6683	0.5721	0.8034	0.7638
B_2	**0.9990**	0.9575	0.9982	0.9912	0.9793	0.9983	0.9969
B_3	**1.0000**	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$C_{1.5}$	0.0004	**0.2606**	0.0057	0.1734	0.2442	0.007	0.0147
C_2	0.0000	**0.7711**	0.0064	0.6052	0.7140	0.0084	0.0264

Table 5: Power comparisons of the tests for $n = 50$ at the significance level 0.05.

A, has the most power. Also, for large sample sizes the test TE_n, against alternative A, has a good performance in compared to the other tests. Against alternative B the test TD_n has the most power. The power differences between the test TD_n and the other tests are substantial. Against alternative C the test TQ_n has the most power and the power differences between

this test and the other tests are substantial.
Generally, we can conclude that the proposed tests TA_n and TE_n have the most power against alternative A, for small and large sample sizes, respectively. Also, the proposed tests TD_n and TQ_n have the most power against alternatives B and C, respectively.

3.3. Applications to real data

In this section, the proposed test procedures are applied to a real data set for illustration.

Example 3.1. We consider the data set discussed in Illowsky and Dean (2018) in Page 317, Table 5.1. The data set consist of smiling times of 55 babies measured in seconds. The data originally follows a uniform distribution U(0,23). We standardize the data to U(0,1). For the transformed data the values of the proposed test statistics are obtained as

$$TD_n = -0.4579\,;\ TQ_n = \ -0.6339;\ TB_n = -0.4896;\ TE_n = -0.5280;$$
$$TC_n = -0.5771;\ TN_n = -0.4880;\ TA_n = -0.4416,$$

which belongs to the acceptance region. Hence, we accept the null hypothesis that the data follows U(0,1).

4. Conclusions

In this article, we have presented seven tests for uniformity based on maximum extropy. We have investigated the extropy estimators and then using them we constructed seven test statistics for uniformity. The properties of the new tests have investigated. Percentage points and power values of the proposed tests for different sample sizes against seven alternatives were reported.
The power simulations for the new tests based on sample extropy showed that the tests are viable for testing the hypothesis of uniformity. Finally, we have applied the proposed test procedures to a real data set for illustration.

Acknowledgements

The author is grateful to the anonymous referees and the associate editor for providing useful comments on an earlier version of this manuscript.

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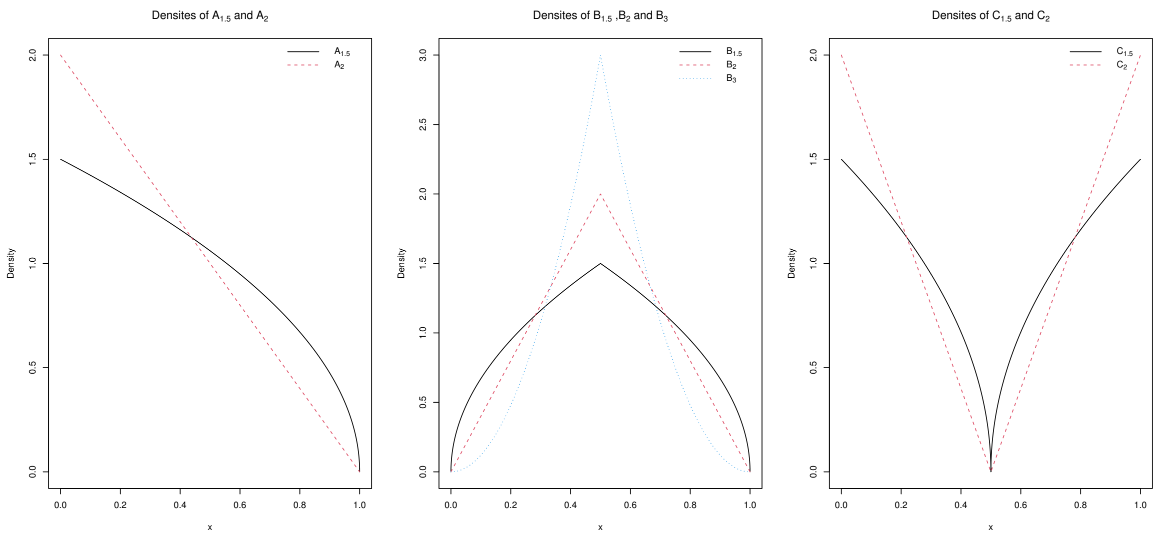

Figure 1: Densities of A_k, B_k and C_k family.

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