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A symmetric transformation of Tukey's $g - h$ family of distributions

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Abstract:

- This class of distributions is a special case of Tukey's $g - h$ family of distributions whose shape is symmetric, which allows approximating several different symmetric distributions. Many researchers consider that the class of Tukey's h distributions depend only on the parameter h, which controls the tails heaviness. We include the parameter g to obtain the symmetric transformation of Tukey's $g - h$ distribution, which enables approximating the even moments (e.g. variance, kurtosis). In this paper, we derive a closed-form expression for the probability density function of the symmetric transformation Tukey's $g - h$ family of distributions, which facilitates the computation of probabilities, central moments, and related measures.

Keywords:

- *Tukey's $g - h$ family of distributions; hyperbolic functions; Johnson's S_U distribution; kurtosis; Lambert's function; Taylor's series.*

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1. Introduction

Statistical data often exhibits symmetry, indicating some kind of kurtosis. In real-life situations where the simple distribution does not adequately fit the data or is unsuitable for modeling, researchers must develop new and flexible distributions to manage this problem. In Tukey (1977), the family of g and h distributions was introduced, and due to their flexibility, several researchers apply it in various fields. The subfamily of Tukey's h distributions is of particular interest because it encompasses several well-known symmetric distributions, such as normal, Cauchy, t-student, and Laplace distribution (see Martínez and Iglewicz, 1984, p. 363). Symmetric distributions are important due to their broad applications in numerous fields (see Johnson and Kotz, 1970). Tukey's h family of distributions has been used in the statistical contexts, with the Lambert W function providing an explicit inverse of the transformation of Tukey's h distribution (see Georg, 2015; Jiménez, 2004). Hoaglin (1985) and Headrick (2010, chapter 5) provide detailed studies on the transformation of Tukey's h distribution.

In this paper, we propose a symmetric transformation of Tukey's $g-h$ family of distributions, incorporating the parameter g into the h-transformation. The appeal of this family lies in its flexibility: starting from a symmetric variable with a probability density function (*pdf*), various distributions can be generated using the parameters g and h, which control skewness and tail heaviness, respectively.

The paper is organized as follows: Section 2 presents the Tukey's $g-h$ family of generalized distributions. Section 3 describes the symmetric transformation of Tukey's $g-h$ distribution, Section 4 discusses statistical properties, including the derivation of the *pdf*, nth moment expressions, and quantile-based measures for the even moments. Section 5 outlines parameter estimation using the method of moments. Section 6 demonstrates the adjustment methodology based on theoretical distributions and provides an illustrative example. Finally, the conclusions are presented.

2. Tukey's $g-h$ family of generalized distributions

Tukey's $g-h$ family distributions was introduced by Tukey (1977) which was derived from two nonlinear transformations

$$Y = T_{g,h}(Z) = \begin{cases} \frac{1}{g}\left(\exp(gZ)-1\right)\exp(hZ^2/2) & \text{if} \quad g \neq 0, \\ Z\exp(hZ^2/2) & \text{if} \quad g = 0 \end{cases} \tag{2.1}$$

where $h \in \mathbb{R}$ and Z be a continuous random variable (*crv*) with a standard normal distribution. The second function was introduced by Tukey (1960) under the assumption that $h \geq 0$. Recently Jiménez et al. (2015) introduced a generalization of Tukey's $g-h$ family of distributions, replacing Z in (2.1) by other *crv*, U, with zero mean and unit variance such that its *pdf*, $f_U(\cdot)$, is symmetric around the origin. In Jiménez et al. (2015) the authors assume that U has a Generalized Error Distribution of parameter α, denoted by $U \sim \text{GED}(\alpha)$. It is said that a *crv* $U \sim GED(\alpha)$ if its *pdf* is given by[1] (see Forbes et al., 2011, p. 86)

$$f_U(u;\alpha) = \frac{1}{2\lambda\Gamma(\alpha+1)}\exp\left\{-\left|\frac{u}{\lambda}\right|^{\frac{1}{\alpha}}\right\}, \qquad u \in \mathbb{R},\ 0 < \alpha \leq 1, \tag{2.2}$$

[1]The GED(α) is also known as the error distribution with parameters $a = 0$, $2^\alpha b = \sqrt{\frac{\Gamma(\alpha)}{\Gamma(3\alpha)}}$ and $\alpha = \frac{c}{2}$.

where $\lambda = \sqrt{\frac{\Gamma(\alpha)}{\Gamma(3\alpha)}}$. Thus, they obtained the following transformation

$$Y = T_{g,h}(U) = \begin{cases} \frac{1}{g}\left(\exp(gU)-1\right)\exp(hU^2/2) & \text{if} \quad g \neq 0, \\ U\exp(hU^2/2) & \text{if} \quad g = 0 \end{cases} \tag{2.3}$$

Similarly as in the original transformation the parameters g and h control the skewness and the tails heaviness of the Tukey's $g-h$ generalized distribution, respectively. The first function given in (2.3) when $h = 0$ is known as the Tukey's g generalized distribution and is denoted by $T_{g,0}(U)$. The second function given in (2.3) is known as the Tukey's h generalized distribution, this distribution has the characteristic of being symmetrical but shows an increasingly high stretch tails according the value of the parameter h.

Proposition 2.1. *The g-and-h distributions given in (2.3) can be expressed as*

$$T_{g,h}(U) = T^{(1)}_{g,h}(U) + T^{(2)}_{g,h}(U) \tag{2.4}$$

where

$$T^{(1)}_{g,h}(U) = \frac{\sinh(gU)}{g}\exp\left\{\frac{1}{2}hU^2\right\} \quad \text{and} \quad T^{(2)}_{g,h}(U) = \frac{\cosh(gU)-1}{g}\exp\left\{\frac{1}{2}hU^2\right\}. \tag{2.5}$$

When $g \to 0$, these transformations reduce to[2]

$$T^{(1)}_{0,h}(U) = U\exp(hU^2/2) \qquad \text{and} \qquad T^{(2)}_{0,h}(U) = 0.$$

Proof: We can rewrite the expression given in (2.3) in terms of hyperbolic functions as follows

$$\begin{aligned} Y = T_{g,h}(U) &= \frac{1}{g}\left(\sinh(gU) + \cosh(gU) - 1\right)\exp(hU^2/2) \\ &= \underbrace{\frac{1}{g}\sinh(gU)\exp(hU^2/2)}_{T^{(1)}_{g,h}(U)} + \underbrace{\frac{1}{g}\left(\cosh(gU)-1\right)\exp(hU^2/2)}_{T^{(2)}_{g,h}(U)}, \end{aligned}$$

These transformations are useful for modeling data that exhibit both skewness and heavy tails, characteristics that cannot be adequately captured by a symmetric *crv* alone. □

Remark 2.1. Note that we can obtain the same transformation as $T^{(1)}_{g,h}(U)$ when $g < 0$, because $T^{(1)}_{-g,h}(U) = T^{(1)}_{g,h}(U)$, and also it satisfied that $T^{(1)}_{g,h}(-U) = -T^{(1)}_{g,h}(U)$. The expressions (2.3) and the first one given in (2.5) are related as:

$$T_{2g,h}(U) = e^{gU}T^{(1)}_{g,h}(U) \qquad \text{for } g \neq 0.$$

Remark 2.2. Furthermore, these transformations satisfy that

$$\left(T^{(1)}_{g,h}(U)\right)^2 - \left(T^{(2)}_{g,h}(U)\right)^2 = \frac{2}{g}T^{(2)}_{g,h}(U)\exp(hU^2/2). \tag{2.6}$$

[2]Note that $T^{(1)}_{0,h}(U)$ coincides with the second function given in (2.3).

2.1. Parametric representation of random variables using the Tukey's $g - h$ distribution

If we wish to approximate any *crv* X by using the transformation given in (2.3), we use the linear model given in Hoaglin and Peters (1979):

$$X = A + BY \qquad \text{with} \qquad Y = T_{g,h}(U), \tag{2.7}$$

where A and B are location and scale parameters, respectively. Since U is symmetric about zero, we should estimate four parameters, which must also satisfy one of the following conditions:

$$x_q = A + BT_{g,h}(u_q) \qquad \text{and} \qquad x_{1-q} = A + BT_{g,h}(-u_q), \tag{2.8}$$

where $q > 0.5$ and x_q, u_q are the qth quantiles of the *crv* X and U, respectively. For any univariate cumulative distribution function (*cdf*) $F_U(\cdot)$, and for $0 < q < 1$, the qth quantile is the quantity

$$F_U^{-1}(q) = u_q = \inf\{u : F_U(u) \geq q\}.$$

Remark 2.3. When we assume that $U \sim \mathrm{GED}(1/2)$ and $h = 0$ in equation (2.7), its inverse is

$$U = T_{g,0}^{-1}\left(\frac{X - A}{B}\right) = \frac{1}{g}\ln\left(\frac{X - \theta}{\eta}\right) \qquad X > \theta \tag{2.9}$$

where $\theta = A - \frac{B}{g}$ and $\eta = \frac{B}{g}$, these parameters are taken by convention to be positive. This expression coincides with the lognormal system of distributions (S_L) presented in Johnson (1949, p. 155).

3. The symmetric transformation of Tukey's $g - h$ distribution

The $g - h$ generalized distribution defined by the function given in (2.3) is a nonlinear transform of a *crv* U and is parameterized by g and h. The second function given in (2.3) includes distributions that increases kurtosis on the increasing magnitude of the parameter h. This subfamily of distributions to help them get to have great importance in the statistical analysis to be a suitable means to study symmetric distributions. Its distributional form includes only the parameter h which fixes the amount of kurtosis. Many researchers consider that the parameter g in (2.1) controls the skewness and $g = 0$ corresponds to symmetry, however, we include the parameter g to obtain the symmetric transformation of Tukey's $g - h$ family of generalized distributions.

Proposition 3.1. *Let X and $Y = T(U)$ be random variables, if there exists a relationship of the form*

$$X = \mathrm{Me}(X) + BT_{g,h}^{(1)}(U), \qquad \textit{with} \quad h \in \mathbb{R} \tag{3.1}$$

where $\mathrm{Me}(X) = x_{0.5}$ and $T_{g,h}^{(1)}(U)$ is given in (2.5), then the transformation $T_{g,h}^{(1)}(U) = \widetilde{T}_{g,h}(U)$ is said to be a symmetric transformation of the Tukey's $g - h$ family of distributions.

Proof: According to Loynes (1966) when the distribution of a *crv* X is symmetric about some point x_0, i.e. has center of symmetry x_0, then

$$x_q - x_0 = x_0 - x_{1-q} \qquad 0.5 < q < 1. \tag{3.2}$$

Let X be a variable which we want to approximate by the transformation $g-h$, by replacing expressions (2.8) in equation (3.2), we obtain

$$A + B\big(T^{(1)}_{g,h}(u_q) + T^{(2)}_{g,h}(u_q)\big) - x_0 = x_0 - \big[A + B\big(T^{(1)}_{g,h}(-u_q) + T^{(2)}_{g,h}(-u_q)\big)\big].$$

Since $T^{(1)}_{g,h}(-u_q) = -T^{(1)}_{g,h}(u_q)$ and $T^{(2)}_{g,h}(-u_q) = T^{(2)}_{g,h}(u_q)$, after some calculations and simplifying we have

$$x_0 = A + BT^{(2)}_{g,h}(u_q),$$

note that if $g = 0$, the value x_0 is constant for all q. Since if a *crv* X has a *pdf* symmetric and its expected value exists, this expected value equals the center of symmetry (see Schay, 2016, p. 176), then

$$\mathbb{E}(X) = A + B\underbrace{\mathbb{E}\big[T^{(1)}_{g,h}(U)\big]}_{0} + B\mathbb{E}\big[T^{(2)}_{g,h}(U)\big] = \mathbb{E}(x_0).$$

Therefore

$$\begin{aligned} x_q &= \underbrace{A + B\mathbb{E}\big[T^{(2)}_{g,h}(u_{0.5})\big]} + BT^{(1)}_{g,h}(u_q) \\ &= \qquad \mathrm{Me}(X) \qquad + \frac{B}{g}\sinh(gu_q)\exp\left\{\frac{1}{2}hu_q^2\right\} \\ &= \begin{cases} \mathrm{Me}(X) + \frac{B}{g}\sinh(gu_q)\exp\left\{\frac{1}{2}hu_q^2\right\} & \text{if} \quad g \neq 0 \\ \mathrm{Me}(X) + Bu_q\exp\left\{\frac{1}{2}hu_q^2\right\} & \text{if} \quad g = 0 \end{cases} \end{aligned} \tag{3.3}$$

This expression enables the approximation of the quantiles of a symmetric *crv* X using $T^{(1)}_{g,h}(U)$. □

Remark 3.1. When we assume that $U \sim \mathrm{GED}(1/2)$ and $h = 0$ in equation (3.1), its inverse is

$$U = \delta\sinh^{-1}\left(\frac{X - \mathrm{Me}(X)}{\eta}\right) \qquad \text{with} \quad \delta = \frac{1}{g} \quad \text{and} \quad \eta = \frac{B}{g}. \tag{3.4}$$

This expression coincides with the unbounded system of symmetrical distributions (S_U) presented in Johnson (1949, p. 158), which are obtained when the shape parameter $\gamma = 0$. In general, the inverse of the symmetric transformation of $\widetilde{T}_{g,h}(U)$ can be obtained numerically.

Remark 3.2. Note that the second expression of (3.3) can also be obtained using the Maclaurin's series expansion of $\frac{\sinh(gu)}{g}$ (see details in McMahon, 1906, p. 23), for that, we rewrite the expression (3.1) in terms of quantiles as

$$x_q = \mathrm{Me}(X) + B\exp\left\{\frac{1}{2}hu_q^2\right\}\sum_{k=0}^{\infty} g^{2k}\frac{u_q^{2k+1}}{(2k+1)!}.$$

Here, when $g \to 0$, only the first term of the expansion above is non-zero, and we obtain

$$x_q = \mathrm{Me}(X) + Bu_q\exp\left\{\frac{1}{2}hu_q^2\right\},$$

this last equation coincides with expression (2.8) when $A = \mathrm{Me}(X)$ and $y_q = T_{0,h}(u_q)$.

4. Statistical properties of the symmetric transformation of Tukey's $g-h$ family

In this section, we discuss the statistical properties of the symmetric transformation of Tukey's $g-h$ family of distributions.

4.1. Probability density function

In Jiménez (2004) by using the inverse function theorem obtained the following relation

$$(F^{-1})'(F(u_q)) = \frac{d}{dq}u_q = \frac{1}{F'(u_q)} = \frac{1}{f(u_q)} \tag{4.1}$$

where u_q denotes the q-th quantile of the *crv* U, i.e., $F_U(u_q) = q$, and $f_U(\cdot)$ is the *pdf* of the *crv* U. Under the transformation in (3.1), the *pdf* of $\widetilde{T}_{g,h}(U)$ is obtained by using the following result

$$t_{g,h}(y_q) = \frac{f_U(u_q)}{\widetilde{T}'_{g,h}(u_q)} \qquad \text{whenever} \qquad |h|u_q\frac{\tanh(gu_q)}{g} < 1, \tag{4.2}$$

where $h < 0$ and y_q denotes the q-th quantile of the symmetric transformation:

$$\widetilde{T}_{g,h}(U) = T^{(1)}_{g,h}(U) = \begin{cases} \frac{\sinh(gU)}{g}\exp\left\{\frac{1}{2}hU^2\right\} & \text{if} \quad g \neq 0, \\ U\exp\{\frac{1}{2}hU^2\} & \text{if} \quad g = 0 \end{cases} \tag{4.3}$$

and $\widetilde{T}'_{g,h}(u_q)$ satisfies the following Ordinary Differential Equation (ODE):

$$\widetilde{T}'_{g,h}(U) = \begin{cases} hU\widetilde{T}_{g,h}(U) + \cosh(gU)\exp\{\frac{1}{2}hU^2\} & \text{if} \quad g \neq 0 \\ (1 + hU^2)\exp\{\frac{1}{2}hU^2\} & \text{if} \quad g = 0 \end{cases} \tag{4.4}$$

subject to the initial condition

$$\widetilde{T}_{g,h}(0) = 0.$$

Remark 4.1. By Equation (3.1) we have that $x_q = \text{Me}(X) + B\widetilde{T}_{g,h}(u_q)$ and by using the expression (4.1) then the *pdf* for the *crv* X is set as follows

$$f_X(x_q) = f_X\left(\text{Me}(X) + B\widetilde{T}_{g,h}(u_q)\right) = \frac{1}{|B|}t_{g,h}(y_q). \tag{4.5}$$

If $U \sim \text{GED}(1/2)$, we can consider the following special cases

1. When $h = 0$, from (4.2) we have immediately

$$\begin{aligned} f_X(x) &= \frac{1}{|B|}t_{g,0}(y) = \frac{1}{\sqrt{B^2}}\frac{\varphi\left(\frac{1}{g}\sinh^{-1}\left(\frac{x-\text{Me}(X)}{\eta}\right)\right)}{\cosh\left(\sinh^{-1}\left(\frac{x-\text{Me}(X)}{\eta}\right)\right)} \\ &= \frac{1}{\sqrt{(\eta g)^2}}\frac{\varphi\left(\frac{1}{g}\ln\left(\frac{x-\text{Me}(X)}{\eta} + \sqrt{\left(\frac{x-\text{Me}(X)}{\eta}\right)^2 + 1}\right)\right)}{\cosh\left(\ln\left(\frac{x-\text{Me}(X)}{\eta} + \sqrt{\left(\frac{x-\text{Me}(X)}{\eta}\right)^2 + 1}\right)\right)} \end{aligned}$$

$$(4.6)\qquad = \frac{1}{\sqrt{2\pi(\eta g)^2}} \frac{\exp\left\{-\frac{1}{2g^2}\ln^2\left(\frac{x-\mathrm{Me}(X)}{\eta} + \sqrt{\left(\frac{x-\mathrm{Me}(X)}{\eta}\right)^2+1}\right)\right\}}{\sqrt{\left(\frac{x-\mathrm{Me}(X)}{\eta}\right)^2+1}},$$

this last expression coincides with the *pdf* of S_U given in Johnson (1949, p. 162).

2. When $g=0$, by substituting into (4.2), we have

$$f_X(x) = \frac{1}{|B|}t_{0,h}(y) = \frac{1}{\sqrt{B^2}}\frac{1}{1+u^2}\varphi\left(\sqrt{\frac{1+h}{h}}u\right)$$

$$(4.7)\qquad = \frac{1}{\sqrt{2\pi B^2}(1+u^2)}\exp\left\{-\frac{1}{2}\frac{1+h}{h}u^2\right\},$$

where u^2 is expressed in terms of the Lambert W function and it is given by (see Georg, 2015, p. 6)

$$u^2 = \mathrm{W}\left(h\left(\frac{x-\mathrm{Me}(X)}{B}\right)^2\right).$$

4.2. Cumulative distribution function

The following proposition provides an analytical expression for evaluate the *cdf* of the symmetric transformation of Tukey's $g-h$ distributions.

Proposition 4.1. *The cumulative distribution function* (cdf) *of the symmetric transformation of Tukey's $g-h$ family of distributions, denoted by $\mathrm{F}_{g,h}(y)$, satisfies the following:*

$$(4.8)\qquad \int_a^b t_{g,h}(u)du = \int_{\widetilde{T}^{-1}_{g,h}(a)}^{\widetilde{T}^{-1}_{g,h}(b)} f_U(v)\,dv = F_U\left(\widetilde{T}^{-1}_{g,h}(b)\right) - F_U\left(\widetilde{T}^{-1}_{g,h}(a)\right),$$

where $\widetilde{T}^{-1}_{g,h}(\cdot)$ is the inverse function of the transformation given in (4.3).

Proof: From the expression (4.2), it follows that

$$\int_a^b t_{g,h}(y)dy = \int_a^b \frac{f_U\left(\widetilde{T}^{-1}_{g,h}(y)\right)}{\widetilde{T}'_{g,h}\left(\widetilde{T}^{-1}_{g,h}(y)\right)}dy.$$

We will solve this by making the following change of variable

$$(4.9)\qquad v = \widetilde{T}^{-1}_{g,h}(y) \qquad\qquad dv = \frac{dy}{\widetilde{T}'_{g,h}\left(\widetilde{T}^{-1}_{g,h}(y)\right)},$$

then

$$\int_{\widetilde{T}^{-1}_{g,h}(a)}^{\widetilde{T}^{-1}_{g,h}(b)} f_U(v)\,dv = F_U\left(\widetilde{T}^{-1}_{g,h}(b)\right) - F_U\left(\widetilde{T}^{-1}_{g,h}(a)\right).$$

In other words, the *cdf* depends on the inverse of the transformation $\widetilde{T}_{g,h}(U)$ and the *cdf* of the *cvr* U. □

4.3. Moments of the symmetric transformation of Tukey's $g-h$ family of distribution

Since both the *pdf* and the *cdf* can be explicitly defined, parameters estimation for Tukey's $g-h$ family of generalized distributions can be carried out using the method of moments or the quantile-based method proposed by Mahbubul et al. (2008). Recently, Möstel et al. (2019) conducted a comparison of various methods for estimating Tukey's $g-h$ distributions, particularly in cases where the likelihood function is not available in closed form

We use the following theorem to derive and explain the moments of the symmetric transformation of Tukey's $g-h$ family of distributions. Since that X has a *pdf* symmetric (when a distribution is symmetric, the mode, median, and mean are all the same) about $\mathrm{Me}(X)$, then the moments of odd order are zero and we can get the moments of even order as

Theorem 4.1. *Let $X=\mathrm{Me}(X)+B\widetilde{T}_{g,h}(U)$ be the transformation given as in (3.1), then the n-th moments of even order of the* crv *X are given by*

$$\mathbb{E}\big[\big(X-\mathrm{Me}(X)\big)^{2n}\big]=\begin{cases}2\eta^{2n}\int_0^\infty[\sinh^2(gu)]^n\exp\{\widetilde{h}u^2\}f_U(u)du & \text{if}\quad g\neq 0,\\ \int_0^\infty u^{2n}\exp\{\widetilde{h}u^2\}f_U(u)du & \text{if}\quad g=0\end{cases}\tag{4.10}$$

where $\eta=B/g$, $\widetilde{h}=nh$ and

$$\mathbb{E}(X)=A+2\eta\int_0^\infty(\cosh(gu)-1)\exp\left\{\frac{h}{2}u^2\right\}f_U(u)du.\tag{4.11}$$

Proof: Using the expression (3.1) when $g\neq 0$ we obtain

$$\begin{aligned}\mathbb{E}\left[\big(X-\mathrm{Me}(X)\big)^{2n}\right]&=\int_0^1\big(x_q-\mathrm{Me}(X)\big)^{2n}dq\\&=B^{2n}\int_0^1\left[\frac{\sinh(gu_q)}{g}\exp\left\{\frac{h}{2}u_q^2\right\}\right]^{2n}dq.\end{aligned}$$

Now, we proceed with the variable change

$$w=u_q=F_U^{-1}(q)\qquad\qquad dw=du_q=\frac{dq}{F'_U(u_q)},\tag{4.12}$$

here we use the expression given in (4.1), since $F'_U(w)=f_U(w)$, we have

$$\begin{aligned}\underbrace{\mathbb{E}\left[\big(X-\mathrm{Me}(X)\big)^{2n}\right]}_{\mu_{2n}(X)}&=\eta^{2n}\int_{-\infty}^{\infty}[\sinh^2(gw)\exp\{hw^2\}]^n f_U(w)dw\\&=2\eta^{2n}\int_0^\infty[\sinh(gw)]^{2n}\exp\{\widetilde{h}w^2\}f_U(w)dw,\end{aligned}\tag{4.13}$$

where $\widetilde{h}=nh$. In the latter term, we used that $f_U(w)$ is a symmetric function around the origin. □

4.4. Special cases of moments

In general, if U be a *crv* with *pdf* symmetric around the origin then the moment generating function (*mgf*) can be obtain as follows

$$M_U(t) = \mathbb{E}(e^{tU}) = 2\int_0^\infty \cosh(tw) f_U(w)dw.$$

Using the following result from the hyperbolic functions (McMahon, 1906)

$$[2\sinh(gw)]^{2n} = \sum_{k=0}^{n-1}(-1)^k \binom{2n}{k} [2\sinh(\widetilde{g}w)]^2,$$

where $\widetilde{g} = (n-k)g$. By substituting the above expression into (4.13) we get

$$\mu_{2n}(X) = 4\left(\frac{\eta}{2}\right)^{2n} \sum_{k=0}^{n-1}(-1)^k \binom{2n}{k} \int_0^\infty \left[\cosh(2\widetilde{g}w) - 1\right] \exp\{\widetilde{h}w^2\} f_U(w)dw. \tag{4.14}$$

However, it is not always reasonable to assume the existence of certain moments; their existence is guaranteed only within a specific range of values for the parameter h. According to this restriction, we have the following cases:

1. Assume that $U \sim \text{GED}(1/2)$ and $h < \frac{1}{2n}$, then expression (4.11) becomes

$$\mathbb{E}(X) = A + \frac{\eta}{\sqrt{1-h}}\left[\exp\left\{\frac{1}{2}\frac{g^2}{1-h}\right\} - 1\right]. \tag{4.15}$$

From equation (4.14) we obtain

$$\mu_{2n}(X) = \begin{cases} \left(\frac{\eta}{2}\right)^{2n} \frac{2}{\sqrt{1-2\widetilde{h}}} \sum_{k=0}^{n-1}(-1)^k \binom{2n}{k} \left[M_U\left(\frac{2\widetilde{g}}{\sqrt{1-2\widetilde{h}}}\right) - 1\right] & \text{if } g \neq 0, \\ \frac{2B^{2n}}{2^n[1-2\widetilde{h}]^{\frac{2n+1}{2}}} \frac{\Gamma(2n)}{\Gamma(n)} & \text{if } g = 0 \end{cases} \tag{4.16}$$

where $M_U(t)$ denotes the *mgf* of the standard normal *crv* and $\Gamma(\cdot)$ denotes the gamma function.

2. Suppose that $U \sim \text{GED}(1)$ and $h < 0$, then expression (4.11) becomes

$$\begin{aligned}\mathbb{E}(X) =& A + \eta\sqrt{\frac{\pi}{|h|}}\left[\exp\left\{\frac{1}{2}\left[\frac{g+\sqrt{2}}{\sqrt{|h|}}\right]^2\right\}\Phi\left[-\frac{g+\sqrt{2}}{\sqrt{|h|}}\right] + \exp\left\{\frac{1}{2}\left[\frac{\sqrt{2}-g}{\sqrt{|h|}}\right]^2\right\}\right. \\ &\left. - \exp\left\{\frac{1}{2}\left(\frac{\sqrt{2}-g}{\sqrt{|h|}}\right)^2\right\}\Phi\left(\frac{\sqrt{2}-g}{\sqrt{|h|}}\right) - 2\exp\left\{\frac{1}{|h|}\right\}\Phi\left(-\sqrt{\frac{2}{|h|}}\right)\right].\end{aligned}$$

From equation (4.14) we obtain[3]

$$\mu_{2n}(X) = \begin{cases} \left(\frac{\eta}{2}\right)^{2n} \frac{2}{\sqrt{n|h|}} \sum_{k=0}^{n-1}(-1)^k \binom{2n}{k} \left\{\left[\exp\left\{\frac{\alpha_{n,k}^2}{2}\right\}\Phi(\alpha_{n,k})\right.\right. \\ \qquad \left.\left. + \exp\left\{\frac{\beta_{n,k}^2}{2}\right\}\Phi(\beta_{n,k})\right] - 2e^{\frac{1}{2n|h|}}\Phi\left(\frac{-1}{\sqrt{n|h|}}\right)\right\} & \text{if } g \neq 0, \\ \frac{2^n B^{2n}}{\sqrt{2n|h|}} \left(\frac{1}{2n|h|}\right)^{2n} e^{\frac{1}{2n|h|}} \sum_{k=0}^{2n}(-1)^k \binom{2n}{k} \left[\Gamma\left(\frac{k+1}{2}\right)\right. \\ \qquad \left. - \int_0^{\frac{1}{2n|h|}} \sqrt{u^{k-1}} e^{-u} du\right] & \text{if } g = 0 \end{cases} \tag{4.17}$$

[3] We use the Table I of Fourier transforms (Oberhettinger, 1973, expression (79)).

where $\alpha_{n,k}$ and $\beta_{n,k}$ are the smaller and larger roots, respectively, of the quadratic equation given below

$$\left[\sqrt{n|h|}r+1\right]^2-2(n-k)^2g^2=0. \tag{4.18}$$

5. Estimation of the parameters

In this section, we explain the estimation of the parameters of this class of symmetric transformation, which we can be accomplished using the method of moments. The $2n$-th standardized moment of the *crv* X is given by

$$\begin{aligned}\beta_n(X) &= \frac{\mathbb{E}[(X-\mathrm{Me}(X))^{2n}]}{\left(\mathbb{E}[(X-\mathrm{Me}(X))^2]\right)^n}=\frac{\mu_{2n}(X)}{\mu_2^n(X)} \qquad n\geq 1\\ &= \sum_{k=0}^{n-1}\frac{(-1)^k}{2^{n-1}}\binom{2n}{k}\frac{\int_0^\infty[2\sinh(\widetilde{g}w)]^2\exp\{\widetilde{h}w^2\}f_U(w)dw}{\left[\int_0^\infty[2\sinh(gw)]^2\exp\{hw^2\}f_U(w)dw\right]^n}.\end{aligned} \tag{5.1}$$

Note that the above expression of the $2n$-th standardized moment does not depend on the parameter B. The formulas for calculating the fourth standardized moment, kurtosis ($\beta_2(X)$), and $\beta_3(X)$, are used to determine the values of g and h.

Remark 5.1. If $U\sim\mathrm{GED}(1/2)$, from expression (4.16), then the variance of the *crv* X is given by

$$\sigma_X^2=\begin{cases}\frac{\eta^2}{\sqrt{1-2h}}\sinh\left(\frac{g^2}{1-2h}\right)\exp\left\{\frac{g^2}{1-2h}\right\}, & \text{if} \quad g\neq 0,\ h<\frac{1}{2}\\ \frac{B^2}{\sqrt{(1-2h)^3}} & \text{if} \quad g=0,\ h<\frac{1}{2}.\end{cases} \tag{5.2}$$

On the other hand, the $2n$-th standardized moment of the *crv* X are given by

$$\beta_n(X)=\kappa_n(h)\sum_{k=0}^{n-1}(-1)^k\binom{2n}{k}\frac{4\sinh\left(\frac{(n-k)^2g^2}{1-2nh}\right)\exp\left\{\frac{(n-k)^2g^2}{1-2nh}\right\}}{\left[4\sinh\left(\frac{g^2}{1-2h}\right)\exp\left\{\frac{g^2}{1-2h}\right\}\right]^n}, \quad n\geq 1 \tag{5.3}$$

where the coefficient is given by

$$\kappa_n(h)=\sqrt{\frac{(1-2h)^n}{1-2nh}} \qquad n=1,2,\ldots.$$

Using the definition of hyperbolic sine function we can rewrite the expression (5.3) as follows:

$$\beta_n(X)=\frac{\kappa_n(h)}{2^{n-1}}\sum_{k=0}^{n-1}(-1)^k\binom{2n}{k}\frac{\exp\left\{\frac{2(n-k)^2g^2}{1-2nh}\right\}-1}{\left[\exp\left\{\frac{2g^2}{1-2h}\right\}-1\right]^n}, \qquad n\geq 1, \tag{5.4}$$

subject to $h<\frac{1}{2n}$. To determine the values of g and h, we can be solved the following system of nonlinear equations

$$\begin{aligned}\frac{2\beta_2(X)}{\kappa_2(h)} &= \left(\frac{w_2(g,h)-1}{w_1(g,h)-1}\right)^2\left(w_2^2(g,h)+2w_2(g,h)+3\right)\\ \frac{4\beta_3(X)}{\kappa_3(h)} &= \left(\frac{w_3(g,h)-1}{w_1(g,h)-1}\right)^3\left(w_3^6(g,h)+3w_3^5(g,h)+6w_3^4(g,h)+10w_3^3(g,h)\right.\\ &\qquad \left.+15w_3^2(g,h)+15w_3(g,h)+10\right)\end{aligned} \tag{5.5}$$

where

$$w_n(g,h) = \exp\left\{\frac{2g^2}{1-2nh}\right\} \qquad n = 1, 2, \ldots$$

and we have that the scale parameter is given by

$$B = \begin{cases} \frac{|g|\sigma_X \sqrt[4]{1-2h}}{\sqrt{\sinh\left(\frac{g^2}{1-2h}\right)}} \exp\left\{-\frac{1}{2}\frac{g^2}{1-2h}\right\} & \text{if} \quad g \neq 0, \\ (1-2h)^{\frac{3}{4}}\sigma_X & \text{if} \quad g = 0. \end{cases}$$

Remark 5.2. When $U \sim \mathrm{GED}(1/2)$ and $h = 0$ from (5.2) and (5.4) we get

$$2\sigma_X^2 = \eta^2(w-1)(w+1) \qquad \text{and} \qquad 2\beta_2(X) = w^4 + 2w^2 + 3 \tag{5.6}$$

where $w = e^{g^2}$, these expressions coincide with the variance and kurtosis for distributions of systems S_U given in Johnson (1949, p. 163), then the value of g is established as the solution of

$$\left(\exp\left\{2g^2\right\} + 1 - \sqrt{2(\beta_2(X)-1)}\right)\left(\exp\left\{2g^2\right\} + 1 + \sqrt{2(\beta_2(X)-1)}\right) = 0$$

i.e., if the kurtosis of the data set is known, the value g can be obtained by the solution of

$$2g^2 = \ln\left(\sqrt{2(\beta_2(X)-1)} - 1\right). \tag{5.7}$$

Note that if $\beta_2(X) = 3$, then $g = 0$. The scale parameter (if $g \neq 0$) is given by

$$\eta = \frac{B}{|g|} = \sigma_X\left[\frac{\sqrt{2(\beta_2(X)-1)}}{2} - 1\right]^{-\frac{1}{2}}, \qquad \beta_2(X) \neq 3. \tag{5.8}$$

It is straightforward to show that (4.15) is equivalent to

$$\mathbb{E}(X) = x_{0.5} + \sqrt{2}\frac{\sqrt[4]{\sqrt{2(\beta_2(X)-1)} - 1} - 1}{\sqrt{\sqrt{2(\beta_2(X)-1)} - 2}}\sigma_X, \qquad \beta_2(X) \neq 3 \tag{5.9}$$

i.e., it depends only on the median, standard deviation and kurtosis.

5.1. Parameter identification procedure based on quantiles

In Hoaglin (1985, p. 469) the qth lower half-spread and upper half-spread are defined as

$$LHS_q = X(u_{0.5}) - X(-u_q) \qquad \text{and} \qquad UHS_q = X(u_q) - X(u_{0.5}) \tag{5.10}$$

where $X(u_q) = A + BT_{g,h}(u_q)$ with $q > 0.5$. By substituting expression (2.8), we obtain

$$UHS_q + LHS_q = X(u_q) - X(-u_q) = 2BT_{g,h}^{(1)}(u_q)$$

it is called the quantile spread and is denoted as $\mathrm{QS}_X(q)$. In David and Johnson (1956, p. 15), a skewness function based on order statistics was proposed. When $q = k/(n+1)$, the qth sample quantile corresponds to the kth order statistic. Based on this, the skewness measure for the transformation given in (2.4), is defined as follows:

$$b_2(q) = \frac{UHS_q - LHS_q}{UHS_q + LHS_q} = \frac{T_{g,h}^{(2)}(u_q)}{T_{g,h}^{(1)}(u_q)} = \frac{e^{gu_q} - 1}{e^{gu_q} + 1}.$$

This expression relates the David-Johnson skewness to the parameter g. It reduces to Bowley's coefficient when $q = \frac{3}{4}$. For fixed $q > 0.5$, the values of $b_2(q)$ will be within the interval $(-1, 1)$, where 1 represents extreme right skewness ($g \to \infty$) and -1 represents extreme left skewness ($g \to -\infty$). Figure 1 shows David-Johnson coefficient as a monotonic increasing function of g.

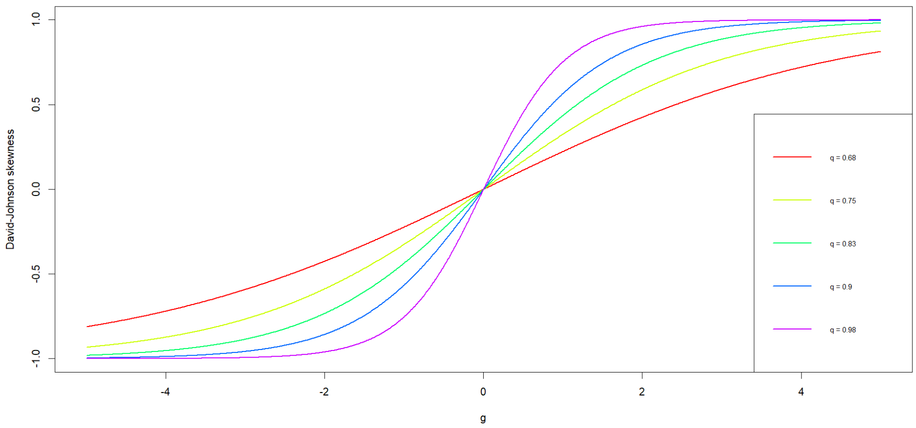

Figure 1: Relation between David-Johnson skewness and parameter g.

On the other hand, following a methodology similar to that of Slifker and Shapiro (1980, p. 240), we derive a simple criterion that can be used to select the transformation given in (2.4). To do this, we establish the following quantities:

$$\frac{m}{p} = \frac{X(3u_q) - X(u_q)}{\mathrm{QS}_X(q)} = \begin{cases} \frac{(2\cosh(gu)+1)e^{4hu_q^2} - e^{-gu_q}}{e^{-gu_q}+1} e^{gu_q} & \text{if } g \neq 0 \text{ and } h \neq 0 \\ \frac{3e^{4hu_q^2}-1}{2} & \text{if } g = 0 \text{ and } h \neq 0 \\ e^{2gu_q} & \text{if } g \neq 0 \text{ and } h = 0 \end{cases}$$

$$\frac{n}{p} = \frac{X(-u_q) - X(-3u_q)}{\mathrm{QS}_X(q)} = \begin{cases} \frac{(2\cosh(gu)+1)e^{4hu_q^2} - e^{gu_q}}{e^{gu_q}+1} e^{-gu_q} & \text{if } g \neq 0 \text{ and } h \neq 0 \\ \frac{3e^{4hu_q^2}-1}{2} & \text{if } g = 0 \text{ and } h \neq 0 \\ e^{-2gu_q} & \text{if } g \neq 0 \text{ and } h = 0 \end{cases}$$

where $u_q \in (0.51, 0.99)$, and its selection will depend on the amount of data. It follows that

$$\frac{m}{p}\frac{n}{p} = \begin{cases} \frac{(2\cosh(gu)+1)e^{4hu_q^2} - e^{gu_q}}{e^{gu_q}+1} \, \frac{(2\cosh(gu)+1)e^{4hu_q^2} - e^{-gu_q}}{e^{-gu_q}+1} & \text{if } g \neq 0 \text{ and } h \neq 0 \\ \frac{(3e^{4hu_q^2}-1)^2}{4} & \text{if } g = 0 \text{ and } h \neq 0 \\ 1 & \text{if } g \neq 0 \text{ and } h = 0. \end{cases} \tag{5.11}$$

Expression (5.11) can be used to choose between the two transformations depending on the result obtained, i.e., using the percentiles of the ordered data, if we compute with these values mn/p^2 and we obtain that the result is equal to 1 then $g \neq 0$ and $h = 0$. When the result is less than 1, $g = 0$ and $h < 0$, in the case where $h < 0$, the existence of the moments of the transformations is always guaranteed. In general, the second condition can be validated as

$$\frac{m}{p}-\frac{n}{p}=\begin{cases}\left(\left(2\cosh(gu_q)+1\right)^2 e^{4hu_q^2}-1\right)b_2(q) & \text{if} \quad g\neq 0 \text{ and } h\neq 0\\ 0 & \text{if} \quad g=0 \text{ and } h\neq 0\\ 2\sinh\left(2gu_q\right) & \text{if} \quad g\neq 0 \text{ and } h=0.\end{cases}$$

Remark 5.3. When the distribution of a *crv* X is defined for positive values, in accordance with Jiménez and Martínez (2006), if it is satisfied that

$$x_q x_{1-q}=x_{0.5}^2 \qquad 0.5<q<1, \tag{5.12}$$

then $B=Ag$ and $h=0$. We will briefly indicate how this can be proved. Let X be a variable which we want to approximate using the transformation $g-h$. By substituting expressions (2.8) into equation (5.12), we obtain

$$\left[A+B\big(T_{g,h}^{(1)}(u_q)+T_{g,h}^{(2)}(u_q)\big)\right]\left[A+B\big(T_{g,h}^{(1)}(-u_q)+T_{g,h}^{(2)}(-u_q)\big)\right]=x_{0.5}^2 \qquad g>0.$$

By means of the quantile technique it is known that $A=x_{0.5}$, after some calculations and simplifying we have

$$B^2\big(T_{g,h}^{(2)}(u_q)\big)^2-B^2\big(T_{g,h}^{(1)}(u_q)\big)^2+2ABT_{g,h}^{(2)}(u_q)=0$$
$$2BT_{g,h}^{(2)}(u_q)\left(A-\eta\exp(hu_q^2/2)\right)=0.$$

Here, we substitute the expression (2.6) in the last equation. This is satisfied when $B=Ag$ and $h=0$. Expression (5.12) and the third condition in (5.11) are equivalent, both allowing us to establish that $h=0$.

6. Illustration

In this section we discuss some examples and applications of the results presented with two examples. In the first example, we consider the symmetric distributions studied in Martínez and Iglewicz (1984, p. 363) and Pearson et al. (1977, p. 240). In the second example, we examined the available dataset from the web address `https://lib.stat.cmu.edu/datasets/pollen.data`.

6.1. Symmetric distributions

We assume that $U\sim \text{GED}(1/2)$ and we use the values of $\beta_2(X)$ and $\beta_3(X)$ provided in Talacko (1956) and Pearson et al. (1977, p. 240) to estimate the parameters g and h. Since both $\beta_2(X)$ and $\beta_3(X)$ for these distributions are known, the parameters of symmetric transformation of Tukey's $g-h$ distributions can be determined by numerically solving the equations given in (5.5). In Table 1, we present the estimated values of the three parameters (B,g,h) for a selected set of well-known symmetrical distributions. Our interest is to compare the results from Martínez and Iglewicz (1984, p. 363) with our proposed model. It is important to note that the authors assume $g=0$ for symmetric distributions, whereas in our approach using the symmetric transformation of Tukey's $g-h$ distributions, the parameter g yields a nonzero value.

Table 1: Values for the parameters of the symmetric transformation of Tukey's $g-h$ distributions.

Density Function	**Standard moments**		**Estimates values**		
	$\beta_2(X)$	$\beta_3(X)$	B	g	h
Uniform (0,1)	$\frac{9}{5}$	$\frac{27}{7}$	$\frac{1434}{3701}$	$\frac{39}{1052}$	$-\frac{13}{54}$
Beta (2,2)	$\frac{15}{7}$	$\frac{125}{21}$	$\frac{4339}{6251}$	$\frac{17}{1065}$	$-\frac{27}{227}$
Normal (0,1)	3	15	1	0	0
SU $(\eta=1,\delta=3)$	3.5287	24.776	$\frac{1}{3}$	$\frac{1}{3}$	0
HyperCsc	4	34	$\frac{3471}{3895}$	$\frac{9659}{13210}$	$-\frac{891}{9640}$
$t_{(10)}$	4	40	$\frac{437}{430}$	$\frac{83}{4847}$	$\frac{56}{939}$
Logistic	$\frac{21}{5}$	$\frac{279}{7}$	$\frac{7045}{8008}$	$\frac{8072}{11255}$	$-\frac{280}{3537}$
HyperSec	5	61	$\frac{989}{1220}$	$\frac{2550}{2701}$	$-\frac{1029}{7558}$
Laplace $\left(0,\frac{1}{\sqrt{2}}\right)$	6	90	$\frac{9374}{12849}$	$\frac{6337}{5391}$	$-\frac{1308}{6467}$

Source: own elaboration

The distributions given in table 1 are characterized by their symmetry and heavier tails compared to the normal distribution, making them flexible for modeling real-world data with moderate extremes, especially in fields such as actuarial science, survival analysis and growth modeling. In addition, the cases where $h<0$ indicates that the respective *crv* has higher moments.

Similar to the other *pdf*, we look for a two-normal mixture distribution (NMD) with equal means and different variance given by

$$h(x;\Lambda)=\omega\varphi\big(x;\mu_1,\sigma_1^2\big)+(1-\omega)\varphi\big(x;\mu_2,\sigma_2^2\big) \qquad\qquad x\in\mathbb{R},$$

where the parameter vector of the mixture model $\Lambda=(\mu_1,\sigma_1^2;\mu_2,\sigma_2^2;\omega)$ and $\varphi\big(x;\mu_k,\sigma_k^2\big)$ denotes the *pdf* of an univariate normal variable with mean μ_k and variance σ_k^2. Here we wish to observe that our model allows to approach the quantile of the mixture of two normal distribution with equal means and different variance. In Table 2, we present the estimated values of the three parameters (B,g,h) for a mixture of two univariate normal distributions.

Table 2: Values for the parameters of the NMD(Λ)

Mixture *pdf*	**Parameters**			**Standard moments**		**Estimates values**		
	(μ_1,σ_1)	(μ_2,σ_2)	ω	$\beta_2(X)$	$\beta_3(X)$	B	g	h
Normal	$(0,1)$	$(0,3)$	0.8	$\frac{1275}{169}$	$\frac{274875}{2197}$	$\frac{361}{395}$	$\frac{3345}{2002}$	$-\frac{1499}{3965}$
Normal	$(0,1)$	$(0,3)$	0.9	$\frac{25}{3}$	$\frac{5125}{27}$	$\frac{5287}{6438}$	$\frac{873}{616}$	$-\frac{2156}{8385}$

Source: own elaboration

6.2. Pollen data

We now use a set of 3848 observations of the variable "density" included in the data set Pollen of the R package HistData. This dataset was used by Pewsey et al. (2012) and explored also by Gómez-Déniz et al. (2019). In Table 3, we present a summary of some descriptive statistics.

In Table 4, we assume that $U\sim GED(1/2)$ and present the estimated parameters of the Tukey's h and $\widetilde{T}_{g,0}(U)$ distributions; our interest is to compare with Pewsey et al. (2012) with our proposed model.

Table 3: Summary of descriptive statistics for the pollen density dataset

Mean	Median	Variance	Skewness	Kurtosis
1.6629×10^{-4}	-0.03045	9.8872	0.10979	3.19338

Source: own elaboration

Table 4: Estimates for adjusting the transformation of Tukey's $g-h$ distributions.

Density Function	Estimated parameters				Test of adjusted
	(A, ϵ)	(B, η)	(g, λ)	(h, γ)	$KS(\%)$
Tukey's h	-0.03045	3.073672	0	0.014938	0.9985
$\widetilde{T}_{g,0}(U)$	-0.03045	14.40308	0.2133641	0	0.9311
SU	-1.701606	11.01016	0.2701436	-0.5470736	0.8126
SN^*	-2.04	3.75	0.93	–	1.1645
Normal	0.0362174	3.156164	0	0	1.5477

*Their parameters are drawn from (Pewsey et al., 2012, p. 12) and (Gómez-Déniz et al., 2019, p. 11)

Source: own elaboration

The lower value of the Kolmogorov-Smirnov (KS) test confirms that the data obey the proposed transformation. The empirical probability density function ($EPDF$) of the data are shown in Figure 2 and compared with Figure 4 of Pewsey et al. (2012, p. 12).

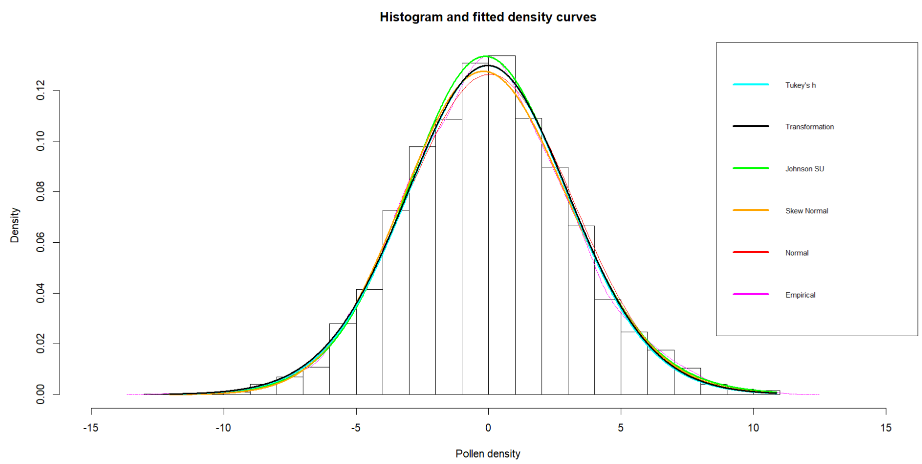

Figure 2: Histogram and fitted density curves for pollen dataset.

7. Conclusions

This paper presents a symmetric transformation of the well-known Tukey's h family of distributions for modeling symmetric data. We derive explicit formulas for the moments, and using these, we estimate the transformation parameters via the method of moments. The proposed model has the advantage that it provides flexibility, particularly when the data distribution does not follow a normal distribution, i.e., when the distribution is leptokurtic. From the proposed model we obtain some special cases of well-known symmetric distributions, and also symmetrical Johnson's S_U distribution (when $h = 0$). Additionally, the model generates a broad class of symmetric probability density functions (*pdf*) by using the g and h parameters of the symmetric transformation of Tukey's $g - h$ family of distributions, which control skewness and tail heaviness, respectively

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