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A two-piece version of modified skew normal distribution

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Abstract:

- Here we consider a new class of two-piece skew normal distributions as a two-piece version of the modified skew normal distribution of Kumar and Anusree (2011) and investigate some of its important structural properties. We discuss the method of maximum likelihood estimation of the parameters of the model and illustrate the procedures with the help of a real life data set. Further, a brief simulation study is conducted for assessing the performance of the estimates of the parameters of the distribution.

Keywords:

- *model selection; reliability measures; simulation; skew normal distribution; two-piece skew normal distribution.*

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1. INTRODUCTION

Azzalini (1985) considered a generalized version of the normal distribution, namely the skew normal distribution (SND) and this distribution has been further studied by several authors such as Azzalini (1986), Henze (1986), Azzalini and Dalla Valle (1996), Branco and Dey (2001) and Arnold and Beaver (2013), Tsujino and Kubokawa (2019), Das et al. (2023), Arnold et al. (2023). Azzalini (1985) defined the SND as given below.

A random variable X is said to follow a skew normal distribution (SND) with skewness parameter $\lambda \in R = (-\infty, \infty)$, denoted by $SND(\lambda)$, if its probability density function (p.d.f.) is of the following form, for $x \in R$.

$$g_1(x; \lambda) = 2f(x)F(\lambda x) \tag{1.1}$$

where $f(\cdot)$ and $F(\cdot)$ are respectively the p.d.f. and cumulative distribution function (c.d.f.) of a standard normal variate. The $SND(\lambda)$ is not suitable for the analysis of plurimodal situations. In order to accommodate plurimodality, Kumar and Anusree (2011) developed a modified version of the $SND(\lambda)$ by considering a generalized mixture of the standard normal and $SND(\lambda)$ through the following p.d.f., in which $x \in R$, $\lambda \in R$ and $\alpha \geq -1$.

$$g_2(x; \lambda, \alpha) = \frac{2}{(\alpha + 2)} f(x)[1 + \alpha F(\lambda x)]. \tag{1.2}$$

For developing a more flexible class of asymmetric normal models, Kim (2005) proposed a two-piece version of the $SND(\lambda)$ through the following p.d.f.

$$g_3(x; \lambda) = \frac{2\pi\ f(x)F(\lambda|x|)}{[\pi + 2\tan^{-1}(\lambda)]}. \tag{1.3}$$

The distribution of a random variable X with p.d.f. (1.2) and (1.3) hereafter are denoted by $MSND(\lambda, \alpha)$ and $TSND(\lambda)$, respectively. As a generalization of the $TSND(\lambda)$, here we introduce a two-piece version of the $MSND(\lambda, \alpha)$ and study some of its important properties. We shall call this new class of two-piece $MSND(\lambda, \alpha)$ as "the two-piece modified skew normal distribution $[TMSND(\lambda, \alpha)]$". The rest of the paper is organized as follows: The definition of $TMSND(\lambda, \alpha)$ and some of its important properties are discussed in Section 2. In Section 3, we obtain expressions for certain reliability measures and discuss some important aspect of the distribution with respect to the mode of the distribution. In Section 4, the maximum likelihood estimation of the parameters of the distribution is presented. In Section 5, practical usefulness of the class is illustrated with the help of a real life data set and a simulation study is undertaken for examining the performance of the estimators.
We need the following notation in the sequel. For any $a \in R$ and $b > 0$, define

$$\xi(a; b) = \int_a^\infty f(x) \int_0^{bx} f(y)\,dy\,dx, \tag{1.4}$$

so that

$$\xi(0, \lambda) = \frac{tan^{-1}(\lambda)}{2\pi} \tag{1.5}$$

and for any reals a, b and s such that $bx + s > 0$,

$$\xi_s(a;b) = \int_a^{\infty} f(x) \int_0^{bx+s} f(y)dydx. \tag{1.6}$$

2. DEFINITION AND PROPERTIES

In this section we first present the definition of a two-piece modified skew normal distribution and discuss some of its important properties.

Definition 2.1. A random variable Z is said to follow a two-piece modified skew normal distribution with parameters $\lambda \in R$ and $\alpha \geq -1$ if its p.d.f. $h(z;\lambda,\alpha)$ is of the following form:

$$h(z;\lambda,\alpha) = \begin{cases} \frac{2}{(\alpha+2)} f(z)[1 + D(\lambda,\alpha)F(-\lambda z)], & z < 0 \\ \frac{2}{(\alpha+2)} f(z)[1 + D(\lambda,\alpha)F(\lambda z)], & z \geq 0 \end{cases} \tag{2.1}$$

where $D(\lambda,\alpha) = \alpha\pi[\pi + 2\tan^{-1}(\lambda)]^{-1}$.

The distribution of a random variable Z with p.d.f. (2.1) is denoted by $TMSND(\lambda,\alpha)$. For some particular choices of λ and α the p.d.f. given in (2.1) of $TMSND(\lambda,\alpha)$ is shown in Figure 1.

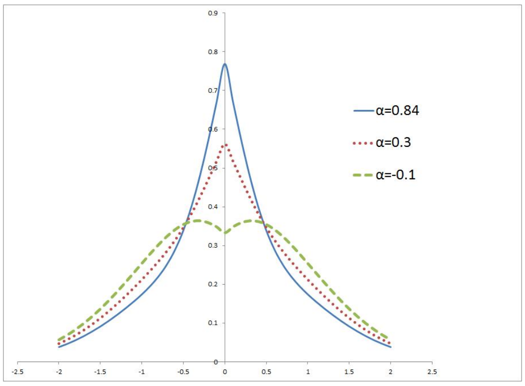

Figure 1: Probability plots of $TMSND(2.5,\alpha)$ for different choices of $\alpha = 0.84, 0.3, -0.1$.

Clearly, the $TMSND(\lambda,\alpha)$ contains the following special cases.

a) $TMSND(\lambda, 0)$ or $TMSND(0,\alpha)$ or the limiting case of the $TMSND(\lambda,\alpha)$ when $\lambda \to -\infty$, $\alpha = 0$ or $\lambda \to \infty$, is the standard normal distribution;

b) the limiting case of the $TMSND(\lambda,\alpha)$ when $\alpha \to \infty$ is the $TSND(\lambda)$;

c) the limiting case of the $TMSND(\lambda,\alpha)$ when $\lambda \to -\infty$ and $\alpha = -1$ is the standard half normal distribution.

Now we derive certain structural properties of the $TMSND(\lambda, \alpha)$ through the following results.

Result 2.1. If Z follows $TMSND(\lambda, \alpha)$, then $Y_1 = -Z$ also follows $TMSND(\lambda, \alpha)$.

Proof: Let $h\,(z; \lambda, \alpha)$ be the p.d.f. of Z. Then, by the symmetric property of f, for any $y_1 \in R$, the p.d.f. $h_1(y_1; \lambda, \alpha)$ of Y_1 is given by

$$\begin{aligned} h_1\,(y_1; \lambda, \alpha) &= h\,(-y_1; \lambda, \alpha) \left|\frac{dz}{dy_1}\right| \\ &= \begin{cases} \frac{2}{(\alpha+2)} f(y_1)[1 + D(\lambda, \alpha)F(-\lambda y_1)], y_1 \leq 0 \\ \frac{2}{(\alpha+2)} f(y_1)[1 + D(\lambda, \alpha)F(\lambda y_1)], y_1 > 0, \end{cases} \end{aligned}$$

which shows that $Y_1 = -Z$ follows $TMSND(\lambda, \alpha)$. □

Result 2.2. If Z follows $TMSND(\lambda, \alpha)$ then, $Y_2 = Z^2$ has p.d.f. (2.2).

Proof: Let $h\,(z; \lambda, \alpha)$ be the p.d.f. of Z. Then, the p.d.f. $h_2(y_2; \lambda, \alpha)$ of Y_2 is given by

$$\begin{aligned} h_2\,(y_2; \lambda, \alpha) &= h\,(z; \lambda, \alpha) \left|\frac{dz}{dy_2}\right| \\ &= h\,(-\sqrt{y_2}; \lambda, \alpha) \left|\frac{dz}{dy_2}\right| + h\,(\sqrt{y_2}; \lambda, \alpha) \left|\frac{dz}{dy_2}\right| \\ &= \frac{1}{(\alpha+2)} \left(\frac{f(\sqrt{y_2})}{2\sqrt{y_2}}\right) [2 + D(\lambda, \alpha)\,(F(\lambda\sqrt{y_2}) + F(-\lambda\sqrt{y_2})]\,. \end{aligned} \tag{2.2}$$

□

Remark 2.1. When $\lambda = 0$, the p.d.f. given in (2.2) reduces to the p.d.f. of a Chi-square variate with one degree of freedom.

In order to find the distribution function of $TMSND(\lambda, \alpha)$, we need the following result.

Result 2.3. If Z is a $TMSND(\lambda, \alpha)$, then for any real numbers d_1, d_2 such that $d_1 \leq d_2$, where $\xi(a, b)$ is as given in (1.4).

$$P(d_1 \leq Z \leq d_2) = \begin{cases} \frac{2}{(\alpha+2)}[F(d_2) - F(d_1)] + \frac{D(\lambda,\alpha)}{(\alpha+2)}[F(d_2) - F(d_1)] + \\ \frac{D(\lambda,\alpha)}{(\alpha+2)}\,[2\xi(d_1, -\lambda) - 2\xi(d_2, -\lambda)]\,,\; d_1 \leq d_2 < 0 \\ \\ \frac{2}{(\alpha+2)}[F(d_2) - F(d_1)] + \frac{D(\lambda,\alpha)}{(\alpha+2)}[F(d_2) - F(d_1)] + \\ \frac{D(\lambda,\alpha)}{(\alpha+2)}\,[2\xi(d_1, \lambda) - 2\xi(d_2, \lambda)]\,,\; 0 \leq d_1 \leq d_2 \end{cases} \tag{2.3}$$

Proof: For any $d_1 \leq d_2 < 0$, by definition,

$$
\begin{aligned}
P(d_1 \leq Z \leq d_2) &= \int_{d_1}^{d_2} h(z;\lambda,\alpha)dz \\
&= \int_{d_1}^{d_2} \left[\frac{2}{(\alpha+2)} f(z) + \frac{D(\lambda,\alpha)}{(\alpha+2)} 2f(z)F(-\lambda z)\right] dz \\
&= \frac{2}{(\alpha+2)}[F(d_2) - F(d_1)] + \frac{D(\lambda,\alpha)}{(\alpha+2)} * \\
& \quad [G(d_2,-\lambda) - G(d_1,-\lambda)]
\end{aligned}
\tag{2.4}
$$

where $G(.,\lambda)$ is the distribution function of the $SND(\lambda)$.
Now, for the case $0 \leq d_1 \leq d_2$,

$$
\begin{aligned}
P(d_1 \leq Z \leq d_2) &= \int_{d_1}^{d_2} h(z;\lambda,\alpha)dz \\
&= \int_{d_1}^{d_2} \left[\frac{2}{(\alpha+2)} f(z) + \frac{D(\lambda,\alpha)}{(\alpha+2)} 2f(z)F(\lambda z)\right] dz \\
&= \frac{2}{(\alpha+2)}[F(d_2) - F(d_1)] + \frac{D(\lambda,\alpha)}{(\alpha+2)} * \\
& \quad [G(d_2,\lambda) - G(d_1,\lambda)].
\end{aligned}
\tag{2.5}
$$

Thus, from (2.4) and (2.5) we get (2.3). □

Result 2.4. The c.d.f. $H(z)$ of a random variable Z following $TMSND(\lambda,\alpha)$ with p.d.f. (2.1) is the following, in which $\xi(a;b)$ is as defined in (1.4).

$$
H(z) = \frac{2}{(\alpha+2)} \begin{cases} F(z) + \frac{D(\lambda,\alpha)}{2}[F(z) - 2\xi(z,-\lambda)], z < 0 \\ \\ F(z) + \frac{D(\lambda,\alpha)}{2}[F(z) - 2\xi(z,\lambda) + 2\frac{tan^{-1}(\lambda)}{\pi}], z \geq 0 \end{cases}
\tag{2.6}
$$

Proof: Let Z be a random variable with p.d.f. (2.1) and c.d.f. $H(z)$.
Then by definition, we have,

$$
H(z) = \begin{cases} L_1, \; z < 0 \\ L_2, \; z \geq 0, \end{cases}
\tag{2.7}
$$

where

$$
\begin{aligned}
L_1 &= \frac{2}{(\alpha+2)} \int_{-\infty}^{z} f(t)dt + \frac{D(\lambda,\alpha)}{(\alpha+2)} \int_{-\infty}^{z} 2f(t)F(-\lambda t)dt \\
&= \frac{2}{(\alpha+2)} F(z) + \frac{D(\lambda,\alpha)}{(\alpha+2)} G(z,-\lambda) \\
&= \frac{2}{(\alpha+2)} F(z) + \frac{D(\lambda,\alpha)}{(\alpha+2)} [F(z) - 2\xi(z,-\lambda)] \qquad (2.8)
\end{aligned}
$$

and

$$L_2 = \int_{-\infty}^{0} \frac{2}{(\alpha+2)} f(t)[1 + D(\lambda,\alpha)F(-\lambda t)]dt + \int_{0}^{z} \frac{2}{(\alpha+2)} f(t)[1 + D(\lambda,\alpha)F(\lambda t)]dt$$

$$= \frac{2}{(\alpha+2)} \left[F(z) + \frac{D(\lambda,\alpha)}{2} G(0,-\lambda) + \frac{D(\lambda,\alpha,\rho)}{2} \left(G(z,\lambda) - G(0,\lambda)\right) \right]$$

$$= \frac{2}{(\alpha+2)} \left[F(z) + \frac{D(\lambda,\alpha)}{2} \left(\frac{1}{2} - 2\xi(0,-\lambda) \right) + \frac{D(\lambda,\alpha)}{2} \left(F(z) - 2\xi(z,\lambda) - \frac{1}{2} - 2\xi(0,\lambda) \right) \right]. \tag{2.9}$$

Considering $\xi(0,\cdot)$ is as defined in (1.5), by substituting (2.8) and (2.9) in (2.7) we get (2.6). □

Result 2.5. The r^{th} raw moment of $TMSND(\lambda,\alpha)$ with (2.1) is

$$\mu_r' = \frac{2}{\alpha+2}\gamma_r' + \frac{D(\lambda,\alpha)}{\alpha+2}\eta_r' + \frac{D(\lambda,\alpha)}{\alpha+2}\zeta_r'$$

where γ_r' and η_r' are the respective r^{th} raw moments of normal distribution and $SND(-\lambda)$ and $\zeta_r' = \int_0^{\infty} x^r f(z)[F(\lambda z) - F(-\lambda z)]dz$

Proof: By the definition, the r^{th} raw moment of $TMSND(\lambda,\alpha)$ is given by

$$\mu_r' = E(X^r)$$

$$= \int_{-\infty}^{0} \frac{2}{\alpha+2} x^r f(z)[1 + D(\lambda,\alpha)F(-\lambda z)]dz + \int_{0}^{\infty} \frac{2}{\alpha+2} x^r f(z)[1 + D(\lambda,\alpha)F(\lambda z)]dz$$

$$= \int_{-\infty}^{\infty} \frac{2}{\alpha+2} x^r f(z) dz + \int_{-\infty}^{0} \frac{2D(\lambda,\alpha)}{\alpha+2} x^r f(z) F(-\lambda z) dz + \int_{0}^{\infty} \frac{2D(\lambda,\alpha)}{\alpha+2} x^r f(z) F(\lambda z) dz$$

$$= \int_{-\infty}^{\infty} \frac{2}{\alpha+2} x^r f(z) dz + \int_{-\infty}^{\infty} \frac{2D(\lambda,\alpha)}{\alpha+2} x^r f(z) F(-\lambda z) dz - \int_{0}^{\infty} \frac{2D(\lambda,\alpha)}{\alpha+2} x^r f(z) F(-\lambda z) dz + \int_{0}^{\infty} \frac{2D(\lambda,\alpha)}{\alpha+2} x^r f(z) F(\lambda z) dz$$

$$= \frac{2}{\alpha+2}\gamma_r^{'} + \frac{D(\lambda,\alpha)}{\alpha+2}\eta_r^{'} + \frac{D(\lambda,\alpha)}{\alpha+2}\zeta_r^{'}.$$

□

Corollary 2.2. The mean and variance of TMSND(λ, α) with p.d.f.(2.1) is

$$Mean, m = \frac{2}{\alpha+2}\gamma_1^{'} + \frac{D(\lambda,\alpha)}{\alpha+2}\eta_1^{'} + \frac{D(\lambda,\alpha)}{\alpha+2}\zeta_1^{'}$$

and

$$Variance = \frac{2}{\alpha+2}\gamma_2^{'} + \frac{D}{\alpha+2}\eta_2^{'} + \frac{D}{\alpha+2}\zeta_2^{'} - m^2$$

where $\gamma_i^{'}$, $\eta_i^{'}$ for $i = 1, 2$ are respectively the first and second raw moments of normal and skew normal distributions. $\zeta_i^{'}$ for $i = 1, 2$ can be evaluated using Mathematical Softwares like MATHCAD, MATHEMATICA and MATHLAB using the expression given in Result 2.4 .

Result 2.6. The Fisher Information Measure of TMSND(λ, α) with p.d.f.(2.1) is

$$I(\lambda,\alpha) = \begin{bmatrix} Var(U_1+U_2) & Cov\left(U_1+U_2, V_1+V_2-\frac{2}{\alpha+2}\right) \\ Cov\left(U_1+U_2, V_1+V_2-\frac{2}{\alpha+2}\right) & Var\left(V_1+V_2-\frac{2}{\alpha+2}\right) \end{bmatrix} \tag{2.10}$$

where

$$U_1 = u_1(Z) = \left[-\frac{D(\lambda,\alpha)Z^{-1}f(-\lambda Z)}{1+D(\lambda,\alpha)F(-\lambda Z)} + \frac{2(1+\lambda^2)^{-1}D(\lambda,\alpha)F(-\lambda Z)}{1+D(\lambda,\alpha)F(-\lambda Z)}\right]$$
$$U_2 = u_2(Z) = \left[\frac{D(\lambda,\alpha)Z^{-1}f(\lambda Z)}{1+D(\lambda,\alpha)F(\lambda Z)} + \frac{2(1+\lambda^2)^{-1}D(\lambda,\alpha)F(\lambda Z)}{1+D(\lambda,\alpha)F(\lambda Z)}\right]$$

$$V_1 = v_1(Z) = [\alpha + \alpha D^{-1}(\lambda,\alpha)F^{-1}(-\lambda Z)]^{-1}$$
$$V_2 = v_2(Z) = [\alpha + \alpha D^{-1}(\lambda,\alpha)F^{-1}(\lambda Z)]^{-1}.$$

Proof: The Fisher Information Matrix, $I(\lambda, \alpha)$ of TMSND (λ, α) with p.d.f. (2.1) is given in terms of the variance covariance matrix as,

$$I(\lambda,\alpha) = \left[Cov\left\{\frac{\partial}{\partial\lambda}\log h(z;\lambda,\alpha), \frac{\partial}{\partial\alpha}\log h(z;\lambda,\alpha)\right\}\right] \tag{2.11}$$

in which

$$\begin{aligned}
\frac{\partial}{\partial\lambda}[\log h(z;\lambda,\alpha)] &= \frac{\partial}{\partial\lambda}\log\left\{\frac{2}{\alpha+2}f(z)[1+D(\lambda,\alpha)F(-\lambda z)]\right\}\\
&\quad+\frac{\partial}{\partial\lambda}\log\left\{\frac{2}{\alpha+2}f(z)[1+D(\lambda,\alpha)F(\lambda z)]\right\}\\
&= \frac{\partial}{\partial\lambda}[2\log 2+2\log f(z)-2\log(\alpha+2)]+\\
&\quad\frac{\partial}{\partial\lambda}\log[1+D(\lambda,\alpha)F(-\lambda z)]+\frac{\partial}{\partial\lambda}\log[1+D(\lambda,\alpha)F(\lambda z)]\\
&= \frac{D(\lambda,\alpha)}{[1+D(\lambda,\alpha)F(-\lambda z)]}\left[-\frac{1}{z}f(-\lambda z)+\frac{2}{1+\lambda^2}F(-\lambda z)\right]\\
&\quad+\frac{D(\lambda,\alpha)}{[1+D(\lambda,\alpha)F(\lambda z)]}\left[-\frac{1}{z}f(\lambda z)+\frac{2}{1+\lambda^2}F(\lambda z)\right]\\
&= u_1(z)+u_2(z). \qquad (2.12)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{\partial}{\partial\alpha}\log h(z;\lambda,\alpha) &= \frac{\partial}{\partial\alpha}\log\left\{\frac{2}{\alpha+2}f(z)[1+D(\lambda,\alpha)F(-\lambda z)]\right\}\\
&\quad+\frac{\partial}{\partial\alpha}\log\left\{\frac{2}{\alpha+2}f(z)[1+D(\lambda,\alpha)F(\lambda z)]\right\}\\
&= \frac{\partial}{\partial\alpha}[2\log 2+2\log f(z)-2\log(\alpha+2)]\\
&\quad+\frac{\partial}{\partial\alpha}\log[1+D(\lambda,\alpha)F(-\lambda z)]+\frac{\partial}{\partial\alpha}\log[1+D(\lambda,\alpha)F(\lambda z)]\\
&= -\frac{1}{\alpha+2}+\frac{F(-\lambda z)}{[1+D(\lambda,\alpha)F(-\lambda z)]}\frac{\partial}{\partial\alpha}D(\lambda,\alpha)\\
&\quad-\frac{1}{\alpha+2}\frac{F(\lambda z)}{[1+D(\lambda,\alpha)F(\lambda z)]}\frac{\partial}{\partial\alpha}D(\lambda,\alpha)\\
&= -\frac{1}{\alpha+2}+\frac{F(-\lambda z)}{[1+D(\lambda,\alpha)F(-\lambda z)]}\frac{1}{\alpha}D(\lambda,\alpha)\\
&\quad-\frac{1}{\alpha+2}+\frac{F(\lambda z)}{[1+D(\lambda,\alpha)F(\lambda z)]}\frac{1}{\alpha}D(\lambda,\alpha)\\
&= -\frac{2}{\alpha+2}+v_1(z)+v_2(z). \qquad (2.13)
\end{aligned}$$

Putting (2.13) and (2.12) into (2.11) we obtain (2.10). □

Result 2.7. The characteristic function, $\phi_Z(t)$ of a random variable Z following $TMSND(\lambda,\alpha)$ with (2.1) is the following, for any $t\in\mathbb{R}$ and $i^2=-1$.

$$\begin{aligned}
\phi_Z(t) &= \frac{2}{(\alpha+2)}e^{-\frac{t^2}{2}}[1+D(\lambda,\alpha)F(-i\delta t)]\\
&\quad-\frac{2}{(\alpha+2)}D(\lambda,\alpha)e^{-\frac{t^2}{2}}[\xi_k(-it,-\lambda)-\xi_{-k}(-it,\lambda)] \qquad (2.14)
\end{aligned}$$

where $\delta=\frac{\lambda}{\sqrt{1+\lambda^2}}$ and $\xi_s(a,b)$ is defined in (1.6).

Proof: Let Z follows $TMSND(\lambda,\alpha)$ with p.d.f. (2.1). By the definition of characteristic function, for any $t\in\mathbb{R}$ we have

$$\phi_Z(t) = E\left(e^{itZ}\right) = \int_{-\infty}^{\infty} e^{itz} h(z;\lambda,\alpha)dz. \tag{2.15}$$

On substituting (2.1) in (2.15), we obtain the following.

$$\phi_Z(t) = \frac{2}{(\alpha+2)} e^{-\frac{t^2}{2}} \left\{ 1 + D(\lambda,\alpha)\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-it)^2}{2}} F(-\lambda z)dz - D(\lambda,\alpha)\int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-it)^2}{2}} F(-\lambda z)dz + D(\lambda,\alpha)\int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-it)^2}{2}} F(\lambda z)dz \right\}. \tag{2.16}$$

If we put $z-it=x$ in (2.16), we get

$$\begin{aligned}
\phi_Z(t) &= \frac{2}{(\alpha+2)} e^{-\frac{t^2}{2}} \left[1 + D(\lambda,\alpha)\int_{-\infty}^{\infty} \frac{e^{-\frac{x^2}{2}} F(-\lambda(x+it))dx}{\sqrt{2\pi}} \right] \\
&\quad - D(\lambda,\alpha)\int_{-it}^{\infty} \frac{e^{-\frac{x^2}{2}} F(-\lambda(x+it))dx}{\sqrt{2\pi}} + D(\lambda,\alpha)\int_{-it}^{\infty} \frac{e^{-\frac{x^2}{2}} F(\lambda(x+it))dx}{\sqrt{2\pi}} \\
&= \frac{2}{(\alpha+2)} e^{-\frac{t^2}{2}} \left\{ 1 + D(\lambda,\alpha)F(-i\delta t) - D(\lambda,\alpha)\int_{-it}^{\infty} f(x)F(-\lambda(x+it))dx \right. \\
&\quad \left. + D(\lambda,\alpha)\int_{-it}^{\infty} f(x)F(\lambda(x+it))dx \right\}.
\end{aligned}$$

Now, rearrange the terms and put $k=-\lambda it$ in the above equation to get the following.

$$\begin{aligned}
\phi_Z(t) &= \frac{2}{(\alpha+2)} e^{-\frac{t^2}{2}} [1 + D(\lambda,\alpha)F(-i\delta t)] - \\
&\quad \frac{2D(\lambda,\alpha)}{(\alpha+2)} e^{-\frac{t^2}{2}} \int_{-it}^{\infty} f(x) \left[\int_{-\infty}^{0} f(u)du + \int_{0}^{-\lambda x+k} f(u)du \right] dx + \\
&\quad + \frac{2D(\lambda,\alpha)}{(\alpha+2)} e^{-\frac{t^2}{2}} \int_{-it}^{\infty} f(x) \left[\int_{-\infty}^{0} f(u)du + \int_{0}^{\lambda x-k} f(u)du \right] dx,
\end{aligned}$$

which implies (2.14). □

Result 2.8. If Z follows $TMSND(\lambda,\alpha)$, then $X=\mu+\sigma Z$ is said to have a location scale extension of the two-piece modified skew normal distribution with location parameter

μ and scale parameter σ which is denoted as $ETMSND(\mu,\sigma;\lambda,\alpha)$. The p.d.f. of X is given by

$$h(x;\mu,\sigma,\lambda,\alpha)=\begin{cases}\frac{2}{(\alpha+2)\sigma}f\left(\frac{x-\mu}{\sigma}\right)\left[1+D(\lambda,\alpha)F\left(-\lambda\frac{x-\mu}{\sigma}\right)\right],x<\mu\\ \frac{2}{(\alpha+2)\sigma}f\left(\frac{x-\mu}{\sigma}\right)\left[1+D(\lambda,\alpha)F\left(\lambda\frac{x-\mu}{\sigma}\right)\right],x\geq\mu,\end{cases}\tag{2.17}$$

in which μ, $\lambda\in\mathbb{R}$, $\sigma>0$ and $\alpha\geq-1$.

Result 2.9. $ETMSND(\mu,\sigma;\lambda,0)$ or $ETMSND(\mu,\sigma;0,\alpha)$ or the limiting case of the $ETMSND(\mu,\sigma;\lambda,\alpha)$ when $\lambda\to-\infty$, $\alpha=0$ or $\lambda\to\infty$, is the normal distribution with parameters μ and σ.

Result 2.10. The limiting case of the $ETMSND(\mu,\sigma;\lambda,\alpha)$ when $\alpha\to\infty$ is the $ETSND(\mu,\sigma;\lambda)$.

Result 2.11. The limiting case of the $ETMSND(\mu,\sigma;\lambda,\alpha)$ when $\lambda\to-\infty$ and $\alpha=-1$ is the half-normal distribution with parameters μ and σ.

Result 2.12. The characteristic function $\phi_X(t)$ of a random variable X having $ETMSND(\mu,\sigma;\lambda,\alpha)$ is the following, in which $k=-\lambda it$, $\delta=\frac{\lambda}{\sqrt{1+\lambda^2}}$. For $i^2=-1$ and $t\in\mathbb{R}$,

$$\begin{aligned}\phi_X(t)&=\frac{2}{(\alpha+2)}e^{i\mu t-\frac{t^2\sigma^2}{2}}[1+D(\lambda,\alpha)F(-i\delta\sigma t)]\\&-\frac{2}{(\alpha+2)}D(\lambda,\alpha)e^{i\mu t-\frac{t^2\sigma^2}{2}}\left[\xi_{k\sigma}(-it\sigma,-\lambda)-\xi_{-k\sigma}(-it\sigma,\lambda)\right].\end{aligned}$$

Result 2.13. The c.d.f. $H^*(t)$ of a random variable X with p.d.f. (2.17) is the following

$$H^*(t)=\frac{2}{(\alpha+2)}\begin{cases}F\left(\frac{t-\mu}{\sigma}\right)+\frac{D(\lambda,\alpha)}{2}\left[F\left(\frac{t-\mu}{\sigma}\right)-2\xi\left(\left(\frac{t-\mu}{\sigma}\right),-\lambda\right)\right)],\ t<\mu\\ F\left(\frac{t-\mu}{\sigma}\right)+\frac{D(\lambda,\alpha)}{2}\left[F\left(\frac{t-\mu}{\sigma}\right)-2\xi\left(\left(\frac{t-\mu}{\sigma}\right),\lambda\right)\right.\\ \left.+2\frac{tan^{-1}(\lambda)}{\pi}\right],\ t\geq\mu.\end{cases}$$

3. RELIABILITY MEASURES AND MODE

Here first we present some reliability aspects of the $TMSND(\lambda,\alpha)$ and derive some important results with respect to the mode of the distribution.

Result 3.1. The reliability function $R(t)$ of a random variable Z having the $TMSND(\lambda,\alpha)$ is the following, in which $\xi(t,.)$ is as defined in (1.4).

$$R(t) = 1 - \frac{2}{(\alpha+2)}\begin{cases} F(t) + \frac{D(\lambda,\alpha)}{2}\left[F(t) - 2\xi(t,-\lambda)\right], t<0 \\ F(t) + \frac{D(\lambda,\alpha)}{2}\left[F(t) + \frac{2}{\pi}\tan^{-1}(\lambda) - 2\xi(t,\lambda)\right]. \end{cases}$$

Proof follows from the definition of reliability function $R(t) = 1 - H(t)$, where $H(\cdot)$ is the c.d.f. of Z as obtained in Result 2.4.

Result 3.2. The failure rate $r(t)$ of a random variable Z following the $TMSND(\lambda,\alpha)$ with p.d.f. $h(z;\lambda,\alpha)$ is

$$r(t) = \begin{cases} \frac{2f(t)[1+D(\lambda,\alpha)F(-\lambda t)]}{(\alpha+2)-2(F(t)+D(\lambda,\alpha)[F(t)-2\xi(t,-\lambda)])}, t<0 \\ \frac{2f(t)[1+D(\lambda,\alpha)F(\lambda t)]}{(\alpha+2)-2\left(F(t)+D(\lambda,\alpha)\left[F(t)-2\xi(t,\lambda)+\frac{2}{\pi}\tan^{-1}(\lambda)\right]\right)}, t\geq 0. \end{cases}$$

Proof follows from the definition of failure rate $r(t) = \frac{h(t;\lambda,\alpha)}{R(t)}$, where $R(t)$ is the reliability function as given in Result 3.1.

Result 3.3. The mean residual life function, $\mu(t)$ of a random variable Z with p.d.f. (2.17) is the following, in which $\delta = \frac{\lambda}{\sqrt{1+\lambda^2}}$.

$$\mu(t) = \frac{2}{R(t)(\alpha+2)}\begin{cases} f(t) + D(\lambda,\alpha)f(t)F(-\lambda t) + \frac{-\delta}{2\sqrt{2\pi}}D(\lambda,\alpha) \\ -\frac{\delta}{\sqrt{2\pi}}D(\lambda,\alpha)F\left(t\sqrt{1+\lambda^2}\right) + \frac{\delta}{2\sqrt{2\pi}}D(\lambda,\alpha) - t,\ t\leq 0 \\ f(t) + D(\lambda,\alpha)f(t)F(\lambda t) + \frac{\delta}{2\sqrt{2\pi}}D(\lambda,\alpha) \\ \frac{\delta}{\sqrt{2\pi}}D(\lambda,\alpha)F\left(t\sqrt{1+\lambda^2}\right) - t,\ t>0. \end{cases} \tag{3.1}$$

Proof: By definition, the mean residual life function of Z following the $TMSND(\lambda,\alpha)$ is given by

$$\mu(t) = E(Z|Z>t) - t$$

$$= \frac{2}{R(t)(\alpha+2)}\begin{cases} \int_t^0 zf(z)\left[1+D(\lambda,\alpha)F(-\lambda z)\right]dz + \int_0^\infty zf(z)dz \\ -D(\lambda,\alpha)\int_0^\infty zf(z)F(\lambda z)dz - t,\ t\leq 0 \\ \int_t^\infty zf(z)\left[1+D(\lambda,\alpha)F(\lambda z)\right]dz - t,\ t>0. \end{cases} \tag{3.2}$$

Now for any $t < 0$,

$$\begin{aligned}\int_t^0 zf(z)[1+D(\lambda,\alpha)F(-\lambda z)]dz &= -\int_t^0 f'(z)[1+D(\lambda,\alpha)F(-\lambda z)]dz \\ &= f(t) - \frac{1}{\sqrt{2\pi}} - D(\lambda,\alpha)\left[\frac{1}{2\sqrt{(2\pi)}} - F(-\lambda t)f(t)\right] \\ &\quad + D(\lambda,\alpha)\lambda\int_t^0 f(-\lambda z)f(z)dz \\ &= f(t) - \frac{1}{\sqrt{2\pi}} - \frac{D(\lambda,\alpha)}{2\sqrt{2\pi}} + D(\lambda,\alpha)F(-\lambda t)f(t) \\ &\quad - \frac{\lambda D(\lambda,\alpha)\left[\frac{1}{2} - F(t\sqrt{1+\lambda^2})\right]}{\sqrt{2\pi}\sqrt{1+\lambda^2}} \end{aligned} \tag{3.3}$$

and for any $t > 0$,

$$\begin{aligned}\int_t^\infty zf(z)[1+D(\lambda,\alpha)F(\lambda z)]dz &= -\int_t^\infty f'(z)[1+D(\lambda,\alpha)F(\lambda z)]dz \\ &= f(t) + D(\lambda,\alpha)F(\lambda t)f(t) + \frac{\lambda D(\lambda,\alpha)}{\sqrt{2\pi}\sqrt{1+\lambda^2}} \\ &\quad - D(\lambda,\alpha)\frac{\lambda}{\sqrt{2\pi}\sqrt{1+\lambda^2}}F(t\sqrt{1+\lambda^2}). \end{aligned} \tag{3.4}$$

In particular, when $t = 0$ in (3.4) we have

$$\int_0^\infty zf(z)[1+D(\lambda,\alpha)F(\lambda z)]dz = \frac{1}{\sqrt{(2\pi)}} + \frac{D(\lambda,\alpha)}{2\sqrt{2\pi}} + \frac{\lambda D(\lambda,\alpha)}{2\sqrt{2\pi}\sqrt{1+\lambda^2}}. \tag{3.5}$$

Now on substituting (3.3), (3.4) and (3.5) in (3.2), we get (3.1). □

Result 3.4. The reliability function, $R^*(t)$ of a random variable X following $ETMSND(\mu,\sigma;\lambda,\alpha)$ is

$$R^*(t) = 1 - \frac{2}{(\alpha+2)}\begin{cases} F\left(\frac{t-\mu}{\sigma}\right) + \frac{D(\lambda,\alpha)}{2}\left[F\left(\frac{t-\mu}{\sigma}\right) - 2\xi\left(\frac{t-\mu}{\sigma},-\lambda\right)\right], & t < \mu \\ F\left(\frac{t-\mu}{\sigma}\right) + \frac{D(\lambda,\alpha)}{2}\left[F\left(\frac{t-\mu}{\sigma}\right) - 2\xi\left(\frac{t-\mu}{\sigma},\lambda\right)\right] + \\ \frac{D(\lambda,\alpha)}{2}\left[\frac{2}{\pi}\tan^{-1}(\lambda)\right], & t \geq \mu. \end{cases}$$

Result 3.5. The failure rate $r_1(t)$ of a random variable X following $ETPMSND(\mu,\sigma;\lambda,\alpha)$ is

$$r_1(t) = \begin{cases} \frac{2f\left(\frac{t-\mu}{\sigma}\right)\left[1+D(\lambda,\alpha)F\left(\lambda_1\frac{t-\mu}{\sigma}\right)\right]}{\sigma(\alpha+2)-2\sigma\left(F\left(\frac{t-\mu}{\sigma}\right)+D(\lambda,\alpha)\left[F\left(\frac{t-\mu}{\sigma}\right)-2\xi\left(\frac{t-\mu}{\sigma},-\lambda\right)\right]\right)}, & t < \mu \\ \frac{2f\left(\frac{t-\mu}{\sigma}\right)\left[1+D(\lambda,\alpha)F\left(\lambda_2\frac{t-\mu}{\sigma}\right)\right]}{\sigma(\alpha+2)-2\sigma\left(F\left(\frac{t-\mu}{\sigma}\right)+D(\lambda,\alpha)\left[F\left(\frac{t-\mu}{\sigma}\right)-2\xi\left(\frac{t-\mu}{\sigma},\lambda\right)+\frac{2}{\pi}\tan^{-1}(\lambda)\right]\right)}, & t \geq \mu. \end{cases}$$

Result 3.6. The p.d.f. of $TMSND(\lambda, \alpha)$ is bimodal with unimodes in the regions: $z \in (-\infty, 0)$ and $z \in [0, \infty)$ subject to the conditions given below, in which for $i = 0, 1$.

$$a_i = \frac{D^i(\lambda, \alpha)(\lambda z)^{1-i} f^i(-\lambda z)}{[1 + D(\lambda, \alpha)F(-\lambda z)]^{i+1}},$$
$$b_i = \frac{D^i(\lambda, \alpha)(\lambda z)^{1-i} f^i(\lambda z)}{[1 + D(\lambda, \alpha)F(\lambda z)]^{i+1}}.$$

Region $z \in (-\infty, 0)$: (i). For all $\alpha \geq 0$ for which either $\lambda > 0$ or $\lambda < 0$ with $|a_0| < a_1$ and
(ii). for all $\alpha < 0$ for which either $\lambda \leq 0$ or $\lambda > 0$ with $|a_1| > a_0$.
Region $z \in [0, \infty)$: (i). For all $\alpha \geq 0$ for which either $\lambda > 0$ or $\lambda \leq 0$ with $|b_0| < b_1$ and
(ii). for all $\alpha < 0$ for which either $\lambda < 0$ or $\lambda > 0$ with $|b_1| > b_0$.

Proof: In order to show that there exists unimodes in regions of $z \in (-\infty, 0)$ and $z \in [0, \infty)$, it is enough to show that the second derivative of $h(z; \lambda, \alpha)$ is negative for all α, λ in the respective regions.
For $z \in (-\infty, 0)$, we have

$$\frac{d^2}{dz^2}\{\ln[h(z; \lambda, \alpha)]\} = -1 + \lambda^2 D(\lambda, \alpha) f(-\lambda z)[a_0 - a_1] \tag{3.6}$$

and for $z \in [0, \infty)$, we have

$$\frac{d^2}{dz^2}\{\ln[h(z; \lambda, \alpha)]\} = -1 - \lambda^2 D(\lambda, \alpha) f(\lambda z)[b_0 + b_1]. \tag{3.7}$$

Note that $D(\lambda, \alpha)$ is positive or negative according to α being positive or negative. Now for the region $z \in (-\infty, 0)$, a_0 is positive or negative according as the value of λ is negative or positive and a_1 is positive or negative according as the value of α is positive or negative. If $\alpha \geq 0$, then (3.6) is negative either for $\lambda > 0$ or for $\lambda < 0$ if $|a_0| < a_{11}$ and for $\alpha < 0$, (3.6) is negative either for $\lambda < 0$ or for $\lambda > 0$ with $|a_1| > a_0$. Hence the p.d.f. given in (2.1) is log-concave and thus unimodal under these cases [Ibagimove (1956)].
Now for the region $z \in [0, \infty)$, b_0 is positive or negative according as the value of λ is positive or negative and b_1 is positive or negative according as the value of α is positive or negative. If $\alpha \geq 0$, then (3.7) is negative either for $\lambda > 0$ or for $\lambda < 0$ if $|b_0| < b_1$. Now for $\alpha < 0$, (3.7) is negative either for $\lambda < 0$ or $\lambda > 0$ if $|b_1| > b_0$. Thus the p.d.f. given in (2.1) is log-concave and hence unimodal under these conditions. □

As a consequence of Result 3.6 we obtain the following result.

Result 3.7. The p.d.f. of $TMSND(\lambda_1, \lambda_2, \alpha)$ is plurimodal in the regions given below:
Region $z \in (-\infty, 0)$: (i). For all $\alpha \geq 0$ for which $\lambda < 0$ with $|a_0| > a_1$ and
(ii). for all $\alpha < 0$ for which $\lambda > 0$ with $|a_1| < a_0$ provided
$|\lambda^2 D(\lambda, \alpha) f(\lambda z)(a_0 - a_1)| > 1$ in both the cases.
Region $z \in [0, \infty)$: (i). for all $\alpha \geq 0$ for which $\lambda \leq 0$ with $|b_0| > b_1$ and
(ii). for all $\alpha < 0$ for which $\lambda \leq 0$ with $|b_1| < b_0$ provided
$|\lambda^2 D(\lambda, \alpha) f(\lambda z)(b_0 + b_1)| > 1$ in both the cases .

4. MAXIMUM LIKELIHOOD ESTIMATION

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from $ETMSND(\mu, \sigma; \lambda, \alpha)$ with p.d.f. (2.17). Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be the ordered sample. Assume that $X_{(r)} < \mu < X_{(r+1)}$, for a particular $r = 1, 2, \ldots, n$. Then the log-likelihood function (l) of the sample is the following, in which Σ_{I_j}, $j = 1, 2$ denote the summation over the set I_j such that $I_1 = \{i : X_{(i)} < \mu \ , \ i = 1, 2, \ldots, r\}$ and $I_2 = \{i : X_{(i)} \geq \mu \ , \ i = r+1, \ldots, n\}$.

$$\begin{aligned} l = & \; n \ln\left(\frac{2}{\sigma(\alpha+2)}\right) + \sum_{I_1} \ln f\left(\frac{x_i - \mu}{\sigma}\right)\left[1 + D(\lambda,\alpha)F\left(\frac{-\lambda(x_i-\mu)}{\sigma}\right)\right] + \\ & \sum_{I_2} \ln f\left(\frac{x_i - \mu}{\sigma}\right)\left[1 + D(\lambda,\alpha)F\left(\frac{\lambda(x_i-\mu)}{\sigma}\right)\right]. \end{aligned} \tag{4.1}$$

On differentiating (4.1) with respect to the parameters μ, σ, λ and α and equating to zero, we obtain the following likelihood equations:

$$\sum_{I_1} \frac{(x_i - \mu)}{\sigma} + \sum_{I_2} \frac{(x_i - \mu)}{\sigma} = D(\lambda,\alpha)\lambda\left[\sum_{I_1} -p\left(x_i\right) + \sum_{I_2} q\left(x_i\right)\right], \tag{4.2}$$

$$\begin{aligned} \frac{n}{2\sigma^2} = & \; \frac{1}{2}\sum_{I_1} \frac{(x_i-\mu)^2}{\sigma^4} + \frac{1}{2}\sum_{I_2} \frac{(x_i-\mu)^2}{\sigma^4} \\ & + \frac{D(\lambda,\alpha)\lambda}{2\sigma^3}\left(\sum_{I_1} p\left(x_i\right)\left(x_i-\mu\right) + \sum_{I_2} q\left(x_i\right)\left(x_i-\mu\right)\right), \end{aligned} \tag{4.3}$$

$$\frac{D(\lambda,\alpha)}{\sigma}\left[\sum_{I_1} p(x_i)(x_i-\mu) + \sum_{I_2} q(x_i)(x_i-\mu)\right] - = 0 \tag{4.4}$$

and

$$D(\lambda,\alpha)\sum_{I_1} P(x_i) + D(\lambda,\alpha)\sum_{I_2} Q(x_i) = \frac{n\alpha}{(\alpha+2)}, \tag{4.5}$$

in which

$$p(x_i) = \frac{f\left(\lambda\frac{(x_i-\mu)}{\sigma}\right)}{\left[\left[1 + D(\lambda,\alpha,\rho)F\left(\lambda\frac{(x_i-\mu)}{\sigma}\right)\right]\right]},$$

$$q(x_i) = \frac{f\left(\rho\lambda\frac{(x_i-\mu)}{\sigma}\right)}{\left[\left[1 + D(\lambda,\alpha,\rho)F\left(\rho\lambda\frac{(x_i-\mu)}{\sigma}\right)\right]\right]},$$

$$P(x_i) = \frac{F\left(\lambda\frac{(x_i-\mu)}{\sigma}\right)}{\left[\left[1 + D(\lambda,\alpha,\rho)F\left(\lambda\frac{(x_i-\mu)}{\sigma}\right)\right]\right]}$$

and

$$Q(x_i) = \frac{F\left(\rho\lambda\frac{(x_i-\mu)}{\sigma}\right)}{\left[\left[1 + D(\lambda,\alpha,\rho)F\left(\rho\lambda\frac{(x_i-\mu)}{\sigma}\right)\right]\right]}.$$

Solving the non-linear system of equations (4.2) to (4.5) by simultaneous solution method using some mathematical softwares such as *MATHCAD*, *MATLAB*, *MATHEMATICA* etc., the maximum likelihood estimates (MLE) of the parameters of $ETMSND(\mu, \sigma; \lambda, \alpha)$ can be obtained.

5. NUMERICAL ILLUSTRATION

For illustrating the usefulness of the $ETMSND(\mu, \sigma; \lambda, \alpha)$, here we considered the following IQ data set for 87 white males hired by a large insurance company in 1971 given in Roberts (1988).
85 94 94 97 98 100 100 101 102 102 103 103 103 103 104 104 106 106 106 106 106 107 107 108 108 108 108 108 108 108 109 109 111 111 112 112 112 112 112 112 112 112 112 112 113 113 113 113 113 113 113 113 114 114 115 116 116 116 116 117 117 117 118 118 118 119 120 120 120 121 121 121 122 122 122 122 122 122 124 124 125 129 131 132 135 136 140.
Here we fitted the proposed model to the data set and compared with the existing models such as the normal distribution $N(\mu, \sigma)$, the extended skew normal distriution $ESND(\mu, \sigma, \lambda)$, the $EMSND(\mu, \sigma; \lambda, \rho)$, and the $ETMSND(\mu, \sigma; \lambda, \alpha)$. For model comparison, we have computed the information criterions such as the Akaike′s Information Criterion (AIC), the Bayesian Information Criterion (BIC) and the corrected Akaike′s Information Criterion (AICc). For numerical evaluation, we have used the MATHCAD softwares and the results obtained are included in Table 1 along with the computed values of the log likelihood "l".

Distri bution:	N (μ, σ)	ESND (μ, σ, λ)	$EMSND$ $(\mu, \sigma; \lambda, \alpha)$	$ETMSND$ $(\mu, \sigma; \lambda, \alpha)$
$\hat{\mu}$	112.86	105.78	107.92	108.43
$\hat{\sigma}$	9.58	11.94	11.31	11.02
$\hat{\lambda}$	-	1.14	0.75	0.49,-0.49
$\hat{\alpha}$	-	-	0.75	-0.91
$\hat{\rho}$	-	-	-	-
l	-319.6	-319.29	-315	-314
AIC	643.2	644.57	639	636
BIC	648.14	651.97	648	644
AICc	643.35	644.86	639.45	637.04

Table 1: Computed values of MLE and the l, AIC, BIC and AICc of the parameters for various fitted models.

From Table 1, it can be seen that the $ETMSND(\mu, \sigma; \lambda, \alpha)$ gives the best fit compared to other existing models.

6. SIMULATION

For examining the asymptotic properties of the maximum likelihood estimators of parameters of the distribution $ETMSND(\mu, \sigma; \lambda, \alpha)$, we carried out a simulation study by generating observations from the distribution with the help of R software for two sets of parameters($\mu = 0.3, \sigma = 0.3, \lambda = 0.3, \alpha = 0.3$ *and* $\mu = 0.3, \sigma = 0.4, \lambda = 0.3, \alpha = 0.8$). We have considered 200 bootstrap samples of sizes 50, 100 and 150. To compare the performance of the estimators with respect to their actual bias and mean square error (MSE). The results ,thus obtained are presened in Table 2.

n	sample size						
	parameters	Est	Bias	MSE	Est	Bias	MSE
50	μ	0.32	0.14	0.00161	0.319	0.139	0.0016
	σ	0.242	-0.634	0.012396	0.335	-0.555	0.00513
	λ	0.342	-0.242	0.00211	0.346	-0.161	0.000363
	α	0.732	0.061	0.00011	0.73	0.059	0.000114
100	μ	0.253	0.672	0.01817	0.282	0.654	0.00719
	σ	0.204	-0.612	0.01160	0.321	-0.579	0.00547
	λ	-0.535	-1.106	0.03301	-0.607	-1.116	0.02908
	α	0.724	0.056	0.00018	0.72	0.05	6.5E-05
150	μ	0.156	0.388	0.00593	0.148	0.368	0.00478
	σ	0.123	-0.567	0.01109	0.235	-0.542	0.00464
	λ	-1.549	-2.126	0.14394	-1.231	-1.845	0.07705
	α	0.12	-0.52	0.01381	0.154	-0.486	0.00486

Table 2: Estimate of the parameters and the corresponding bias and mean square errors of the maximum likelihood estimates of the parameters of the $ETMSND(\mu, \sigma; \lambda, \alpha)$.

From Table 2, it can be observed that the bias approaches to zero and MSEs are in the decreasing order as the sample size increases.

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