REVSTAT – Statistical Journal
Volume 7, Number 2, June 2009, 171–187

A BAYESIAN SHRINKAGE APPROACH IN WEIBULL TYPE-II CENSORED DATA USING PRIOR POINT INFORMATION

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Received: June 2008 Revised: March 2009 Accepted: May 2009

Abstract:

- In the present paper we study the performance of the Bayes Shrinkage estimators for the scale parameter of the Weibull distribution under the squared error loss and the LINEX loss functions in the presence of a prior point information of the scale parameter when Type-II censored data are available. The properties of the minimax estimators are also discussed.

Key-Words:

- *Bayes shrinkage estimator; Uniformly Minimum Variance Unbiased (UMVU) estimator; Minimax estimator and LINEX loss function.*

AMS Subject Classification:

- 62A15, 62C10, 62C20.

1. INTRODUCTION

The 'time of failure' and 'average life' of a component, measured from some specified time until it fails, is represented by a continuous random variable. Extensively in recent years, one distribution that has been used as a model to deal with such problems for product life is the Weibull distribution. Its applications in life-testing problems and survival analysis have been widely advocated (Weibull, 1951; Berrettoni, 1964). It has been used as model with diverse types of items such as ball bearing (Lieblein & Zelen, 1956), vacuum tube (Kao, 1959) and electrical isolation (Nelson, 1972). Mittnik & Reachev (1993) found that the Weibull distribution might be an adequate statistical model for stock returns. Mann (1968) gave a variety of situations in which the distribution is used for other types of failure data. Whittemore & Altschuler (1976) used it as a model in biomedical applications.

The probability density function of the Weibull distribution is given by

$$f(x;v,\theta) = \frac{v}{\theta}\, x^{v-1} e^{-\frac{x^v}{\theta}} , \qquad x>0, \quad v>0, \quad \theta>0 , \tag{1.1}$$

where the parameters v and θ are referred to as the shape and scale parameters of the distribution, respectively.

For the special case $v=1$, the Weibull distribution is the Exponential distribution. For $v=2$, is the Rayleigh distribution. For shape parameter values in the range $3 \le v \le 4$, the shape of the Weibull distribution is close to that of the Normal distribution and for a large values of v, say $v \ge 10$, the shape of the Weibull distribution is close to that of a smallest extreme value distribution.

Pandey (1983), Pandey *et al.* (1989), Pandey & Singh (1993) considered the estimation of the Weibull shape parameter in censored data. The prediction problems in the Weibull distribution have been discussed by Engelhardt & Bain (1973), Nigm (1989), Dellaportas & Wright (1991) and others. Montanari *et al.* (1997), Singh & Shukla (2000), Hisada & Arizino (2002), Singh *et al.* (2002), Tsionas (2002) and others considered the Weibull distribution in different contexts.

In many situations, the experimenter has some prior information about the parameter in the form of a point guess value. To utilize this guess value, the shrinkage estimators have been discussed by a number of authors, for details see the article Casella & Lehmann (1998), Prakash & Singh (2006, 2008), Singh *et al.* (2007). The shrinkage estimator performs better than the usual estimator when a guess value is approximately the true value of the parameter and sample size is small. A shrinkage estimator (Thompson, 1968) for the parameter θ when a prior point guess value θ_0 of θ is available, is defined as

$$S = k\,\hat{\theta} + (1-k)\,\theta_0\ , \qquad 0 \le k \le 1\ . \tag{1.2}$$

Here $\hat{\theta}$ is any usual estimator of the parameter θ. The shrinkage procedure has been applied in numerous problems, including mean survival time in epidemiological studies (Harries & Shakarki, 1979), forecasting of the money supply (Tso, 1990), estimating mortality rates (Marshall, 1991) and improved estimation in sample surveys (Wooff, 1985).

When positive and negative errors have different consequences, the use of squared error loss function (SELF) in Bayesian estimation may not be appropriate. To overcome this difficulty, Varian (1975) and Zellner (1986) proposed an asymmetric loss function known as the LINEX loss function (LLF). The invariant version of LLF for any parameter θ is given by

$$L(\Delta) = e^{a\Delta} - a\,\Delta - 1\ , \qquad a \neq 0 \quad \text{and} \quad \Delta = \frac{\hat{\theta}-\theta}{\theta}\ . \tag{1.3}$$

The sign and magnitude of 'a' represents the direction and degree of asymmetry respectively. The positive (negative) value of 'a' is used when overestimation is more (less) serious than underestimation. The loss function (1.3) is approximately square error and almost symmetric if $|a|$ is near to zero. A number of authors have discussed the estimation procedures under LLF criterion. A Few recent works under the Bayesian and/or the LLF criterions are Nigm *et al.* (2003), Bellhouse (2004), Xu & Shi (2004), Ahmadi *et al.* (2005), Prakash & Singh (2006), Son & Oh (2006), Singh *et al.* (2007), Ahmad *et al.* (2007), Prakash & Singh (2008), among others.

Let $x_1, x_2, ..., x_n$ be the life times of n items put to test under model (1.1). The maximum likelihood estimate of θ (when v is known) is given by

$$\hat{\theta} = \sum_{i=1}^{n} \frac{x_i^v}{n}\ . \tag{1.4}$$

Consider Type-II censored sampling, where the test terminates as soon as the r^{th} item fails ($r \le n$). Let $x_1, x_2, ..., x_r$ be the observed failure times for the first r components. Then the likelihood function for the r failure items is

$$L\big(x_1, x_2, ..., x_r \,|\, \theta\big) = \frac{v^r}{\theta^r} \prod_{i=1}^{r} x_i^{v-1} \exp\left\{-\frac{r\,T_r}{\theta}\right\}, \tag{1.5}$$

where $T_r = \frac{1}{r}\left\{\sum_{i=1}^{r} x_i^v + (n-r)\,x_{(r)}^v\right\}$ is a UMVU estimator (Sinha, 1986) of the parameter θ and $\frac{2\,r\,T_r}{\theta} \backsim \chi^2_{2r}$.

The risks under the SELF and the LLF for T_r are given as

$$R_{(S)}(T_r) = \frac{\theta^2}{r} \qquad \text{and} \qquad R_{(L)}(T_r) = e^{-a}\left(\frac{r}{r-a}\right)^r - 1\ ,$$

where suffix S and L respectively, denote the risk taken under the SELF and the LLF criterions.

If parameter v is known, the natural family of conjugate prior of θ is taken as the inverted Gamma distribution with probability density function

$$g_1(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\, \theta^{-\alpha-1} e^{-\frac{\beta}{\theta}} , \qquad \alpha > 0, \quad \beta > 0 . \tag{1.6}$$

In the situation where the researchers have no prior information about the parameter θ, one may use the uniform, quasi or improper prior. A family of priors is given by

$$g_2(\theta) = \theta^{-d}\, e^{-\frac{c\,d}{\theta}} , \qquad d > 0, \quad c > 0 . \tag{1.7}$$

If $d = 0$, we get a diffuse prior and if $d = 1$, $c = 0$ a non-informative prior is obtained. For a set of values of d and c, that satisfies the equality $\Gamma(d-1) = (c\,d)^{d-1}$, makes $g_2(\theta)$ as a proper prior.

If both of the parameters θ and v are unknown in model (1.1), the joint prior distribution (Sinha, 1986) is considered as

$$g(\theta, v) = g_1(\theta)\,.\,h(v), \quad h(v) = \frac{1}{\vartheta} , \qquad 0 < v < \vartheta, \quad \vartheta > 0 . \tag{1.8}$$

In the present paper, we suggest some Bayes shrinkage estimators for the scale parameter of the two-parameter Weibull distribution in presence of a prior point information when Type-II censored data is available under the SELF and the LLF. The properties have been studied in terms of the relative efficiencies when compared with the UMVU estimator. The properties of the minimax estimator are also discussed in the last section.

2. THE BAYES SHRINKAGE ESTIMATORS (KNOWN SHAPE PARAMETER)

The posterior density of the parameter θ under prior density $g_1(\theta)$, is

$$Z_1(\theta) = \frac{(r\,T_r + \beta)^{\alpha+r}}{\Gamma(\alpha+r)}\, e^{-\frac{(r\,T_r+\beta)}{\theta}}\, \theta^{-\alpha-r-1} , \tag{2.1}$$

which is again an inverted Gamma distribution with the parameters $(\alpha + r)$ and $(r\,T_r + \beta)$. The Bayes estimator of the parameter θ under the SELF is obtained as

$$\hat{\theta}_1 = E_p(\theta) = \varphi_1(r\,T_r + \beta) , \qquad \varphi_1 = (\alpha + r - 1)^{-1} . \tag{2.2}$$

Here, the suffix p indicates that the expectation is taken under posterior density. We choose the parameters of the prior distribution $g_1(\theta)$ such that $E(\hat{\theta}_1) = \theta_0$, where θ_0 is the point guess value of θ. This gives $\beta = (\alpha - 1)\theta_0$. Substituting β in (2.2), we obtain the Bayes estimator for θ as

$$\bar{\theta}_1 = \lambda_1 T_r + (1-\lambda_1)\theta_0 , \qquad \lambda_1 = r\varphi_1, \quad (\alpha + r) > 1 . \tag{2.3}$$

This is similar to the shrinkage estimator defined in (1.2). We termed $\bar{\theta}_1$ as the Bayes shrinkage estimator.

The Bayes estimate of θ under the LLF (1.3) is obtained by simplifying the equality

$$E_p\left(\frac{1}{\theta}\, e^{\frac{a\,\hat{\theta}_2}{\theta}}\right) = e^a\, E_p\left(\frac{1}{\theta}\right) \implies \\ \implies \hat{\theta}_2 = \varphi_2(r\,T_r + \beta), \quad \varphi_2 = \frac{1}{a}\left(1 - \exp\left(-\frac{a}{\alpha + r + 1}\right)\right). \tag{2.4}$$

Again,

$$E(\hat{\theta}_2) = \theta_0 \implies \beta = \theta_0\,(1 - r\,\varphi_2)\,\varphi_2^{-1} .$$

Hence, the Bayes shrinkage estimator for θ under the LLF with this choice of constant is given by

$$\bar{\theta}_2 = \lambda_2 T_r + (1-\lambda_2)\theta_0 , \qquad \lambda_2 = r\,\varphi_2 . \tag{2.5}$$

The expressions of the risks of these estimators under the SELF and the LLF are obtained as

$$R_{(S)}(\hat{\theta}_i) = r\,\theta^2\,\varphi_i^2 + \big(\theta\,(r\,\varphi_i - 1) + \beta\,\varphi_i\big)^2 , \tag{2.6}$$

$$R_{(L)}(\hat{\theta}_i) = \exp\left(a\left(\frac{\varphi_i\,\beta}{\theta} - 1\right)\right)\,(1 - a\,\varphi_i)^{-r} - 1 - a\left(r + \frac{\beta}{\theta} - 1\right), \tag{2.7}$$

$$R_{(S)}(\bar{\theta}_i) = \theta^2\left\{\lambda_i^2\left(\frac{r+1}{r} + \delta\,(\delta - 2)\right) + (1-\delta)^2\,(1 - 2\,\lambda_i)\right\} \tag{2.8}$$

and

$$R_{(L)}(\bar{\theta}_i) = e^{a((1-\lambda_i)\,\delta - 1)}\,(1 - \frac{a\,\lambda_i}{r})^{-r} - 1 + a\,(1-\delta)\,(1-\lambda_i) , \tag{2.9}$$

where $\delta = \frac{\theta}{\theta_0}$, $i = 1, 2$.

The posterior density of θ corresponding to $g_2(\theta)$ is given as

$$Z_2(\theta) = \frac{(r\,T_r + c\,d)^{r+d-1}}{\Gamma(r+d-1)}\, e^{-\frac{(r\,T_r + c\,d)}{\theta}}\,\theta^{-r-d} . \tag{2.10}$$

This posterior distribution has the same form as the posterior (2.1). The only change is that in the place of α and β there are $d-1$ and $c\,d$, respectively. All the results discussed in Section 3 hold if we substitute $d = (\alpha + 1)$ and $c = \frac{\beta}{(\alpha+1)}$.

3. NUMERICAL ANALYSIS

The relative efficiencies of the Bayes shrinkage estimator $\bar{\theta}_i$ $(i = 1, 2)$ relative to the UMVU estimator T_r under the SELF and the LLF criterions are defined as

$$RE_{(S)}(\bar{\theta}_i, T_r) = \frac{R_{(S)}(T_r)}{R_{(S)}(\bar{\theta}_i)}$$

and

$$RE_{(L)}(\bar{\theta}_i, T_r) = \frac{R_{(L)}(T_r)}{R_{(L)}(\bar{\theta}_i)}, \qquad i = 1, 2 .$$

The expressions of relative efficiencies are the functions of r, a, δ and α whereas $RE_{(S)}(\bar{\theta}_i, T_r)$ is independent with 'a'. For the selected set of values $r = 04\,(02)10$; $a = 0.50(0.50)2.00$; $\delta = 0.25\,(0.25)1.75$ and $\alpha = 1.25, 1.50, 2.50, 5.00, 10, 20$, the relative efficiencies have been calculated and presented in Tables 1–4, respectively. The numerical findings are presented here only for $r = 04$ when risk criterion is the LLF.

Table 1: $RE_{(S)}(\bar{\theta}_1, T_r)$.

r	δ	α					
		1.25	1.50	2.50	5.00	10.00	20.00
04	0.25	1.1191	1.2226	1.4362	1.2308	0.8525	0.6387
	0.50	1.1245	1.2462	1.6575	2.0000	1.7423	1.4032
	0.75	1.1278	1.2607	1.8264	3.2000	4.6621	4.9788
	1.00	1.1289	1.2656	1.8906	4.0000	10.562	33.062
	1.25	1.1278	1.2607	1.8264	3.2000	4.6621	4.9788
	1.50	1.1245	1.2462	1.6575	2.0000	1.7423	1.4032
	1.75	1.1191	1.2226	1.4362	1.2308	0.8525	0.6387
06	0.25	1.0787	1.1467	1.2903	1.1111	0.7273	0.4983
	0.50	1.0823	1.1615	1.4286	1.6667	1.4286	1.0823
	0.75	1.0844	1.1706	1.5267	2.3810	3.3898	3.6470
	1.00	1.0851	1.1736	1.5625	2.7778	6.2500	17.361
	1.25	1.0844	1.1706	1.5267	2.3810	3.3898	3.6470
	1.50	1.0823	1.1615	1.4286	1.6667	1.4286	1.0823
	1.75	1.0787	1.1467	1.2903	1.1111	0.7273	0.4983
08	0.25	1.0588	1.1094	1.2175	1.0588	0.6744	0.4317
	0.50	1.0614	1.1202	1.3175	1.5000	1.2788	0.9275
	0.75	1.0630	1.1267	1.3858	2.0000	2.7656	2.9816
	1.00	1.0635	1.1289	1.4102	2.2500	4.5156	11.390
	1.25	1.0630	1.1267	1.3858	2.0000	2.7656	2.9816
	1.50	1.0614	1.1202	1.3175	1.5000	1.2788	0.9275
	1.75	1.0588	1.1094	1.2175	1.0588	0.6744	0.4317
10	0.25	1.0469	1.0872	1.1739	1.0316	0.6497	0.3947
	0.50	1.0490	1.0957	1.2521	1.4000	1.1934	0.8389
	0.75	1.0502	1.1008	1.3042	1.7818	2.3967	2.5827
	1.00	1.0506	1.1025	1.3225	1.9600	3.6100	8.4100
	1.25	1.0502	1.1008	1.3042	1.7818	2.3967	2.5827
	1.50	1.0490	1.0957	1.2521	1.4000	1.1934	0.8389
	1.75	1.0469	1.0872	1.1739	1.0316	0.6497	0.3947

Table 1 shows that the Bayes shrinkage estimator $\bar{\theta}_1$ performs uniformly well for small $\alpha \leq 5.00$ with respect to the UMVU estimator T_r under the SELF. The effective interval (the interval in which the relative efficiency is more than one) decreases with the sample size r as well as α increases under the SELF. The efficiency attains maximum at the point $\delta = 1.00$ and the gain in efficiency decreases as r increases for all considered values of δ when other parametric values are fixed. Further, the gains in efficiencies increase as α increases in the interval $0.75 \leq \delta \leq 1.25$ with other fixed parametric values.

On the other hand, when the risk criterion is the LLF (Table 2) the estimator $\bar{\theta}_1$ performs uniformly well with respect to T_r when sample size is small $r\,(\leq 06)$ for all considered values of parametric space but for a large sample size, this property holds in the interval $0.50 \leq \delta \leq 1.50$. The gain in efficiency increases when 'a' increases for all considered values of δ with small sample size $r\,(\leq 06)$ and in the interval $\delta \leq 1.25$ otherwise, under other fixed parametric values. Other properties are similar to the SELF-criterion.

Table 2: $RE_{(L)}(\bar{\theta}_1, T_r)$.

$r = 04$		α					
a	δ	1.25	1.50	2.50	5.00	10.00	20.00
0.50	0.25	1.1374	1.2835	1.6460	1.4888	1.0422	1.0173
	0.50	1.1448	1.2954	1.8404	2.3580	2.0777	1.6762
	0.75	1.1501	1.2968	1.9546	3.6144	5.4006	5.7685
	1.00	1.1534	1.3007	1.9569	4.2400	11.414	36.177
	1.25	1.1281	1.2613	1.8306	3.2413	4.8409	5.2728
	1.50	1.1169	1.2311	1.6228	1.9921	1.7805	1.4433
	1.75	1.1039	1.1738	1.3638	1.2135	1.1517	1.1352
1.00	0.25	1.1540	1.3182	1.9506	1.8884	1.3370	1.0178
	0.50	1.1702	1.3510	2.0770	2.9135	2.6054	2.1057
	0.75	1.1847	1.3757	2.1064	4.2662	6.5833	7.0413
	1.00	1.1975	1.3914	2.1565	4.6862	12.955	41.695
	1.25	1.1363	1.2782	1.8866	3.4259	5.2801	5.8795
	1.50	1.1172	1.2321	1.6349	2.0697	1.9090	1.5603
	1.75	1.1068	1.1814	1.3725	1.2466	1.1890	1.1618
1.50	0.25	1.1845	1.3822	2.3035	2.5672	1.8402	1.4108
	0.50	1.2102	1.4372	2.4297	3.8543	3.5120	2.8440
	0.75	1.2348	1.4861	2.4822	5.3708	8.6319	9.2565
	1.00	1.2582	1.5275	2.5244	5.5108	15.782	51.739
	1.25	1.1580	1.3225	2.0301	3.8550	6.1813	7.0553
	1.50	1.1307	1.2594	1.7207	2.2881	2.1938	1.8118
	1.75	1.1128	1.1942	1.4223	1.3606	1.1996	1.1738
2.00	0.25	1.2425	1.5052	2.7492	3.8948	2.8309	2.1846
	0.50	1.2790	1.5879	3.0769	5.6791	5.3001	4.3013
	0.75	1.3153	1.6677	3.2667	7.1635	12.675	13.647
	1.00	1.3512	1.7432	3.2835	7.4945	21.480	71.977
	1.25	1.2060	1.4211	2.3516	4.7975	8.0839	9.4870
	1.50	1.1695	1.3370	1.9500	2.7951	2.8120	2.3530
	1.75	1.1332	1.2540	1.5866	1.6381	1.2402	1.1918

The Bayes shrinkage estimator $\bar{\theta}_2$ performs well for all considered values of the parametric space when $\alpha \leq 10.00$ with respect to T_r under the SELF. The gain in efficiency increases when 'a' increases in the interval $0.75 \leq \delta \leq 1.25$ for all considered parametric values when $\alpha \leq 10.00$. Other properties of the estimator $\bar{\theta}_2$ are similar to the estimator $\bar{\theta}_1$ under the SELF.

Table 3: $RE_{(S)}(\bar{\theta}_2, T_r)$.

$r = 04$		α					
a	δ	1.25	1.50	2.50	5.00	10.00	20.00
	0.25	1.4051	1.3797	1.2604	1.0111	1.0132	0.6178
	0.50	1.8994	1.9341	1.9980	1.9079	1.6312	1.3636
	0.75	2.4076	2.5485	3.0790	4.0782	4.8817	4.9465
0.50	1.00	2.6433	2.8504	3.7566	6.5691	14.537	39.850
	1.25	2.4076	2.5485	3.0790	4.0782	4.8817	4.9465
	1.50	1.8994	1.9341	1.9980	1.9079	1.6312	1.3636
	1.75	1.4051	1.3797	1.2604	1.0111	1.0132	0.6178
	0.25	1.3786	1.3502	1.2295	1.0921	1.0766	0.6157
	0.50	1.9353	1.9611	2.0000	1.8922	1.6203	1.3596
	0.75	2.5542	2.6919	3.2053	4.1529	4.8971	4.9425
1.00	1.00	2.8589	3.0737	4.0111	6.9016	15.026	40.651
	1.25	2.5542	2.6919	3.2053	4.1529	4.8971	4.9425
	1.50	1.9353	1.9611	2.0000	1.8922	1.6203	1.3596
	1.75	1.3786	1.3502	1.2295	1.0921	1.0766	0.6157
	0.25	1.3482	1.3183	1.1988	1.1738	1.0991	0.6137
	0.50	1.9626	1.9806	1.9978	1.8760	1.6096	1.3557
	0.75	2.7013	2.8351	3.3291	4.2240	4.9113	4.9383
1.50	1.00	3.0888	3.3113	4.2797	7.2479	15.528	41.465
	1.25	2.7013	2.8351	3.3291	4.2240	4.9113	4.9383
	1.50	1.9626	1.9806	1.9978	1.8760	1.6096	1.3557
	1.75	1.3482	1.3183	1.1988	1.1738	1.0991	0.6137
	0.25	1.3153	1.2850	1.1687	1.2563	1.1524	0.6117
	0.50	1.9820	1.9930	1.9919	1.8595	1.5991	1.3518
	0.75	2.8480	2.9772	3.4498	4.2914	4.9241	4.9341
2.00	1.00	3.3336	3.5638	4.5629	7.6084	16.044	42.293
	1.25	2.8480	2.9772	3.4498	4.2914	4.9241	4.9341
	1.50	1.9820	1.9930	1.9919	1.8595	1.5991	1.3518
	1.75	1.3153	1.2850	1.1687	1.2563	1.1524	0.6117

Under the LLF criterion (Table 4), the estimator $\bar{\theta}_2$ also performs well for $\alpha \leq 10.00$ with respect to T_r and the gain in efficiency increases as 'a' increases for all considered values of parametric space. Other properties of $\bar{\theta}_2$ are similar to the Bayes shrinkage estimator $\bar{\theta}_1$ under the LLF criterion.

The gain in efficiency is larger for the Bayes shrinkage estimator $\bar{\theta}_2$ under the LLF-criterion with respect to the SELF-criterion.

Remark 3.1. One may obtain the results for the complete sample case by replacing only the censored sample size r with the complete sample size n.

Table 4: $RE_{(L)}(\bar{\theta}_2, T_r)$.

$r = 04$		α					
a	δ	1.25	1.50	2.50	5.00	10.00	20.00
	0.25	1.6696	1.6475	1.5223	1.2316	1.0475	0.7625
	0.50	2.1863	2.2395	2.3499	2.2710	1.9461	1.6295
	0.75	2.6513	2.8215	3.4666	4.6905	5.6626	5.7285
0.50	1.00	2.7676	2.9917	3.9753	7.0418	15.778	43.665
	1.25	2.4211	2.5656	3.1147	4.1820	5.1063	5.2472
	1.50	1.8671	1.9049	1.9857	1.9310	1.6731	1.4025
	1.75	1.3451	1.3242	1.2403	1.0074	1.0002	0.6137
	0.25	2.0557	2.0308	1.8866	1.5459	1.2075	0.9837
	0.50	2.7072	2.7745	2.9142	2.8197	2.4254	2.0421
	0.75	3.2546	3.4681	4.2746	5.7837	6.9391	6.9833
1.00	1.00	3.2684	3.5344	4.7001	8.3263	18.629	51.436
	1.25	2.6878	2.8427	3.4322	4.5902	5.6420	5.8586
	1.50	1.9502	1.9870	2.0702	2.0351	1.7908	1.5118
	1.75	1.3584	1.3390	1.2455	1.0308	1.0102	0.6363
	0.25	2.7217	2.6918	2.5135	2.0847	1.6518	1.3611
	0.50	3.6007	3.6926	3.8833	3.7633	3.2503	2.7518
	0.75	4.2805	4.5695	5.6579	7.6678	9.1522	9.1678
1.50	1.00	4.3134	4.6534	5.9421	10.564	23.661	65.268
	1.25	3.1686	3.3496	4.0386	5.4076	6.7074	7.0433
	1.50	2.1639	2.2055	2.3070	2.3004	2.0581	1.7507
	1.75	1.4374	1.4204	1.3358	1.1282	1.0425	0.7064
	0.25	4.0363	3.9961	3.7493	3.1458	2.5266	2.1040
	0.50	5.3596	5.5001	5.7924	5.6237	4.8782	4.1525
	0.75	6.2841	6.7235	8.3740	11.384	13.536	13.598
2.00	1.00	6.7483	6.8356	8.3882	14.978	33.669	92.906
	1.25	4.1355	4.3749	5.2875	7.1208	8.9341	9.4941
	1.50	2.6588	2.7148	2.8614	2.9034	2.6444	2.2679
	1.75	1.6843	1.6704	1.5919	1.3743	1.0963	0.8737

4. THE BAYES SHRINKAGE ESTIMATORS (UNKNOWN SHAPE PARAMETER)

When both parameters of the model (1.1) are unknown, the joint posterior density with respect to $g(\theta, v)$, in (1.8), is obtained as

$$(4.1)\qquad Z_3(\theta, v) = \frac{v'\,\theta^{-(\alpha+r+1)}\,e^{-\frac{(r\,T_r+\beta)}{\theta}}}{\Gamma(\alpha+r)\displaystyle\int_0^{\vartheta} v'(r\,T_r+\beta)^{-\alpha-r}\,dv}\,, \qquad v' = v^r \prod_{i=1}^{r} x_i^{v-1}\,.$$

The marginal density of θ is obtained as

$$(4.2)\qquad Z_4(\theta) = \frac{\theta^{-(\alpha+r+1)} \displaystyle\int_0^{\vartheta} v' e^{-\frac{(r\,T_r+\beta)}{\theta}}\, dv}{\Gamma(\alpha+r) \displaystyle\int_0^{\vartheta} v' (r\,T_r+\beta)^{-\alpha-r}\, dv} .$$

Hence, the Bayes estimate of the parameter θ under the SELF is obtained as

$$(4.3)\qquad \hat{\theta}_3 = \frac{I\big(v', (\alpha+r-1)\big)}{(\alpha+r-1)\; I\big(v', (\alpha+r)\big)} ,$$

where $I(\omega_1, \omega_2) = \int_0^{\vartheta} (\omega_1)\,(r\,T_r+\beta)^{-\omega_2}\, dv$.

Similarly, the Bayes estimate of the parameter θ under the LLF is

$$(4.4)\qquad I\big(v'', (\alpha+r-1)\big) = e^{a}\, I\big(v'', (\alpha+r)\big) , \qquad v'' = v' \left(1 - \frac{a\,\hat{\theta}_4}{r\,T_r+\beta}\right)^{-\alpha-r+1} .$$

The Bayes estimates of the parameter θ under the SELF and the LLF criterions do not exist in the close form. Therefore, the risks under both risks criterion also do not exist. Hence, the Bayes shrinkage estimator is not obtained in this case. However, the numerical findings of the Bayes estimates and their risks under both risks criterion are presented here by using the following example.

Example 4.1. Mann and Fertig (1973) give failure times of airplane components subjected to a life test. The Weibull distribution has often been found a suitable model in such situations. The data are Type-II censored: 13 components were placed on test and test was terminated at time of 10^{th} failure. Failure times (in hours) of the 10 components that failed were

0.22 0.50 0.88 1.00 1.32 1.33 1.54 1.76 2.50 3.00 .

The expressions of the Bayes estimates of θ and their risks under the SELF and LLF risk criterion involve a, α, β, θ, n, ϑ and r. For the similar set of selected values of 'a' and α as considered earlier with $\beta = 0.50, 2.00, 5.00, 10.00, 20.00$; $\theta = 02, 04, 06$ and $\vartheta = 02, 04, 06, 10$, the numerical findings have been obtained and we present them in Table 5–6.

Table 5 presents the numerical values of the Bayes estimate $\hat{\theta}_3$ (SELF) only for $\theta = 2.00$ and their risks under the SELF and the LLF for $(\theta, \vartheta, a) = 2.00$ only. It is observed form the table that the magnitude of the risks under both risk criterion increase (decrease) as $\beta(\alpha)$ increases when other considered values of the parametric space are fixed. It also has been seen that the risks under the LLF-criterion increases when 'a' increase with other fixed parametric values. It is noted that there is a smaller magnitude of the risks under the LLF-criterion than under the SELF-criterion.

Table 5: The values of the Bayes estimate $\hat{\theta}_3$.

ϑ	$\beta\downarrow$ $\alpha\rightarrow$	1.25	1.50	2.50	5.00	10.00	20.00
2.00	0.50	3.0896	2.9711	2.5581	1.8471	1.1469	0.6381
	2.00	3.3315	3.2086	2.7774	2.0228	1.2637	0.7056
	5.00	3.7889	3.6581	3.1954	2.3646	1.4946	0.8398
	10.00	4.4901	4.3479	3.8411	2.9064	1.8722	1.0612
	20.00	5.7463	5.5829	4.9988	3.8992	2.6010	1.4982
4.00	0.50	3.2633	3.1129	2.6198	1.8538	1.1469	0.6380
	2.00	3.5547	3.3931	2.8622	2.0336	1.2637	0.7059
	5.00	4.1258	3.9425	3.3383	2.3879	1.4950	0.8400
	10.00	5.0506	4.8327	4.1115	2.9660	1.8744	1.0611
	20.00	6.8305	6.5474	5.6050	4.0890	2.6171	1.4983
6.00	0.50	3.2631	3.1126	2.6195	1.8548	1.1446	0.6430
	2.00	3.5546	3.3929	2.8618	2.0341	1.2629	0.7096
	5.00	4.1260	3.9426	3.3382	2.3873	1.4966	0.8390
	10.00	5.0507	4.8328	4.1118	2.9657	1.8749	1.0581
	20.00	6.8306	6.5474	5.6047	4.0894	2.6163	1.5010
10.00	0.50	3.2469	3.0962	2.6104	1.8782	1.1311	0.6470
	2.00	3.5399	3.3760	2.8443	2.0500	1.2632	0.7035
	5.00	4.1247	3.9370	3.3183	2.3802	1.5201	0.8204
	10.00	5.0715	4.8522	4.1168	2.9371	1.8978	1.0412
	20.00	6.8295	6.5524	5.6308	4.1048	2.5871	1.5249

The values of risks of the Bayes estimator							
$(a,\vartheta)=2.00$	$\beta\downarrow$ $\alpha\rightarrow$	1.25	1.50	2.50	5.00	10.00	20.00
$R_{(S)}(\hat{\theta}_3)$	0.50	64.744	63.185	57.786	48.327	38.039	29.313
$R_{(L)}(\hat{\theta}_3)$		2.9978	2.9517	2.7875	2.4814	2.1149	1.7671
$R_{(S)}(\hat{\theta}_3)$	2.00	67.093	65.451	59.765	49.816	39.013	29.874
$R_{(L)}(\hat{\theta}_3)$		3.0694	3.0215	2.8510	2.5331	2.1525	1.7912
$R_{(S)}(\hat{\theta}_3)$	5.00	71.919	70.104	63.826	52.862	41.000	31.011
$R_{(L)}(\hat{\theta}_3)$		3.2130	3.1615	2.9784	2.6369	2.2278	1.8396
$R_{(S)}(\hat{\theta}_3)$	10.00	80.342	78.222	70.896	58.142	44.421	32.956
$R_{(L)}(\hat{\theta}_3)$		3.4530	3.3957	3.1915	2.8106	2.3540	1.9205
$R_{(S)}(\hat{\theta}_3)$	20.00	98.617	95.818	86.170	69.469	51.678	37.022
$R_{(L)}(\hat{\theta}_3)$		3.9354	3.8663	3.6201	3.1603	2.6083	2.0837

Table 6 presents the numerical values of the Bayes estimate $\hat{\theta}_4$ (LLF) only for $\theta = 2.00$, $\vartheta = 2.00, 10.00$ and $a = 0.50, 2.00$ and their risks under the SELF and the LLF for $(\theta, \vartheta, a) = 2.00$ only. It is observed form the table that the magnitudes of the risks under both risk criterions increase when 'a' increases with other fixed parametric values. Other properties are similar to the Bayes estimator $\hat{\theta}_3$.

Table 6: The values of the Bayes estimate $\hat{\theta}_4$.

ϑ	a	$\beta\downarrow$ $\alpha\rightarrow$	1.25	1.50	2.50	5.00	10.00	20.00
2.00	0.50	0.50	2.1562	2.2002	2.3762	2.8162	3.6963	5.4564
		2.00	2.0055	2.0465	2.2102	2.6195	3.4381	5.0752
		5.00	1.7637	1.7997	1.9436	2.3036	3.0234	4.4632
		10.00	1.4743	1.5044	1.6248	1.9256	2.5274	3.7309
		20.00	1.1174	1.1402	1.2314	1.4595	1.9156	2.8278
	2.00	0.50	0.5390	0.5500	0.5940	0.7040	0.9241	1.3641
		2.00	0.5014	0.5116	0.5525	0.6549	0.8595	1.2688
		5.00	0.4409	0.4499	0.4859	0.5759	0.7559	1.1158
		10.00	0.3686	0.3761	0.4062	0.4814	0.6318	0.9327
		20.00	0.2794	0.2851	0.3079	0.3649	0.4789	0.7069
10.0	0.50	0.50	2.6221	2.6756	2.8896	3.4247	4.4950	6.6354
		2.00	2.4649	2.5152	2.7164	3.2195	4.2255	6.2377
		5.00	2.2106	2.2557	2.4362	2.8874	3.7897	5.5942
		10.00	1.9024	1.9412	2.0965	2.4848	3.2613	4.8143
		20.00	1.5133	1.5442	1.6677	1.9766	2.5942	3.8296
	2.00	0.50	0.6555	0.6689	0.7224	0.8562	1.1237	1.6589
		2.00	0.6162	0.6288	0.6791	0.8049	1.0564	1.5594
		5.00	0.5527	0.5639	0.6091	0.7218	0.9474	1.3986
		10.00	0.4756	0.4853	0.5241	0.6212	0.8153	1.2036
		20.00	0.3783	0.3860	0.4169	0.4941	0.6486	0.9574

The values of risks of the Bayes estimator							
$(a,\vartheta)=2.00$	$\beta\downarrow$ $\alpha\rightarrow$	1.25	1.50	2.50	5.00	10.00	20.00
$R_{(S)}(\hat{\theta}_4)$	0.50	13.445	13.395	13.198	12.713	11.774	10.022
$R_{(L)}(\hat{\theta}_4)$		0.9928	0.9899	0.9785	0.9501	0.8941	0.7853
$R_{(S)}(\hat{\theta}_4)$	2.00	13.550	13.502	13.313	12.846	11.939	10.241
$R_{(L)}(\hat{\theta}_4)$		0.9989	0.9961	0.9852	0.9580	0.9042	0.7995
$R_{(S)}(\hat{\theta}_4)$	5.00	13.734	13.690	13.514	13.079	12.232	10.634
$R_{(L)}(\hat{\theta}_4)$		1.0095	1.0070	0.9969	0.9717	0.9219	0.8246
$R_{(S)}(\hat{\theta}_4)$	10.00	13.983	13.944	13.786	13.395	12.632	11.179
$R_{(L)}(\hat{\theta}_4)$		1.0237	1.0215	1.0125	0.9901	0.9457	0.8586
$R_{(S)}(\hat{\theta}_4)$	20.00	14.341	14.308	14.177	13.853	13.215	11.987
$R_{(L)}(\hat{\theta}_4)$		1.0440	1.0421	1.0348	1.0164	0.9798	0.9078

5. THE MINIMAX ESTIMATORS AND THEIR PROPERTIES

The basic principle of this approach is to minimize the loss. The derivation depends primarily on a theorem, which is due to Hodge & Lehmann (1950) and can be stated as follows.

Let $\tau = \{F_\theta \colon \theta \in \Theta\}$ be a family of distribution functions and D be a class of estimators of the parameter θ. Suppose that $d^* \in D$ is a Bayes estimator against a prior distribution $\pi(\theta)$ on the parameter space Θ. Then the Bayes estimator d^* is said to be the Minimax estimator if the risk function of d^* is independent on Θ.

When the shape parameter v is considered to be known, the Bayes estimator for the parameter θ corresponding to the SELF and LLF are given respectively in equations (2.2) and (2.4). Further, the expressions of the risk for these Bayes estimators corresponding to the considered loss criterion are given in equations (2.6) and (2.7) respectively.

Both expressions of the risk involve the parameter θ. Hence, the Bayes estimators $\hat{\theta}_1$ and $\hat{\theta}_2$ are not Minimax estimators. Thus, under the natural family of the conjugate prior the Minimax estimators do not exist.

Now, the Bayes estimators corresponding to the posterior $Z_2(\theta)$, given in (2.10), are obtained respectively under both loss criteria as

$$\hat{\theta}_5 = \varphi_5\, r\, T_r\ , \qquad \varphi_5 = (d+r-2)^{-1} \tag{5.1}$$

and

$$\hat{\theta}_6 = \varphi_6\, r\, T_r\ , \qquad \varphi_6 = \frac{1}{a}\left(1-\exp\left(-\frac{a}{d+r}\right)\right) . \tag{5.2}$$

The risks of these Bayes estimators corresponding to the SELF and the LLF are given respectively as

$$R_{(S)}(\hat{\theta}_i) \;=\; \theta^2\left(r\,(r+1)\,\varphi_i^2 + 1 - 2\,r\,\varphi_i\right) \tag{5.3}$$

and

$$R_{(L)}(\hat{\theta}_i) \;=\; e^{-a}\,(1-a\,\varphi_i)^{-r} - 1 - a\,(r\,\varphi_i - 1)\ , \qquad i = 5, 6\ . \tag{5.4}$$

It is observed that the Bayes estimators $\hat{\theta}_5$ and $\hat{\theta}_6$ are not the minimax estimators corresponding to the loss criterion SELF. However, the risk of Bayes estimators $\hat{\theta}_5$ and $\hat{\theta}_6$ are independent of the parameter θ under the LLF criterion. Hence, both estimators $\hat{\theta}_5$ and $\hat{\theta}_6$ are Minimax estimators under the LLF loss criterion.

The following statistical problem (Minimax Estimation) is equivalent to some two person zero sum game between the Statistician (Player-II) and Nature (Player-I). Here the pure strategies of Nature are the different values of θ in the interval $(0,\infty)$ and the mixed strategies of Nature are the prior densities of θ in the interval $(0,\infty)$. The pure strategies of Statistician are all possible decision functions in the interval $(0,\infty)$.

The expected value of the loss function is the risk function and it is the gain of the Player-I. Further, the Bayes risk is defined as

$$R^*(\eta, \hat{\theta}_B) \;=\; E_\theta\, R(\hat{\theta}_B)\ .$$

Here, the expectation has been taken under the prior density of parameter θ. If the loss function is continuous in both the estimator $\hat{\theta}_B$ and the parameter θ, and convex in $\hat{\theta}_B$ for each value of θ then there exist measures η^* and $\hat{\theta}^*_B$ for all θ and $\hat{\theta}_B$ so that, the following relation holds:

$$R^*(\eta, \hat{\theta}^*_B) \leq R^*(\eta^*, \hat{\theta}^*_B) \leq R^*(\eta^*, \hat{\theta}_B) \ .$$

The number $R^*(\eta^*, \hat{\theta}^*_B)$ is known as the value of the game, and η^* and $\hat{\theta}^*_B$ are the corresponding optimum strategies of the Player I and II. In statistical terms η^* is the least favorable prior density of θ and the estimator $\hat{\theta}^*_B$ is the minimax estimator. In fact, the value of the game is the loss of the Player-II. Hence, the optimum strategy of Player-II and the value of game are given as

Optimum Strategy	Corresponding Loss	Value of Game
$\hat{\theta}_5 = \varphi_5\, r\, T_r$	LLF	$e^{-a}(1-a\,\varphi_5)^{-r} - 1 - a\,(r\,\varphi_5 - 1)$
$\hat{\theta}_6 = \varphi_6\, r\, T_r$	LLF	$e^{-a}(1-a\,\varphi_6)^{-r} - 1 - a\,(r\,\varphi_6 - 1)$

ACKNOWLEDGMENTS

The authors acknowledge helpful suggestions by the referee and the editor of the paper that have considerable improved the earlier version.

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