REVSTAT – Statistical Journal
Volume 0, Number 0, Month 0000, 000–000
https://doi.org/00.00000/revstat.v00i0.000

Modified Method of Moments Estimators for the Lomax Distribution

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Received: Month 0000 Revised: Month 0000 Accepted: Month 0000

Abstract:

- Modified method of moments estimators are proposed for the Lomax distribution. These estimators are obtained by replacing the expectation of an exponential function of a random variable by its empirical estimator. The proposed estimators are compared by simulation to the method of moments estimator, L moments estimator, trimmed L moments estimator, probability weighted moments estimator and maximum likelihood estimator. The proposed estimators are shown to be competitors to these estimators for small sample sizes.

Keywords:

- *Covariance; Exponential integral; Variance*

AMS Subject Classification:

- Primary 62E15.

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1. INTRODUCTION

The Lomax distribution was introduced by Lomax (1954) to model business failure data. This distribution has, however, been applied to data from a wide range of areas. Some of the areas include income and wealth data Atkinson and Harrison (1978), receiver operating characteristic curve analysis Campbell and Ratnaparkhi (1993), reliability and survival analysis Hassan and Al-Ghamdi (2009); Hassan et al. (2016); Abujarad and Khan (2021); Ogunde et al. (2023), bladder cancer data Rady et al. (2016), wind speed data Ahsan ul Haq et al. (2020), COVID-19 death cases of Nepal Dhungana and Kumar (2022), insurance claims data Hamed et al. (2022), and wireless channel modeling Sanchez and Lopez-Martinez (2023); to mention but a few.

The Lomax distribution is a special Pareto distribution whose support is shifted to start at zero. Its probability density function is

$$f(x) = \frac{\alpha}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-(\alpha+1)} \tag{1.1}$$

for $x > 0$, $\alpha > 0$ and $\lambda > 0$, where α is a shape parameter and λ is a scale parameter. We shall denote a random variable X having the probability density function (1.1) by $L(\alpha, \lambda)$. If $X \sim L(\alpha, \lambda)$ then its cumulative distribution function is

$$F(x) = 1-\left(1+\frac{x}{\lambda}\right)^{-\alpha}$$

for $x > 0$, $\alpha > 0$, and $\lambda > 0$. The mean and the variance of $X \sim L(\alpha, \lambda)$ are

$$\mathrm{E}\,(X) = \frac{\lambda}{\alpha-1}, \qquad \alpha > 1 \tag{1.2}$$

and

$$\mathrm{Var}\,(X) = \frac{\alpha\lambda^2}{(\alpha-1)^2(\alpha-2)}, \qquad \alpha > 2, \tag{1.3}$$

respectively.

A number of different estimators for α and λ have been proposed in the literature. Abd-Elfattah and Alharbey (2010) derived generalised probability weighted moments estimators. Bayesian estimators have been considered by Nasiri and Hosseini (2012), Ferreira et al. (2020) and He et al. (2023). Giles et al. (2013) studied the bias of maximum likelihood estimators. Shakeel et al. (2017) compared the performance of L moments estimators, trimmed L moments estimators and probability weighted moments estimators. Nombebe et al. (2023) compared the performance of method of moments estimators, maximum likelihood estimators, L moments estimators and some minimum distance estimators.

The maximum likelihood estimators and their bias-corrected versions do not have closed forms. The aim of this paper is to propose closed form estimators based on a modified method of moments. The proposed estimators have the potential to be applied easily to any application area of the Lomax distribution. The advantages of the proposed estimators over others include: i) they do not require numerical solving which can be prone to numerical errors; ii) they do not require numerical solving which can be costly too; iii) they can provide deep insights into the underlying behavior and relationships between parameters, enhancing theoretical understanding; iv) they allow for straightforward sensitivity analysis; v) they can be valuable in educational contexts for demonstrating principles and methods without the distraction of numerical implementation details; vi) they can be used to make precise predictions and to develop models that are both interpretable and actionable; vii) practitioners in some countries may not have access to software to compute estimators taking non-closed forms; viii) many departments in Africa do not have more than a few computers and they lack computer software; ix) there may be no real valued solutions when estimators require solving of non-linear equations; x) even if real valued solutions exist they may not be unique.

Furthermore, the modified method of moments includes a bias correction mechanism, producing estimators that are unbiased even in smaller samples, thus addressing the bias problem effectively. The flexibility of modified method of moments estimators across different sample sizes is another motivating factor. While maximum likelihood and traditional moment estimators may underperform or become inefficient in small samples, the modified method of estimators demonstrate reliability and robustness, providing consistent performance across various sample sizes. Therefore, the motivation for the modified method of estimators lies in its ability to balance simplicity, computational efficiency, and reduced bias, making it an appealing alternative to more complex estimation methods.

Section 3 proposes closed form estimators for α and λ. Large sample properties of these estimators including consistency and asymptotic normality are derived in Section 4. A simulation study is conducted in Section 5 to compare the performance of the proposed estimators with other estimators taking closed forms and based on moments, see Section 2. However, maximum likelihood estimators, which do not have closed form expressions, are also compared to the proposed estimators. Conclusions and future work are discussed in Section 6.

The calculations of this paper involve the generalised exponential integral defined by

$$E_p(z) = z^{p-1} \int_z^{\infty} \frac{e^{-t}}{t^p} dt$$

for $z > 0$.

2. SOME COMMON ESTIMATORS

Throughout, we suppose $X_1, X_2, \ldots, X_n$ are independent and identical copies of $X \sim L(\alpha, \lambda)$. Let $X_{1:n} < X_{2:n} < \cdots < X_{n:n}$ denote the corresponding order statistics.

2.1. MAXIMUM LIKELIHOOD ESTIMATORS

Maximum likelihood estimators for α and λ are the simultaneous solutions of

$$\begin{aligned} & \frac{n}{\alpha} - \sum_{i=1}^{n} \log\left(1+\frac{X_i}{\lambda}\right) = 0, \\ & \frac{\alpha+1}{\lambda} \sum_{i=1}^{n} \frac{X_i}{\lambda+X_i} - \frac{n}{\lambda} = 0. \end{aligned} \tag{2.1}$$

The equations in (2.1) do not have closed form solutions and hence require iterative methods to solve. Nombebe et al. (2023) explained one approach that can be used to carry out the optimisation procedure. It can be shown that

$$\sqrt{n}\begin{pmatrix} \widehat{\alpha}_{MLE}-\alpha \\ \widehat{\lambda}_{MLE}-\lambda \end{pmatrix} \underset{d}{\longrightarrow} N\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \alpha^2(\alpha+1)^2 & \alpha\lambda(\alpha+1)(\alpha+2) \\ \alpha\lambda(\alpha+1)(\alpha+2) & \frac{\lambda^2(\alpha+1)^2(\alpha+2)}{\alpha} \end{pmatrix}\right]$$

as $n \to \infty$, where $\widehat{\alpha}_{MLE}$ and $\widehat{\lambda}_{MLE}$ denote maximum likelihood estimators of α and λ, respectively.

2.2. L MOMENTS ESTIMATORS

The L moments method of parameter estimation was introduced by Hosking (1990) and is based on comparing moments of linear combinations of order statistics to sample versions. Given a random variable X, the rth population L moment is

$$\lambda_r = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k \binom{r-1}{k} \mathrm{E}\left(X_{r-k:r}\right) \tag{2.2}$$

for $r \geq 1$. The corresponding rth sample moment is

$$l_r = \binom{n}{r}^{-1} \sum\sum_{1\leq i_1<\cdots<i_r\leq n}\cdots\sum r^{-1}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k} X_{i_{r-k}:n}$$

for $r = 1, 2, \ldots, n$. The first two sample moments are

$$l_1 = \frac{1}{n}\sum_{i=1}^{n} X_{i:n} = \overline{X} \tag{2.3}$$

and

$$l_2 = \frac{1}{2}\binom{n}{2}^{-1}\sum_{i>j}\sum \left(X_{i:n}-X_{j:n}\right) = \frac{2}{n(n-1)}\sum_{i=1}^{n}(i-1)X_{i:n} - \overline{X}. \tag{2.4}$$

For $X \sim L(\alpha, \lambda)$, the first two population L moments are

$$\lambda_1 = \frac{\lambda}{\alpha-1} \tag{2.5}$$

and

$$\lambda_2 = \frac{\lambda\alpha}{2(\alpha-1)^2}. \tag{2.6}$$

Equating (2.3) and (2.4) to (2.5) and (2.6), respectively, Shakeel et al. (2017) obtained the L moments estimators for the Lomax distribution as

$$\widehat{\alpha}_{LM} = \frac{l_2}{2l_2-1}, \quad \widehat{\lambda}_{LM} = \frac{l_1\left(l_1-l_2\right)}{2l_2-1}.$$

2.3. TRIMMED L MOMENTS ESTIMATORS

The method of trimmed L moments is due to Elamir and Seheult (2003). Given a random variable X, the rth population trimmed L moment is

$$\lambda_r^{(t_1,t_2)} = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}\mathrm{E}\ \left(X_{r-k+t_1:r+t_1+t_2}\right) \tag{2.7}$$

for $r \geq 1$, where t_1 and t_2 are non-negative integers. Unlike L moments, trimmed L moments can be applied to distributions not having finite moments all orders (for example, the Lomax distribution). If $t_1 = t_2 = 0$ then (2.7) reduces to (2.2). If $t_1 = t_2 = t$ then (2.7) reduces to

$$\lambda_r^{(t)} = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}\mathrm{E}\ \left(X_{r-k+t:r+2t}\right). \tag{2.8}$$

An unbiased estimator of (2.8) is

$$l_r^{(t)} = \frac{1}{r}\sum_{k=r+t}^{n-t}\left[\frac{\sum_{j=0}^{r-1}(-1)^j\binom{r-1}{j}\binom{k-1}{r+t-j-1}\binom{n-k}{t+j}}{\binom{n}{r+2t}}\right]X_{k:n}. \tag{2.9}$$

The particular cases of (2.9) for $r = 1, 2$ and $t = 1$ are

$$l_1^{(1)} = 6\sum_{k=2}^{n-1}\frac{(k-1)(n-k)}{n(n-1)(n-2)}X_{k:n} \tag{2.10}$$

and

$$l_2^{(1)} = 6\sum_{k=3}^{n-1}\frac{(k-1)(n-k)(2k-n-1)}{n(n-1)(n-2)(n-3)}X_{k:n}. \tag{2.11}$$

For $X \sim L(\alpha,\lambda)$, the first two population trimmed L moments for $t_1 = t_2 = 1$ are

$$\lambda_1^{(1)} = \frac{\lambda(5\alpha-1)}{(2\alpha-1)(3\alpha-1)} \tag{2.12}$$

and

$$\lambda_2^{(1)} = \frac{6\lambda\alpha^2}{(2\alpha-1)(3\alpha-1)(4\alpha-1)}. \tag{2.13}$$

Equating (2.10) and (2.11) to (2.12) and (2.13), respectively, Shakeel et al. (2017) obtained the trimmed L moments estimators for the Lomax distribution as

$$\widehat{\alpha}_{TLM} = \frac{9l_2^{(1)} + \sqrt{\left(l_2^{(1)}\right)^2 + 24l_1^{(1)}l_2^{(1)}}}{2\left(20l_2^{(1)} - 6l_1^{(1)}\right)}, \quad \widehat{\lambda}_{TLM} = \frac{l_1^{(1)}\left(2\widehat{\alpha}_{TLM}-1\right)\left(3\widehat{\alpha}_{TLM}-1\right)}{5\widehat{\alpha}_{TLM}-1}.$$

2.4. METHOD OF MOMENTS ESTIMATORS

One of the most popular methods of estimation is the method of moments where the estimation is made by comparing population moments to sample moments. The rth moment of $X \sim L(\alpha, \lambda)$ is

$$\mathrm{E}\,\left(X^r\right) = \frac{\lambda^r \Gamma(\alpha - r)\Gamma(1 + r)}{\Gamma(\alpha)} \tag{2.14}$$

for $\alpha > r$ and $r = 1, 2, \ldots$. From (2.14), we obtain the first two population moments as

$$\begin{aligned} \mathrm{E}\,(X) &= \frac{\lambda}{\alpha - 1}, \\ \mathrm{E}\,\left(X^2\right) &= \frac{2\lambda^2}{(\alpha - 1)(\alpha - 2)}. \end{aligned} \tag{2.15}$$

Comparing the population moments in (2.15) to the corresponding sample moments leads to the method of moments estimators Nombebe et al. (2023)

$$\widehat{\alpha}_{MM} = \frac{2\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \overline{X}^2\right)}{\frac{1}{n}\sum_{i=1}^{n} X_i^2 - 2\overline{X}^2},$$

$$\widehat{\lambda}_{MM} = \frac{\overline{X}\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2\right)}{\frac{1}{n}\sum_{i=1}^{n} X_i^2 - 2\overline{X}^2}.$$

2.5. PROBABILITY WEIGHTED MOMENTS ESTIMATORS

Greenwood et al. (1979) introduced a method based on Probability Weighted Moments (PWMs). For a random variable X with cumulative distribution function F_X, the PWM is defined as Greenwood et al. (1979)

$$M_{p,u,v} = \mathrm{E}\,\left\{X^p \left[F_X(X)\right]^u \left[1 - F_X(X)\right]^v\right\},$$

where p, u and v are non-negative integers. An alternative form of the PWM is

$$M_{p,u,v} = \int_0^1 \left[Q\left(F_X\right)\right]^p \left[F_X(X)\right]^u \left[1 - F_X(X)\right]^v dF_X(x)$$

if the inverse cumulative distribution function $Q(\cdot)$ can be written in a closed form. Using this alternative form, Shakeel et al. (2017) showed that for $X \sim L(\alpha, \lambda)$,

$$M_{1,0,0} = \frac{\alpha\lambda}{\lambda - 1} - \lambda \tag{2.16}$$

and

$$M_{1,0,1} = \frac{\alpha\lambda}{\lambda - 1} - \lambda - \frac{\alpha^2\lambda}{(\lambda - 1)(2\lambda - 1)} + \frac{\lambda}{2}. \tag{2.17}$$

Equating (2.16) and (2.17) to their sample counterparts, Shakeel et al. (2017) obtained the probability weighted moments estimators for the Lomax distribution as

$$\widehat{\alpha}_{PWM} = \frac{2\widehat{M}_{1,0,1} - \widehat{M}_{1,0,0}}{4\widehat{M}_{1,0,1} - \widehat{M}_{1,0,0}}, \quad \widehat{\lambda}_{PWM} = \frac{2\widehat{M}_{1,0,0}\widehat{M}_{1,0,1}}{\widehat{M}_{1,0,0} - 4\widehat{M}_{1,0,1}}.$$

3. MODIFIED METHOD OF MOMENTS ESTIMATORS

In this section, we propose modified method of moments estimators for the Lomax distribution. This is based on moments of the exponential function of $X \sim L(\alpha, \lambda)$. Similar approaches have been used by other researchers, see Ng et al. (2003) giving estimators for a two-parameter Birnbaum-Saunders distribution and Ghitany et al. (2017) giving estimators for a two-parameter weighted Lindley distribution.

Let $X \sim L(\alpha, \lambda)$ then we can easily show that

$$\mathrm{E}\left(e^{-X}\right) = \alpha e^{\lambda} E_{\alpha+1}(\lambda) \tag{3.1}$$

and

$$\mathrm{E}\left(Xe^{-X}\right) = \alpha\lambda e^{\lambda}\left[E_{\alpha}(\lambda) - E_{\alpha+1}(\lambda)\right]. \tag{3.2}$$

Using integration by parts, we can also easily show that

$$E_{\alpha+1}(\lambda) = \frac{1}{\alpha}\left[e^{-\lambda} - \lambda E_{\alpha}(\lambda)\right]. \tag{3.3}$$

(3.3) can be used to rewrite (3.1) and (3.2) as

$$\mathrm{E}\left(e^{-X}\right) = 1 - \lambda e^{\lambda} E_{\alpha}(\lambda) \tag{3.4}$$

and

$$\mathrm{E}\left(Xe^{-X}\right) = \alpha\lambda\left(1 + \frac{\lambda}{\alpha}\right) e^{\lambda} E_{\alpha}(\lambda) - \lambda = (\alpha+\lambda)\lambda e^{\lambda} E_{\alpha}(\lambda) - \lambda, \tag{3.5}$$

respectively. Substituting (3.4) into (3.5) and simplifying gives

$$\mathrm{E}\left(Xe^{-X}\right) = \alpha\left[1 - \mathrm{E}\left(e^{-X}\right)\right] - \lambda \mathrm{E}\left(e^{-X}\right). \tag{3.6}$$

Solving (1.2) and (3.6) simultaneously gives

$$\alpha = \frac{AB - C}{AB + B - 1}, \quad \lambda = \frac{A(1 - B - C)}{AB + B - 1},$$

where $A = \mathrm{E}(X)$, $B = \mathrm{E}\left(e^{-X}\right)$ and $C = \mathrm{E}\left(Xe^{-X}\right)$. Using the weak law of large numbers, we can replace the expectations $A = \mathrm{E}(X)$, $B = \mathrm{E}\left(e^{-X}\right)$ and $C = \mathrm{E}\left(Xe^{-X}\right)$ by sample versions to obtain the proposed estimators for α and λ given by

$$\begin{aligned} \widehat{\alpha} &= \frac{\overline{X}\left(\frac{1}{n}\sum_{i=1}^{n} e^{-X_i}\right) - \frac{1}{n}\sum_{i=1}^{n} X_i e^{-X_i}}{\overline{X}\left(\frac{1}{n}\sum_{i=1}^{n} e^{-X_i}\right) + \frac{1}{n}\sum_{i=1}^{n} e^{-X_i} - 1}, \\ \widehat{\lambda} &= \frac{\overline{X}\left(1 - \frac{1}{n}\sum_{i=1}^{n} e^{-X_i} - \frac{1}{n}\sum_{i=1}^{n} X_i e^{-X_i}\right)}{\overline{X}\left(\frac{1}{n}\sum_{i=1}^{n} e^{-X_i}\right) + \frac{1}{n}\sum_{i=1}^{n} e^{-X_i} - 1}. \end{aligned} \tag{3.7}$$

4. LARGE SAMPLE PROPERTIES

In this section, we show that the proposed estimators are strongly consistent and asymptotically normal. We also derive their asymptotic variances.

Theorem 4.1 The estimators $\widehat{\alpha}$ and $\widehat{\lambda}$ given in (3.7) are strongly consistent for α and λ, respectively.

Proof Consider a random sample of size n from $X \sim L(\alpha, \lambda)$. Let $\overline{P}$, $\overline{Q}$ and $\overline{R}$ be the means of X, e^{-X} and Xe^{-X}, respectively. From (1.2), (3.4) and (3.5), we have

$$\begin{aligned} &\mathrm{E}\,(X) = \frac{\lambda}{\alpha - 1}, \\ &\mathrm{E}\,\left(e^{-X}\right) = 1 - \lambda e^{\lambda} E_{\alpha}(\lambda), \\ &\mathrm{E}\,\left(Xe^{-X}\right) = (\alpha + \lambda)\lambda e^{\lambda} E_{\alpha}(\lambda) - \lambda. \end{aligned}$$

By the strong law of large numbers

$$\begin{pmatrix} \overline{P} \\ \overline{Q} \\ \overline{R} \end{pmatrix} \to \begin{pmatrix} \dfrac{\lambda}{\alpha - 1} \\ 1 - \lambda e^{\lambda} E_{\alpha}(\lambda) \\ (\alpha + \lambda)\lambda e^{\lambda} E_{\alpha}(\lambda) - \lambda \end{pmatrix}$$

almost surely as $n \to \infty$. Define $h(p, q, r) = \frac{pq - r}{pq + q - 1}$ and $g(p, q, r) = \frac{p(1 - q - r)}{pq + q - 1}$. Note that $h(p, q, r)$ and $g(p, q, r)$ are continuous at

$$\left(\frac{\lambda}{\alpha - 1}, 1 - \lambda e^{\lambda} E_{\alpha}(\lambda), (\alpha + \lambda)\lambda e^{\lambda} E_{\alpha}(\lambda) - \lambda \right).$$

Applying the continuous mapping theorem implies that $\widehat{\alpha}$ and $\widehat{\lambda}$ converge almost surely to α and λ, respectively, as $n \to \infty$. The proof is complete. □

Theorem 4.2 requires evaluation of some expectations and variances involving $\overline{P}$, $\overline{Q}$ and $\overline{R}$. Using (1.3), we have

$$\mathrm{Var}\,\left(\sqrt{n}\,\overline{P}\right) = \mathrm{Var}\,(X) = \frac{\alpha\lambda^2}{(\alpha - 1)^2(\alpha - 2)}. \tag{4.1}$$

We can show that

$$\begin{aligned} \mathrm{E}\,\left(e^{-X}\right)^2 &= \alpha(2\lambda)^{\alpha} e^{2\lambda} \int_{2\lambda}^{\infty} \frac{e^{-t}}{t^{\alpha+1}} dt \\ &= \alpha e^{2\lambda} E_{\alpha+1}(2\lambda) \\ &= \alpha e^{2\lambda} \left[\frac{e^{-2\lambda}}{\alpha} - \frac{2\lambda}{\alpha} E_{\alpha}(2\lambda) \right] \\ &= 1 - 2\lambda e^{2\lambda} E_{\alpha}(2\lambda) \end{aligned}$$

and therefore

$$\mathrm{Var}\,\left(\sqrt{n}\,\overline{Q}\right) = \mathrm{Var}\,\left(e^{-X}\right) = 1 - 2\lambda e^{2\lambda} E_{\alpha}(2\lambda) - \left[1 - \lambda e^{\lambda} E_{\alpha}(\lambda)\right]^2. \tag{4.2}$$

We can also show that

$$\begin{aligned}
\mathrm{E}\ \left(Xe^{-X}\right)^2 &= \alpha\lambda^2 e^{2\lambda}\left[E_{\alpha-1}(2\lambda) - 2E_\alpha(2\lambda) + E_{\alpha+1}(2\lambda)\right] \\
&= \alpha\lambda^2 e^{2\lambda}\left[\frac{e^{-2\lambda}}{2\lambda} + \frac{(1-\alpha)}{2\lambda}E_\alpha(2\lambda) - 2E_\alpha(2\lambda) + \frac{e^{-2\lambda}}{\alpha} - \frac{2\lambda}{\alpha}E_\alpha(2\lambda)\right] \\
&= \lambda\left[\frac{\alpha}{2} + \lambda + \left(\frac{\alpha(1-\alpha)}{2} - 2\alpha\lambda - 2\lambda^2\right)e^{2\lambda}E_\alpha(2\lambda)\right]
\end{aligned}$$

and hence

$$\mathrm{Var}\ \left(\sqrt{n}\ \overline{R}\right) = \lambda\left[\frac{\alpha}{2} + \lambda + \left(\frac{\alpha(1-\alpha)}{2} - 2\alpha\lambda - 2\lambda^2\right)e^{2\lambda}E_\alpha(2\lambda)\right] - \left[(\alpha+\lambda)\lambda e^{\lambda}E_\alpha(\lambda) - \lambda\right]^2. \tag{4.3}$$

Using (1.2), (3.4) and (3.5), we can show that

$$\mathrm{Cov}\ \left(\sqrt{n}\ \overline{P}, \sqrt{n}\ \overline{Q}\right) = \mathrm{Cov}\ \left(X, e^{-X}\right) = \frac{\alpha\lambda}{\alpha-1}\left[(\alpha+\lambda-1)e^{\lambda}E_\alpha(\lambda) - 1\right]. \tag{4.4}$$

We can show that

$$\begin{aligned}
\mathrm{E}\ \left(X^2 e^{-X}\right) &= \alpha\lambda^2 e^{\lambda}\left[E_{\alpha-1}(\lambda) - 2E_\alpha(\lambda) + E_{\alpha+1}(\lambda)\right] \\
&= \alpha\lambda^2 e^{\lambda}\left[\frac{e^{-\lambda}}{\lambda} + \frac{(1-\alpha)}{\lambda}E_\alpha(\lambda) - 2E_\alpha(\lambda) + \frac{e^{-\lambda}}{\alpha} - \frac{\lambda}{\alpha}E_\alpha(\lambda)\right] \\
&= \lambda\left[\alpha + \lambda + \left\{\alpha(1-\alpha) - 2\alpha\lambda - \lambda^2\right\}e^{\lambda}E_\alpha(\lambda)\right]
\end{aligned}$$

and then use (1.2) along with (3.5) to show that

$$\mathrm{Cov}\ \left(\sqrt{n}\ \overline{P}, \sqrt{n}\ \overline{R}\right) = \mathrm{Cov}\ \left(X, Xe^{-X}\right) = \frac{\alpha\lambda}{\alpha-1}\left[\alpha + \lambda - 1 + \left\{(1-\alpha)(\alpha-1) - 2\alpha\lambda + \lambda - \lambda^2\right\}e^{\lambda}E_\alpha(\lambda)\right]. \tag{4.5}$$

Similarly, we can show that

$$\begin{aligned}
\mathrm{E}\ \left(Xe^{-2X}\right) &= \alpha\lambda e^{2\lambda}\left[E_\alpha(2\lambda) - E_{\alpha+1}(2\lambda)\right] \\
&= \alpha\lambda e^{2\lambda}\left[E_\alpha(2\lambda) - \frac{e^{-2\lambda}}{\alpha} + \frac{2\lambda}{\alpha}E_\alpha(2\lambda)\right] \\
&= \lambda\left[(\alpha+2\lambda)\,e^{2\lambda}E_\alpha(2\lambda) - 1\right]
\end{aligned}$$

and then use (3.4) together with (3.5) to show that

$$\begin{aligned}
\mathrm{Cov}\ \left(\sqrt{n}\ \overline{Q}, \sqrt{n}\ \overline{R}\right) &= \mathrm{Cov}\ \left(e^{-X}, Xe^{-X}\right) \\
&= \lambda\left[(\alpha+2\lambda)e^{2\lambda}E_\alpha(2\lambda) - 1 - \left\{1 - \lambda e^{\lambda}E_\alpha(\lambda)\right\}\left\{(\alpha+\lambda)e^{\lambda}E_\alpha(\lambda) - 1\right\}\right]. \qquad (4.6)
\end{aligned}$$

Theorem 4.2 The estimators $\widehat{\alpha}$ and $\widehat{\lambda}$ given in (3.7) are asymptotically normally distributed as

$$\sqrt{n}\left(\widehat{\alpha} - \alpha, \widehat{\lambda} - \lambda\right) \xrightarrow{d} N\left(\mathbf{0}_2, \mathbf{M\Sigma M}^T\right)$$

as $n \to \infty$, where

$$\mathbf{M} = \frac{1}{D}\begin{pmatrix} \frac{(\alpha-1)^2}{\lambda}\left[1 - \lambda e^{\lambda}E_\alpha(\lambda)\right] & \frac{(\alpha-1)(\alpha+\lambda)}{\lambda} & \frac{\alpha-1}{\lambda} \\ (\alpha-1)^2 e^{\lambda}E_\alpha(\lambda) & \alpha+\lambda & 1 \end{pmatrix},$$

$$\mathbf{\Sigma} = \begin{pmatrix} \frac{\alpha\lambda^2}{(\alpha-1)^2(\alpha-2)} & \frac{\alpha\lambda}{\alpha-1}\left[(\alpha+\lambda-1)e^{\lambda}E_\alpha(\lambda)-1\right] & \text{Cov }(P,R) \\ \frac{\alpha\lambda}{\alpha-1}\left[(\alpha+\lambda-1)e^{\lambda}E_\alpha(\lambda)-1\right] & 1-2\lambda e^{2\lambda}E_\alpha(2\lambda)-\left[1-\lambda e^{\lambda}E_\alpha(\lambda)\right]^2 & \text{Cov }(Q,R) \\ \text{Cov }(P,R) & \text{Cov }(Q,R) & \text{Var }(R) \end{pmatrix},$$

$$\text{Cov }(P,R) = \frac{\alpha\lambda}{\alpha-1}\left[\alpha+\lambda-1+\left\{(1-\alpha)(\alpha-1)-2\alpha\lambda+\lambda-\lambda^2\right\}e^{\lambda}E_\alpha(\lambda)\right],$$

$$\text{Cov }(Q,R) = \lambda\left[(\alpha+2\lambda)e^{2\lambda}E_\alpha(2\lambda)-1-\left\{1-\lambda e^{\lambda}E_\alpha(\lambda)\right\}\left\{(\alpha+\lambda)e^{\lambda}E_\alpha(\lambda)-1\right\}\right],$$

$$\text{Var }(R) = \lambda\left[\frac{\alpha}{2}+\lambda+\left(\frac{\alpha(1-\alpha)}{2}-2\alpha\lambda-2\lambda^2\right)e^{2\lambda}E_\alpha(2\lambda)\right]-\left[(\alpha+\lambda)\lambda e^{\lambda}E_\alpha(\lambda)-\lambda\right]^2$$

and $D = (\alpha+\lambda-1)e^{\lambda}E_\alpha(\lambda)-1$ provided that $\alpha > 2$.

Proof By the Central Limit Theorem

$$\sqrt{n}\left[\left(\overline{P},\overline{Q},\overline{R}\right)-(p_0,q_0,r_0)\right] \xrightarrow{d} N\left(\mathbf{0}_3,\mathbf{\Sigma}\right)$$

as $n \to \infty$, where

$$(p_0,q_0,r_0) = \left(\frac{\lambda}{\alpha-1}, 1-\lambda e^{\lambda}E_\alpha(\lambda), (\alpha+\lambda)\lambda e^{\lambda}E_\alpha(\lambda)-\lambda\right),$$

$$\mathbf{\Sigma} = \begin{pmatrix} \text{Var }(P) & \text{Cov }(P,Q) & \text{Cov }(P,R) \\ \text{Cov }(P,Q) & \text{Var }(Q) & \text{Cov }(Q,R) \\ \text{Cov }(P,R) & \text{Cov }(Q,R) & \text{Var }(R) \end{pmatrix}$$

and the entries of $\mathbf{\Sigma}$ are given by (4.1)-(4.6). Using the delta method Kelley (1928)

$$\sqrt{n}\left(\widehat{\alpha}-\alpha,\widehat{\lambda}-\lambda\right) \xrightarrow{d} N\left(\mathbf{0}_2,\mathbf{M\Sigma M}^T\right)$$

as $n \to \infty$, where

$$\mathbf{M} = \begin{pmatrix} \frac{\partial h}{\partial p} & \frac{\partial h}{\partial q} & \frac{\partial h}{\partial r} \\ \frac{\partial g}{\partial p} & \frac{\partial g}{\partial q} & \frac{\partial g}{\partial r} \end{pmatrix},$$

where the partial derivatives in $\mathbf{M}$ are evaluated at

$$(p_0,q_0,r_0) = \left(\frac{\lambda}{\alpha-1}, 1-\lambda e^{\lambda}E_\alpha(\lambda), (\alpha+\lambda)\lambda e^{\lambda}E_\alpha(\lambda)-\lambda\right).$$

Evaluating the partial derivatives gives the stated $\mathbf{M}$. The proof is complete. □

5. COMPARISON OF ESTIMATORS

In this section, we compare variances and absolute biases of the method of moments estimators, L moments estimators, trimmed L moments estimators, probability weighted moments estimators, maximum likelihood estimators and our estimators for finite n less than or equal to 50. The comparison was performed through a simulation scheme as follows.

i) select values for α and λ;

ii) set $n = 5$;

iii) simulate 10,000 random samples each of size n from the Lomax distribution;

iv) compute the estimators of α and λ using the six methods for each of the 10,000 samples;

v) use the values in step iv) to compute the variances and absolute biases;

vi) repeat steps iii) to v) for $n = 6, 7, \ldots, 50$.

This scheme was executed for a wide range of values of α and λ. Figure 1 shows how the variances compare versus n for $\alpha = 0.1$ and $\lambda = 0.001$. Figure 2 shows how the variances compare versus n for $\alpha = 0.1$ and $\lambda = 0.0001$. Figure 3 shows how the variances compare versus n for $\alpha = 0.1$ and $\lambda = 0.005$. Figure 4 shows how the absolute biases compare versus n for $\alpha = 0.1$ and $\lambda = 0.001$. Figure 5 shows how the absolute biases compare versus n for $\alpha = 0.1$ and $\lambda = 0.0001$. Figure 6 shows how the absolute biases compare versus n for $\alpha = 0.1$ and $\lambda = 0.005$. Since the y axes are in log scale, only the positive differences in variances / absolute biases appear in the figures.

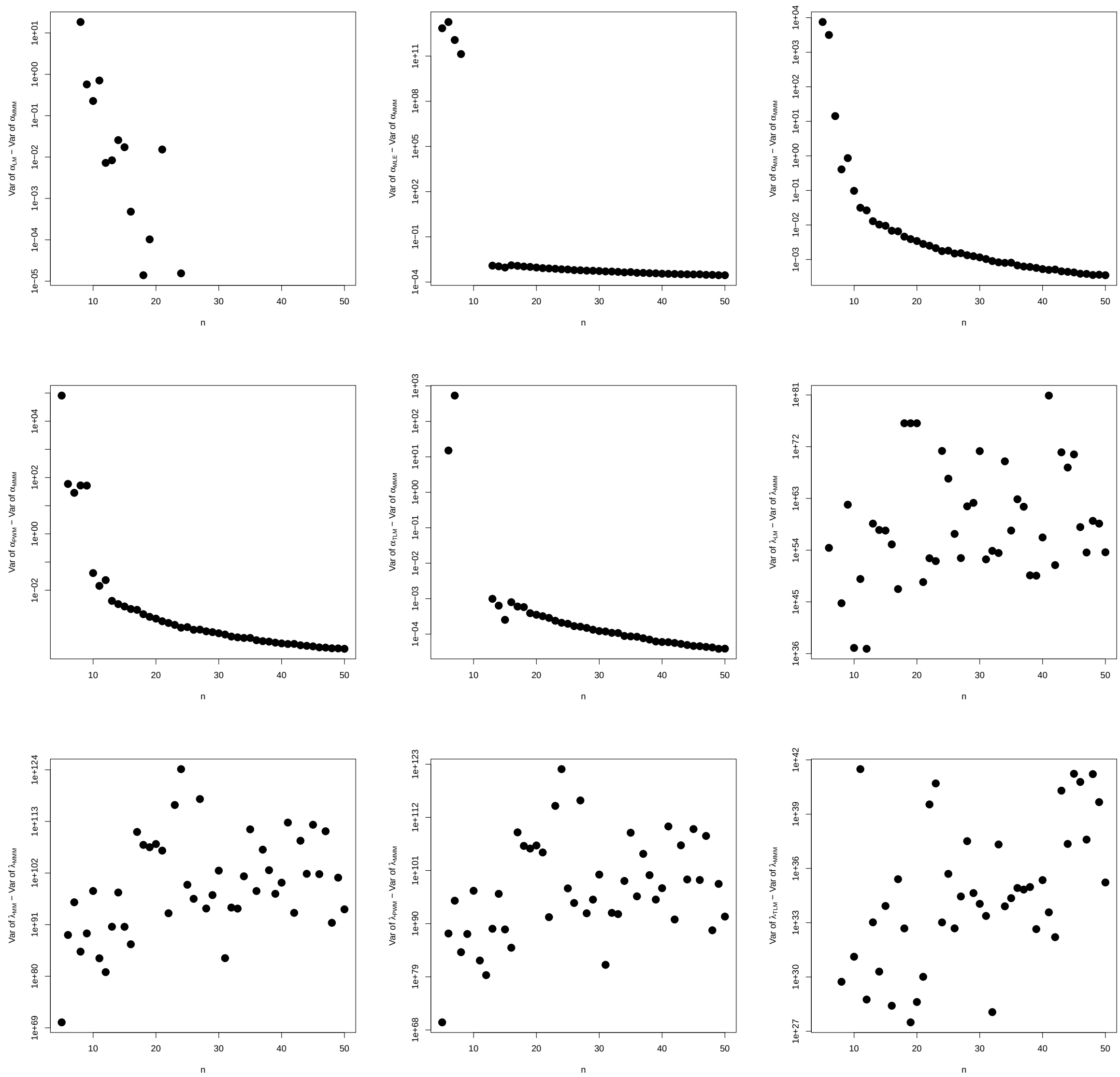

Figure 1: Comparison of the variances for $\alpha = 0.1$ and $\lambda = 0.001$: Var $(\widehat{\alpha}_{LM})-$Var $(\widehat{\alpha}_{MMM})$ in the left of first row; Var $(\widehat{\alpha}_{MLE})-$ Var $(\widehat{\alpha}_{MMM})$ in the middle of first row; Var $(\widehat{\alpha}_{MM})$ $-$ Var $(\widehat{\alpha}_{MMM})$ in the right of first row; Var $(\widehat{\alpha}_{PWM})$ $-$ Var $(\widehat{\alpha}_{MMM})$ in the left of second row; Var $(\widehat{\alpha}_{TLM})$ $-$ Var $(\widehat{\alpha}_{MMM})$ in the middle of second row; Var $\left(\widehat{\lambda}_{LM}\right)$ $-$ Var $\left(\widehat{\lambda}_{MMM}\right)$ in the right of second row; Var $\left(\widehat{\lambda}_{MM}\right)$ $-$ Var $\left(\widehat{\lambda}_{MMM}\right)$ in the left of third row; Var $\left(\widehat{\lambda}_{PWM}\right)$ $-$ Var $\left(\widehat{\lambda}_{MMM}\right)$ in the middle of third row; Var $\left(\widehat{\lambda}_{TLM}\right)$ $-$ Var $\left(\widehat{\lambda}_{MMM}\right)$ in the right of third row. The y axes are in log scale.

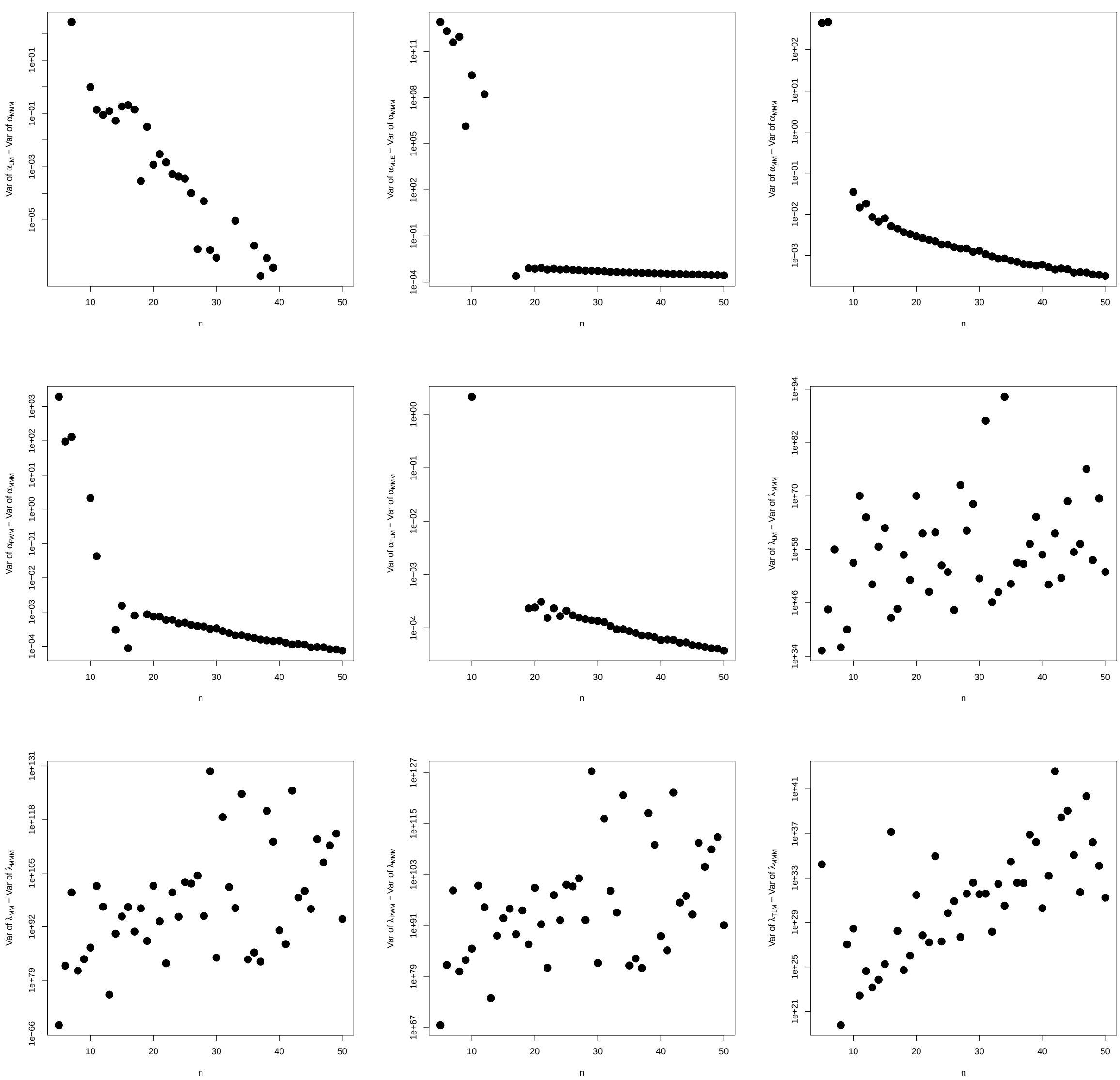

Figure 2: Comparison of the variances for $\alpha = 0.1$ and $\lambda = 0.0001$: Var $(\widehat{\alpha}_{LM})-$Var $(\widehat{\alpha}_{MMM})$ in the left of first row; Var $(\widehat{\alpha}_{MLE})-$ Var $(\widehat{\alpha}_{MMM})$ in the middle of first row; Var $(\widehat{\alpha}_{MM})$ $-$ Var $(\widehat{\alpha}_{MMM})$ in the right of first row; Var $(\widehat{\alpha}_{PWM})$ $-$ Var $(\widehat{\alpha}_{MMM})$ in the left of second row; Var $(\widehat{\alpha}_{TLM})$ $-$ Var $(\widehat{\alpha}_{MMM})$ in the middle of second row; Var $\left(\widehat{\lambda}_{LM}\right)$ $-$ Var $\left(\widehat{\lambda}_{MMM}\right)$ in the right of second row; Var $\left(\widehat{\lambda}_{MM}\right)$ $-$ Var $\left(\widehat{\lambda}_{MMM}\right)$ in the left of third row; Var $\left(\widehat{\lambda}_{PWM}\right)$ $-$ Var $\left(\widehat{\lambda}_{MMM}\right)$ in the middle of third row; Var $\left(\widehat{\lambda}_{TLM}\right)$ $-$ Var $\left(\widehat{\lambda}_{MMM}\right)$ in the right of third row. The y axes are in log scale.

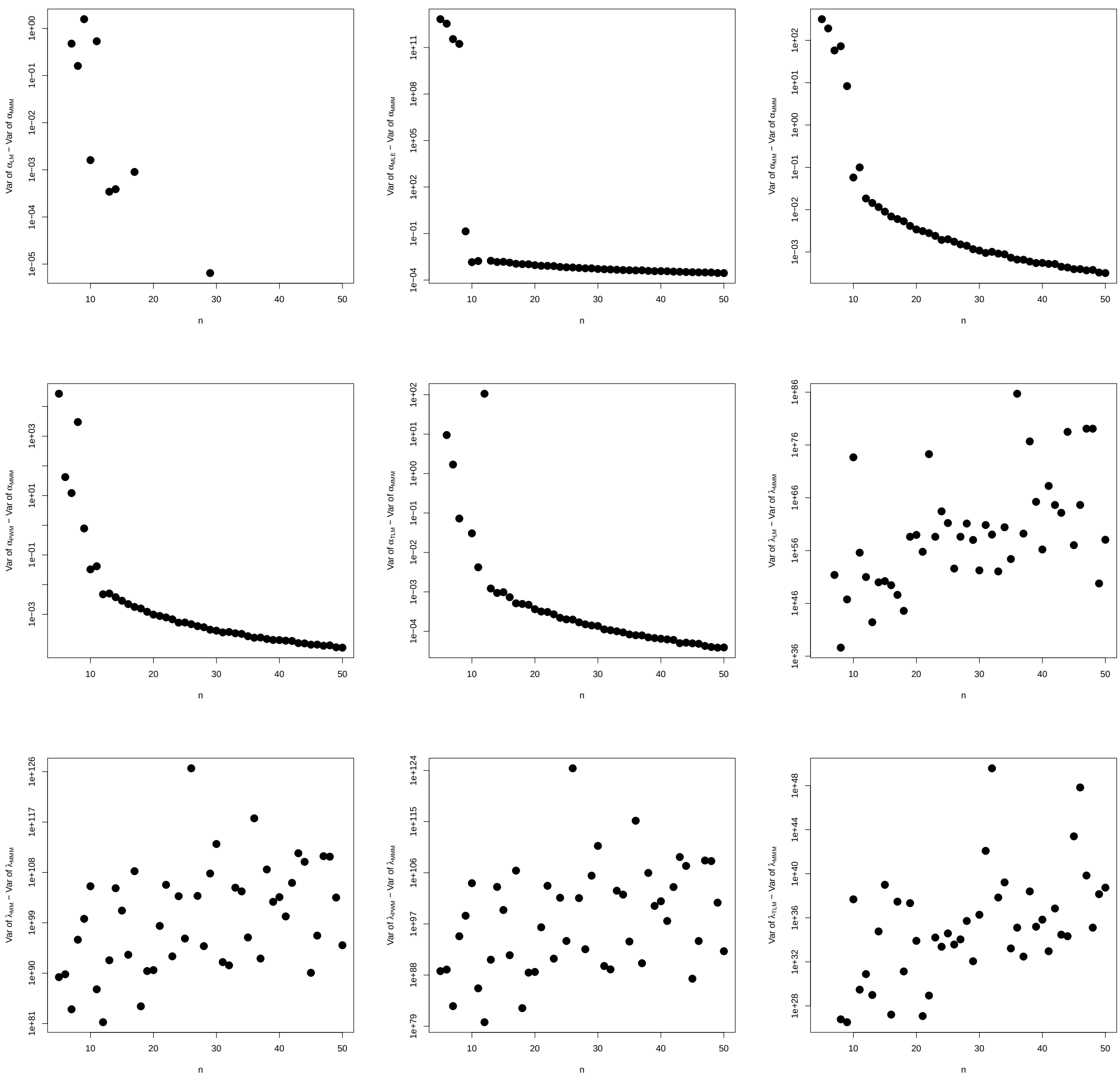

Figure 3: Comparison of the variances for $\alpha = 0.1$ and $\lambda = 0.005$: Var $(\widehat{\alpha}_{LM})-$Var $(\widehat{\alpha}_{MMM})$ in the left of first row; Var $(\widehat{\alpha}_{MLE})-$ Var $(\widehat{\alpha}_{MMM})$ in the middle of first row; Var $(\widehat{\alpha}_{MM})$ $-$ Var $(\widehat{\alpha}_{MMM})$ in the right of first row; Var $(\widehat{\alpha}_{PWM})$ $-$ Var $(\widehat{\alpha}_{MMM})$ in the left of second row; Var $(\widehat{\alpha}_{TLM})$ $-$ Var $(\widehat{\alpha}_{MMM})$ in the middle of second row; Var $\left(\widehat{\lambda}_{LM}\right)$ $-$ Var $\left(\widehat{\lambda}_{MMM}\right)$ in the right of second row; Var $\left(\widehat{\lambda}_{MM}\right)$ $-$ Var $\left(\widehat{\lambda}_{MMM}\right)$ in the left of third row; Var $\left(\widehat{\lambda}_{PWM}\right)$ $-$ Var $\left(\widehat{\lambda}_{MMM}\right)$ in the middle of third row; Var $\left(\widehat{\lambda}_{TLM}\right)$ $-$ Var $\left(\widehat{\lambda}_{MMM}\right)$ in the right of third row. The y axes are in log scale.

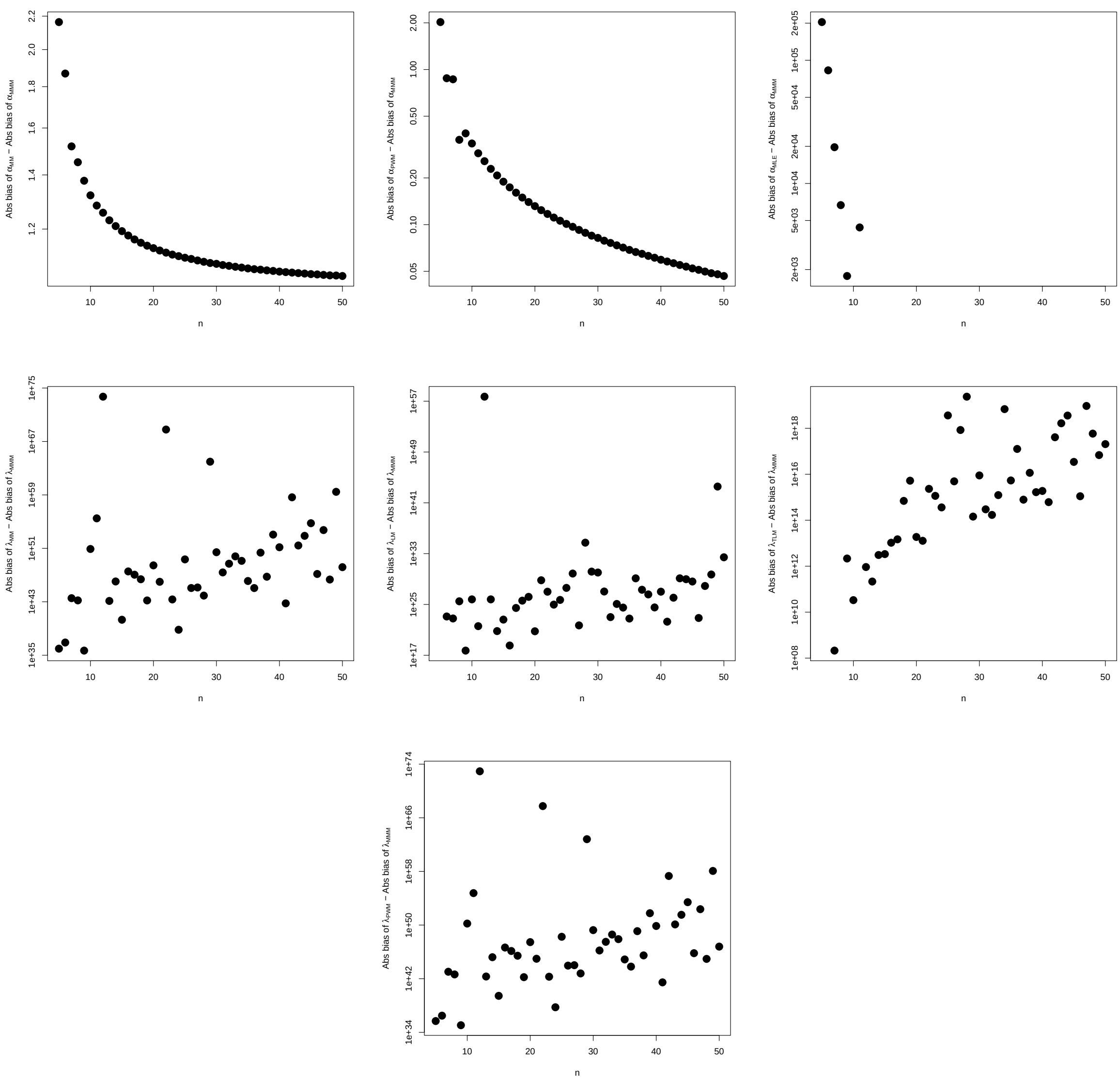

Figure 4: Comparison of the absolute biases for $\alpha = 0.1$ and $\lambda = 0.001$: $|\text{Bias}\ (\widehat{\alpha}_{MM})| - |\text{Bias}\ (\widehat{\alpha}_{MMM})|$ in the left of first row; $|\text{Bias}\ (\widehat{\alpha}_{PWM})| - |\text{Bias}\ (\widehat{\alpha}_{MMM})|$ in the middle of first row; $|\text{Bias}\ (\widehat{\alpha}_{MLE})| - |\text{Bias}\ (\widehat{\alpha}_{MMM})|$ in the right of first row; $\left|\text{Bias}\ \left(\widehat{\lambda}_{MM}\right)\right| - \left|\text{Bias}\ \left(\widehat{\lambda}_{MMM}\right)\right|$ in the left of second row; $\left|\text{Bias}\ \left(\widehat{\lambda}_{LM}\right)\right| - \left|\text{Bias}\ \left(\widehat{\lambda}_{MMM}\right)\right|$ in the middle of second row; $\left|\text{Bias}\ \left(\widehat{\lambda}_{TLM}\right)\right| - \left|\text{Bias}\ \left(\widehat{\lambda}_{MMM}\right)\right|$ in the right of second row; $\left|\text{Bias}\ \left(\widehat{\lambda}_{PWM}\right)\right| - \left|\text{Bias}\ \left(\widehat{\lambda}_{MMM}\right)\right|$ in third row. The y axes are in log scale.

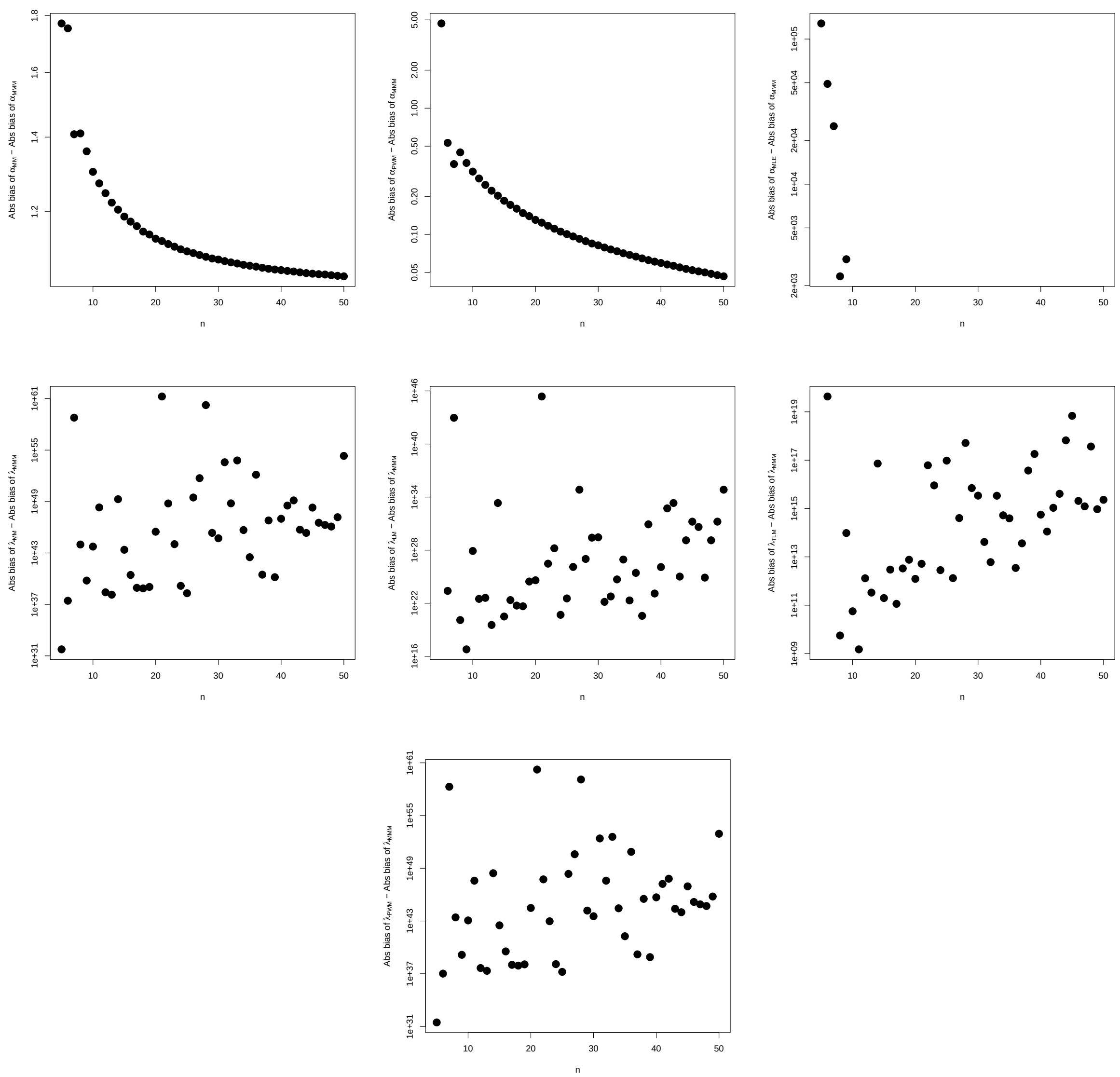

Figure 5: Comparison of the absolute biases for $\alpha = 0.1$ and $\lambda = 0.0001$: $|\text{Bias } (\widehat{\alpha}_{MM})| - |\text{Bias } (\widehat{\alpha}_{MMM})|$ in the left of first row; $|\text{Bias } (\widehat{\alpha}_{PWM})| - |\text{Bias } (\widehat{\alpha}_{MMM})|$ in the middle of first row; $|\text{Bias } (\widehat{\alpha}_{MLE})| - |\text{Bias } (\widehat{\alpha}_{MMM})|$ in the right of first row; $\left|\text{Bias } \left(\widehat{\lambda}_{MM}\right)\right| - \left|\text{Bias } \left(\widehat{\lambda}_{MMM}\right)\right|$ in the left of second row; $\left|\text{Bias } \left(\widehat{\lambda}_{LM}\right)\right| - \left|\text{Bias } \left(\widehat{\lambda}_{MMM}\right)\right|$ in the middle of second row; $\left|\text{Bias } \left(\widehat{\lambda}_{TLM}\right)\right| - \left|\text{Bias } \left(\widehat{\lambda}_{MMM}\right)\right|$ in the right of second row; $\left|\text{Bias } \left(\widehat{\lambda}_{PWM}\right)\right| - \left|\text{Bias } \left(\widehat{\lambda}_{MMM}\right)\right|$ in third row. The y axes are in log scale.

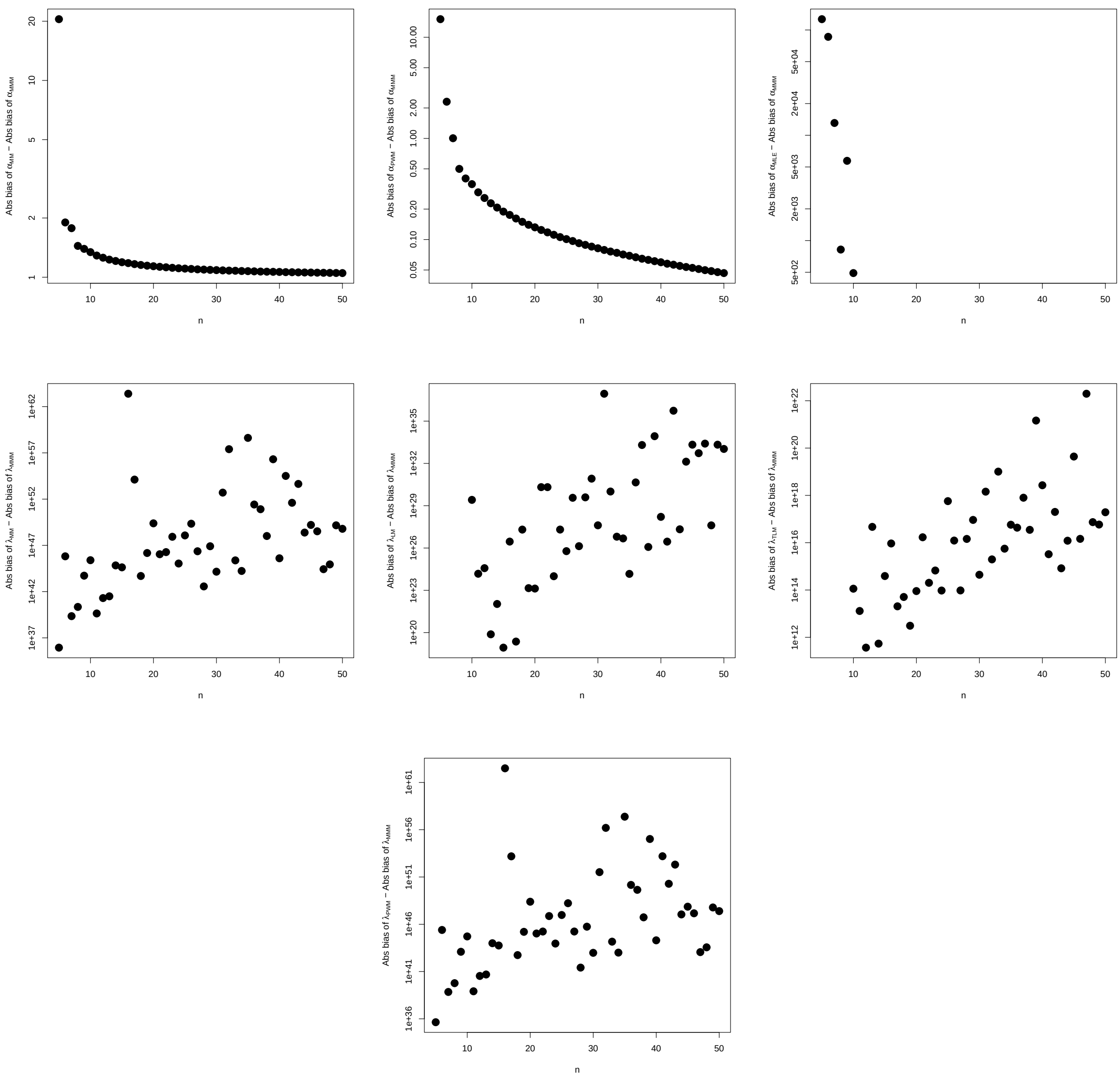

Figure 6: Comparison of the absolute biases for $\alpha = 0.1$ and $\lambda = 0.005$: $|\text{Bias } (\widehat{\alpha}_{MM})| - |\text{Bias } (\widehat{\alpha}_{MMM})|$ in the left of first row; $|\text{Bias } (\widehat{\alpha}_{PWM})| - |\text{Bias } (\widehat{\alpha}_{MMM})|$ in the middle of first row; $|\text{Bias } (\widehat{\alpha}_{MLE})| - |\text{Bias } (\widehat{\alpha}_{MMM})|$ in the right of first row; $\left|\text{Bias } \left(\widehat{\lambda}_{MM}\right)\right| - \left|\text{Bias } \left(\widehat{\lambda}_{MMM}\right)\right|$ in the left of second row; $\left|\text{Bias } \left(\widehat{\lambda}_{LM}\right)\right| - \left|\text{Bias } \left(\widehat{\lambda}_{MMM}\right)\right|$ in the middle of second row; $\left|\text{Bias } \left(\widehat{\lambda}_{TLM}\right)\right| - \left|\text{Bias } \left(\widehat{\lambda}_{MMM}\right)\right|$ in the right of second row; $\left|\text{Bias } \left(\widehat{\lambda}_{PWM}\right)\right| - \left|\text{Bias } \left(\widehat{\lambda}_{MMM}\right)\right|$ in third row. The y axes are in log scale.

We see from Figure 1 that the proposed estimator for α is superior to L moments estimator (with variance reducing by a factor of 10) for all $n \leq 25$, maximum likelihood estimator (with variance reducing by a factor of 10^{11}) for all $n \leq 50$, method of moments estimator (with variance reducing by a factor of 10^4) for all $n \leq 50$, probability weighted moments estimator (with variance reducing by a factor of 10^4) for all $n \leq 50$ and trimmed L moments estimator (with variance reducing by a factor of 10^3) for all $n \leq 50$. The proposed estimator for λ is superior to L moments estimator (with variance reducing by a factor of 10^{81}) for all $n \leq 50$, method of moments estimator (with variance reducing by a factor of 10^{124}) for all $n \leq 50$, probability weighted moments estimator (with variance reducing by a factor of 10^{123}) for all $n \leq 50$ and trimmed L moments estimator (with variance reducing by a factor of 10^{42}) for all $n \leq 50$. Only the maximum likelihood estimator for λ performed better than the proposed estimator for λ for all $n \leq 50$.

We see from Figure 2 that the proposed estimator for α is superior to L moments estimator (with variance reducing by a factor of 10) for all $n \leq 40$, maximum likelihood estimator (with variance reducing by a factor of 10^{11}) for all $n \leq 50$, method of moments estimator (with variance reducing by a factor of 10^{2}) for all $n \leq 50$, probability weighted moments estimator (with variance reducing by a factor of 10^{3}) for all $n \leq 50$ and trimmed L moments estimator (with variance reducing by a factor of 10) for all $n \leq 50$. The proposed estimator for λ is superior to L moments estimator (with variance reducing by a factor of 10^{94}) for all $n \leq 50$, method of moments estimator (with variance reducing by a factor of 10^{131}) for all $n \leq 50$, probability weighted moments estimator (with variance reducing by a factor of 10^{127}) for all $n \leq 50$ and trimmed L moments estimator (with variance reducing by a factor of 10^{41}) for all $n \leq 50$. Only the maximum likelihood estimator for λ performed better than the proposed estimator for λ for all $n \leq 50$.

We see from Figure 3 that the proposed estimator for α is superior to L moments estimator (with variance reducing by a factor of 1) for all $n \leq 29$, maximum likelihood estimator (with variance reducing by a factor of 10^{11}) for all $n \leq 50$, method of moments estimator (with variance reducing by a factor of 10^{2}) for all $n \leq 50$, probability weighted moments estimator (with variance reducing by a factor of 10^{3}) for all $n \leq 50$ and trimmed L moments estimator (with variance reducing by a factor of 10^{2}) for all $n \leq 50$. The proposed estimator for λ is superior to L moments estimator (with variance reducing by a factor of 10^{86}) for all $n \leq 50$, method of moments estimator (with variance reducing by a factor of 10^{126}) for all $n \leq 50$, probability weighted moments estimator (with variance reducing by a factor of 10^{124}) for all $n \leq 50$ and trimmed L moments estimator (with variance reducing by a factor of 10^{48}) for all $n \leq 50$. Only the maximum likelihood estimator for λ performed better than the proposed estimator for λ for all $n \leq 50$.

We see from Figure 4 that the proposed estimator for α is superior to method of moments estimator (with absolute bias reducing by a factor of 2) for all $n \leq 50$, probability weighted moments estimator (with absolute bias reducing by a factor of 2) for all $n \leq 50$ and maximum likelihood estimator (with absolute bias reducing by a factor of $2 \cdot 10^{5}$) for all $n \leq 10$. The L moments and trimmed L moments estimators for α performed better than the proposed estimator for α for all $n \leq 50$. The proposed estimator for λ is superior to method of moments estimator (with absolute bias reducing by a factor of 10^{75}) for all $n \leq 50$, L moments estimator (with absolute bias reducing by a factor of 10^{57}) for all $n \leq 50$, trimmed L moments estimator (with absolute bias reducing by a factor of 10^{18}) for all $n \leq 50$ and probability weighted moments estimator (with absolute bias reducing by a factor of 10^{74}) for all $n \leq 50$. Only the maximum likelihood estimator for λ performed better than the proposed estimator for λ for all $n \leq 50$.

We see from Figure 5 that the proposed estimator for α is superior to method of moments estimator (with absolute bias reducing by a factor of 1.8) for all $n \leq 50$, probability weighted moments estimator (with absolute bias reducing by a factor of 5) for all $n \leq 50$

and maximum likelihood estimator (with absolute bias reducing by a factor of 10^5) for all $n \leq 9$. The L moments and trimmed L moments estimators for α performed better than the proposed estimator for α for all $n \leq 50$. The proposed estimator for λ is superior to method of moments estimator (with absolute bias reducing by a factor of 10^{61}) for all $n \leq 50$, L moments estimator (with absolute bias reducing by a factor of 10^{46}) for all $n \leq 50$, trimmed L moments estimator (with absolute bias reducing by a factor of 10^{19}) for all $n \leq 50$ and probability weighted moments estimator (with absolute bias reducing by a factor of 10^{61}) for all $n \leq 50$. Only the maximum likelihood estimator for λ performed better than the proposed estimator for λ for all $n \leq 50$.

We see from Figure 6 that the proposed estimator for α is superior to method of moments estimator (with absolute bias reducing by a factor of 20) for all $n \leq 50$, probability weighted moments estimator (with absolute bias reducing by a factor of 10) for all $n \leq 50$ and maximum likelihood estimator (with absolute bias reducing by a factor of $5 \cdot 10^4$) for all $n \leq 10$. The L moments and trimmed L moments estimators for α performed better than the proposed estimator for α for all $n \leq 50$. The proposed estimator for λ is superior to method of moments estimator (with absolute bias reducing by a factor of 10^{62}) for all $n \leq 50$, L moments estimator (with absolute bias reducing by a factor of 10^{35}) for all $n \leq 50$, trimmed L moments estimator (with absolute bias reducing by a factor of 10^{22}) for all $n \leq 50$ and probability weighted moments estimator (with absolute bias reducing by a factor of 10^{61}) for all $n \leq 50$. Only the maximum likelihood estimator for λ performed better than the proposed estimator for λ for all $n \leq 50$.

The conclusions were similar to those based on Figures 1 to 6 for other small values of α and λ. All computations in this section were performed using the base package of the R software (R Core Team, 2024). None of the contributed packages in the R software were used.

6. CONCLUSIONS

We have proposed new closed form estimators for the shape and scale parameters of the Lomax distribution. We have established their strong consistency and asymptotic normality, deriving among others asymptotic variances of the estimators. We have conducted a simulation study to compare their performance versus method of moments estimators taking closed forms, L moments estimators taking closed forms, trimmed L moments estimators closed forms, probability weighted moments estimators taking closed forms and maximum likelihood estimators not taking closed forms. The performance was assessed in terms of absolute biases and variances for sample sizes less than or equal to 50. The proposed estimator for the shape parameter performed better than all of the estimators but the L moments and trimmed L moments estimators in terms of absolute bias. The proposed estimator for the shape parameter performed better than all of the estimators in terms of variance. The proposed estimator for the scale parameter performed better than all of the estimators but the maximum likelihood estimator in terms of absolute bias. The proposed estimator for the scale parameter performed better than all of the estimators but the maximum likelihood estimator in terms of variance. The relative performance of the proposed estimators over others were often of several orders of magnitude.

The mean squared error is the sum of variance and the square of absolute bias. Hence, the conclusions are the same with respect to mean square error. That is, the proposed estimator for the shape parameter is better than all of the estimators but the L moments and trimmed L moments estimators in terms of mean squared error; the proposed estimator for the scale parameter is better than all of the estimators but the maximum likelihood estimator in terms of mean square error.

In summary, the proposed modified method of moments estimators offer a valuable alternative for parameter estimation in the Lomax distribution, especially when dealing with small sample sizes or in situations where computational resources are limited. Their closed form nature makes them particularly attractive for quick and reliable estimation in practical applications.

Future work is to derive new closed form estimators for the parameters of bivariate Lomax distributions, multivariate Lomax distributions, matrix variate Lomax distributions and complex variate Lomax distributions. We could explore the performance of the proposed estimators in various applied contexts such as reliability analysis, income modeling, or survival data. Additionally, investigating the robustness of the estimators to outliers or model misspecification could provide further insights into their practical utility.

ACKNOWLEDGMENTS

The authors would like to thank the Editor and the referee for careful reading and comments which greatly improved the paper.

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