REVSTAT – Statistical Journal
Volume 0, Number 0, Month 0000, 000–000
https://doi.org/00.00000/revstat.v00i0.000

Record-Based Transmuted Unit-Omega Distribution: Different Methods of Estimation and Applications

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Received: Month 0000 Revised: Month 0000 Accepted: Month 0000

Abstract:

- Dombi et al. (2019) introduced a three parameter omega distribution and showed that its asymptotic distribution is the Weibull model. We propose a new record-based transmuted generalization of the unit omega distribution by considering Balakrishnan and He (2021) approach. We call it the RTUOMG distribution. We derive expressions for some statistical quantities, like, probability density function, distribution, hazard function, quantile function, moments, incomplete moments, inverted moments, moment generating function, Lorenz curve, and Bonferroni curve of the proposed distribution. The numerical values of various measures of central tendency and coefficient of skewness and kurtosis are also presented. Concepts of stochastic ordering and some results related to ordered statistics of the RTUOMG distribution are discussed. The parameters of the RTUOMG distribution are estimated using five distinct estimators. Additionally, the Monte Carlo simulations are performed to assess the performance of these estimators. Finally, two real data sets are analyzed to demonstrate the utility of the RTUOMG distribution.

Keywords:

- *Omega distribution; Record values; Transmuted distributions; Gauss hypergeometric function; Incomplete gauss hypergeometric function; Estimation methods.*

AMS Subject Classification:

- 60E05, 62F10, 62E15, 65C05, 33C05

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1. INTRODUCTION

Proportion data is frequently encounter in diverse disciplines such as economics, finance, reliability, medical, biology, chemistry, etc. It is known that when the values of a random variable are reported in percentage or fraction of the whole, it is referred to as proportion data. Such variables encompass all possible values within the unit interval and require an appropriate probability distribution to effectively model the observed data. Modeling approaches on the bounded interval have received considerable attention since they are related to specific issues such as the acceptance rate, recovery rate, mortality rate, scores, proportion of the educational measurements, etc. In the last decade, a lot of research have been done for developing distributions that are defined on a bounded interval. Various techniques are used to generate these distributions such as random variable transformation, function composition, and generation of new families of distributions.

Beta distribution is one of the popularly known examples of the bounded distribution which has different density shapes and is widely used in different areas of applied sciences. Several alternative models have been defined on the bounded interval in the statistical distribution literature in order to provide better results. Topp-Leone and Kumaraswamy distributions are also well known example of the unit distribution introduced by Topp and Leone (1955) and Kumaraswamy (1980), respectively. These models have received great attention in the past years in diverse areas of the applied sciences. However, these models do not have a closed expression for the moments and are mathematically less tractable. Due to growing interest in study of the bounded distributions, some new families of the unit distributions have been proposed and studied in the recent literature. Some new among these are, namely, unit-inverse gaussian distribution Ghitany et al. (2019), unit-Lindley distribution Mazucheli et al. (2019), unit Weibull Mazucheli et al. (2020), unit log-log distribution Ribeiro-Reis (2021), unit-bimodal Martínez-Flórez et al. (2024), unit Muth distribution Maya et al. (2024), Korkmaz and Korkmaz (2023), and unit generalized half-normal distribution Mazucheli et al. (2023).

Recently, by means of the omega function, Dombi et al. (2019) proposed a new probability distribution on the bounded domain and discussed its applications in reliability theory. A random variable U is said to have omega distribution with parameters α, β, and d, denoted by $\mathrm{OMG}(\alpha, \beta, d)$ if its density function is given by

$$f_U(x; \alpha, \beta, d) = 2\alpha\beta x^{\beta-1} \frac{d^{2\beta}}{d^{2\beta} - x^{2\beta}} \omega_d^{(-2\alpha,\beta)}(x),\ 0 < x < d, \tag{1.1}$$

where $\alpha > 0$, $\beta > 0$, $d > 0$, and $\omega_d^{(\alpha,\beta)}(x) = \left(\frac{d^\beta + x^\beta}{d^\beta - x^\beta}\right)^{\alpha d^\beta/2}$ denotes the omega function. The hazard function of the omega distribution can be monotonic, constant, and bathtub shapes. Therefore, it can used in modeling diverse class of real phenomena. Dombi et al. (2019) showed that the limiting omega distribution is just the Weibull distribution and can be used in place of the Weibull distribution. For some more developments and discussion about omega distribution one can refer to Okorie and Nadarajah (2019), Alsubie et al. (2021), Birbiçer and Genç (2023), Jónás and Bakouch (2022), and Özbilen and Genç (2022). Specially, when $d = 1$, the omega distribution corresponds to the unit omega distribution (see Prataviera and Cordeiro (2024)) supported on the unit interval. Its density and distribution functions (DF) are given by

$$f_U(x;\alpha,\beta) = 2\alpha\beta x^{\beta-1}\frac{1}{1-x^{2\beta}}\omega_1^{(-2\alpha,\beta)}(x),\ 0<x<1 \tag{1.2}$$

and

$$F_U(x;\alpha,\beta) = 1-\omega_1^{(-2\alpha,\beta)}(x),\ 0<x<1, \tag{1.3}$$

respectively. Birbiçer and Genç (2023) showed that the unit omega distribution is actually a unit exponentiated half logistic distribution. That is, the random variable $X = e^{-Z}$ has the unit omega distribution when Z is the exponentiated half logistic random variable. The distribution function of the unit omega is mathematically simple and independent special functions and anticipate ease in exploring its important statistical properties and statistical inference in the model. It can have different shapes including U-shaped, J-shaped, reversed J-shaped, left and right skewed and are used in various applications related to reliability.

By the means of the quadratic rank transformation, Shaw and Buckley (2007) introduced a new class of transmuted distributions using baseline distribution. We say that a random variable Y has the transmuted distribution if its distribution is expressed as

$$G_\theta(y) = (1+\theta)H(y) - \theta H^2(y),\ y\in\mathbb{R}, \tag{1.4}$$

where $H(\cdot)$ is the baseline distribution and $|\theta|\le 1$. One can see that the transmuted distribution defined in (1.4) can be obtained from the combination of distribution functions of the smallest and largest order statistics from the baseline distribution in a sample of size 2. Transmutation of the baseline distribution is a powerful tool to construct the skewed probability distribution and have been used by several researchers in the recent years. Granzotto et al. (2017) proposed the cubic rank transmuted distribution and discussed its important statistical properties. In addition to these publications, there are several papers in the literature that discussed the fundamental characteristics of various new quadratic and cubic transmuted distributions, readers can see Merovci (2013), Elbatal (2013), Tian et al. (2014), Khan and King (2014), Kemaloglu and Yilmaz (2017), Alizadeh et al. (2018), Kharazmi and Balakrishnan (2021), Chhetri et al. (2022), and Taniş and Saraçoğlu (2023). Recently, Balakrishnan and He (2021) formulated a record-based transmuted map to generate new class of probability models from the baseline distribution. They showed that the record-based transmuted distribution are expressed via the relation

$$F_p(x) = H(x) + p\overline{H}(x)\log\overline{H}(x),\ x\in\mathbb{R}, \tag{1.5}$$

where $0\le p\le 1$, and $H(\cdot)$, $\overline{H}(\cdot)$ denotes the distribution function (DF) and survival function (SF) of the baseline distribution, respectively. Balakrishnan and He (2021) also introduced some new record-based transmuted (RT) probability distribution, namely, RT-exponential distribution, RT-Weibull distribution, and RT-Linear exponential distribution. Taniş and Saraçoğlu (2022) discussed the different estimation for parameters estimation of the RT-Weibull distribution and demonstrated its real application. The RT-generalized linear exponential distribution is proposed and studied by Arshad et al. (2024). They also analyzed the lifetime data sets using it.

This paper aims to introduce a new record-based transmuted generalization of the unit omega distribution and study its important properties. The proposed distribution is a more broad family of probability distributions and includes the unit omega distribution as a submodel. Its density and hazard function can be used to handle a vast class of data that arises in a variety of domains. These functions have various shapes, including as increasing, decreasing, and bathtub shapes.

The organization of the article is as follows: Section 2 presents the mathematical framework of the proposed RTUOMG distribution. We derive the expressions for the DF, density, hazard function, quantile function, moments, incomplete moments, inverted moments, moment generating function, lorenz curve, and bonferroni curve of the RTUOMG distribution. Section 3 presents some concepts of stochastic ordering and some results related to ordered statistic of the proposed distribution. In Section 4, the maximum likelihood estimators, least squares and weighted least squares estimators, Cramér-von Mises estimators, and Anderson-Darling estimators of the parameters are explored. Section 5 presents the detailed Monte Carlo simulation study to validate the performances of the estimators through the absolute bias and mean squared error measures. Finally, in Section 6, two real data sets are analyzed to show the applicability of RTUOMG distribution in real situations.

2. RECORD-BASED TRANSMUTED UNIT-OMEGA DISTRIBUTION

Let $Y_1, Y_2, \ldots$ be a sequence of independent and identically distributed (iid) random variables having DF $H(\cdot)$. Let $Y_{U(1)}$ and $Y_{U(2)}$ denote the first two upper records from the sequence of iid random variables. Now, define a random variable X as

$$X = \begin{cases} Y_{U(1)}, & \text{with probability } 1-p \\ Y_{U(2)}, & \text{with probability } p, \end{cases}$$

where $0 \leq p \leq 1$. The DF of X is obtained as follows (see Balakrishnan and He (2021)):

$$\begin{aligned} F_X(x) =& (1-p)P(Y_{U(1)} \leq x) + pP(Y_{U(2)} \leq x) \\ =& (1-p)H(x) + p\left[1 - \overline{H}(x)\sum_{r=0}^{1}\frac{(-\log \overline{H}(x))^r}{r!}\right] \\ =& (1-p)H(x) + p[1 - \overline{H}(x)(1 - \log \overline{H}(x))] \\ =& H(x) + p\overline{H}(x)\log \overline{H}(x), \; x \in \mathbb{R}, \end{aligned} \tag{2.1}$$

where $\overline{H}(x) = 1 - H(x)$ is SF of the baseline distribution $H(x)$. The density function and hazard function (HF) of X are respectively, given by

$$f_X(x) = h(x)[1 - p - p\log \overline{H}(x)], \; x \in \mathbb{R} \tag{2.2}$$

and

$$r_X(x) = r(x)\frac{1 - p - p\log \overline{H}(x)}{1 - p\log \overline{H}(x)}, \quad x \in \mathbb{R}, \tag{2.3}$$

where $h(x)$ is the density function of the baseline distribution, and $r(x)$ denotes the HF of the baseline distribution and is defined as $r(x) = h(x)/\overline{H}(x)$.

We say that a random variable X follows the record-based transmuted unit omega distribution (denoted by $X \sim \text{RTUOMG}(\alpha, \beta, p)$) if its distribution function is given by

$$F_X(x; \alpha, \beta, p) = 1 - \omega_1^{(-2\alpha,\beta)}(x) + p\omega_1^{(-2\alpha,\beta)}(x)\log\left(\omega_1^{(-2\alpha,\beta)}(x)\right), \tag{2.4}$$

where $0 < x < 1$, $\alpha > 0$, $\beta > 0$ and $p \in [0, 1]$. The corresponding density function of the RTUOMG distribution is given by

$$f_X(x; \alpha, \beta, p) = \left(\frac{2\alpha\beta x^{\beta-1}}{1 - x^{2\beta}} \right) \omega_1^{(-2\alpha,\beta)}(x) \left[1 - p - p \log \left(\omega_1^{(-2\alpha,\beta)}(x) \right) \right]. \tag{2.5}$$

It may be seen that when $p = 0$, the RTUOMG distribution reduces to OMG distribution. Also, one can easily show that $F_X(x) \leq H(x)$, $\forall x \in (0, 1)$. The HF of the the RTUOMG distribution is

$$r_X(x; \alpha, \beta, p) = \frac{2\alpha\beta x^{\beta-1}}{(1 - x^{2\beta})} \left\{ \frac{1 - p - p \log \left(\omega_1^{(-2\alpha,\beta)}(x) \right)}{1 + p \log \left(\omega_1^{(-2\alpha,\beta)}(x) \right)} \right\}. \tag{2.6}$$

For different values of the model parameters α, β, and p, the plots of the probability density function (PDF) and HF are shown in the Figure 1 and Figure 2, respectively. From these figures, we see that the PDF and HF of the RTUOMG distribution takes different shapes and suggest applicability of the model in diverse areas.

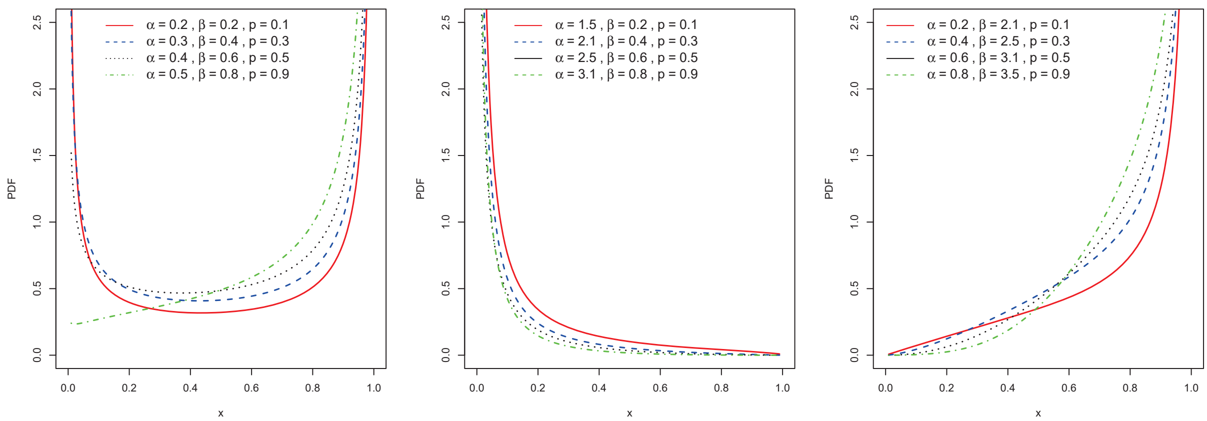

Figure 1: PDF plots of RTUOMG distribution.

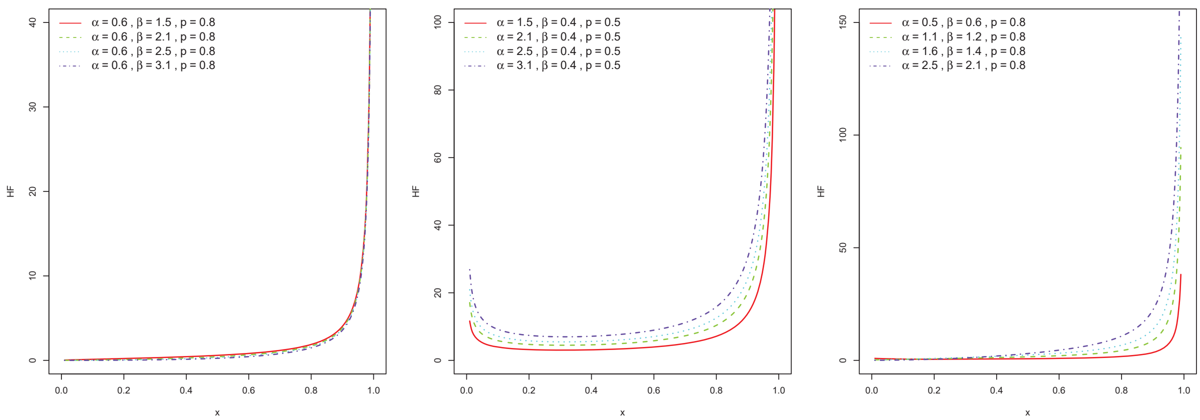

Figure 2: HF plots of RTUOMG distribution.

2.1. Quantile function

The quantile function of the RTUOMG distribution can be derived in terms of the Lambert W function which is defined as

$$W(\xi)e^{W(\xi)} = \xi,$$

where ξ is a complex number. When $\xi \geq -1/e$, it has two branches, namely, principal branch (W_o) and negative branch (W_{-1}). For a detailed discussion about the Lambert W function one can refer to a recent paper by Jodrá and Arshad (2022).

Theorem 2.1. *Let X be a random variable having RTUOMG(α, β, p) distribution. Then, the quantile function $Q(u)$ is*

$$Q(u) = \left[\frac{e^{\frac{-1}{\alpha}\left\{W_{-1}\left(\frac{u-1}{pe^{1/p}}\right)+\frac{1}{p}\right\}} - 1}{e^{\frac{-1}{\alpha}\left\{W_{-1}\left(\frac{u-1}{pe^{1/p}}\right)+\frac{1}{p}\right\}} + 1} \right]^{\frac{1}{\beta}}, \quad 0 < u < 1. \tag{2.7}$$

Proof: For $u \in (0, 1)$, the solution of $F(x) = u$ yields the quantile function. Now, we have

$$\left\{1 - p\log\left(\frac{1+x^\beta}{1-x^\beta}\right)^{-\alpha}\right\}\left(\frac{1+x^\beta}{1-x^\beta}\right)^{-\alpha} = 1 - u$$

$$\left\{\frac{-1}{p} + \log\left(\frac{1+x^\beta}{1-x^\beta}\right)^{-\alpha}\right\} e^{\log\left(\frac{1+x^\beta}{1-x^\beta}\right)^{-\alpha} - \frac{1}{p}} = \frac{u-1}{pe^{\frac{1}{p}}}.$$

It can be verified that $(u-1)/pe^{1/p} \in [-1/e, 0)$, and $\frac{-1}{p} + \log\left(\frac{1+x^\beta}{1-x^\beta}\right)^{-\alpha} \in (-\infty, -1]$. Now, using the W_{-1} function in the above equation, we get

$$W_{-1}\left(\frac{u-1}{pe^{\frac{1}{p}}}\right) = \frac{-1}{p} + \log\left(\frac{1+x^\beta}{1-x^\beta}\right)^{-\alpha}. \tag{2.8}$$

On solving (2.8) for x, completes the proof of the theorem. ∎

Next, we have the following lemmas which are needed to derive various types of the moments.

Lemma 2.1. *For $\lambda > 0$ and $\mu > 0$, we have (see Lee et al. (2011))*

$$\int_0^1 x^{\lambda-1}(1-x)^{\mu-1}(1-\beta x)^{-\nu}dx = B(\lambda, \mu)\ {}_2F_1(\nu, \lambda; \lambda + \mu; \beta), \tag{2.9}$$

where $B(\alpha, \beta)$ and ${}_2F_1(\alpha, \beta; \gamma; x)$ are the beta function and Gauss hypergeometric function, respectively, defined by

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt$$

and

$${}_2F_1(\alpha, \beta; \gamma; x) = \sum_{i=0}^{\infty} \frac{(\alpha)_i(\beta)_i}{(\gamma)_i}\frac{x^i}{i!},$$

where $(n)_i = n(n+1)(n+2)\ldots(n+i-1)$ denotes the falling factorial.

Lemma 2.2. *[Formula 1.513.1 in Gradshteyn and Ryzhik (2014)]. If $y^2 < 1$, then*

$$\log\left(\frac{1+y}{1-y}\right) = 2\sum_{k=1}^{\infty}\frac{1}{2k-1}y^{2k-1}. \tag{2.10}$$

Lemma 2.3. *For $Re(a) > Re(b) > 0, |arg(1-x)| < \pi$, we have (see Özarslan and Ustaoğlu (2019))*

$$\int_0^{\lambda} y^{b-1}(1-y)^{a-b-1}(1-xy)^{-\alpha}dy = B(b, a-b)\ {}_2\delta_1(\alpha, [b, a; \lambda], x), \tag{2.11}$$

where $B(\alpha, \beta)$ is the beta function defined in Lemma 2.1 and ${}_2\delta_1(\alpha, [b, a; \lambda], x)$ represents the incomplete gauss hypergeometric function defined by

$${}_2\delta_1(\alpha, [b, a; \lambda], x) = \sum_{i=0}^{\infty}(\alpha)_i[b, a; \lambda]_i\frac{x^i}{i!},$$

where $[b, a; \lambda]_i$ denotes the incomplete Pochhammer ratio which is introduced in terms of the incomplete beta function as follows

$$[b, a; \lambda]_i = \frac{B_\lambda(b+i, a-b)}{B(b, a-b)},$$

where $B_\lambda(m, n)$ is known as incomplete beta function and given by

$$B_\lambda(m, n) = \int_0^{\lambda} t^{m-1}(1-t)^{n-1}dt,\ \ Re(m) > 0,\ \ Re(n) > 0,\ \ 0 \le \lambda < 1.$$

2.2. Moments and measures

The rth order moment is defined by

$$\mu_r' = \mathbb{E}[X^r] = \int x^r f_X(x)dx.$$

Proposition 2.1. *Let $X \sim RTUOMG(\alpha, \beta, p)$. Then*

$$\begin{aligned}\mu_r' =& 2\alpha(1-p)B\Big(\frac{r}{\beta}+1, \alpha\Big)\ {}_2F_1\Big(\alpha+1, \frac{r}{\beta}+1; \frac{r}{\beta}+\alpha+1; -1\Big)\\ &+ 4p\alpha^2\sum_{k=1}^{\infty}\left[\frac{1}{2k-1}B\Big(\frac{r}{\beta}+2k, \alpha\Big)\ {}_2F_1\Big(\alpha+1, \frac{r}{\beta}+2k; \frac{r}{\beta}+2k+\alpha; -1\Big)\right].\end{aligned} \tag{2.12}$$

Proof: We have

$$\begin{aligned}\mu_r' &= \int_0^1 x^r\left(\frac{2\alpha\beta x^{\beta-1}}{1-x^{2\beta}}\right)\omega_1^{(-2\alpha,\beta)}(x)\left[1-p-p\log\left(\omega_1^{(-2\alpha,\beta)}(x)\right)\right]dx\\ &= \int_0^1 x^r\frac{2\alpha\beta x^{\beta-1}}{1-x^{2\beta}}\left(\frac{1+x^\beta}{1-x^\beta}\right)^{-\alpha}\left[1-p+p\alpha\log\left(\frac{1+x^\beta}{1-x^\beta}\right)\right]dx.\end{aligned}$$

Setting $y = x^\beta$, we get

$$\mu'_r = \int_0^1 y^{\frac{r}{\beta}} \frac{2\alpha}{1-y^2} \left(\frac{1+y}{1-y}\right)^{-\alpha} \left[1 - p + p\alpha \log\left(\frac{1+y}{1-y}\right)\right] dy.$$

Using Lemma 2.2, we get

$$\mu'_r = 2\alpha \int_0^1 \frac{y^{\frac{r}{\beta}}}{(1+y)(1-y)} \left(\frac{1+y}{1-y}\right)^{-\alpha} \left[1 - p + 2p\alpha \sum_{k=1}^{\infty} \frac{1}{2k-1} y^{2k-1}\right] dy.$$

By an use of the Lemma 2.1 and simple calculation completes the proof. □

The coefficient of Skewness (CS) and the coefficient of Kurtosis (CK) using first four moment are respectively, defined as

$$CS = \frac{\mu'_3 - 3\mu'_2\mu'_1 + 2{\mu'_1}^3}{{\mu_2}^{\frac{3}{2}}}$$

and

$$CK = \frac{\mu'_4 - 4\mu'_1\mu'_3 + 6{\mu'_1}^2\mu'_2 - 3{\mu'_1}^4}{{\mu_2}^2},$$

where μ_2 is variance and is defined as $\mu_2 = \mu'_2 - (\mu'_1)^2$.

In Table 1, we find a compilation of statistical moments, variance, CS, and CK for various parameter values of the RTUOMG(α, β, p) distribution. Additionally, the Figure 3 provides a visual representation of how skewness and kurtosis change across different parameter configurations.

2.3. Moment generating function

For a random variable X, the moment generating function (MGF) $M_X(t)$ is defined as $M_X(t) = E(e^{tX})$ provided the expectation exists in some neighborhood of the origin.
Let $X \sim$ RTUOMG(α, β, p). Then, the MGF $M_{RTUOMG}(t)$ of X is

$$\begin{aligned} M_{RTUOMG}(t) &= \int_0^1 e^{tx} f_X(x)dx = \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_0^1 x^r f_X(x)dx \\ &= 2\alpha(1-p) \sum_{r=0}^{\infty} \left\{ \frac{t^r}{r!} B\left(\frac{r}{\beta}+1, \alpha\right) {}_2F_1\left(\alpha+1, \frac{r}{\beta}+1; \frac{r}{\beta}+1+\alpha; -1\right) \right\} \\ &+ 4p\alpha^2 \sum_{r=0}^{\infty} \left\{ \frac{t^r}{r!} \sum_{k=1}^{\infty} \left[\frac{1}{2k-1} B\left(\frac{r}{\beta}+2k, \alpha\right) {}_2F_1\left(\alpha+1, \frac{r}{\beta}+2k; \frac{r}{\beta}+2k+\alpha; -1\right) \right] \right\}. \end{aligned}$$

Next, we present the incomplete moments for the RTUOMG distribution.

2.4. Incomplete moment

The rth incomplete moment is defined as

$$\phi_r(z) = \int_0^z x^r f_X(x)dx.$$

Proposition 2.2. *Let $X \sim RTUOMG(\alpha, \beta, p)$. Then*

$$\begin{aligned}\phi_r(z) &= 2\alpha(1-p)B\Big(\frac{r}{\beta}+1, \alpha\Big)\ {}_2\delta_1\Big(\alpha+1, \Big[\frac{r}{\beta}+1, \alpha+\frac{r}{\beta}+1; z^{\beta}\Big], -1\Big)\\ &+ 4p\alpha^2\sum_{k=1}^{\infty}\Bigg[\frac{1}{2k-1}B\Big(\frac{r}{\beta}+2k, \alpha\Big)\ {}_2\delta_1\Big(\alpha+1, \Big[\frac{r}{\beta}+2k, \alpha+\frac{r}{\beta}+2k; z^{\beta}\Big], -1\Big)\Bigg].\end{aligned} \tag{2.13}$$

One can easily establish the proof of the Proposition 2.2 with simple calculation and using the Lemma 2.2-2.3.

2.5. Inverted moments

The rth inverted moment of RTUOMG distribution defined by

$$\mu_r^* = \int x^{-r} f_X(x)dx$$

Proposition 2.3. *Let $X \sim RTUOMG(\alpha, \beta, p)$. Then, the rth inverted moment is*

$$\begin{aligned}\mu_r^* &= 2\alpha(1-p)B\Big(1-\frac{r}{\beta}, \alpha\Big)\ {}_2F_1\Big(\alpha+1, 1-\frac{r}{\beta}; 1-\frac{r}{\beta}+\alpha; -1\Big)\\ &+ 4p\alpha^2\sum_{k=1}^{\infty}\Bigg[\frac{1}{2k-1}B\Big(2k-\frac{r}{\beta}, \alpha\Big)\ {}_2F_1\Big(\alpha+1, 2k-\frac{r}{\beta}; 2k-\frac{r}{\beta}+\alpha; -1\Big)\Bigg].\end{aligned} \tag{2.14}$$

2.6. Lorenz and Bonferroni curve

The Lorenz curve measure the uncertainty in data. It is defined in terms of the incomplete moment and mean as

$$L(x) = \frac{\phi_1(z)}{\mathbb{E}(X)}. \tag{2.15}$$

For the RTUOMG distribution, the expression of the Lorenz curve L can be obtained with the help of (2.12) and (2.13).
The Bonferroni curve is obtained using the Lorenz curve as

$$B(x) = \frac{L(x)}{F_X(x)}. \tag{2.16}$$

For the RTUOMG distribution, we can calculate $B(x)$ by values of the its component.

3. STOCHASTIC ORDERING AND ORDER STATISTICS

In distribution theory and statistics, a stochastic ordering quantifies the concept of one random variable being bigger than another. A random variable X is said to be smaller than a random variable Y if

Table 1: Mean, variance, coefficient of skewness (CS), and coefficient of kurtosis (CK) for different values of parameters α, β, and p.

β	p	α	μ_1'	μ_2'	μ_3'	μ_4'	μ_2	CS	CK
0.7	0.2	0.50	0.5542510	0.4285096	0.3644273	0.3239216	0.1213154	-0.1787491	1.5419750
		1.25	0.2901651	0.1567514	0.1047541	0.0777581	0.0725556	0.8782454	2.6772931
		2.00	0.1805073	0.0711336	0.0378631	0.0235777	0.0385507	1.4672175	4.6838320
		2.75	0.1247350	0.0373275	0.0160948	0.0084739	0.0217687	1.8706299	6.7571180
		3.50	0.0923828	0.0217244	0.0077187	0.0034674	0.0131898	2.1617793	8.6740660
		4.25	0.0718269	0.0136545	0.0040625	0.0015718	0.0084954	2.3771614	10.355870
		5.00	0.0578708	0.0091041	0.0023016	0.0007740	0.0057551	2.5393587	11.790823
		5.75	0.0479085	0.0063591	0.0013840	0.0004081	0.0040639	2.6634355	12.997680
		6.50	0.0405136	0.0046111	0.0008741	0.0002279	0.0029698	2.7597558	14.006293
α	p	β							
1.5	0.5	0.50	0.2310710	0.1156176	0.0734855	0.0524786	0.0622238	1.1605455	3.368963
		1.25	0.4610186	0.2805956	0.1949608	0.1463780	0.0680574	0.1604957	1.978038
		2.00	0.5881271	0.4001152	0.2959190	0.2310710	0.0542216	-0.2517045	2.168528
		2.75	0.6672679	0.4872244	0.3768347	0.3033692	0.0419780	-0.4994430	2.523453
		3.50	0.7210560	0.5528266	0.4420729	0.3645541	0.0329048	-0.6698945	2.881894
		4.25	0.7599340	0.6037974	0.4954208	0.4165299	0.0262978	-0.7959642	3.211055
		5.00	0.7893271	0.6444595	0.5397056	0.4610186	0.0214221	-0.8936171	3.505502
		5.75	0.8123212	0.6776175	0.5769850	0.4994245	0.0177518	-0.9717678	3.767074
		6.50	0.8307971	0.7051561	0.6087628	0.5328590	0.0149322	-1.0358663	3.999452
α	β	p							
0.5	0.3	0.1	0.3710223	0.2728817	0.2271800	0.1993273	0.1352242	0.5146700	1.679386
		0.2	0.4031998	0.3038938	0.2565056	0.2271873	0.1413237	0.3766815	1.533736
		0.3	0.4353719	0.3349045	0.2858282	0.2550473	0.1453558	0.2427470	1.437843
		0.4	0.4675487	0.3659152	0.3151508	0.2829073	0.1473134	0.1116919	1.386651
		0.5	0.4997255	0.3969259	0.3444733	0.3107672	0.1472004	-0.0177101	1.377406
		0.6	0.5319022	0.4279366	0.3737959	0.3386272	0.1450166	-0.1465645	1.409213
		0.7	0.5640815	0.4589473	0.4031185	0.3664872	0.1407593	-0.2757944	1.482790
		0.8	0.5962576	0.4899580	0.4324410	0.3943472	0.1344348	-0.4060726	1.600227
		0.9	0.6284332	0.5209687	0.4617636	0.4222072	0.1260404	-0.5373839	1.764014

1. Stochastic order $X \le_{st} Y$ if $F_X(x) \ge F_Y(x)\ \forall\ x$,
2. Hazard rate order $X \le_{hr} Y$ if $h_X(x) \ge h_Y(x)\ \forall\ x$,
3. Mean residual life order $X \le_{mrl} Y$ if $m_X(x) \ge m_Y(x)\ \forall\ x$,
4. Likelihood ratio order $X \le_{lr} Y$ if $\frac{f_X(x)}{f_Y(x)}$ decreases in x.

Shaked and Shanthikumar (2007) discussed the following interconnections
$X \le_{lr} Y \Rightarrow X \le_{hr} Y \Rightarrow X \le_{mrl} Y \Rightarrow X \le_{st} Y$.
Next, we have the following result.

Theorem 3.1. *Let $X \sim RTUOMG(\alpha_1, \beta_1, p_1)$ and $Y \sim RTUOMG(\alpha_2, \beta_2, p_2)$. If $\alpha_1 > \alpha_2$, $\beta_1 = \beta_2 = \beta$, and $p_1 = p_2 = p$, then $X \le_{lr} Y$. Hence, $X \le_{hr} Y, X \le_{mrl} Y$ and $X \le_{st} Y$.*

Proof: We have

$$\frac{f_X(x)}{f_Y(x)} = \frac{\frac{2\alpha_1\beta_1 x^{\beta_1-1}}{1-x^{2\beta_1}}\left(\omega_1^{(-2\alpha_1,\beta_1)}(x)\right)\left\{1-p_1\left(1+\log\left(\omega_1^{(-2\alpha_1,\beta_1)}(x)\right)\right)\right\}}{\frac{2\alpha_2\beta_2 x^{\beta_2-1}}{1-x^{2\beta_2}}\left(\omega_1^{(-2\alpha_2,\beta_2)}(x)\right)\left\{1-p_2\left(1+\log\left(\omega_1^{(-2\alpha_2,\beta_2)}(x)\right)\right)\right\}}.$$

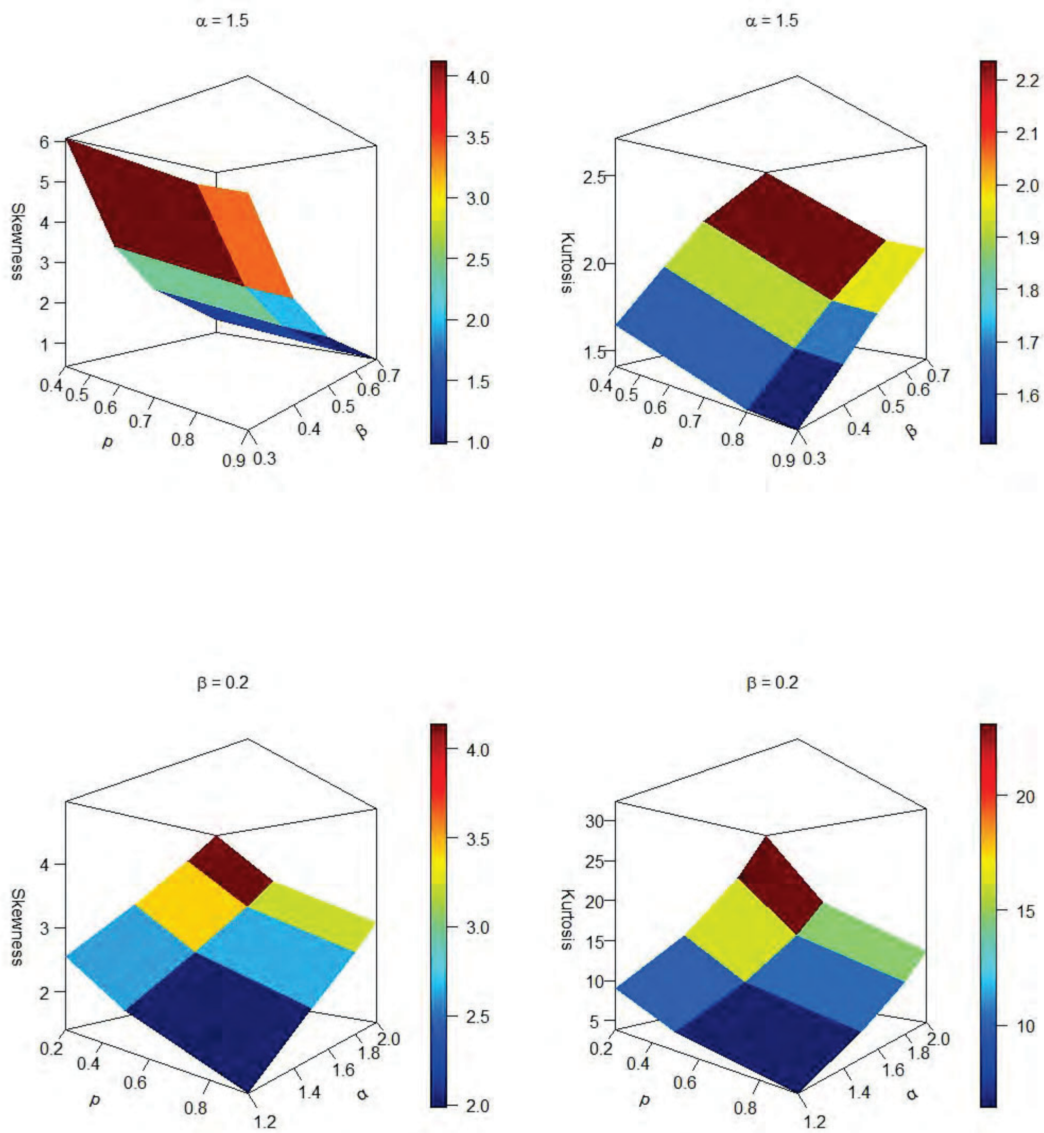

Figure 3: Graph of Skewness and Kurtosis for different parameters.

Taking log and differentiating both sides with respect to x, we get

$$\frac{d}{dx}\log\left\{\frac{f_X(x)}{f_Y(x)}\right\} = \frac{\beta_1-1}{x} - \frac{\beta_2-1}{x} - \frac{2\alpha_1\beta_1 x^{\beta_1-1}}{1-x^{2\beta_1}} + \frac{2\alpha_2\beta_2 x^{\beta_2-1}}{1-x^{2\beta_2}} - \frac{2\beta_2 x^{2\beta_2-1}}{1-x^{2\beta_2}} + \frac{2\beta_1 x^{2\beta_1-1}}{1-x^{2\beta_1}}$$
$$+ \frac{\frac{2\alpha_1\beta_1 p_1 x^{\beta_1-1}}{1-x^{2\beta_1}}}{\left\{1-p_1\left(1+\log\left(\frac{1+x^{\beta_1}}{1-x^{\beta_1}}\right)^{-\alpha_1}\right)\right\}} - \frac{\frac{2\alpha_2\beta_2 p_2 x^{\beta_2-1}}{1-x^{2\beta_2}}}{\left\{1-p_2\left(1+\log\left(\frac{1+x^{\beta_2}}{1-x^{\beta_2}}\right)^{-\alpha_2}\right)\right\}}.$$

Now, if $\beta_1 = \beta_2 = \beta, p_1 = p_2 = p$, then
(3.1)

$$= \frac{2\beta x^{\beta-1}}{1-x^{2\beta}}(\alpha_2-\alpha_1)+\frac{2\beta p x^{\beta-1}}{1-x^{2\beta}}\left\{\frac{\alpha_1}{\left\{1-p\left(1+\log\left(\frac{1+x^\beta}{1-x^\beta}\right)^{-\alpha_1}\right)\right\}}-\frac{\alpha_2}{\left\{1-p\left(1+\log\left(\frac{1+x^\beta}{1-x^\beta}\right)^{-\alpha_2}\right)\right\}}\right\}.$$

Now, we can see from equation (3.1), if $\beta_1 = \beta_2 = \beta, p_1 = p_2 = p$, and $\alpha_1 > \alpha_2$, then $\frac{d}{dx}\log\left\{\frac{f_X(x)}{f_Y(x)}\right\} \le 0$ implies that $X \le_{lr} Y$. Consequently, the other relations also holds. □

3.1. Order statistics

Let us consider a random sample $X_1, X_2, \ldots, X_n$ taken from the RTUOMG distribution, with corresponding order statistics $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$. The PDF of the r^{th} order statistic $X_{(r)}$ (where $r = 1, 2, \ldots, n$) is given by

$$f_{X_{(r)}}(x) = \frac{f(x)}{B(r, n-r+1)}\sum_{i=0}^{r-1}\binom{r-1}{i}(-1)^i\,(1-F(x))^{n+i-r}. \tag{3.2}$$

The CDF of r^{th} order statistics (X_r) are given by (see P. Singh and Dhar Das (2023))

$$F_{X_{(r)}}(x) = \sum_{j=r}^{n}\binom{n}{j}F^j(x)[1-F(x)]^{n-j}.$$

This expression can also be written as

$$F_{X_{(r)}}(x) = \sum_{j=r}^{n}\sum_{l=0}^{n-j}\binom{n}{j}\binom{n-j}{l}(-1)^l F^{j+l}(x). \tag{3.3}$$

The PDF and CDF of r^{th} order statistics of RTUOMG distribution are respectively, given as

(3.4)

$$f_{X_{(r)}}(x) = \frac{f(x)}{B(r, n-r+1)}\sum_{i=0}^{r-1}\binom{r-1}{i}(-1)^i\left[\omega_1^{(-2\alpha,\beta)}(x) - p\omega_1^{(-2\alpha,\beta)}(x)\log\left\{\omega_1^{(-2\alpha,\beta)}(x)\right\}\right]^{n+i-r}$$

and
(3.5)

$$F_{X_{(r)}}(x) = \sum_{j=r}^{n}\sum_{l=0}^{n-j}\binom{n}{j}\binom{n-j}{l}(-1)^l\left[1-\omega_1^{(-2\alpha,\beta)}(x) + p\omega_1^{(-2\alpha,\beta)}(x)\log\left\{\omega_1^{(-2\alpha,\beta)}(x)\right\}\right]^{j+l}.$$

In addition, the PDFs of the smallest and largest order statistics are

$$f_{X_{(1)}}(x) = nf(x)\left[\omega_1^{(-2\alpha,\beta)}(x) - p\omega_1^{(-2\alpha,\beta)}(x)\log\left\{\omega_1^{(-2\alpha,\beta)}(x)\right\}\right]^{n-1}$$

and

$$f_{X_{(n)}}(x) = nf(x)\sum_{i=0}^{n-1}\binom{n-1}{i}(-1)^i\left[\omega_1^{(-2\alpha,\beta)}(x) - p\omega_1^{(-2\alpha,\beta)}(x)\log\left\{\omega_1^{(-2\alpha,\beta)}(x)\right\}\right]^{i},$$

respectively.

3.2. Record statistics

Let $X_1, X_2, \ldots, X_n$ be a sequence of random variables from the RTUOMG distribution, and let $U_1, U_2, \ldots, U_n$ and $L_1, L_2, \ldots, L_n$ be the first n upper and lower record statistics, respectively, observed from the sequence $X_1, X_2, \ldots, X_n$. Then, the PDF of n^{th} upper (U_n) and lower (L_n) record statistic are respectively, given by (see Sakthivel and Nandhini (2022))

$$f_{U_n}(u_n) = \frac{(-\log[1 - F(u_n)])^{(n-1)}}{(n-1)!} f(u_n), \quad u_n > 0 \tag{3.6}$$

and

$$f_{L_n}(l_n) = \frac{(-\log[F(l_n)])^{(n-1)}}{(n-1)!} f(l_n), \quad l_n > 0. \tag{3.7}$$

Furthermore, by substituting the CDF of the RTUOMG distribution, we get

$$f_{U_n}(u_n) = \frac{\left(-\log\left[\omega_1^{(-2\alpha,\beta)}(u_n) - p\omega_1^{(-2\alpha,\beta)}(u_n)\log\omega_1^{(-2\alpha,\beta)}(u_n)\right]\right)^{(n-1)}}{(n-1)!} f(u_n), \quad u_n > 0$$

and

$$f_{L_n}(l_n) = \frac{\left(-\log\left[1 - \omega_1^{(-2\alpha,\beta)}(l_n) + p\omega_1^{(-2\alpha,\beta)}(l_n)\log\omega_1^{(-2\alpha,\beta)}(l_n)\right]\right)^{(n-1)}}{(n-1)!} f(l_n), \quad l_n > 0.$$

Moreover, the joint PDF of first n upper records $\mathbf{R} = (R_1, R_2, \ldots R_n)$ is given by

$$f_R(r) = \prod_{j=1}^{n-1} h(r_j) f(r_n).$$

Hence,

$$f_R(r) = \left\{ \prod_{j=1}^{n-1} \left(\frac{2\alpha\beta r_j^{\beta-1}}{1 - r_j^{2\beta}} \right) \left[\frac{1 - p - p\log\omega_1^{(-2\alpha,\beta)}(r_j)}{1 + p\log\omega_1^{(-2\alpha,\beta)}(r_j)} \right] \right\} f(r_n),$$

where $\mathbf{r} = (r_1, r_2, \ldots, r_n)$ denotes the observed value of $\mathbf{R} = (R_1, R_2, \ldots R_n)$ with $r_1 < r_2 < \ldots < r_n$, and $f(.)$ denotes the pdf in equation (2.5).

4. PARAMETERS ESTIMATION

This section examines the estimation of unknown parameters of RTUOMG(α, β, p) distribution. Several methods of point estimation such as maximum likelihood (ML), ordinary least square (OLS), weighted least square (WLS), Cramér-von Mises (CvM), and Anderson Darling (AD) are applied to calculate the estimators for unknown parameters of the proposed distribution.

4.1. Maximum likelihood (ML) estimation

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n taken from the RTUOMG(α, β, p) distribution. Then, the log-likelihood function can be written as

$$L(\alpha,\beta,p|\mathbf{x}) = n\log 2\alpha + n\log\beta + (\beta-1)\sum_{i=1}^{n}\log x_i - \sum_{i=1}^{n}\log(1-x_i^{2\beta}) - \alpha\sum_{i=1}^{n}\log\left(\frac{1+x_i^{\beta}}{1-x_i^{\beta}}\right) + \sum_{i=1}^{n}\log\left\{1-p+p\alpha\log\left(\frac{1+x_i^{\beta}}{1-x_i^{\beta}}\right)\right\},$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. The maximum likelihood (ML) estimators, $\hat{\alpha}_{ML}, \hat{\beta}_{ML}$, and $\hat{p}_{ML}$ of α, β, and p are calculated by simultaneously solving the following non-linear equations.

$$\frac{\partial L(\alpha,\beta,p|\mathbf{x})}{\partial\alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n}\log\left(\frac{1+x_i^{\beta}}{1-x_i^{\beta}}\right) + p\sum_{i=1}^{n}\left[\frac{\log\left(\frac{1+x_i^{\beta}}{1-x_i^{\beta}}\right)}{1-p+p\alpha\log\left(\frac{1+x_i^{\beta}}{1-x_i^{\beta}}\right)}\right] = 0,$$

$$\frac{\partial L(\alpha,\beta,p|\mathbf{x})}{\partial\beta} = \frac{n}{\beta} + \sum_{i=1}^{n}\log x_i + 2\sum_{i=1}^{n}\frac{x_i^{2\beta}\log x_i}{1-x_i^{2\beta}} - 2\alpha\sum_{i=1}^{n}\frac{x_i^{\beta}\log x_i}{1-x_i^{2\beta}} + 2p\alpha\sum_{i=0}^{n}\left[\frac{\frac{x_i^{\beta}\log x_i}{1-x_i^{2\beta}}}{1-p+p\alpha\log\left(\frac{1+x_i^{\beta}}{1-x_i^{\beta}}\right)}\right] = 0, \tag{4.1}$$

$$\frac{\partial L(\alpha,\beta,p|\mathbf{x})}{\partial p} = \sum_{i=1}^{n}\left[\frac{\alpha\log\left(\frac{1+x_i^{\beta}}{1-x_i^{\beta}}\right)-1}{1-p+p\alpha\log\left(\frac{1+x_i^{\beta}}{1-x_i^{\beta}}\right)}\right] = 0.$$

There are several numerical techniques such as Broyden-Flecther-Goldfarb-Shanno (BFGS) and Nelder-Mead (NM) that can be use to solve the above non-linear equations obtained in (4.1). These techniques can be easily applied using optim() function in R-programming.

4.2. Ordinary least squares (OLS) and weighted least squares (WLS) estimation:

It is common practice to estimate the distribution parameters using the relationship between the empirical cumulative distribution function and order statistics. These ideas form the foundation of the weighted least squares and ordinary least squares estimation concepts. Let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution $F(x)$. Then, the empirical distribution function is defined as

$$F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I(X_i \le x),$$

where I denotes the indicator function. According to Swain et al. (1988), the OLS estimators of unknown parameters can be obtained by minimizing the sum of squares differences between the vector of uniformized order statistics and the corresponding vector of expected values. Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ denote the order statistics of a random sample $X_1, X_2, \ldots, X_n$ of size n taken from RTUOMG distribution. Then, the mean and variance of the transformed variable $F(X_{(i)})$ are as follows

$$E(F(X_{(i)})) = \frac{i}{n+1}; i = 1, 2, \ldots, n$$

and

$$V(F(X_{(i)})) = \frac{i(n-i+1)}{(n+1)^2(n+2)}; i = 1, 2, \ldots, n.$$

Based on the mean and variance of $F(X_{(i)})$, we can employ the following two methods of least squares estimation.

Ordinary least squares (OLS) estimation: The OLS estimators, $\hat{\alpha}_{OLS}$, $\hat{\beta}_{OLS}$, and $\hat{p}_{OLS}$ of RTUOMG(α, β, p) distribution can be obtained by minimizing the following equation

$$Z(\alpha, \beta, p) = \sum_{i=1}^{n} \left(F(X_{(i)}) - \frac{i}{n+1} \right)^2. \tag{4.2}$$

By considering the distribution function given in equation (2.4) and equation (4.2), we can derive the following three non-linear equations by taking partial derivatives with respect to parameters α, β, and p

$$\frac{\partial}{\partial \alpha} Z(\alpha, \beta, p) = \sum_{i=1}^{n} \left(\frac{\partial F(X_i, \alpha, \beta, p)}{\partial \alpha} \right) \left(F(X_{(i)}) - \frac{i}{n+1} \right) = 0, \tag{4.3}$$

$$\frac{\partial}{\partial \beta} Z(\alpha, \beta, p) = \sum_{i=1}^{n} \left(\frac{\partial F(X_i, \alpha, \beta, p)}{\partial \beta} \right) \left(F(X_{(i)}) - \frac{i}{n+1} \right) = 0, \tag{4.4}$$

$$\frac{\partial}{\partial p} Z(\alpha, \beta, p) = \sum_{i=1}^{n} \left(\frac{\partial F(X_i, \alpha, \beta, p)}{\partial p} \right) \left(F(X_{(i)}) - \frac{i}{n+1} \right) = 0. \tag{4.5}$$

These equations capture the relationship between the observed data and the parameters of interest. The solution of equations (4.3)-(4.5) can be obtained by using a non-linear equation solver technique such as Broyden-Flecther-Goldfarb-Shanno (BFGS) and Nelder-Mead (NM) technique etc.

Weighted least squares (WLS) estimation: The WLS estimator follows a similar procedure to the OLS estimator, where the objective is to minimize the weighted sum of squares differences. The WLS estimators, $\hat{\alpha}_{WLS}$, $\hat{\beta}_{WLS}$, and $\hat{p}_{WLS}$ for the unknown parameters α, β, and p can be computed by minimizing

$$W(\alpha, \beta, p) = \sum_{i=1}^{n} \eta_i \left(F(X_{(i)}) - \frac{i}{n+1} \right)^2,$$

with respect to the unknown parameters. This function is minimized by adjusting the values of α, β, and p such that the sum of squared differences between the observed values $F(X_{(i)})$ and the corresponding expected values $\frac{i}{n+1}$ is minimized through weight factor η_i. The factor η_i is derived from the inverse of the variance of $F(X_{(i)})$ and depends on the specific distribution being considered.

4.3. Cramér-von Mises (CvM) estimation:

Cramér-von Mises (CvM) estimator is one type of goodness-of-fit estimator and based on Cramér-von Mises statistics (see Dey et al. (2018)). This estimator is basically based on the difference between the empirical distribution function and the theoretical distribution function. Therefore, the CvM estimators, $\hat{\alpha}_{CvM}$, $\hat{\beta}_{CvM}$, and $\hat{p}_{CvM}$ of unknown parameters can be obtained by minimizing the following equation with respect to parameters α, β, and p

$$C(\alpha, \beta, p) = \frac{1}{12n} + \sum_{i=1}^{n} \left(F(X_{(i)}) - \frac{2i-1}{2n} \right)^2 .$$

4.4. Anderson-Darling (AD) estimation:

Anderson and Darling (1952) suggested an estimator based on Anderson-Darling statistic which minimizes the Anderson-Darling distance between the empirical and theoretical distribution function. The AD estimators, $\hat{\alpha}_{AD}$, $\hat{\beta}_{AD}$, and $\hat{p}_{AD}$ of unknown parameters can be computed by minimizing the following equation with respect to parameters α, β, and p

$$Q(\alpha, \beta, p) = -n - \frac{1}{n} \sum_{i=1}^{n} (2i-1) \left[\log F(X_{(i)}) + \log \bar{F}(X_{(n+1-i)}) \right] ,$$

where $\bar{F}(x) = 1 - F(x)$ denotes the SF of RTUOMG distribution. For more details of this technique, one may refer to Boos (1982) and Arshad et al. (2024).

5. SIMULATION STUDY

In this section, a well-organized Monte Carlo simulation study is carried out to evaluate the performance of ML, OLS, WLS, CvM, and AD estimators of unknown parameters of proposed RTUOMG(α, β, p) distribution. A well known Broyden-Fletcher-Goldfarb-Shanno (BFGS) technique introduced by Broyden (1970), Fletcher (1970), Goldfarb (1970), and Shanno (1970) is used to obtain the estimates of population parameters α, β, and p. This optimization is performed using optim$(\cdot)$ function which involves the BFGS technique. For a detailed information about this algorithm, one can refer to 'stats' package of R (version 4.2.1) programming library. In present simulation, we generate the data of different sample sizes (n) such as $25, 50, 100, 150, 200$, and 250 with different combinations of parameter values (α, β, p) using inverse cumulative distribution function method. The absolute biases and mean squared errors (MSEs) of considered estimators are calculated using 500 repetitions for each sample sizes. The functional form of these measures are as follows; absolute bias= $N^{-1} \sum_{i=1}^{N} |(\hat{\eta}_i - \eta)|$, mean squared errors (MSEs)= $N^{-1} \sum_{i=1}^{N} (\hat{\eta}_i - \eta)^2$, where $\eta = (\alpha, \beta, p)$ is the true value of the parameter, $\hat{\eta}_i = (\hat{\alpha}_i, \hat{\beta}_i, \hat{p}_i)$ is the estimated value of the parameter η for the ith repeated sample, and N be the number of repeated samples. The main findings of this simulation study are reported in Table 2-5. Table 2 presents the absolute biases and MSEs of the estimators for fix settings of parameters $\eta = (0.3, 0.4, 0.8)$. From Table 2, it can be observe that the absolute biases and MSEs of all the considered estimators are decreases as the sample sizes increases. Therefore, we can say that all considered estimators are consistent. The similar performance of considered estimators are obtained in case of other settings of parameter values. One more observation, we analyze that the ML estimation technique performs well at the estimation of small setting of parameters i.e. $\forall\ \alpha, \beta, p \in (0,1)$ while AD estimation technique works better for the estimation of $\alpha \geq 1$ and $\beta,\ p \in (0,1)$. Moreover, in case of $\beta \geq 1$ and $\alpha,\ p \in (0,1)$ or $\alpha,\ \beta \geq 1$ and $p \in (0, 0.5)$, AD estimation technique works better for the estimation of α, β and ML technique performs better to estimate the parameter p in terms of absolute bias and MSEs. The CvM technique works worse in all situations. Based on the results of the simulation study, we recommended the maximum likelihood (ML), Anderson-Darling (AD), and weighted least squares (WLS) estimation technique for the estimation of parameters of the proposed distribution.

Table 2: Absolute biases and MSEs of the estimators for settings of parameters $\eta = (0.3, 0.4, 0.8)$.

$N = 500$	↓Sample	Absolute Bias			MSE		
Estimators	(n)	$\hat{\alpha}$	$\hat{\beta}$	$\hat{p}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{p}$
ML	25	0.0575402	0.5636601	0.3744900	0.0051212	0.7029732	0.2106772
	50	0.0460293	0.4086689	0.3399031	0.0033725	0.2802979	0.1763369
	100	0.0360543	0.3303454	0.2885562	0.0021142	0.1552839	0.1273787
	150	0.0303397	0.3070048	0.2578554	0.0015002	0.1224424	0.0988758
	200	0.0262740	0.3058071	0.2467511	0.0010812	0.1132115	0.0823941
	250	0.0229363	0.3081646	0.2442920	0.0007905	0.1094451	0.0744624
OLS	25	0.0684853	0.3996601	0.4082226	0.0071143	0.4539321	0.2519106
	50	0.0579523	0.3564157	0.3997664	0.0051239	0.2856817	0.2362091
	100	0.0425999	0.2906173	0.3190678	0.0030105	0.1527906	0.1524894
	150	0.0334161	0.2740858	0.2768188	0.0018788	0.1192181	0.1120023
	200	0.0278311	0.2632146	0.2493752	0.0013178	0.1026786	0.0881649
	250	0.0242371	0.2575848	0.2375982	0.0009229	0.0908743	0.0742529
WLS	25	0.0669899	0.4007772	0.4041653	0.0067799	0.4072548	0.2491423
	50	0.0541343	0.3419191	0.3724363	0.0045822	0.2584859	0.2145772
	100	0.0402377	0.2854043	0.3030421	0.0027186	0.1350776	0.1414393
	150	0.0312444	0.2712038	0.2613837	0.0016816	0.1098474	0.1024181
	200	0.0255624	0.2651793	0.2359697	0.0010919	0.0973379	0.0782198
	250	0.0223360	0.2699729	0.2281872	0.0007785	0.0926324	0.0681823
CvM	25	0.0636285	0.5163060	0.3927234	0.0063881	0.7009315	0.2354711
	50	0.0542338	0.4067139	0.3903516	0.0045268	0.3618528	0.2260682
	100	0.0402142	0.3101414	0.3128003	0.0026977	0.1739882	0.1457470
	150	0.0315371	0.2878147	0.2728989	0.0016897	0.1308638	0.1080854
	200	0.0264091	0.2739417	0.2463895	0.0012001	0.1104116	0.0858663
	250	0.0230677	0.2661871	0.2354335	0.0008426	0.0966527	0.0726858
AD	25	0.0610353	0.4128566	0.3822879	0.0057994	0.4160725	0.2267281
	50	0.0506099	0.3458641	0.3624216	0.0041280	0.2184971	0.2018953
	100	0.0387137	0.2924843	0.2995147	0.0024893	0.1366311	0.1349263
	150	0.0305515	0.2820945	0.2624722	0.0015762	0.1141870	0.1003933
	200	0.0254312	0.2755060	0.2394694	0.0010539	0.1018982	0.0786142
	250	0.0221263	0.2774801	0.2309920	0.0007644	0.0963062	0.0690289

Table 3: Absolute biases and MSEs of the estimators for settings of parameters $\eta = (1.5, 0.4, 0.8)$.

$N = 500$	↓Sample	Absolute Bias			MSE		
Estimators	(n)	$\hat{\alpha}$	$\hat{\beta}$	$\hat{p}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{p}$
ML	25	0.3497333	0.2236051	0.5125836	0.19385894	0.06425815	0.3318941
	50	0.2597831	0.1979366	0.4998509	0.09421299	0.04653150	0.3204776
	100	0.2012286	0.1674090	0.4492008	0.05820005	0.03379105	0.2773317
	150	0.1763568	0.1426037	0.3938933	0.04577178	0.02613773	0.2301613
	200	0.1439343	0.1229506	0.3423611	0.03364347	0.02047264	0.1833536
	250	0.1175313	0.1062177	0.3001589	0.02341726	0.01611237	0.1463373
OLS	25	0.4566141	0.1802872	0.7527976	0.26447902	0.05098879	0.5877450
	50	0.4078215	0.1784580	0.7482363	0.20304339	0.04229995	0.5834461
	100	0.3974076	0.1773041	0.7411818	0.18009355	0.03623652	0.5748995
	150	0.3867420	0.1744949	0.7320833	0.16853860	0.03349793	0.5649600
	200	0.3831334	0.1725529	0.7240970	0.16285928	0.03189547	0.5550598
	250	0.3961742	0.1784412	0.7531013	0.16773712	0.03318493	0.5836874
WLS	25	0.3860936	0.1761375	0.6553208	0.21006981	0.04876736	0.5005673
	50	0.3308242	0.1740738	0.6410506	0.14698090	0.03912981	0.4807757
	100	0.3087392	0.1753061	0.6455267	0.12307668	0.03483763	0.4748319
	150	0.3066162	0.1774102	0.6589561	0.11804590	0.03419037	0.4836404
	200	0.3157362	0.1813900	0.6782137	0.12059840	0.03468639	0.4995112
	250	0.3336347	0.1888402	0.7200899	0.12782737	0.03683236	0.5432789
CvM	25	0.4082337	0.2211056	0.7438294	0.23055472	0.07281502	0.5790463
	50	0.3582042	0.1981327	0.7386132	0.16498067	0.05151793	0.5736056
	100	0.3711385	0.1861582	0.7329617	0.16097120	0.03948357	0.5664762
	150	0.3794644	0.1830302	0.7413456	0.16140139	0.03630204	0.5745171
	200	0.3780736	0.1792293	0.7321075	0.15716656	0.03411290	0.5631307
	250	0.3880864	0.1830108	0.7549924	0.16130300	0.03469556	0.5851894
AD	25	0.2909188	0.1619203	0.4006342	0.14709415	0.04011143	0.2089489
	50	0.2072535	0.1499430	0.4088768	0.06392817	0.03124702	0.2141149
	100	0.1627509	0.1383655	0.3970670	0.03965682	0.02430970	0.2038295
	150	0.1396941	0.1368092	0.4086397	0.02952598	0.02346276	0.2107771
	200	0.1340009	0.1400162	0.4203310	0.02649028	0.02333142	0.2161384
	250	0.1246115	0.1459661	0.4383566	0.02279590	0.02426058	0.2238062

Table 4: Absolute biases and MSEs of the estimators for setting of parameters $\eta = (0.5, 1.4, 0.7)$.

$N = 500$	↓Sample	Absolute Bias			MSE		
Estimators	(n)	$\hat{\alpha}$	$\hat{\beta}$	$\hat{p}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{p}$
ML	25	0.1120347	0.4778450	0.3510794	0.0192453	0.4553152	0.1653206
	50	0.0882323	0.4759241	0.3538716	0.0115647	0.3755231	0.1699393
	100	0.0730091	0.4901289	0.3268296	0.0082510	0.3552174	0.1474419
	150	0.0644513	0.5374551	0.3064261	0.0065860	0.3798693	0.1310115
	200	0.0584768	0.5666936	0.2965667	0.0053447	0.3882113	0.1181616
	250	0.0520851	0.6055517	0.2937361	0.0041196	0.4195092	0.1085803
OLS	25	0.1169415	0.5490052	0.3767365	0.0207611	0.5735533	0.1939783
	50	0.0955362	0.4938152	0.3775485	0.0136563	0.3935783	0.1906785
	100	0.0742208	0.4622377	0.3252509	0.0087476	0.3337550	0.1456954
	150	0.0597227	0.4525198	0.2898479	0.0059388	0.2942829	0.1148964
	200	0.0489766	0.4411952	0.2581814	0.0041646	0.2647838	0.0910280
	250	0.0439092	0.4310606	0.2446191	0.0032173	0.2441803	0.0794551
WLS	25	0.1137814	0.5097649	0.3640515	0.0199774	0.4917461	0.1856189
	50	0.0897559	0.4689411	0.3441709	0.0123696	0.3905042	0.1703281
	100	0.0703048	0.4475743	0.3138069	0.0079280	0.2959510	0.1348477
	150	0.0572977	0.4514203	0.2805834	0.0056517	0.2817592	0.1106532
	200	0.0467773	0.4539823	0.2526273	0.0037037	0.2667118	0.0868828
	250	0.0408495	0.4603980	0.2358324	0.0027702	0.2676532	0.0752283
CvM	25	0.1153831	0.5903472	0.3593871	0.0212673	0.7178636	0.1758703
	50	0.0933515	0.5122568	0.3659089	0.0129054	0.4440887	0.1778830
	100	0.0710206	0.4625675	0.3228048	0.0080142	0.3361106	0.1407267
	150	0.0573306	0.4648073	0.2886636	0.0054565	0.3091565	0.1126145
	200	0.0471177	0.4544770	0.2583327	0.0038578	0.2761777	0.0900179
	250	0.0420647	0.4416537	0.2438570	0.0029645	0.2553325	0.0782985
AD	25	0.1082465	0.4530197	0.3458671	0.0182808	0.3655762	0.1621560
	50	0.0881673	0.4225620	0.3487991	0.0118716	0.2853728	0.1611526
	100	0.0696419	0.4106469	0.3230672	0.0077281	0.2279476	0.1324356
	150	0.0562887	0.4263870	0.3067596	0.0053736	0.2224153	0.1148221
	200	0.0489789	0.4455409	0.2952685	0.0040292	0.2305750	0.1036650
	250	0.0434670	0.4665140	0.2900217	0.0030678	0.2463197	0.0952949

Table 5: Absolute biases and MSEs of the estimators for setting of parameters $\eta = (2.5, 3, 0.2)$.

$N = 500$	↓Sample	Absolute Bias			MSE		
Estimators	(n)	$\hat{\alpha}$	$\hat{\beta}$	$\hat{p}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{p}$
ML	25	0.8061627	0.4544024	0.1484790	1.4487436	0.3635223	0.0359888
	50	0.4895446	0.3169428	0.1301799	0.4510390	0.1720310	0.0260875
	100	0.3099033	0.2278899	0.1101981	0.1644552	0.0861656	0.0161722
	150	0.2312480	0.1948946	0.1026410	0.0930790	0.0607583	0.0125469
	200	0.1849125	0.1793090	0.0922127	0.0583091	0.0469182	0.0093837
	250	0.1519173	0.1785981	0.0936273	0.0380139	0.0435949	0.0093413
OLS	25	0.6721036	0.5160953	0.1965019	0.9517571	0.4045551	0.0566855
	50	0.4528296	0.3629825	0.1719837	0.3543042	0.2104053	0.0432459
	100	0.2923195	0.2512083	0.1513627	0.1444090	0.0976153	0.0327181
	150	0.2309553	0.1999917	0.1387837	0.0850821	0.0607940	0.0274345
	200	0.1967073	0.1628344	0.1302848	0.0587973	0.0396424	0.0227732
	250	0.1877359	0.1322615	0.1191776	0.0522507	0.0278735	0.0187148
WLS	25	0.6634569	0.4632552	0.2016422	0.8629090	0.3438677	0.0626860
	50	0.4572236	0.3234613	0.1733783	0.3559206	0.1814369	0.0474915
	100	0.3155410	0.2245553	0.1436479	0.1673690	0.0811050	0.0315608
	150	0.2360105	0.1821378	0.1233989	0.0900462	0.0535859	0.0214425
	200	0.2038839	0.1476566	0.1091343	0.0651935	0.0329917	0.0147964
	250	0.1805628	0.1366309	0.1015546	0.0500695	0.0285151	0.0120991
CvM	25	0.8235136	0.4981632	0.1845208	1.6105885	0.4277520	0.0491581
	50	0.4949517	0.3507950	0.1657580	0.4490723	0.2131639	0.0397025
	100	0.3015803	0.2441061	0.1494132	0.1613265	0.0950858	0.0316030
	150	0.2275366	0.1979540	0.1366198	0.0856153	0.0597272	0.0261653
	200	0.1919231	0.1591142	0.1271712	0.0570228	0.0378875	0.0212320
	250	0.1851113	0.1315436	0.1184449	0.0511737	0.0274468	0.0182570
AD	25	0.6703474	0.4422730	0.1777463	0.9582408	0.3274166	0.0476750
	50	0.4453855	0.3134355	0.1599620	0.3374060	0.1693089	0.0382549
	100	0.3046488	0.2203649	0.1362785	0.1534672	0.0796048	0.0256305
	150	0.2329083	0.1824005	0.1257978	0.0882921	0.0528154	0.0203533
	200	0.1950100	0.1504189	0.1149436	0.0590851	0.0343346	0.0154094
	250	0.1768263	0.1350550	0.1087862	0.0466696	0.0284675	0.0129945

6. DATA ANALYSIS

In this section, we have analyzed two different datasets to assess the usefulness of proposed distribution in modeling real life data. Considered data set are also fitted to some well known distributions such as unit omega (UOMG) Prataviera and Cordeiro (2024), complementary unit gompertz (CUG), complementary unit lomax (CUL) Guerra et al. (2021), and unit weibull (UW) distribution Mazucheli et al. (2020). The probability density and distribution functions of these fitted distributions are presented in Table 6. First of all, we have

Table 6: Probability density and distribution function of considered distributions.

Distribution	Density Function	Distribution Function
UOMG	$\frac{2\alpha\beta x^{\beta-1}}{(1-x^{2\beta})}\left(\frac{1+x^{\beta}}{1-x^{\beta}}\right)^{-\alpha}$	$1-\left(\frac{1+x^{\beta}}{1-x^{\beta}}\right)^{-\alpha}$; $\alpha>0,\ \beta>0,\ x\in(0,1)$
CUG	$\frac{\beta\log 2}{(1-\mu)^{-\beta}-1}(1-x)^{-(\beta+1)}2^{\left(\frac{(1-x)^{-\beta}-1}{1-(1-\mu)^{-\beta}}\right)}$	$1-2^{\left(\frac{(1-x)^{-\beta}-1}{1-(1-\mu)^{-\beta}}\right)}$; $\beta>0,\ \mu\in(0,1),\ x\in(0,1)$
CUL	$\frac{\log 2}{\beta(1-x)}\left[\log\left(1-\beta^{-1}\log(1-\mu)\right)\right]^{-1}$ $\times\left[1-\beta^{-1}\log(1-x)\right]^{\left(\frac{-\log 2}{\log[1-\beta^{-1}\log(1-\mu)]}\right)-1}$	$1-\left[1-\beta^{-1}\log(1-x)\right]^{\left(\frac{-\log 2}{\log[1-\beta^{-1}\log(1-\mu)]}\right)}$; $\beta>0,\ \mu\in(0,1),\ x\in(0,1)$
UW	$\frac{1}{x}\alpha\beta(-\log x)^{\beta-1}e^{-\alpha(-\log x)^{\beta}}$	$e^{-\alpha(-\log x)^{\beta}}$; $\alpha>0,\ \beta>0,\ x\in(0,1)$

performed the exploratory data analysis and obtain the MLEs of the RTUOMG distribution with four other taken distributions. Then, we have done a comparative study of RTUOMG distribution with UOMG, CUG, CUL, and UW distribution. To determine the performance and appropriateness of fitted distributions, we use some selection statistics such as the value of -2log-likelihood function of fitted distribution, Akaike's information criteria (AIC) of fitted model and some goodness of fit test statistics; Kolmogorov-Smirnov (KS), Cramer von Mises (CvM), and Anderson-Darling (AD) along with their p-values. The goodness of fit statistic with smaller value and highest p-value gives the better fit of distribution. The functional form of these measures and their performing algorithm are easily available in R-programming library.

Data Analysis 1: Here, we consider a bladder cancer data (see Lee and Wang (2003)) about the remission times (in months) of 128 patients. To avoid the biased results, first, we have performed the outlier analysis using boxplot technique and extract one extreme observation 79.05* from data set. Since, the aim of this analysis is to verify the usefulness of proposed distribution with support $(0,1)$. Therefore, the final data set of 127 observations divided by 100 are as follows; {0.0008, 0.0209, 0.0348, 0.0487, 0.0694, 0.0866, 0.1311, 0.2363, 0.0020, 0.0223, 0.0352, 0.0498, 0.0697, 0.0902, 0.1329, 0.0040, 0.0226, 0.0357, 0.0506, 0.0709, 0.0922, 0.1380, 0.2574, 0.0050, 0.0246, 0.0364, 0.0509, 0.0726, 0.0947, 0.1424, 0.2582, 0.0051, 0.0254, 0.0370, 0.0517, 0.0728, 0.0974, 0.1476, 0.2631, 0.0081, 0.0262, 0.0382, 0.0532, 0.0732, 0.1006, 0.1477, 0.3215, 0.0264, 0.0388, 0.0532, 0.0739, 0.1034, 0.1483, 0.3426, 0.0090, 0.0269, 0.0418, 0.0534, 0.0759, 0.1066, 0.1596, 0.3666, 0.0105, 0.0269, 0.0423, 0.0541, 0.0762, 0.1075, 0.1662, 0.4301, 0.0119, 0.0275, 0.0426, 0.0541, 0.0763, 0.1712, 0.4612, 0.0126, 0.0283, 0.0433, 0.0549, 0.0766, 0.1125, 0.1714, 0.0135, 0.0287, 0.0562, 0.0787, 0.1164, 0.1736, 0.0140, 0.0302, 0.0434, 0.0571, 0.0793, 0.1179, 0.1810, 0.0146, 0.0440, 0.0585, 0.0826, 0.1198, 0.1913, 0.0176, 0.0325, 0.0450, 0.0625, 0.0837, 0.1202, 0.0202, 0.0331, 0.0451, 0.0654, 0.0853, 0.1203, 0.2028, 0.0202, 0.0336, 0.0676, 0.1207, 0.2173, 0.0207, 0.0336, 0.0693, 0.0865, 0.1263, 0.2269}. The basic information of bladder cancer data are presented in Table 7. The reported value of skewness and kurtosis indicate that data is positively skewed with high kurtosis. For modelling this positive skewed and high kurtosis data a distribution defined on support (0,1) is needed.

We have fit the RTUOMG distribution with four other considered distributions for this

Table 7: The descriptive statistic for bladder cancer data set.

Minimum	1st Quartile	Median	Mean	3rd Quartile	Maximum	Skewness	Kurtosis
0.00080	0.03335	0.06250	0.08817	0.11715	0.46120	2.08009	5.09506

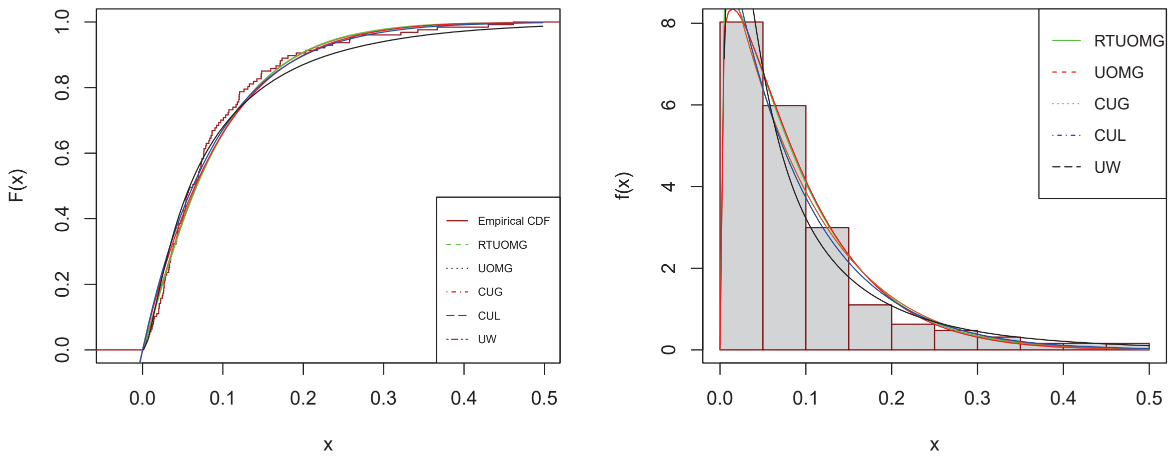

Figure 4: Fitted CDFs and PDFs for bladder cancer data.

data set and calculated MLEs with their standard errors are reported in Table 8. As well as, the fitted empirical and theoretical CDFs and PDFs plots using MLEs are displayed in Figure 4. These fitted figures reveals a good fit of bladder cancer data set with RTUOMG distribution. Further, we have evaluated the values of $-2\log L$ and $AIC = 2k - 2\log L$, where k is the number of parameters and L denotes the maximized value of the likelihood function. To analyze the appropriateness of RTUOMG distribution, three other selection statistics, namely, KS, CvM, and AD along with their corresponding p-values are also calculated and reported in Table 9. The smallest value of AIC and goodness of fit statistics (with their high p-value) supports the best fit of RTUOMG distribution among other considered distributions.

Data Analysis 2: Here, we consider a failure times (in weeks) data set of 50 components (see Tanış and Saraçoğlu (2022)). The values of data set divided by 100 are as follows; {0.00013, 0.00065, 0.00111, 0.00111, 0.00163, 0.00309, 0.00426, 0.00535, 0.00684, 0.00747, 0.00997, 0.01284, 0.01304, 0.01647, 0.01829, 0.02336, 0.02838, 0.03269 0.03977, 0.03981, 0.04520, 0.04789, 0.04849, 0.05202, 0.05291, 0.05349 0.05911, 0.06018, 0.06427, 0.06456, 0.06572, 0.07023, 0.07087, 0.07291 0.07787, 0.08596, 0.09388, 0.10261, 0.10713, 0.11658,

Table 8: The MLEs and (standard error*) for bladder cancer data set.

Distribution	MLEs (standard errors*)		
RTUOMG	$\hat{\alpha} = 7.08299$ (1.79777*)	$\hat{\beta} = 1.11156$ (0.076196*)	$\hat{p} = 0.01$ (0.19155*)
UOMG	$\hat{\alpha} = 7.33820$ (1.26419*)	$\hat{\beta} = 1.13426$ (0.07418*)	
CUG	$\hat{\mu} = 0.06533$ (0.00660*)	$\hat{\beta} = 0.01000$ (0.77332*)	
CUL	$\hat{\mu} = 0.06255$ (0.00646*)	$\hat{\beta} = 1.42330$ (2.04066*)	
UW	$\hat{\alpha} = 0.03608$ (0.00928*)	$\hat{\beta} = 2.84277$ (0.18084*)	

Table 9: Model selection statistics for bladder cancer data set.

$n = 127$			KS		CvM		AD	
Model	$-2 \log L$	AIC	Statistic	P-value	Statistic	P-value	Statistic	P-value
RTUOMG	-365.0942	-359.0942	0.0655	0.6468	0.1077	0.5492	0.6613	0.5917
UOMG	-362.3548	-358.3548	0.0693	0.5759	0.1145	0.5188	0.6694	0.5846
CUG	-362.3154	-358.3154	0.0786	0.4121	0.1557	0.3730	1.0273	0.3431
CUL	-362.9674	-358.9674	0.0891	0.2654	0.1701	0.3341	1.1275	0.2967
UW	-358.6090	-354.6090	0.0679	0.6011	0.1546	0.3764	1.0428	0.3354

0.13006, 0.13388, 0.13842, 0.17152, 0.17283, 0.19418, 0.23471, 0.24777, 0.32795, 0.48105}. The descriptive statistics of failure times data set are given in Table 10. The MLEs estimators with their standard errors and model selection statistics fitted to the failure times data are reported in Table 11-12. Figure 5 shows the graph of fitted CDFs and PDFs obtained from the estimates in Table 11. The RTUOMG distribution fits this data better than the UOMG, CUG, CUL, and UW distribution in terms of lowest values of AIC and goodness of fit statistics (with their high p-values).

Table 10: The descriptive statistic for failure times data set.

Minimum	1st Quartile	Median	Mean	3rd Quartile	Maximum	Skewness	Kurtosis
0.00013	0.01390	0.05320	0.07821	0.10043	0.48105	2.37799	7.22886

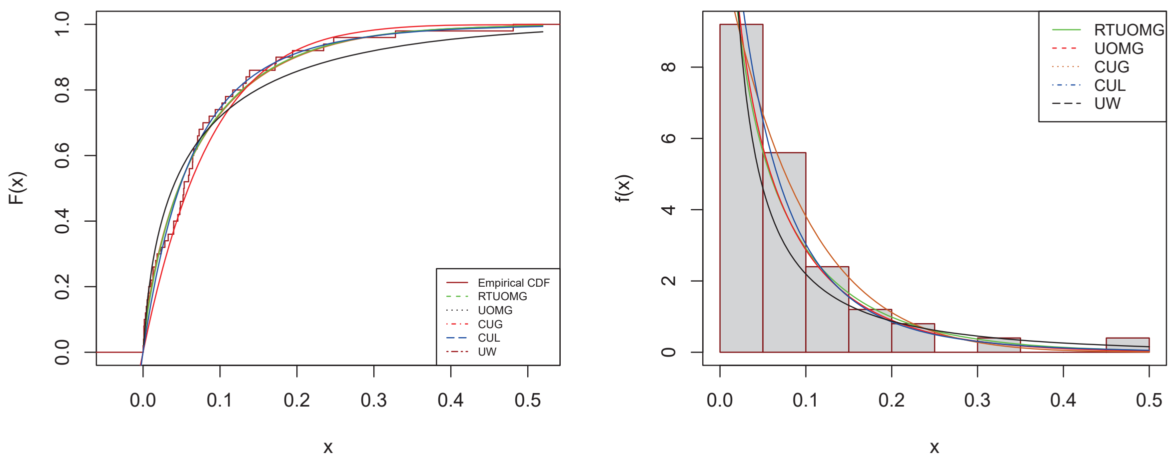

Figure 5: Fitted CDFs and PDFs for failure times data.

Table 11: The MLEs and (standard error*) for failure times data set.

Distribution	MLEs (standard errors*)		
RTUOMG	$\hat{\alpha} = 4.84868$ (1.11256*)	$\hat{\beta} = 0.687588$ (0.15471*)	$\hat{p} = 0.526608$ (0.422478*)
UOMG	$\hat{\alpha} = 4.27178$ (1.03678*)	$\hat{\beta} = 0.818416$ (0.08980*)	
CUG	$\hat{\mu} = 0.05885$ (0.00965*)	$\hat{\beta} = 0.01000$ (1.03830*)	
CUL	$\hat{\mu} = 0.04708$ (0.00915*)	$\hat{\beta} = 0.21444$ (0.18292*)	
UW	$\hat{\alpha} = 0.05624$ (0.02112*)	$\hat{\beta} = 2.12460$ (0.22026*)	

Table 12: Model selection statistics for failure times data set.

$n = 50$			KS		CvM		AD	
Model	$-2 \log L$	AIC	Statistic	P-value	Statistic	P-value	Statistic	P-value
RTUOMG	-159.8693	-153.8693	0.1003	0.6960	0.0638	0.7920	0.3342	0.9100
UOMG	-156.5532	-152.5532	0.0973	0.7306	0.0676	0.7684	0.4231	0.8248
CUG	-152.4296	-148.4296	0.1207	0.4597	0.1702	0.3343	1.6746	0.1399
CUL	-156.3623	-152.3623	0.0892	0.8212	0.0889	0.6443	0.8247	0.4629
UW	-154.2329	-150.2329	0.1483	0.2214	0.1817	0.3064	0.9602	0.3784

7. CONCLUSION

In this paper, we have proposed a new record-based transmuted generalization of the unit omega distribution and call it RTUOMG distribution. We derived mathematical expressions for the various important statistical quantities and graphically demonstrated the behavior of the density and hazard function. We adopted the five different types of estimation techniques for estimating the unknown parameters of the proposed distribution. Using the Monte Carlo algorithm, a well organized simulation study has been performed to understand the behavior of the considered estimators for the RTUOMG distribution with respect to absolute biases and MSEs under different setups of parameters. Based on simulation results, we recommend to use ML, AD, and WLS estimation technique for estimating the parameters of RTUOMG distribution. Moreover, the utility of the RTUOMG distribution has been also compared with some other considered distributions by analyzing two real life data sets. A dominating nature of RTUOMG distribution shows the more effectiveness and flexibility in real life modeling.

ACKNOWLEDGMENTS

The authors are thankful to the editor and referee for their constructive and helpful comments which have significantly improved the article. The first author would like to express his gratitude to DST-FIST grant with letter no. SR/FST/MS-I/2021/104. M. Arshad would like to express his gratitude to the Science and Engineering Research Board (SERB), India for financial support under the Core Research Grant CRG/2023/001230. A. Ali would like to thank DST Inspire Fellowship [IF190682], Government of India, for providing financial support.

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