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Zero-inflated Poisson integer valued GARCH model with periodic coefficients

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Abstract:

- In this paper we introduce the zero-inflated Poisson integer-valued generalized autoregressive conditional heteroskedastic model with periodic coefficients, to address two critical aspects: the zero-inflated nature of count time series and the presence of periodic features hidden in the autocorrelation function. The study focuses on analyzing the fundamental probabilistic and statistical properties of this class. Specifically, the conditions for the existence of higher order moments are obtained and their explicit formulas in terms of the model parameters are derived. In particular, the periodic stationarity conditions for the first and second moments are established, and their closed-form expressions are derived based on the obtained conditions. Furthermore, the research examines the periodic autocovariance structure and provides a closed-form expression for the periodic autocorrelation function. To estimate the underlying parameters, the Conditional Maximum Likelihood (CML) method is applied, using the Expectation Maximization (EM) algorithm. The effectiveness of this method is assessed through a simulation study. Additionally, the paper illustrates the practical application of the proposed model by analyzing the daily number of COVID-19 deaths in Finland.

Keywords:

- *Zero inflated Poisson distribution; Higher order moments; EM algorithm; COVID-19 deaths in Finland.*

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1. INTRODUCTION

In the literature of count time series analysis, researchers frequently observe a specific type of time series known as ”zero-inflated” time series in real-world applications. The concept of zero inflation arises when a significant part of the observations is exactly equal to zero, and this deviation from the expected distribution can have influences for modeling and analysis. Ignoring zero inflation can have two primary effects. Firstly, the estimated parameters and standard errors may be biased. Secondly, the excessive presence of null values can lead to overdispersion, a phenomenon explained by Zuur et al. (2009). Several fields provide examples of this phenomenon. For instance, in finance, the transaction volume of a stock may exhibit zero inflation during non-trading periods like weekends or holidays. In epidemiology, disease counts may have an excess of zeros during periods of low transmission or when a particular disease is absent in a population. Additionally, even in biomedical and public health fields, certain rare diseases with low infection rates can give rise to count time series with a significant number of zeros.

In recent years, the analysis of zero-inflation count data has received increasing attention, leading to the development of several proposed models. The concept of zero inflation was initially introduced by Neyman (1939) and Feller (1943) to solve the problem of excessive zeros. Since then, extensive research has been devoted to developing models specifically adapted to zero-inflation count data. Notable examples include the hurdle model, introduced by Mullahy (1986), the zero-inflated Poisson model, studied by Lambert (1992), and the two-part model, also known as the zero-altered model, studied by Heilbron (1994). These models have made an important contribution to the field, and have made it possible to solve the complex problems associated with zero-inflated count data. Jazi et al. (2012) conducted a recent study that introduced a new stationary first-order integer-valued autoregressive process which is characterized by the zero inflated Poisson innovations distribution. Furthermore, Piancastelli and Barreto-Souza (2019) explored the inferential aspects of the zero-infated Poisson INAR(1) process. Even more, Yang et al. (2013) and Yang et al. (2015) proposed a zero-inflated autoregressive model in which the autocorrelation is expressed as a function of previous responses. Particularly for observation-driven processes, Zhu (2012) proposed zero inflated Poisson and negative binomial INGARCH models to effectively represent count time series data that exhibit an excess of zero values.

These models offer enhanced flexibility and are considered as valuable extensions of the Poisson and negative binomial INGARCH models, respectively. Furthermore, Lee et al. (2021) investigate statistical inferences, such as estimating parameters and conducting change tests for count time series models where the conditional distribution of current observations given past information is modeled using a zero-inflated one-parameter exponential family. More generally, a zero-inflated model can be understood as a combination of two distributions: one with a concentrated probability mass at zero, representing the excess zeros, and another non-degenerate distribution, such as the Poisson or negative binomial distribution, representing the remaining non-zero values.

Gladyshev (1961) introduced the concept of time series with periodic variations in mean, variance and covariance, commonly known as periodically correlated processes, whose presence in time series is reinforced by real-life applications in many practical fields. Indeed, we can mention without claiming to be exhaustive the analysis of births in Quebec conducted by Manaa and Bentarzi (2021b), the investigation of daytime road accidents by Manaa and Bentarzi (2021a) and Souakri and Bentarzi (2022), the examination of the number of cases of campylobacterosis infections by Ouzzani and Bentarzi (2019), and Manaa and Bentarzi (2021c). The recent study of monthly short-term disability claims by Bentarzi and Souakri (2023), and Manaa and Souakri (2024) presented an analysis of the daily number of COVID-19 deaths in Finland.

However, although many non-negative integer-valued time series found in diverse fields such as epidemiology, economics, environment, criminology and others exhibit periodic patterns in their autocovariance structures, the zero inflated Poisson INGARCH model, which provides advantages such as positivity, discreteness nature, time-varying volatility, overdispersion and excess zeros, fails to capture this periodicity. This characteristic, which cannot be adequately represented or described by integer-valued time series models with time-invariant parameters, is still not taken into account. As a consequence, a good reason and motivation were given to extend this class of time-invariant models to the Poisson INGARCH model with zero-inflated periodic coefficients which generalizes the periodic Poisson INGARCH model introduced by Bentarzi and Bentarzi (2017).

In the following section, we present the definition and certain properties of the zero-inflated Poisson distribution. The basic notations and definitions related to a periodic zero-inflated Poisson integer-valued generalized autoregressive conditional heteroskedastic model, which is denoted as PZIP-INGARCH for brevity, are presented in Section 3. In Section 4, we investigate the existence of higher moments and provide a precise expression using model parameters for their calculation. Also, we examine the periodic stationarity problem of the proposed model and both first and second order conditions for stationarity are established. Section 5 focuses on the analysis of the autocovariance structure of the underlying periodic model while presenting an explicit expression for the autocorrelation function. Based on the EM algorithm introduced by Dempster et al. (1977), the unknown periodic parameters of our models are estimated using the conditional maximum likelihood estimation method, in Section 6. In Section 7, we examine the performance of the proposed estimation method through a simulation study. Also an application on the daily number of COVID-19 deaths in Finland data set is provided in Section 8. Section 9 summarizes the conclusions drawn from our study. Finally, in Section 10, we include an Appendix that provides additional calculations and demonstrations.

2. ZERO INFLATED POISSON DISTRIBUTION

For the purposes of this study, we recall the definition of the zero-inflated distribution. A stochastic process $\{\varepsilon_t;\ t \in \mathbb{Z}\}$ is said to follow a zero-inflated Poisson distribution with periodic parameters ϱ_t and λ_t, which we denote by $\mathcal{ZIP}\left(\varrho_t, \lambda_t\right)$, if its probability mass function can be expressed as follows :

$$P\left(\varepsilon_t = k\right) = \begin{cases} \varrho_t + \left(1-\varrho_t\right) e^{-\lambda_t}, & \text{if } k = 0, \\ \left(1-\varrho_t\right) \dfrac{e^{-\lambda_t}\lambda_t^k}{k!}, & \text{if } k = 1, 2, ... \end{cases}$$

or equivalently,

$$P(\varepsilon_t = k) = \varrho_t \delta_{k,0} + (1-\varrho_t)\frac{e^{-\lambda_t}\lambda_t^k}{k!}, \text{ for } k = 0, 1, 2, \ldots$$

with, $\lambda_t > 0$, and $\varrho_t \in \left(\max\left[-1, -\frac{e^{-\lambda_t}}{1-e^{-\lambda_t}}\right], 1\right)$ controls the inflation or deflation of zeros for $\varrho_t > 0$ and $\varrho_t < 0$, respectively, where $\delta_{k,0}$ is the Kronecker delta, i.e., $\delta_{k,0}$ is 1 when $k = 0$ and is 0 when $k \neq 0$. The Poisson distribution is obtained as a particular case by choosing $\varrho_t = 0$. It is important to note that this probability mass function of zero inflated Poisson with periodic parameters is a version of the zero inflated Poisson distribution, with time-invariant parameters and which is presented by Jazi et al. (2012) and it was given by the following probability mass function :

$$P(\varepsilon_t = k) = \begin{cases} \varrho + (1-\varrho)e^{-\lambda}, & \text{if } k = 0, \\ (1-\varrho)\frac{e^{-\lambda}\lambda^k}{k!}, & \text{if } k = 1, 2, \ldots \end{cases}$$

or equivalently,

$$P(\varepsilon_t = k) = \varrho\delta_{k,0} + (1-\varrho)\frac{e^{-\lambda}\lambda^k}{k!}, \text{ for } k = 0, 1, 2, \ldots$$

Given that $\lambda > 0$ and $\varrho \in \left(\max\left[-1, -\frac{e^{-\lambda}}{1-e^{-\lambda}}\right], 1\right)$, the parameter ϱ controls the inflation or deflation of zeros for $\varrho > 0$ and $\varrho < 0$, respectively. Additionally, $\delta_{k,0}$ represents the Kronecker delta, i.e., $\delta_{k,0}$ is 1 when $k = 0$ and is zero when $k \neq 0$. The Poisson distribution is obtained as a particular case by choosing $\varrho = 0$. To review some important properties of the zero inflated Poisson distribution, and as it indicated in Zhu (2012), the probability generating function is

$$g(z) = \varrho + (1-\varrho)e^{\lambda(z-1)}.$$

Also, from Lemma 1 in Ferland et al. (2006) we know that the uncentered moments of ε_t satisfy

$$\mathbb{E}(\varepsilon_t^m) = (1-\varrho)\sum_{i=0}^{m} \left\{ {m \atop i} \right\} \lambda^i,$$

where $\left\{ {m \atop i} \right\}$ is the Stirling number of the second kind. From this relation, we can obtain easily the mean and the variance,

$$\mathbb{E}(\varepsilon_t) = \lambda(1-\varrho), \; Var(\varepsilon_t) = \lambda(1-\varrho)(1+\lambda\varrho).$$

It should be noted that, according to Fisher's dispersion index, the variance is greater than the mean, i.e.,

$$I_\varepsilon = \frac{\sigma_\varepsilon^2}{\mu_\varepsilon} = \frac{\lambda(1-\varrho)(1+\lambda\varrho)}{\lambda(1-\varrho)} = 1 + \lambda\varrho > 1.$$

3. PERIODIC ZERO INFLATED POISSON INGARCH MODEL

As Zhu (2012) mentions, an integer-valued stochastic process $\{y_t;\ t \in \mathbb{Z}\}$ satisfies the zero inflated Poisson integer-valued generalized autoregressive conditional heteroskedastic

ZIP-INGARCH(p,q) model, at orders p and q, conditional on past information $\mathcal{F}_{t-1}$, if it is provided by:

$$\begin{array}{c} y_t|\,\mathcal{F}_{t-1} \rightsquigarrow \mathcal{ZIP}\left(\varrho,\lambda_t\right), \\ \lambda_t = \alpha_0 + \sum_{i=1}^{p} \alpha_i y_{t-i} + \sum_{j=1}^{q} \beta_j \lambda_{t-j}, \end{array} \tag{3.1}$$

where, $\alpha_0 > 0$, $\alpha_i \geq 0$, and $\beta_j \geq 0$, $i = 1, ..., p$, $j = 1, ..., q$, $p \geq 0$, $q \geq 0$, and $\mathcal{F}_{t-1}$ is the σ-field generated by $\{y_{t-1}, y_{t-2}, \ldots\}$. In the sense of Gladyshev (1961), a periodically correlated integer-valued process $\{y_t;\ t \in \mathbb{Z}\}$ is said to satisfy a periodic zero inflated Poisson integer-valued generalized autoregressive conditional heteroskedastic model, with period S and orders p and q, noted in short by PZIP-INGARCH(p, q), if it takes the following form:

$$\begin{array}{c} y_t|\,\mathcal{F}_{t-1} \rightsquigarrow \mathcal{ZIP}\left(\varrho_t,\lambda_t\right), \\ \lambda_t = \alpha_{0,t} + \sum_{i=1}^{p} \alpha_{i,t} y_{t-i} + \sum_{j=1}^{q} \beta_{j,t} \lambda_{t-j}, \end{array} \tag{3.2}$$

where, $0 < \varrho_t < 1$, and $\mathcal{F}_{t-1}$ denotes, as usually, the σ-field generated by $\{y_{t-1}, y_{t-2}, \ldots\}$. The parameters $\alpha_{i,t}$, $i = 0, 1, ..., p$, $\beta_{j,t}$, $j = 1, ..., q$, and ϱ_t are periodic in time t, with period S, i.e., $\alpha_{i,t+vS} = \alpha_{i,t}$, $\beta_{j,t+vS} = \beta_{j,t}$ and $\varrho_{t+vS} = \varrho_t$, $t, v \in \mathbb{Z}$. The following conditions for $\alpha_{i,t}$'s must be imposed : $\alpha_{0,t} > 0$, and $\alpha_{i,t} \geq 0$, $i = 1, ..., p$ and $\beta_{j,t} \geq 0$, $j = 1, ..., q$, $t \in \mathbb{Z}$, to guarantee that zero or negative conditional variances are not possible. A particular case, the periodic ZIP-INGARCH$(1, 1)$ model, i.e., the case when $p = q = 1$, is the main focus of this paper:

$$\begin{array}{c} y_t|\,\mathcal{F}_{t-1} \rightsquigarrow \mathcal{ZIP}\left(\varrho_t,\lambda_t\right), \\ \lambda_t = \alpha_{0,t} + \alpha_{1,t} y_{t-1} + \beta_t \lambda_{t-1}, \end{array} \tag{3.3}$$

where, the parameters ϱ_t, $\alpha_{i,t}$, $i = 0, 1$, and β_t are periodic in t, with period S, i.e., $\varrho_{i,t+vS} = \varrho_{i,t}$, $\alpha_{i,t+vS} = \alpha_{i,t}$, $i = 0, 1$, and $\beta_{t+vS} = \beta_t$, $t, v \in \mathbb{Z}$. Moreover, these parameters are such that : $\alpha_{0,t} > 0$, $\alpha_{1,t} \geq 0$, and $\beta_t \geq 0$, $t \in \mathbb{Z}$. The conditional mean and conditional variance of y_t are given, respectively, by

$$\mathbb{E}\left(y_t|\,\mathcal{F}_{t-1}\right) = \lambda_t\left(1 - \varrho_t\right), \, Var\left(y_t|\,\mathcal{F}_{t-1}\right) = \lambda_t\left(1 - \varrho_t\right)\left(1 + \lambda_t \varrho_t\right).$$

Letting $t = s + \tau S$, $s = 1, ..., S$ and $\tau \in \mathbb{Z}$, the last model can be rewritten in the equivalent form

$$\begin{array}{c} y_{s+\tau S}|\,\mathcal{F}_{s-1+\tau S} \rightsquigarrow \mathcal{ZIP}\left(\varrho_s,\lambda_{s+\tau S}\right), \\ \lambda_{s+\tau S} = \alpha_{0,s} + \alpha_{1,s} y_{s-1+\tau S} + \beta_s \lambda_{s-1+\tau S}. \end{array} \tag{3.4}$$

This model extends the following time-invariant ZIP-INGARCH$(1, 1)$ model, studied by Zhu (2012), to the time periodic case :

$$\begin{array}{c} y_t|\,\mathcal{F}_{t-1} \rightsquigarrow \mathcal{ZIP}\left(\varrho,\lambda_t\right) \\ \lambda_t = \alpha_0 + \alpha_1 y_{t-1} + \beta \lambda_{t-1}. \end{array} \tag{3.5}$$

Note that if $\varrho_s = 0$, $s = 1, ..., S$, the zero inflated Poisson distribution becomes the Poisson distribution so that the periodic ZIP-INGARCH$(1, 1)$ model turns into the periodic INGARCH$(1, 1)$ model introduced by Bentarzi and Bentarzi (2017).

4. HIGHER-ORDER MOMENTS: EXISTENCE AND CALCULATION

In this section, we aim to identify the conditions under which the m-th order moment $\mathbb{E}\left(y_t^m\right)$ exists and to derive its explicit formula in terms of the model parameters. Specifically, we present the periodic stationarity conditions for the first and second moments and derive their closed-form expressions. Also, we calculate the skewness and kurtosis coefficients using specific cases such as $\mathbb{E}\left(y_t^4\right), \mathbb{E}\left(y_t^3\right), \mathbb{E}\left(y_t^2\right)$ and $\mathbb{E}\left(y_t\right)$.

4.1. Higher-order moment calculation of λ_t and y_t

In this paragraph, we aim to determine the condition for the existence the m-th order moment $\mathbb{E}\left(y_t^m\right)$, for the model satisfying (3.3). Furthermore, under this condition, we are able to derive the explicit expressions for $\mathbb{E}\left(\lambda_t^m\right)$ and $\mathbb{E}\left(y_t^m\right)$. To present the main result, we need to introduce three m-column vectors with the following definitions, $\underline{\Lambda}_t^{(m)} = \left(\lambda_t^m, \lambda_t^{m-1}, ..., \lambda_t\right)'$, $\underline{\alpha}_{0,t}^{(m)} = \left(\alpha_{0,t}^m, \alpha_{0,t}^{m-1}, ..., \alpha_{0,t}\right)'$, $\underline{\mu}_{y,t}^{(m)} = \left(\mathbb{E}\left(y_t^m\right), \mathbb{E}\left(y_t^{m-1}\right), ..., \mathbb{E}\left(y_t\right)\right)'$, and two squared $m \times m$ matrices $\Theta_t^{(m)}$ and $\Omega_t^{(m)}$ with elements given, respectively, for $i, j = 1, ..., m$, by

$$\Theta_t^{(m)} = \begin{cases} \psi_{m-i+1,t} & \text{if } i = j, \\ \phi_{m-j+1,t}^{(m-i+1)} & \text{if } i < j, \\ 0 & \text{otherwise.} \end{cases}, \quad \Omega_t^{(m)} = \begin{cases} \omega_{m-i+1,t}^{(m-i+1)} & \text{if } i = j, \\ \omega_{m-j+1,t}^{(m-i+1)} & \text{if } i < j, \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

with,

$$\begin{aligned} \psi_{m,i} &= \left(1 - \varrho_{i-1}\right) \sum\nolimits_{j=1}^{m} \tbinom{m}{j} \alpha_{1,i}^j \beta_i^{m-j} + \beta_i^m, \quad \omega_{k,t}^{(m)} = \left(1 - \varrho_t\right) \left\{ \begin{matrix} m \\ k \end{matrix} \right\}, \\ \phi_{i,t}^{(m)} &= \tbinom{m}{i} \alpha_{0,t}^{m-i} \psi_{i,t} + \sum\nolimits_{j=i+1}^{m} \sum\nolimits_{k=j-i}^{j-1} \mathcal{K}_{j,k+1,j-i}^{(m,t)}, \end{aligned} \tag{4.2}$$

and,

$$\mathcal{K}_{i,j,k}^{(m,t)} = \tbinom{m}{i} \tbinom{i}{j} \left\{ \begin{matrix} j-1 \\ k \end{matrix} \right\} \left(1 - \varrho_{t-1}\right) \alpha_{0,t}^{m-i} \alpha_{1,t}^j \beta_t^{i-j}.$$

Proposition 4.1. *The unconditional m-th order moment $E\left(\lambda_t^m\right)$, exists and is finite if and only if :*

$$\prod\nolimits_{i=1}^{S} \psi_{m,i} < 1, \tag{4.3}$$

where $\psi_{m,i}$ is given in (4.2). Then, the closed form of the unconditional vector m-th order moment $E\left(\underline{\Lambda}_t^{(m)}\right)$ is, under this condition, given by :

$$\mathbb{E}\left(\underline{\Lambda}_{s+\tau S}^{(m)}\right) = \left(I_m - \Psi_{s,S}^{(m)}\right)^{-1} \sum\nolimits_{i=1}^{S} \Psi_{s,i-1}^{(m)} \underline{\alpha}_{0,s-i+1}^{(m)}, \; s = 1, ..., S,$$

where $\Psi_{s,j}^{(m)} = \prod_{i=1}^{j} \Theta_{s-i+1}^{(m)}$, and the elements of the $m \times m$ matrix $\Theta_s^{(m)}$ are given in (4.1).

Proof of Proposition 4.1: see Appendix □

Remark 4.1. For any periodic parameter χ with a negative index, it can be equivalently redefined using its corresponding positive index as follows:

$$\chi_{-i} = \chi_{S-i}, \; i = 0, 1, ..., S-1.$$

In the case of time-invariant model (3.5), i.e. $S = 1$, the results of **Proposition** 4.1 can be presented by the following corollary. First, we need also to introduce these notations

$$\Theta^{(m)} = \begin{cases} \psi_{m-i+1} & \text{if } i = j, \\ \phi_{m-i+1,m-j+1} & \text{if } i < j, \\ 0 & \text{otherwise.} \end{cases}, \quad \Omega^{(m)} = \begin{cases} \omega_{m-i+1,m-i+1} & \text{if } i = j, \\ \omega_{m-i+1,m-j+1} & \text{if } i < j, \\ 0 & \text{otherwise.} \end{cases}, \tag{4.4}$$

with,

$$\begin{aligned} \psi_m &= (1-\varrho)\sum\nolimits_{j=1}^{m} \tbinom{m}{j}\alpha_1^j\beta^{m-j} + \beta^m, \quad \omega_{m,k} = (1-\varrho)\left\{ \begin{matrix} m \\ k \end{matrix} \right\}, \\ \phi_{i,m} &= \tbinom{m}{i}\alpha_0^{m-i}\psi_i + \sum\nolimits_{j=i+1}^{m}\sum\nolimits_{k=j-i}^{j-1} \mathcal{K}_{j,k+1,j-i}^{(m)}, \end{aligned} \tag{4.5}$$

and,

$$\mathcal{K}_{i,j,k}^{(m)} = \tbinom{m}{i}\tbinom{i}{j}\left\{ \begin{matrix} j \\ j-k \end{matrix} \right\}(1-\varrho)\,\alpha_0^{m-i}\alpha_1^j\beta^{i-j}.$$

Corollary 4.1. *The unconditional m-th order moment $E\left(\lambda_t^m\right)$ exists and is finite, if and only if*

$$\psi_m < 1, \tag{4.6}$$

with ψ_m given in (4.5). Then, the unconditional vector moment $E\left(\underline{\Lambda}_t^{(m)}\right)$ is, under this condition, given by :

$$\mathbb{E}\left(\underline{\Lambda}_t^{(m)}\right) = \left(I_m - \Theta^{(m)}\right)^{-1}\underline{\alpha}_0^{(m)},$$

where the elements of the matrix $\Theta^{(m)}$ are given in (4.4).

Once the unconditional vector moments $\mathbb{E}\left(\underline{\Lambda}_t^{(m)}\right)$ have been calculated, the following lemma gives the unconditional vector moments $\underline{\mu}_{y,s}^{(m)}$.

Lemma 4.1. *The unconditional vector moment $\underline{\mu}_{y,s}^{(m)}$ of the process $\{y_t;\ t \in \mathbb{Z}\}$ is, under condition (4.3), given in terms of the unconditional vector moment $E\left(\underline{\Lambda}_s^{(m)}\right)$, by the vector form below :*

$$\underline{\mu}_{y,s}^{(m)} = \Omega_s^{(m)}\mathbb{E}\left(\underline{\Lambda}_s^{(m)}\right),$$

where the elements of the $m \times m$ matrix $\Omega_s^{(m)}$ are given in (4.1).

Proof: The proof is straightforward. □

Corollary 4.2. *The unconditional vector moment $\underline{\mu}_{y,s}^{(m)}$ of the periodically correlated process $\{y_t;\ t \in \mathbb{Z}\}$ is, under the condition (4.3), given by*

$$\underline{\mu}_{y,s}^{(m)} = \Omega_s^{(m)}\left(I - \Psi_{s,S}^{(m)}\right)^{-1}\sum\nolimits_{i=1}^{S}\Psi_{s,i-1}^{(m)}\underline{\alpha}_{0,s-i+1}^{(m)},$$

where, $\Psi_{s,j}^{(m)} = \prod_{i=1}^{j}\Theta_{s-i+1}^{(m)}$ and the $m \times m$ matrices $\Theta_s^{(m)}$ and $\Omega_s^{(m)}$ are defined in (4.1).

Proof: The proof is straightforward. □

In the case of time-invariant model (3.5), i.e. $S = 1$, the results of Lemma 4.1 can be presented by the following corollary.

Corollary 4.3. *The unconditional vector moment $\underline{\mu}_y^{(m)}$ of the process $\{y_t;\ t \in \mathbb{Z}\}$, satisfying the model (3.5), is, under the condition (4.6), given by*

$$\underline{\mu}_y^{(m)} = \Omega^{(m)} \mathbb{E}\left(\underline{\Lambda}_t^{(m)}\right),$$

where the elements of the $m \times m$ matrix $\Omega^{(m)}$ are given in (4.4).

4.2. Periodic stationarity conditions

This paragraph is devoted to present the explicit expressions for the periodic mean and variance, with respect to the first and second-order moments, which are special cases of Proposition 4.1.

Proposition 4.2. *The periodically correlated non negative integer-valued process $\{y_t;\ t \in \mathbb{Z}\}$, satisfying the model (3.3), is periodically stationary in mean, if and only if,*

$$\prod_{i=1}^{S} \left((1 - \varrho_{i-1})\, \alpha_{1,i} + \beta_i \right) < 1. \tag{4.7}$$

Then, the periodic mean $\mu_{y,s} = E(y_s) = (1 - \varrho_s) E(\lambda_s)$, $s = 1, ..., S$, is, under this condition, provided by:

$$\mu_{y,s} = \frac{1 - \varrho_s}{1 - \prod_{i=1}^{S} \left(\alpha_{1,i} (1 - \varrho_{i-1}) + \beta_i \right)} \sum_{j=1}^{S} \left(\prod_{i=1}^{j-1} \left(\alpha_{1,s-i+1} + \beta_{s-i+1} \right) \right) \alpha_{0,s-j+1},$$

with the convention $\prod_{i=1}^{j} x_i = 1$ if $j < 1$.

Proof of Proposition 4.2: see Appendix □

In what follows, we suppose that the process $\{y_t;\ t \in \mathbb{Z}\}$ follows a PZIP-INARCH(1) model (i.e., $q = 0$), then the corollary below provides the mean periodic stationarity condition.

Corollary 4.4. *The periodically correlated non negative integer-valued process $\{y_t;\ t \in \mathbb{Z}\}$, satisfying the periodic ZIP-INARCH(1) model, is periodiccally stationary in mean, if and only if :*

$$\prod_{i=1}^{S} (1 - \varrho_{i-1})\, \alpha_{1,i} < 1. \tag{4.8}$$

Then, under this condition, the closed form of the mean $E(y_s) = \mu_{y,s}$, $s = 1, ..., S$, is given by :

$$\mu_{y,s} = \frac{1 - \varrho_s}{1 - \prod_{i=1}^{S} \alpha_{1,i} (1 - \varrho_{i-1})} \sum_{j=1}^{S} \left(\prod_{i=1}^{j-1} \alpha_{1,s-i+1} (1 - \varrho_{s-i}) \right) \alpha_{0,s-j+1}.$$

For the time-invariant model (3.5), i.e. $S = 1$, the results of **Proposition** 4.2 can be presented by the following corollary.

Corollary 4.5. *The non negative integer-valued process $\{y_t;\ t \in \mathbb{Z}\}$, satisfying the ZIP-INGARCH$(1,1)$ model (3.5), is stationary in mean, if and only if :*

$$(1-\varrho)\,\alpha_1 + \beta < 1.$$

Then, the closed forms of the mean $E(y_t) = \mu_y$ of such process is, under this condition given by :

$$\mu_y = (1 - (\alpha_1 (1-\varrho) + \beta))^{-1}\, \alpha_0 (1-\varrho)\,.$$

In the following proposition, we provide the closed-form expression for the periodic variance under the periodic stationarity condition with respect to the second order moment for the process $\{y_t;\ t \in \mathbb{Z}\}$ satisfying (3.3).

Proposition 4.3. *The periodically correlated non negative integer-valued process $\{y_t;\ t \in \mathbb{Z}\}$ satisfying the model (3.3) is periodically stationary in second order, if and only if,*

$$\prod_{i=1}^{S} \left((1-\varrho_{i-1})\,(\alpha_{1,i} + \beta_i)^2 + \varrho_{i-1}\beta_i^2 \right) < 1. \tag{4.9}$$

Then, under this condition, the closed-form of the variance $Var(y_s) = \gamma_y^{(s)}(0)$, $s = 1, ..., S$, and the variance $Var(\lambda_s) = \gamma_\lambda^{(s)}(0)$ are given by :

$$\gamma_\lambda^{(s)}(0) = \frac{1}{1 - \left(\prod_{i=1}^{S} \psi_{2,i}\right)} \sum_{j=1}^{S} \left(\prod_{i=1}^{j-1} \psi_{2,s-i+1}\right) F_{s-j+1},$$

$$\gamma_y^{(s)}(0) = \frac{1-\varrho_t}{1 - \left(\prod_{i=1}^{S} \psi_{2,i}\right)} \sum_{j=1}^{S} \left(\prod_{i=1}^{j-1} \psi_{2,s-i+1}\right) F_{s-j+1} + \mu_{y,s} + \frac{\varrho_t}{1-\varrho_t}\mu_{y,s}^2.$$

*with $\psi_{2,t} = (1-\varrho_{t-1})\,(\alpha_{1,t} + \beta_t)^2 + \varrho_{t-1}\beta_t^2$, $F_s = \alpha_{1,s}^2 (1-\varrho_{s-1})\,(1 + \varrho_{s-1}\mu_{\lambda,s-1})\,\mu_{\lambda,s-1}$ and $\mu_{y,s} = (1-\varrho_s)\,\mu_{\lambda,s}$ is given in **Proposition** 4.2, with the convention $\prod_{i=1}^{j} x_i = 1$ if $j < 1$.*

Proof of Proposition 4.3: see Appendix □

In particular, we suppose that the process $\{y_t;\ t \in \mathbb{Z}\}$ follows a PZIP-INARCH(1) model (i.e., $q = 0$). The next corollary gives the periodic stationarity in second order.

Corollary 4.6. *The periodically correlated non negative integer-valued process $\{y_t;\ t \in \mathbb{Z}\}$, satisfying the periodic ZIP-INARCH(1) model, is periodically stationary in second order, if and only if*

$$\prod_{i=1}^{S} (1-\varrho_{i-1})\,\alpha_{1,i}^2 < 1. \tag{4.10}$$

Then, the closed-form of the variance $Var(y_s) = \gamma_y^{(s)}(0)$, $s = 1, ..., S$, of such process and the variance $Var(\lambda_s) = \gamma_\lambda^{(s)}(0)$ are, under this condition, given by :

$$\gamma_\lambda^{(s)}(0) = \frac{1}{1 - \left(\prod_{i=1}^{S} \Lambda_{2,i}\right)} \sum_{j=1}^{S} \left(\prod_{i=1}^{j-1} \Lambda_{2,s-i+1}\right) F_{s-j+1},$$

$$\gamma_y^{(s)}(0) = \frac{1-\varrho_s}{1 - \left(\prod_{i=1}^{S} \Lambda_{2,i}\right)} \sum_{j=1}^{S} \left(\prod_{i=1}^{j-1} \Lambda_{2,s-i+1}\right) F_{s-j+1} + \mu_{y,s} + \frac{\varrho_t}{1-\varrho_t}\mu_{y,s}^2,$$

where $\Lambda_{2,s} = \alpha_{1,s}^2 (1-\varrho_{s-1})$, $\digamma_s = \alpha_{1,s}^2 (1-\varrho_{s-1})(1+\varrho_{s-1}\mu_{\lambda,s-1})\mu_{\lambda,s-1}$ and $\mu_{y,s} = (1-\varrho_s)\mu_{\lambda,s}$ *is given in* **Corollary** *4.4, with the convention* $\prod_{i=1}^{j} x_i = 1$ *if* $j<1$.

In the case of time invariant model (3.5), i.e. $S=1$, the results of **Proposition** 4.3 can be presented by the following corollary.

Corollary 4.7. *The non negative integer-valued process* $\{y_t;\ t\in\mathbb{Z}\}$, *satisfying the ZIP-INGARCH*$(1,1)$ *model (3.5), is stationary in second order, if and only if,*

$$(1-\varrho)(\alpha_1+\beta)^2 + \varrho\beta^2 < 1.$$

Then, under this condition, the closed-form of the $Var(y_t) = \gamma_y(0)$ *the variance* $Var(\lambda_t) = \gamma_\lambda(0)$ *are given by*

$$\gamma_\lambda(0) = \frac{\alpha_1^2(1-\varrho)(1+\varrho\mu_\lambda)\mu_\lambda}{1-\left[(1-\varrho)(\alpha_1+\beta)^2+\varrho\beta^2\right]}, \quad \gamma_y(0) = \frac{(1-\varrho)\left(1-\beta^2-2(1-\varrho)\alpha_1\beta\right)(1+\varrho\mu_\lambda)\mu_\lambda}{1-\left[(1-\varrho)(\alpha_1+\beta)^2+\varrho\beta^2\right]}.$$

4.3. Skewness and kurtosis coefficients

In this paragraph, we present the results of calculating the first four moments of the processes $\{\lambda_t;\ t\in\mathbb{Z}\}$ and $\{y_t;\ t\in\mathbb{Z}\}$. These results are necessary to establish a corollary that provides the skewness and kurtosis coefficients.

Corollary 4.8. *The first four unconditional moments exist and their closed-forms are, under the condition (4.3) for* $m=4$, *given by*

$$\mathbb{E}\left(\underline{\Lambda}_{s+\tau S}^{(4)}\right) = \left(I-\Psi_{s,S}^{(4)}\right)^{-1}\sum_{i=1}^{S}\Psi_{s,i-1}^{(4)}\underline{\alpha}_{0,s-i+1}^{(4)},$$

where, $\underline{\alpha}_{0,t}^{(4)} = \left(\alpha_{0,t}^4, \alpha_{0,t}^3, \alpha_{0,t}^2, \alpha_{0,t}\right)'$, $\Psi_{s,i}^{(4)} = \prod_{i=1}^{j}\Theta_{s-i+1}^{(4)}$ *and the periodic* 4×4 *matrix* $\Theta_s^{(4)}$, *for* $s=1,\ldots,S$, *and* $i,j=1,\ldots,4$, *is given by*

$$\Theta_s^{(4)} = \begin{cases} \psi_{4-i+1,s} & \text{if } i=j, \\ \alpha_s^{(i,j)} & \text{if } i<j, \\ 0 & \text{otherwise.} \end{cases}$$

where,

$$\begin{aligned}
\psi_{1,s} &= \beta_s + \alpha_{1,s}(1-\varrho_{s-1}), \quad \psi_{2,s} = \beta_s^2 + \left(2\alpha_{1,s}\beta_s + \alpha_{1,s}^2\right)(1-\varrho_{s-1}), \\
\psi_{3,s} &= \beta_s^3 + \left(3\alpha_{1,s}\beta_s^2 + 3\alpha_{1,s}^2\beta_s + \alpha_{1,s}^3\right)(1-\varrho_{s-1}), \\
\psi_{4,s} &= \beta_s^4 + \left(6\beta_s^2\alpha_{1,s}^2 + 4\beta_s\alpha_{1,s}^3 + \alpha_{1,s}^4 + 4\alpha_{1,s}\beta_s^3\right)(1-\varrho_{s-1}),
\end{aligned}$$

and,

$$\begin{aligned}
\alpha_s^{(1,2)} &= 4\alpha_{0,s}\beta_s^3 + \left(4\alpha_{0,s}\alpha_{1,s}^3 + 12\left(\alpha_{0,s}\alpha_{1,s}\beta_s^2 + \alpha_{1,s}^3\beta_s + \alpha_{0,s}\alpha_{1,s}^2\beta_s\right) + 6\left(\alpha_{1,s}^2\beta_s^2 + \alpha_{1,s}^4\right)\right)(1-\varrho_{s-1}), \\
\alpha_s^{(1,3)} &= 6\alpha_{0,s}^2\beta_s^2 + \left(4\alpha_{1,s}^3\beta_s + 12\left(\alpha_{0,s}\alpha_{1,s}^2\beta_s + \alpha_{0,s}^2\alpha_{1,s}\beta_s + \alpha_{0,s}\alpha_{1,s}^3\right) + 7\alpha_{1,s}^4 + 6\alpha_{0,s}^2\alpha_{1,s}^2\right)(1-\varrho_{s-1}), \\
\alpha_s^{(1,4)} &= 4\alpha_{0,s}^3\beta_s + \left(\alpha_{1,s}^4 + 4\left(\alpha_{0,s}\alpha_{1,s}^3 + \alpha_{0,s}^3\alpha_{1,s}\right) + 6\alpha_{0,s}^2\alpha_{1,s}^2\right)(1-\varrho_{s-1}), \\
\alpha_s^{(2,3)} &= 3\alpha_{0,s}\beta_s^2 + \left(3\alpha_{1,s}^3 + 3\left(\alpha_{1,s}^2\beta_s + \alpha_{0,s}\alpha_{1,s}^2\right) + 6\alpha_{0,s}\alpha_{1,s}\beta_s\right)(1-\varrho_{s-1}), \\
\alpha_s^{(2,4)} &= 3\alpha_{0,s}^2\beta_s + \left(\alpha_{1,s}^3 + 3\alpha_{0,s}\alpha_{1,s}^2 + 3\alpha_{0,s}^2\alpha_{1,s}\right)(1-\varrho_{s-1}), \\
\alpha_s^{(3,4)} &= 2\alpha_{0,s}\beta_s + \left(\alpha_{1,s}^2 + 2\alpha_{0,s}\alpha_{1,s}\right)(1-\varrho_{s-1}).
\end{aligned}$$

Corollary 4.9. *The first four unconditional moments of the periodically correlated process $\{y_t;\ t \in \mathbb{Z}\}$, satisfying the periodic PZIP-INGARCH$(1,1)$ model (3.3) are, under the condition (4.3) for $m = 4$, given by*

$$\underline{\mu}_{y,s}^{(4)} = \Omega_s^{(4)} \left(I - \Psi_{s,S}^{(4)} \right)^{-1} \sum_{i=1}^{S} \Psi_{s,i-1}^{(4)} \underline{\alpha}_{0,s-i+1}^{(4)},$$

where, $\Psi_{s,j}^{(4)} = \prod_{i=1}^{j} \Theta_{s-i+1}^{(4)}$ and the matrix $\Omega_s^{(4)}$ is given by

$$\Omega_s^{(4)} = \begin{pmatrix} (1-\varrho_s) & 6(1-\varrho_s) & 7(1-\varrho_s) & (1-\varrho_s) \\ 0 & (1-\varrho_s) & 3(1-\varrho_s) & (1-\varrho_s) \\ 0 & 0 & (1-\varrho_s) & (1-\varrho_s) \\ 0 & 0 & 0 & (1-\varrho_s) \end{pmatrix},$$

and the matrix $\Theta_s^{(4)}$ is given by (4.4).

Proof: The proof is straightforward. □

Corollary 4.10. *The skewness and the kurtosis coefficients of the periodically correlated process $\{y_t;\ t \in \mathbb{Z}\}$, satisfying the periodic ZIP-INGARCH$(1,1)$ model (3.3) are, under the condition (4.3) for $m = 4$, given, for $s = 1, ..., S$, by :*

$$\mathcal{K}ur_s = \mu_{y,s}^{*(4)} \Big/ \left(\mu_{y,s}^{*(2)} \right)^2 = \left(\mu_{y,s}^{(4)} - 4\mu_{y,s}\mu_{y,s}^{(3)} + 6\mu_{y,s}^2 \mu_{y,s}^{(2)} - 3\mu_{y,s}^4 \right) \Big/ \left(\mu_{y,s}^{(2)} - \mu_{y,s}^2 \right)^2,$$

and

$$\mathcal{S}k_s = \mu_{y,s}^{*(3)} \Big/ \left(\mu_{y,s}^{*(2)} \right)^{3/2} = \left(\mu_{y,s}^{(3)} - 3\mu_{y,s}\mu_{y,s}^{(2)} + 2\mu_{y,s}^3 \right) \Big/ \left(\mu_{y,s}^{(2)} - \mu_{y,s}^2 \right)^{3/2},$$

where, $\mu_{y,s}^{(4)}$, $\mu_{y,s}^{(3)}$, $\mu_{y,s}^{(2)}$, and $\mu_{y,s}$ are given, in terms of the parameters of the model, by ***Corollary*** *4.9.*

5. AUTOCOVARIANCE STRUCTURE

The following proposition establishes the autocovariance structure of the process $\{y_t;\ t \in \mathbb{Z}\}$ satisfying the periodic ZIP-INGARCH$(1,1)$ model.

Proposition 5.1. *The autocovariance structure of the periodically correlated integer-valued processes $\{y_t;\ t \in \mathbb{Z}\}$ and $\{\lambda_t;\ t \in \mathbb{Z}\}$ satisfying the model (3.3) are, under the conditions (4.7) and (4.9), given as follows :*

$$\gamma_y^{(s)}(0) = \frac{1-\varrho_s}{1-\left(\prod_{i=1}^{S}\psi_{2,i}\right)} \sum_{j=1}^{S} \left(\prod_{i=1}^{j-1} \psi_{2,s-i+1} \right) \digamma_{s-j+1} + \frac{\varrho_t}{1-\varrho_t} \mu_{y,s}^2 + \mu_{y,s},$$

$$\gamma_y^{(s)}(h) = \left(\prod_{i=1}^{h-1} \psi_{1,s-i+1} \right) (1-\varrho_s) \left((\alpha_{1,s-h+1} + \beta_{s-h+1}) \gamma_y^{(s-h)}(0) - \beta_{s-h+1} \left(1 + \tfrac{\varrho_{s-h}}{1-\varrho_{s-h}} \mu_{y,s-h} \right) \mu_{y,s-h} \right),$$

$$\gamma_\lambda^{(s)}(0) = \frac{1}{1-\left(\prod_{i=1}^{S}\psi_{2,i}\right)} \sum_{j=1}^{S} \left(\prod_{i=1}^{j-1} \psi_{2,s-i+1} \right) \digamma_{s-j+1}, \quad \gamma_\lambda^{(s)}(h) = \left(\prod_{i=1}^{h} \psi_{1,s-i+1} \right) \gamma_\lambda^{(s-h)}(0),$$

where, $\psi_{1,s} = (1-\varrho_{s-1})\alpha_{1,s} + \beta_s$, $\psi_{2,s} = (1-\varrho_{s-1})(\alpha_{1,s}+\beta_s)^2 + \varrho_{s-1}\beta_s^2$, $\mu_{y,s} = (1-\varrho_s)\mu_{\lambda,s}$ and $\digamma_s = \alpha_{1,s}^2 (1-\varrho_{s-1})(1+\varrho_{s-1}\mu_{\lambda,s-1})\mu_{\lambda,s-1}$ with the convention $\prod_{i=1}^{j} x_i = 1$ if $j < 1$.

Proof of Proposition 5.1: see Appendix □

Corollary 5.1. *The autocorrelation functions of the periodically correlated integer-valued processes $\{y_t;\ t\in\mathbb{Z}\}$ and $\{\lambda_t;\ t\in\mathbb{Z}\}$ satisfying the model (3.3) are, under the condition (4.7), given, for $\nu=1,...,S$ and $k\in\mathbb{N}$, as follows :*

$$\rho_y^{(s)}(\nu+kS)=\sqrt{\gamma_y^{(s-\nu)}(0)\Big/\gamma_y^{(s)}(0)}\left(\prod_{i=1}^{S}\psi_{1,i}\right)^k\left(\prod_{i=1}^{\nu}\psi_{1,s-i+1}\right)\times$$
$$\left((1-\varrho_s)\left((\alpha_{1,s-\nu+1}+\beta_{s-\nu+1})-\left(1+\frac{\varrho_{s-\nu}\mu_{y,s-\nu}}{1-\varrho_{s-\nu}}\right)\beta_{s-\nu+1}\mu_{y,s-\nu}\Big/\gamma_y^{(s-\nu)}(0)\right)\right),$$

$$\rho_\lambda^{(s)}(\nu+kS)=\sqrt{\gamma_\lambda^{(s-\nu)}(0)\Big/\gamma_\lambda^{(s)}(0)}\left(\prod_{i=1}^{S}\psi_{1,i}\right)^k\left(\prod_{i=1}^{\nu}\psi_{1,s-i+1}\right).$$

In the following, we will consider a special cases of **Proposition** 5.1. Suppose that the process $\{y_t;\ t\in\mathbb{Z}\}$ follows a PZIP-INARCH(1) model, (i.e., $q=0$), then the following corollary gives the autocovariance structure.

Corollary 5.2. *The autocovariance structure of the periodically correlated integer-valued processes $\{y_t;\ t\in\mathbb{Z}\}$ and $\{\lambda_t;\ t\in\mathbb{Z}\}$ satisfying the PZIP-INARCH(1) are, the conditions (4.8) and (4.10), given as follows :*

$$\gamma_y^{(s)}(0)=\frac{1-\varrho_s}{1-\left(\prod_{i=1}^{S}\Lambda_{2,i}\right)}\sum_{j=1}^{S}\left(\prod_{i=1}^{j-1}\Lambda_{2,s-i+1}\delta_s\right)F_{s-j+1}+\mu_{y,s}+\frac{\varrho_t}{1-\varrho_t}\mu_{y,s}^2,$$
$$\gamma_\lambda^{(s)}(0)=\frac{1}{1-\left(\prod_{i=1}^{S}\Lambda_{2,i}\right)}\sum_{j=1}^{S}\left(\prod_{i=1}^{j-1}\Lambda_{2,s-i+1}\right)F_{s-j+1},$$
$$\gamma_\lambda^{(s)}(h)=\left(\prod_{i=1}^{h}\Lambda_{1,s-i+1}\right)\gamma_\lambda^{(s-h)}(0),\quad \gamma_y^{(s)}(h)=\left(\prod_{i=1}^{h}\Lambda_{1,s-i+1}\right)\gamma_y^{(s-h)}(0),$$

with $\Lambda_{i,s}=\alpha_{1,s}^{i}(1-\varrho_{s-1}),\ i=1,2.$

Corollary 5.3. *The autocorrelation functions of the periodically correlated integer-valued processes $\{y_t;\ t\in\mathbb{Z}\}$ and $\{\lambda_t;\ t\in\mathbb{Z}\}$ satisfying the PZIP-INARCH(1) model are, under the condition (4.8), given, for $\nu=1,...,S$ and $k\in\mathbb{N}$, as follows :*

$$\rho_y^{(s)}(\nu+kS)=\left(\prod_{i=1}^{S}\Lambda_{1,i}\right)^k\left(\prod_{i=1}^{\nu}\Lambda_{1,s-i+1}\right)\sqrt{\gamma_\lambda^{(s-\nu)}(0)\Big/\gamma_\lambda^{(s)}(0)}.$$

$$\rho_\lambda^{(s)}(\nu+kS)=\left(\prod_{i=1}^{S}\Lambda_{1,i}\right)^k\left(\prod_{i=1}^{\nu}\Lambda_{1,s-i+1}\right)\sqrt{\gamma_\lambda^{(s-\nu)}(0)\Big/\gamma_\lambda^{(s)}(0)}.$$

with $\Lambda_{1,s}=\alpha_{1,s}(1-\varrho_{s-1})$

6. MAXIMUM LIKELIHOOD ESTIMATION

This section discusses the parameter estimation issues of the PZIP-INGARCH(1, 1) model presented in (3.3), using the conditional maximum likelihood (CML) approach. Indeed, the estimation of parameters is straightforward, achieved by maximizing the conditional likelihood function via the EM algorithm introduced by Dempster et al. (1977). In the E-step, the conditional expectation of missing data is computed, while in the M-step, the parameters are obtained by maximizing the log-likelihood function. Suppose that we have an observation $\underline{y} = (y_1, y_2, ..., y_n)$ generated according to model (3.3). Now suppose we have the ability to distinguish between zeros generated by the generate zero process and zeros generated by the Poisson process. In other words, suppose we can observe $Z_t = 1$ when y_t is from the generate zero process and $Z_t = 0$ when y_t is from the Poisson process. The distribution of Z_t is given by $P(Z_t = 1) = 1 - P(Z_t = 0) = \varrho_t$. Let $\underline{Z} = (Z_1, Z_2, ..., Z_n)$ denote the vector containing the unobserved data, and $\underline{\theta}_t = (\alpha_{0,t}, \alpha_{1,t}, \beta_t)$, $\underline{\varphi}_t = (\varrho_t, \underline{\theta}_t)$. The distribution of $\underline{Z}|\,\underline{\varphi}_t$ is given by

$$P\left(\underline{Z}|\,\underline{\varphi}_t\right) = \varrho_t^{Z_t}\left(1-\varrho_t\right)^{1-Z_t},$$

and the distribution of $\underline{y}|\,\underline{Z}, \underline{\varphi}_t$ is

$$P\left(\underline{y}|\,\underline{Z}, \underline{\varphi}_t\right) = \prod_{t=2}^{n}\left(\frac{\lambda_t^{y_t}\exp\left(-\lambda_t\right)}{y_t!}\right)^{1-Z_t}.$$

Then, the complete data conditional likelihood function of the parameter vector $\underline{\varphi}_t$ can be expressed as follows

$$L\left(\underline{y}; \underline{\varphi}_t\right) = \prod_{t=2}^{n}\varrho_t^{Z_t}\left(\left(1-\varrho_t\right)\frac{\lambda_t^{y_t}\exp\left(-\lambda_t\right)}{y_t!}\right)^{1-Z_t},$$

then, the complete data conditional log-likelihood is given by

$$\mathcal{L}\left(\underline{y}; \underline{\varphi}_t\right) = \sum_{t=2}^{n}\left[Z_t\log\left(\varrho_t\right) + \left(1-Z_t\right)\left(\log\left(1-\varrho_t\right) + y_t\log\left(\lambda_t\right) - \lambda_t - \log\left(y_t!\right)\right)\right]. \tag{6.1}$$

Letting, for simplicity of notation, the size of the observation be a multiple of S, i.e., $n = NS$, $N \in \mathbb{N}^*$, and replacing $t = s + \tau S$, with $s = 1, ..., S$ and $\tau = 0, 1, ..., N-1$, then the complete data conditional log-likelihood function can be written in the form :

$$\begin{aligned}\mathcal{L}\left(\underline{y}; \underline{\varphi}_s\right) = &\sum_{\tau=0}^{N-1}\sum_{s=1}^{S}\left[Z_{s+\tau S}\log\left(\varrho_s\right) + \left(1-Z_{s+\tau S}\right)\left(\log\left(1-\varrho_s\right)\right) + \right.\\ &\left.\left(1-Z_{s+\tau S}\right)\left(y_{s+\tau S}\log\left(\lambda_{s+\tau S}\right) - \lambda_{s+\tau S} - \log\left(y_{s+\tau S}!\right)\right)\right].\end{aligned}$$

The first derivatives of the conditional log-likelihood with respect to ϱ_s and $\theta_{i,s}$, $s = 1, 2, ..., S$, $i = 1, 2, 3$, are given as follows:

$$\frac{\partial\mathcal{L}}{\partial\varrho_s} = \sum_{\tau=0}^{N-1}\left(\frac{Z_{s+\tau S}}{\varrho_s} - \frac{1-Z_{s+\tau S}}{1-\varrho_s}\right),$$

$$\frac{\partial\mathcal{L}}{\partial\theta_{i,s}} = \sum_{\tau=0}^{N-1}\left(1-Z_{s+\tau S}\right)\left(\frac{y_{s+\tau S}}{\lambda_{s+\tau S}} - 1\right)\frac{\partial\lambda_{s+\tau S}}{\partial\theta_{i,s}},\quad i = 1, 2, 3.$$

To estimate the parameters by maximizing the log likelihood function, an iterative EM algorithm is employed. It consists of an E-step and an M-step described as follows:

E-step: The parameters $\underline{\varphi}_s$ in the i-th iteration are assumed to be known. The missing data $Z_{s+\tau S}$ are replaced by their conditional expectations on the parameters $\underline{\varphi}_s$ and on the observed data $\underline{y}$, which are denoted by $\pi_{s+\tau S}$, hence

$$\pi_{s+\tau S} = \begin{cases}\dfrac{\varrho_s}{\varrho_s + \left(1-\varrho_s\right)\exp\left(-\lambda_{s+\tau S}\right)} & \text{if } y_{s+\tau S} = 0,\\ 0 & \text{otherwise.}\end{cases}$$

M-step: The missing data $Z_{s+\tau S}$ are assumed now to be known. To estimate the parameters, we have to maximize the log-likelihood function by setting derivatives $\dfrac{\partial \mathcal{L}}{\partial \varrho_s}$ and $\dfrac{\partial \mathcal{L}}{\partial \theta_{i,s}}$, $i = 1, 2, 3$, equal to zero. The solutions are as follows:

$$\widehat{\varrho}_s = \frac{1}{N-1} \sum\nolimits_{\tau=0}^{N-1} \pi_{s+\tau S},$$

$$\sum\nolimits_{\tau=0}^{N-1} (1-\pi_{s+\tau S}) \left(\frac{y_{s+\tau S}}{\lambda_{s+\tau S}} - 1 \right) \frac{\partial \lambda_{s+\tau S}}{\partial \theta_{i,s}} = 0, \text{ for } i = 1, 2, 3. \tag{6.2}$$

Due to the unavailability of closed form solutions for equation (6.2), we employ a standard Newton-Raphson algorithm in order to derive estimates.

$$-\frac{\partial^2 \mathcal{L}}{\partial \theta_{i,s} \partial \theta_{j,s}} = \sum\nolimits_{\tau=0}^{N-1} (1-\pi_{s+\tau S}) \left(\frac{y_{s+\tau S}}{\lambda_{s+\tau S}^2} \frac{\partial \lambda_{s+\tau S}}{\partial \theta_{i,s}} \frac{\partial \lambda_{s+\tau S}}{\partial \theta_{j,s}} - \left(\frac{y_{s+\tau S}}{\lambda_{s+\tau S}} - 1 \right) \frac{\partial^2 \mathcal{L}}{\partial \theta_{i,s} \partial \theta_{j,s}} \right),$$

the estimates value of $\underline{\theta}_s$ can be obtained by the following iterative equation

$$\underline{\theta}_s^{(i+1)} = \underline{\theta}_s^{(i)} - \left[\frac{\partial^2 \mathcal{L}}{\partial \underline{\theta}_s \partial \underline{\theta}'_s} \bigg|_{\underline{\theta}_s^{(i)}} \right]^{-1} \frac{\partial \mathcal{L}}{\partial \underline{\theta}_s} \bigg|_{\underline{\theta}_s^{(i)}}.$$

To estimate $\underline{\varphi}_s$, the iterative process is repeated until convergence is achieved. The EM algorithm employs a criterion to check for convergence, which is:

$$\left| \frac{\varphi_{s,j}^{(i+1)} - \varphi_{s,j}^{(i+1)}}{\varphi_{s,j}^{(i)}} \right| \leq 10^{-5}.$$

Remark 6.1. Our choice to use the EM algorithm is based on its simplicity and practical application. However, an alternative approach is to directly maximize the log-likelihood function. For the PZIP-INGARCH$(1,1)$ model, the likelihood function can be expressed as follows:

$$\prod\nolimits_{t=1}^{n} \left[\varrho_t \delta_{y_t,0} + (1-\varrho_t) \frac{e^{-\lambda_t} \lambda_t^{y_t}}{y_t!} \right],$$

then the log-likelihood function is given by :

$$\sum\nolimits_{t=1}^{n} \log \left(\varrho_t \delta_{y_t,0} + (1-\varrho_t) \frac{e^{-\lambda_t} \lambda_t^{y_t}}{y_t!} \right), \tag{6.3}$$

letting $n = NS$, $N \in N^*$, and replacing $t = s + \tau S$, with $s = 1, ..., S$ and $\tau = 0, 1, ..., N-1$, into equation (6.3), we obtain:

$$\sum\nolimits_{\tau=0}^{N-1} \sum\nolimits_{s=1}^{S} \log \left(\varrho_s \delta_{y_{s+\tau S},0} + (1-\varrho_s) \frac{e^{-\lambda_{s+\tau S}} \lambda_{s+\tau S}^{y_{s+\tau S}}}{y_{s+\tau S}!} \right).$$

7. NUMERICAL SIMULATION

In this section, a Monte Carlo study was conducted to assess the performance of the conditional maximum likelihood (CML) estimation method. Four different time series datasets were used, with sample sizes ranging from small to medium and relatively large. The datasets were generated on the basis of two models: the PZIP-INGARCH$(1,1)$ model and the PZIP-INARCH(1) model, both with a period of $S=4$. To analyze the PZIP-INGARCH$(1,1)$ model, two models, namely Model 1 and Model 2, are considered. Similarly, Model 3 and Model 4 are employed to analyze the PZIP-INARCH(1) model. The selected parameters in each model guarantee the first-order periodic stationarity condition.

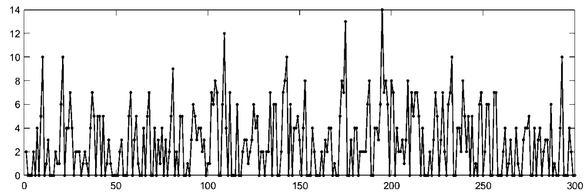
The main objective of this study was to illustrate specific empirical estimation properties. To achieve this, we generated 1000 distinct series for various sample sizes $(180, 300,$ 500, 1000, 1500, 2000, 3000). Each data generation process was repeated 1000 times. The mean estimates and their root mean square error (RMSE) are presented in Tables 1, 2, 3, and 4. In addition, graphical representations illustrating the trajectory of each generated data are provided in Figure 1. The true parameter values of these models are presented below:

PZIP-INGARCH $(1,1)$ **models**
Model 1: $\underline{\alpha}_0=(2.5,1,2,3)\,,\underline{\alpha}_1=(0.5,0.6,0.45,0.4)\,,\underline{\beta}=(0.1,0.3,0.4,0.2)\,,\ \underline{\varrho}=(0.1,0.4,0.5,0.35)\,,$
Model 2: $\underline{\alpha}_0=(3,2,1.5,1)\,,\underline{\alpha}_1=(0.3,0.1,0.5,0.4)\,,\underline{\beta}=(0.6,0.35,0.15,0.25)\,,\ \underline{\varrho}=(0.3,0.5,0.4,0.6)\,.$

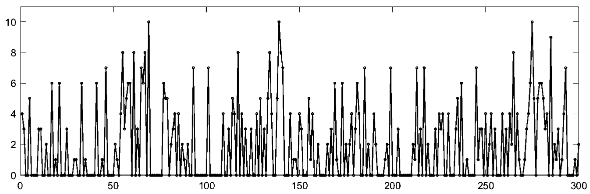
PZIP-INARCH (1) **models**
Model 3 : $\underline{\alpha}_0=(1,3,2,4)\,,\underline{\alpha}_1=(0.3,0.1,0.15,0.25)\,,\underline{\varrho}=(0.2,0.35,0.45,0.1)\,,$
Model 4 : $\underline{\alpha}_0=(2,4,3,5)\,,\underline{\alpha}_1=(0.5,0.4,0.65,0.35)\,,\underline{\varrho}=(0.45,0.55,0.65,0.35)\,.$

Table 1: Sample mean and root mean square error RMSE (in bracket) for Model 1.

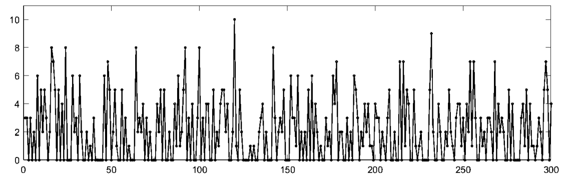
T.V	*Size*	180	300	500	1000	1500	2000	3000
2.5	$\widehat{\alpha}_{0,1}$	2.1107 (1.1222)	2.2269 (0.9103)	2.3699 (0.6381)	2.4643 (0.4845)	2.4679 (0.3905)	2.4831 (0.3464)	2.4842 (0.2824)
1	$\widehat{\alpha}_{0,2}$	1.0439 (0.9117)	1.0740 (0.7999)	1.0021 (0.6775)	1.0077 (0.5119)	1.0111 (0.4286)	1.0088 (0.3860)	1.0048 (0.3016)
2	$\widehat{\alpha}_{0,3}$	1.9890 (1.2763)	1.9497 (1.0316)	2.0218 (0.8262)	2.0054 (0.5741)	2.0039 (0.4392)	1.9926 (0.3848)	1.9886 (0.3071)
3	$\widehat{\alpha}_{0,4}$	2.7693 (1.2575)	2.9082 (1.0069)	2.9462 (0.7584)	2.9961 (0.5082)	2.9895 (0.4563)	2.9871 (0.3689)	2.9906 (0.3097)
0.5	$\widehat{\alpha}_{1,1}$	0.5055 (0.1247)	0.4974 (0.0915)	0.5028 (0.0722)	0.4981 (0.0487)	0.5025 (0.0403)	0.4995 (0.0347)	0.4996 (0.0292)
0.6	$\widehat{\alpha}_{1,2}$	0.6002 (0.1906)	0.6045 (0.1454)	0.5988 (0.1055)	0.6033 (0.0753)	0.6015 (0.0589)	0.6009 (0.0506)	0.6002 (0.0441)
0.45	$\widehat{\alpha}_{1,3}$	0.4548 (0.1868)	0.4560 (0.1438)	0.4484 (0.1044)	0.4511 (0.0736)	0.4478 (0.0587)	0.4518 (0.0517)	0.4505 (0.0413)
0.4	$\widehat{\alpha}_{1,4}$	0.3911 (0.1552)	0.3995 (0.1212)	0.4030 (0.0884)	0.4000 (0.0621)	0.4022 (0.0500)	0.3997 (0.0431)	0.4002 (0.0361)
0.1	$\widehat{\beta}_1$	0.1734 (0.2205)	0.1531 (0.1779)	0.1243 (0.1252)	0.1072 (0.0939)	0.1057 (0.0764)	0.1036 (0.0668)	0.1035 (0.0559)
0.3	$\widehat{\beta}_2$	0.2869 (0.2432)	0.2837 (0.2040)	0.2969 (0.1671)	0.2970 (0.1236)	0.2924 (0.1013)	0.2967 (0.0887)	0.2980 (0.0686)
0.4	$\widehat{\beta}_3$	0.4076 (0.2759)	0.4068 (0.2230)	0.3947 (0.1738)	0.3972 (0.1196)	0.4026 (0.0907)	0.3987 (0.0826)	0.4012 (0.0661)
0.2	$\widehat{\beta}_4$	0.2482 (0.2418)	0.2202 (0.1874)	0.2082 (0.1463)	0.2003 (0.0967)	0.2006 (0.0847)	0.2036 (0.0699)	0.2020 (0.0589)
0.1	$\widehat{\varrho}_1$	0.0932 (0.0495)	0.0966 (0.0387)	0.0973 (0.0304)	0.0992 (0.0207)	0.0990 (0.0172)	0.0991 (0.0141)	0.0995 (0.0123)
0.4	$\widehat{\varrho}_2$	0.3930 (0.0749)	0.3990 (0.0597)	0.3976 (0.0450)	0.3998 (0.0321)	0.3993 (0.0255)	0.3996 (0.0223)	0.4009 (0.0185)
0.5	$\widehat{\varrho}_3$	0.4964 (0.0774)	0.4961 (0.0583)	0.5001 (0.0453)	0.5008 (0.0318)	0.5005 (0.0260)	0.4994 (0.0234)	0.4993 (0.0190)
0.35	$\widehat{\varrho}_4$	0.3514 (0.0743)	0.3471 (0.0575)	0.3508 (0.0436)	0.3477 (0.0301)	0.3500 (0.0251)	0.3495 (0.0219)	0.3502 (0.0176)



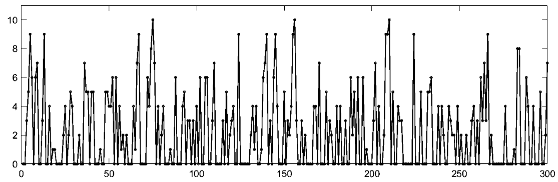
(a) Model 1



(b) Model 2



(c) Model 3



(d) Model 4

Figure 1: The trajectory of data generated by various models.

Table 2: Sample mean and root mean square error RMSE (in bracket) for Model 2.

$T.V$	$Size$	180	300	500	1000	1500	2000	3000
3	$\widehat{\alpha}_{0,1}$	3.0081 (0.9583)	2.9773 (0.8369)	2.9833 (0.6952)	2.9835 (0.5126)	2.9420 (0.4619)	2.9902 (0.3748)	2.9697 (0.3297)
2	$\widehat{\alpha}_{0,2}$	1.9102 (1.5376)	1.8546 (1.4495)	1.9776 (1.3042)	2.0077 (1.0461)	2.0508 (0.8729)	2.0271 (0.8033)	2.0040 (0.6537)
1.5	$\widehat{\alpha}_{0,3}$	1.1089 (1.0450)	1.1613 (0.9897)	1.2037 (0.9651)	1.2741 (0.8851)	1.3149 (0.8013)	1.3373 (0.7411)	1.3900 (0.6605)
1	$\widehat{\alpha}_{0,4}$	0.8716 (0.7427)	0.9202 (0.6589)	0.9711 (0.5626)	0.9492 (0.4515)	1.0137 (0.3672)	0.9946 (0.3216)	1.0042 (0.2472)
0.3	$\widehat{\alpha}_{1,1}$	0.3274 (0.2577)	0.3035 (0.2024)	0.3031 (0.1606)	0.2958 (0.1088)	0.2977 (0.0931)	0.2996 (0.0754)	0.2970 (0.0636)
0.1	$\widehat{\alpha}_{1,2}$	0.1308 (0.1359)	0.1160 (0.1058)	0.1048 (0.0836)	0.1019 (0.0611)	0.0993 (0.0522)	0.1015 (0.0480)	0.0990 (0.0393)
0.5	$\widehat{\alpha}_{1,3}$	0.5076 (0.1726)	0.5064 (0.1355)	0.4968 (0.1012)	0.5000 (0.0699)	0.5014 (0.0555)	0.5019 (0.0491)	0.5006 (0.0391)
0.4	$\widehat{\alpha}_{1,4}$	0.4262 (0.2175)	0.4137 (0.1662)	0.4072 (0.1237)	0.4004 (0.0874)	0.3994 (0.0729)	0.4018 (0.0632)	0.3985 (0.0509)
0.6	$\widehat{\beta}_1$	0.5879 (0.3648)	0.6095 (0.3145)	0.6064 (0.2705)	0.6136 (0.1982)	0.6220 (0.1852)	0.6093 (0.1508)	0.6154 (0.1295)
0.35	$\widehat{\beta}_2$	0.3471 (0.3368)	0.3695 (0.3117)	0.3521 (0.2798)	0.3467 (0.2224)	0.3406 (0.1841)	0.3438 (0.1716)	0.3493 (0.1396)
0.15	$\widehat{\beta}_3$	0.2428 (0.2685)	0.2285 (0.2476)	0.2246 (0.2403)	0.2048 (0.2188)	0.1955 (0.1998)	0.1896 (0.1849)	0.1774 (0.1662)
0.25	$\widehat{\beta}_4$	0.2730 (0.2561)	0.2651 (0.2250)	0.2512 (0.1831)	0.2653 (0.1434)	0.2463 (0.1146)	0.2474 (0.1020)	0.2463 (0.0760)
0.3	$\widehat{\varrho}_1$	0.2946 (0.0705)	0.2942 (0.0568)	0.3002 (0.0421)	0.2989 (0.0294)	0.3001 (0.0238)	0.2992 (0.0208)	0.2998 (0.0176)
0.5	$\widehat{\varrho}_2$	0.4953 (0.0735)	0.4969 (0.0575)	0.4986 (0.0452)	0.4970 (0.0318)	0.5006 (0.0262)	0.4995 (0.0224)	0.4996 (0.0185)
0.4	$\widehat{\varrho}_3$	0.3917 (0.0846)	0.3964 (0.0621)	0.3969 (0.0483)	0.3979 (0.0343)	0.3987 (0.0281)	0.4000 (0.0239)	0.3997 (0.0198)
0.6	$\widehat{\varrho}_4$	0.5864 (0.0877)	0.5905 (0.0669)	0.5962 (0.0502)	0.5970 (0.0367)	0.5980 (0.0291)	0.5986 (0.0248)	0.5981 (0.0206)

By analyzing Tables 1, 2, 3, and 4, it becomes clear that the estimation method used exhibits enhanced performance as the value of n increases for the both PZIP-INGARCH$(1,1)$ and PZIP-INARCH(1) models. This is due to the fact that as the sample size n increases, all parameter estimators converge and the root mean square error (RMSE) decreases. Consequently, our CML-vector estimators $\underline{\widehat{\theta}}_{s,CML}$ are empirically consistent for all the parameters under analysis.

Table 3: Sample mean and root mean square error RMSE (in bracket) for Model 3.

$T.V$	$Size$	180	300	500	1000	1500	2000	3000
1	$\widehat{\alpha}_{0,1}$	0.9953 (0.4922)	1.0065 (0.3335)	0.9885 (0.2539)	0.9944 (0.1800)	0.9898 (0.1494)	0.9975 (0.1252)	1.0001 (0.1073)
3	$\widehat{\alpha}_{0,2}$	2.9777 (0.5181)	2.9540 (0.3601)	2.9709 (0.2927)	2.9888 (0.2002)	2.9910 (0.1694)	3.0007 (0.1477)	2.9941 (0.1219)
2	$\widehat{\alpha}_{0,3}$	1.9523 (0.5373)	1.9445 (0.3702)	1.9795 (0.2887)	1.9940 (0.2090)	1.9987 (0.1633)	1.9952 (0.1457)	1.9957 (0.1155)
4	$\widehat{\alpha}_{0,4}$	3.9726 (0.4230)	3.9837 (0.3131)	3.9945 (0.2562)	3.9961 (0.1747)	3.9999 (0.1447)	3.9997 (0.1252)	3.9989 (0.0999)
0.3	$\widehat{\alpha}_{1,1}$	0.2936 (0.1239)	0.2960 (0.0901)	0.3006 (0.0631)	0.2994 (0.0442)	0.3024 (0.0368)	0.3003 (0.0327)	0.3006 (0.0268)
0.1	$\widehat{\alpha}_{1,2}$	0.0964 (0.2360)	0.1235 (0.1389)	0.1112 (0.1058)	0.1058 (0.0807)	0.1012 (0.0671)	0.1010 (0.0603)	0.1008 (0.0516)
0.15	$\widehat{\alpha}_{1,3}$	0.1533 (0.1994)	0.1610 (0.1256)	0.1524 (0.1001)	0.1489 (0.0699)	0.1490 (0.0574)	0.1494 (0.0502)	0.1517 (0.0430)
0.25	$\widehat{\alpha}_{1,4}$	0.2486 (0.2298)	0.2496 (0.1645)	0.2517 (0.1271)	0.2487 (0.0915)	0.2484 (0.0726)	0.2478 (0.0647)	0.2509 (0.0515)
0.2	$\widehat{\varrho}_1$	0.1862 (0.0854)	0.1882 (0.0667)	0.1957 (0.0519)	0.1964 (0.0359)	0.1975 (0.0290)	0.1978 (0.0238)	0.1983 (0.0212)
0.35	$\widehat{\varrho}_2$	0.3423 (0.0771)	0.3484 (0.0583)	0.3466 (0.0436)	0.3508 (0.0326)	0.3481 (0.0255)	0.3484 (0.0230)	0.3503 (0.0180)
0.45	$\widehat{\varrho}_3$	0.4342 (0.0874)	0.4424 (0.0684)	0.4464 (0.0519)	0.4502 (0.0366)	0.4496 (0.0291)	0.4487 (0.0253)	0.4492 (0.0209)
0.1	$\widehat{\varrho}_4$	0.0976 (0.0475)	0.1004 (0.0373)	0.0999 (0.0291)	0.0999 (0.0199)	0.0993 (0.0169)	0.0996 (0.0142)	0.0996 (0.0120)

Table 4: Sample mean and root mean square error RMSE (in bracket) for Model 4.

$T.V$	$Size$	180	300	500	1000	1500	2000	3000
2	$\widehat{\alpha}_{0,1}$	1.9691 (0.9648)	1.9730 (0.4199)	2.0028 (0.3259)	1.9928 (0.2279)	1.9942 (0.1814)	2.0019 (0.1597)	1.9940 (0.1275)
4	$\widehat{\alpha}_{0,2}$	3.9900 (0.7660)	3.9818 (0.4630)	4.0165 (0.3737)	3.9955 (0.2651)	4.0042 (0.2040)	4.0066 (0.1824)	3.9965 (0.1442)
3	$\widehat{\alpha}_{0,3}$	3.4390 (9.3253)	3.0008 (0.4929)	2.9878 (0.3761)	3.0006 (0.2571)	3.0094 (0.2087)	3.0017 (0.1872)	3.0046 (0.1527)
5	$\widehat{\alpha}_{0,4}$	5.0389 (0.6767)	5.0019 (0.3994)	4.9896 (0.3004)	4.9954 (0.2167)	4.9997 (0.1708)	5.0037 (0.1458)	5.0045 (0.1182)
0.5	$\widehat{\alpha}_{1,1}$	0.5011 (0.2013)	0.5037 (0.1091)	0.4980 (0.0824)	0.5003 (0.0562)	0.5015 (0.0441)	0.5023 (0.0391)	0.5013 (0.0309)
0.4	$\widehat{\alpha}_{1,2}$	0.4232 (0.1973)	0.4015 (0.1671)	0.3945 (0.1257)	0.3985 (0.0949)	0.3983 (0.0731)	0.3969 (0.0620)	0.4019 (0.0509)
0.65	$\widehat{\alpha}_{1,3}$	0.2859 (0.1997)	0.6498 (0.1733)	0.6510 (0.1390)	0.6448 (0.0953)	0.6457 (0.0774)	0.6506 (0.0673)	0.6470 (0.0519)
0.35	$\widehat{\alpha}_{1,4}$	0.3303 (0.1872)	0.3503 (0.1562)	0.3551 (0.1152)	0.3527 (0.0824)	0.3464 (0.0628)	0.3500 (0.0558)	0.3480 (0.0446)
0.45	$\widehat{\varrho}_1$	0.4354 (0.0835)	0.4419 (0.0647)	0.4470 (0.0483)	0.4494 (0.0349)	0.4486 (0.0275)	0.4492 (0.0241)	0.4490 (0.0191)
0.55	$\widehat{\varrho}_2$	0.5497 (0.0753)	0.5494 (0.0588)	0.5498 (0.0430)	0.5499 (0.0314)	0.5499 (0.0258)	0.5510 (0.0223)	0.5495 (0.0180)
0.65	$\widehat{\varrho}_3$	0.6459 (0.0764)	0.6477 (0.0569)	0.6502 (0.0428)	0.6501 (0.0322)	0.6506 (0.0261)	0.6509 (0.0215)	0.6508 (0.0172)
0.35	$\widehat{\varrho}_4$	0.3472 (0.0708)	0.3521 (0.0557)	0.3511 (0.0444)	0.3509 (0.0309)	0.3500 (0.0251)	0.3508 (0.0220)	0.3495 (0.0177)

8. REAL DATA EXAMPLE

This section analyzes a dataset containing 320 recorded instances of daily COVID-19 death rates in Finland from January 1, 2021, to November 16, 2021. The focus is on two key features: zero value excess and periodic patterns. Figure 2 presents a visual representation of the time series, while Table 5 summarizes descriptive statistics. Additionally, Table 6 details the empirical periodic mean, variance, Fisher's dispersion index, and zero inflation index.

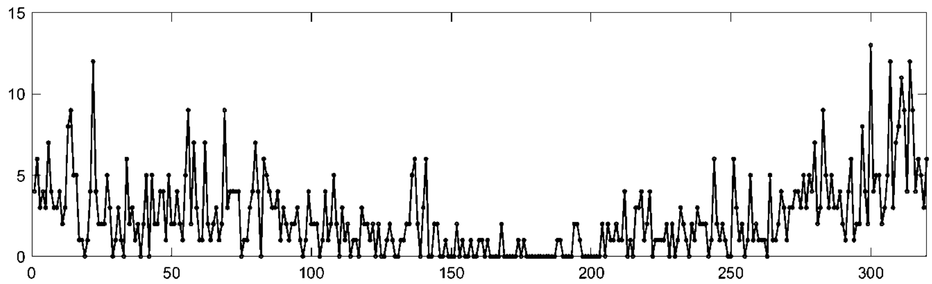


Figure 2: Trajectory of the daily COVID-19 time series.

Table 5: Several descriptive statistics for the COVID-19 time series.

Sample size	Min	Max	Median	Mean	Variance	Fisher index	Skewness	Kurtosis
320	0	13	2	2.3875	5.8807	2.4631	1.5681	6.0471

Table 6: Empirical periodic mean, variance, Fisher index, and zero inflation index for the dataset.

s	1	2	3	4	5	6	7
$\mu_{y,s}$	2.4783	2.3043	2.5217	1.7826	1.6957	3.4000	2.5556
$\sigma^2_{y,s}$	5.0995	4.0386	6.2106	3.9517	3.0609	12.2909	5.4798
$\text{FI}_{y,s}$	2.0577	1.7526	2.4629	2.2168	1.8051	3.6150	2.1442
$p_{0,s}$	0.1522	0.2174	0.2174	0.3478	0.3696	0.2000	0.1333
zi_s	0.2403	0.3377	0.3948	0.4076	0.4130	0.5266	0.2116

Tables 5 and 6 show that the daily count of COVID-19 deaths in the Finland time series exhibits a higher level of dispersion. In addition, Puig and Valero (2006) introduced a zero inflation index zi_s to measure the departure from the Poisson model:

$$zi_s = 1 + \log\left(p_{0,s}\right) / \mu_{y,s}, s = 1, ..., 7,$$

where, $p_{0,s}$ is proportion of 0's and $\mu_{y,s}$ is the mean in each period $s = 1, ..., 7$. Notice that, $zi_s = 0$, if X (X is a count variable) is Poisson distributed and $zi_s > 0$, if X is zero inflate. From Table 6, the zero inflation index is $zi_s > 0$, for $s = 1, ..., 7$, which indicates that there is a zero inflation. Consequently, it appears that the models proposed by the authors in this work, PZIP-INGARCH$(1, 1)$ and PZIP-INARCH(1), are well suited to effectively capture the patterns within the time series, as well as accounting for the presence of zero values.

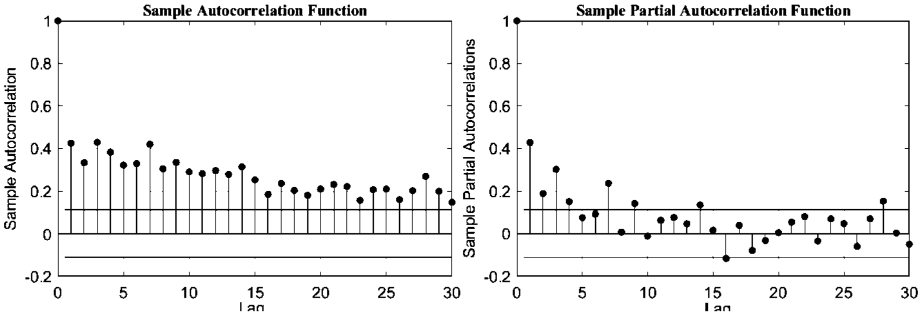
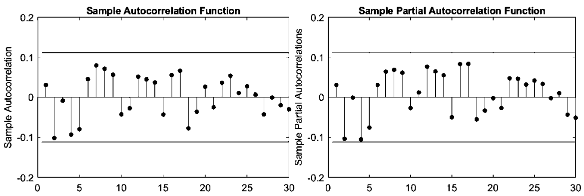
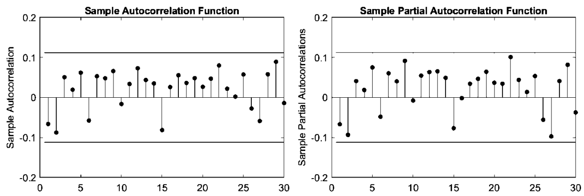


Figure 3: ACF and PACF of the COVID-19 time series.

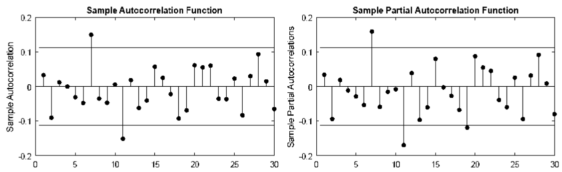
Furthermore, an analysis of the autocorrelation function (ACF) and partial autocorrelation function (PACF), as shown in Figure 3, reveals a clear periodic pattern in the time series with a period of $S = 7$ (peaks marked at lags of $S = 7$, 14, ...). This periodicity is due to the fact that data are collected every day.



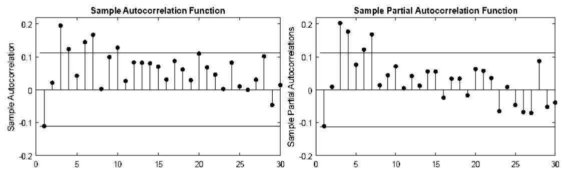
(a) PZIP-INGARCH(1,1) model



(b) PZIP-INARCH(1) model



(c) ZIP-INGARCH(1,1) model



(d) ZIP-INARCH(1) model

Figure 4: ACF and PACF of the residual time series

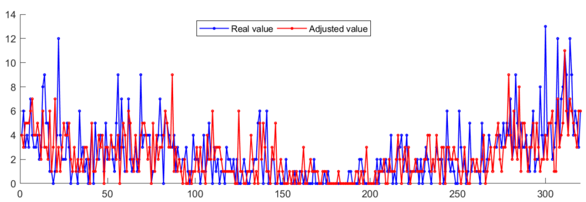
The data is well described by the PZIP-INGARCH(1, 1) and PZIP-INARCH(1) models, as confirmed by the empirical residual autocorrelation function (ACF) and the partial autocorrelation function (PACF). Indeed, the residuals show no statistically significant autocorrelation, as shown in Figure 4. So, the adequacy of the models is not statistically rejected. Also, the periodic behavior of the residual autocorrelation in the fitted PZIP-INGARCH(1, 1) and PZIP-INARCH(1) models has been fully captured, which indicates that these models capture the periodic feature of the data set with a period of $S = 7$. However, the ZIP-INGARCH(1, 1) and ZIP-INARCH(1) models exhibit a significant autocorrelation, and some periodicity in the residual autocorrelation function persists, especially in the 7th lag, which indicates that they don't fully account for the periodic structure of the data.

Furthermore, Table 7 provides the parameter estimation results and the corresponding standard errors, computed numerically using the bootstrap method, for the proposed PZIP-INGARCH(1, 1) and PZIP-INARCH(1) models, as well as for several existing models. This comparison is based on the Akaike information criterion (AIC), the Bayesian information criterion (BIC), and the sum of squared errors (SSE). The proposed models show significant improvements over the existing ones (including, ZIP-INAR(1) model Jazi et al. (2012),

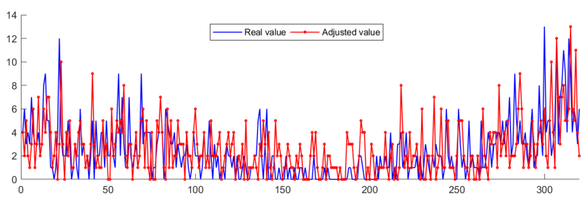
Table 7: Fitting results of different models.

Model	Para.				CML				AIC	BIC	SSE
ZIP-INAR(1)	φ				0.3717 (0.0365)				1274.38	1285.69	2200
	ϱ				0.4045 (0.0473)						
	λ				2.5174 (0.1632)						
ZIP-INARCH(1)	α_0				1.5182 (0.1380)				1299.25	**1310.55**	1990
	α_1				0.4674 (0.0562)						
	ϱ				0.1127 (0.0256)						
ZIP-INGARCH(1, 1)	α_0				0.0212 (0.0275)				1197.63	**1212.70**	1796
	α_1				0.1313 (0.0384)						
	β				0.8683 (0.0456)						
	ϱ				0.0395 (0.0268)						
INAR(1)	φ				0.3181 (0.0508)				1360.79	1368.33	2312
	λ				1.6265 (0.1330)						
	s	1	2	3	4	5	6	7			
$PINAR_7\,(1)$	φ_s	0.2604 (0.1134)	0.4534 (0.1208)	0.4638 (0.0799)	0.4956 (0.1353)	0.2072 (0.1887)	0.2660 (0.1068)	0.3162 (0.1373)	1314.76	1367.52	2075
	λ_s	1.7790 (0.3005)	1.1808 (0.3054)	1.4530 (0.1971)	0.5328 (0.2802)	1.3264 (0.3980)	2.9744 (0.3936)	1.4804 (0.3941)			
PZIP-$INARCH_7\,(1)$	$\alpha_{0,s}$	1.5065 (0.3898)	0.6587 (0.3528)	0.6451 (0.1988)	0.3051 (0.1647)	1.9143 (0.4456)	2.8194 (0.4260)	1.2387 (0.2961)	1244.86	**1316.46**	1903
	$\alpha_{1,s}$	0.3892 (0.1372)	0.7205 (0.1544)	0.8485 (0.1188)	0.5859 (0.0836)	0.2337 (0.1670)	0.7880 (0.2042)	0.3873 (0.0851)			
	ϱ_s	0.0262 (0.0398)	0.0497 (0.0549)	0.0307 (0.0423)	0.0000 (0.0455)	0.2871 (0.0896)	0.1770 (0.0681)	0.0000 (0.0145)			
PZIP-$INGARCH_7\,(1,1)$	$\alpha_{0,s}$	0.0000 (0.2131)	0.0000 (0.1956)	0.0000 (0.1289)	0.0000 (0.0986)	0.3329 (0.2389)	0.1936 (0.2747)	0.0000 (0.0962)	1174.49	**1257.40**	1592
	$\alpha_{1,s}$	0.0378 (0.0881)	0.1528 (0.1052)	0.4727 (0.1459)	0.2995 (0.1010)	0.0000 (0.0932)	0.2765 (0.1112)	0.1203 (0.1012)			
	β_s	0.8962 (0.1542)	0.8063 (0.1764)	0.6216 (0.1741)	0.5310 (0.1348)	1.000 (0.1227)	1.000 (0.1040)	0.7164 (0.1443)			
	ϱ_s	0.0000 (0.0348)	0.0000 (0.0494)	0.0000 (0.0264)	0.0000 (0.0551)	0.2190 (0.0952)	0.0726 (0.0536)	0.0000 (0.0274)			

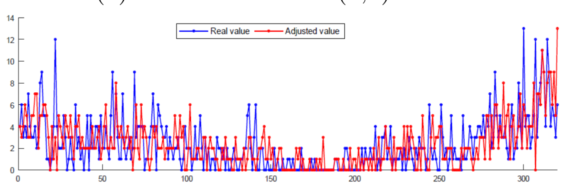
ZIP-INARCH(1), ZIP-INGARCH(1, 1) models Zhu (2012), Poisson INAR(1) model Al-Osh and Alzaid (1987), and the periodic $PINAR_7\,(1)$ model Monteiro et al. (2010)), particularly in terms of AIC and SSE, where they achieve lower values. However, the BIC values for the proposed models are higher compared to the ZIP-INGARCH(1, 1) and ZIP-INARCH(1) models. This difference is mostly due to the higher number of estimated parameters in our models.



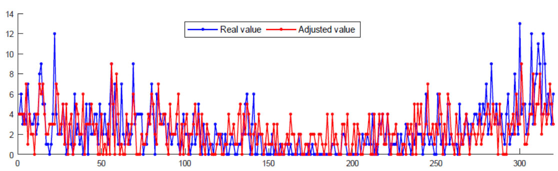
(a) PZIP-INGARCH(1,1) model



(b) PZIP-INARCH(1) model



(c) ZIP-INGARCH(1,1) model



(d) ZIP-INARCH(1) model

Figure 5: Comparison between the adjustments of the different models.

Figure 5 shows the adjusted series for each model, such as the series values are shown in blue, while the red line denotes the adjusted series. The fitted values of the PZIP-INGARCH$(1,1)$ model appear to be quite overlapped with the actual data points in the dataset.

9. CONCLUSION

In this paper, we have introduced a particular version of the PZIP-INGARCH(p,q) model developed to address overdispersion, hidden periodicity in autocovariance structures, and excess zeros in integer-valued time series. The conditions for the existence of higher-order moments and their closed-form expressions are derived, and periodic stationarity for the first and second moments is established. Additionally, we have analyzed the periodic autocovariance structure and derived a closed-form expression for the periodic autocorrelation function. To estimate the unknown periodic parameters, the CML method is employed via EM algorithm. Through a simulation study, the performance of the obtained estimators is assisted. An application to the daily COVID-19 death counts in Finland shows that the model offers better performance and better captures the unique characteristics of the data.

10. APPENDIX

Proof of Proposition 4.1: The conditional mean $\mathbb{E}\left(\lambda_t^m \middle| \mathcal{F}_{t-1}\right)$ is given by

$$\begin{aligned}\mathbb{E}\left(\lambda_t^m \middle| \mathcal{F}_{t-1}\right) &= \sum\nolimits_{i=0}^{m} \binom{m}{i} \alpha_{0,t}^{m-i} \left(\alpha_{1,t} y_{t-1} + \beta_t \lambda_{t-1}\right)^i \\ &= \sum\nolimits_{i=0}^{m} \binom{m}{i} \alpha_{0,t}^{m-i} \sum\nolimits_{j=0}^{i} \binom{i}{j} \alpha_{1,t}^{j} \beta_t^{i-j} \lambda_{t-1}^{i-j} \mathbb{E}\left(y_{t-1}^j \middle| \mathcal{F}_{t-2}\right).\end{aligned}$$

It is well known that the j-th moment of a zero inflated Poisson variable with parameters ϱ_t and λ_t is given, while using the formula of Stirling numbers of the second kind, by : $\mathbb{E}\left(y_t^j \middle| \mathcal{F}_{t-1}\right) = (1-\varrho_t) \sum_{k=1}^{j} \left\{ {j \atop k} \right\} \lambda_t^k$. Then we have :

$$\begin{aligned}\mathbb{E}\left(\lambda_t^m \middle| \mathcal{F}_{t-2}\right) &= \alpha_{0,t}^m + \sum_{i=1}^{m} \binom{m}{i} \alpha_{0,t}^{m-i} \sum_{j=0}^{i} \binom{i}{j} \alpha_{1,t}^{j} \beta_t^{i-j} \lambda_{t-1}^{i-j} \left[\sum_{k=1}^{j} \left\{ {j \atop k} \right\} (1-\varrho_{t-1}) \lambda_{t-1}^k \right] \\ &= \alpha_{0,t}^m + \psi_{m,t} \lambda_{t-1}^m + \sum_{i=1}^{m-1} \binom{m}{i} \alpha_{0,t}^{m-i} \psi_{i,t} \lambda_{t-1}^i + \\ &\quad \sum\nolimits_{i=1}^{m} \sum\nolimits_{j=1}^{i} \sum\nolimits_{k=1}^{j-1} \binom{m}{i} \binom{i}{j} (1-\varrho_{t-1}) \alpha_{0,t}^{m-i} \alpha_{1,t}^{j} \beta_t^{i-j} \left\{ {j \atop k} \right\} \lambda_{t-1}^{i-(j-k)},\end{aligned}$$

which can be written in the form

$$\mathbb{E}\left(\lambda_t^m \middle| \mathcal{F}_{t-2}\right) = \alpha_{0,t}^m + \psi_{m,t} \lambda_{t-1}^m + \sum_{i=1}^{m-1} \binom{m}{i} \alpha_{0,t}^{m-i} \psi_{i,t} \lambda_{t-1}^i + \sum_{i=1}^{m} \sum_{j=1}^{i} \sum_{k=1}^{j-1} \mathcal{K}_{i,j,k}^{(m,t)} \lambda_{t-1}^{i-k},$$

with,

$$\begin{aligned}\mathcal{K}_{i,j,k}^{(m,t)} &= \binom{m}{i} \binom{i}{j} \left\{ {j-1 \atop k} \right\} (1-\varrho_{t-1}) \alpha_{0,t}^{m-i} \alpha_{1,t}^{j} \beta_t^{i-j}, \\ \psi_{m,t} &= (1-\varrho_{t-1}) \sum\nolimits_{j=1}^{m} \binom{m}{j} \alpha_{1,t}^{j} \beta_t^{m-j} + \beta_t^m.\end{aligned}$$

the last two sums in the precedent expression can be rearranged as follows

$$\sum_{i=1}^{m-1}\binom{m}{i}\alpha_{0,t}^{m-i}\psi_{i,t}\lambda_{t-1}^{i}+\sum_{i=1}^{m}\sum_{j=1}^{i}\sum_{k=1}^{j-1}\mathcal{K}_{i,j,k}^{(m,t)}\lambda_{t}^{i-k}=\sum_{i=1}^{m-1}\phi_{i,t}^{(m)}\lambda_{t-1}^{i},$$

where,

$$\phi_{i,t}^{(m)}=\binom{m}{i}\alpha_{0,t}^{m-i}\psi_{i,t}+\sum_{j=i+1}^{m}\sum_{k=j-i}^{j-1}\mathcal{K}_{j,k+1,j-i}^{(m,t)},$$

Hence, we have

$$\mathbb{E}\left(\lambda_t^m\middle|\mathcal{F}_{t-2}\right)=\alpha_{0,t}^m+\psi_{m,t}\lambda_{t-1}^m+\sum_{i=1}^{m-1}\phi_{i,t}^{(m)}\lambda_{t-1}^{i}.$$

Replacing i by m, $m-1$, $m-2$,..., 3, 2, 1, we obtain the following matrix difference equation

$$\mathbb{E}\left(\underline{\Lambda}_t^{(m)}\middle|\mathcal{F}_{t-2}\right)=\Theta_t^{(m)}\underline{\Lambda}_{t-1}^{(m)}+\underline{\alpha}_{0,t}^{(m)},$$

where the elements of the $m\times m$-matrix $\Theta_t^{(m)}$ are given in (4.1). Iterating the last equation n times, we obtain

$$\mathbb{E}\left(\underline{\Lambda}_t^{(m)}\middle|\mathcal{F}_{t-2-n}\right)=\sum_{j=0}^{n}\left(\prod_{i=1}^{j}\Theta_{t-i+1}^{(m)}\right)\underline{\alpha}_{0,t-j}^{(m)}+\left(\prod_{i=1}^{n+1}\Theta_{t-i+1}^{(m)}\right)\underline{\Lambda}_{t-(n+1)}^{(m)}.$$

Letting $n=kS-2$, then we have, while taking account of the matrix $\Theta_t^{(m)}$ and the column vector $\underline{\alpha}_{0,t}^{(m)}$

$$\mathbb{E}\left(\underline{\Lambda}_t^{(m)}\middle|\mathcal{F}_{t-kS}\right)=\sum_{j=0}^{kS-2}\left(\prod_{i=1}^{j}\Theta_{t-i+1}^{(m)}\right)\underline{\alpha}_{0,t-j}^{(m)}+\left(\prod_{i=1}^{kS-1}\Theta_{t-i+1}^{(m)}\right)\underline{\Lambda}_{t-(kS-1)}^{(m)},$$

which can be written in the form

$$\begin{aligned}\mathbb{E}\left(\underline{\Lambda}_t^{(m)}\middle|\mathcal{F}_{t-kS}\right)=&\sum_{l=0}^{k-2}\left(\prod_{i=1}^{S}\Theta_{t-i+1}^{(m)}\right)^{l}\sum_{j=1}^{S}\left(\prod_{i=1}^{j-1}\Theta_{t-i+1}^{(m)}\right)\underline{\alpha}_{0,t-(lS+j)+1}^{(m)}+\\&\left(\prod_{i=1}^{S}\Theta_{t-i+1}^{(m)}\right)^{k-1}\left[\sum_{j=1}^{S-1}\left(\prod_{i=1}^{j-1}\Theta_{t-i+1}^{(m)}\right)\underline{\alpha}_{0,t-(lS+j)+1}^{(m)}+\left(\prod_{i=1}^{S-1}\Theta_{t-i+1}^{(m)}\right)\underline{\Lambda}_{t-(kS-1)}^{(m)}\right].\end{aligned}$$

Replacing t by $s+\tau S$ and taking account of the periodicity, we obtain

$$\begin{aligned}\mathbb{E}\left(\underline{\Lambda}_t^{(m)}\middle|\mathcal{F}_{t-(k-\tau)S}\right)=&\sum_{l=0}^{k-2}\left(\prod_{i=1}^{S}\Theta_{s-i+1}^{(m)}\right)^{l}\sum_{j=1}^{S}\left(\prod_{i=1}^{j-1}\Theta_{s-i+1}^{(m)}\right)\underline{\alpha}_{0,s-j+1}^{(m)}+\\&\left(\prod_{i=1}^{S}\Theta_{s-i+1}^{(m)}\right)^{k-1}\left[\sum_{j=1}^{S-1}\left(\prod_{i=1}^{j-1}\Theta_{s-i+1}^{(m)}\right)\underline{\alpha}_{0,s-j+1}^{(m)}+\left(\prod_{i=1}^{S-1}\Theta_{s-i+1}^{(m)}\right)\underline{\Lambda}_{s-((k-\tau)S-1)}^{(m)}\right].\end{aligned}$$

Since the matrices $\Theta_{s-i+1}^{(m)}$, $i=1,...,S$ are upper-triangular with eigenvalues $\psi_{m,s-i+1}$, $\psi_{m-1,s-i+1}$, ..., $\psi_{2,s-i+1}$, $\psi_{1,s-i+1}$, then a sufficient condition for the matrix $\left(\prod_{i=1}^{S}\Theta_{s-i+1}^{(m)}\right)^{k-1}$ to converge, as $k\to\infty$, to the null matrix is that

$$\prod_{i=1}^{S}\psi_{m,i}<1,\ \text{with}\ \psi_{m,t}=\sum_{j=1}^{m}\binom{m}{j}\alpha_{1,t}^{j}\beta_t^{m-j}+\beta_t^m.$$

Under this condition, we have, the closed-form of the vector

$$\begin{aligned}\mathbb{E}\left(\underline{\Lambda}_t^{(m)}\right)&=\lim_{k\to\infty}\mathbb{E}\left(\underline{\Lambda}_t^{(m)}\middle|\mathcal{F}_{t-(k-\tau)S}\right)=\sum_{l=0}^{\infty}\left(\prod_{i=1}^{S}\Theta_{s-i+1}^{(m)}\right)^{l}\sum_{j=1}^{S}\left(\prod_{i=1}^{j-1}\Theta_{s-i+1}^{(m)}\right)\underline{\alpha}_{0,s-j+1}^{(m)},\\&=\left(I-\Psi_{s,S}^{(m)}\right)^{-1}\sum_{i=1}^{S}\Psi_{s,i-1}^{(m)}\underline{\alpha}_{0,s-i+1}^{(m)},\ \text{where}\ \Psi_{s,j}^{(m)}=\prod_{i=1}^{j}\Theta_{s-i+1}^{(m)}.\end{aligned}$$

□

Proof of Proposition 4.2: The proof follows directly from **Proposition** 4.1 by setting $m = 1$. □

Proof of Proposition 4.3: The proof follows directly from **Proposition** 4.1 by setting $m = 1$ and $m = 2$. □

Proof of Proposition 5.1: The variances $\gamma_{\lambda}^{(s)}(0)$ and $\gamma_{y}^{(s)}(0)$ were established in **Proposition** 4.3. The autocovariance $\gamma_{y}^{(s)}(1)$, can be calculated as follows :

$$\begin{aligned}\gamma_{y}^{(s)}(1) &= Cov\left(y_s; y_{s-1}\right), \\ &= \left(\alpha_{1,s}+\beta_s\right)\left(1-\varrho_s\right)\gamma_{y}^{(s-1)}(0)-\beta_s\left(1-\varrho_s\right)\left(1+\frac{\varrho_{s-1}}{1-\varrho_{s-1}}\mu_{y,s-1}\right)\mu_{y,s-1},\end{aligned}$$

More generally, let us calculate the autocovariance $\gamma_{y}^{(s)}(h)$, $h \geq 2$,

$$\gamma_{y}^{(s)}(h) = Cov\left(y_s; y_{s-h}\right) = \frac{1-\varrho_s}{1-\varrho_{s-1}}\psi_{1,s}\gamma_{y}^{(s-1)}(h-1), \; h \geq 2.$$

Iterating the last equation m times and replacing m by $h-1$ then using (4.1), we obtain

$$\begin{aligned}\gamma_{y}^{(s)}(h) &= \left(\prod_{i=1}^{h-1}\psi_{1,s-i+1}\right)\frac{1-\varrho_s}{1-\varrho_{s-h+1}}\gamma_{y}^{(s-h)}(1) = \left(\prod_{i=1}^{h-1}\psi_{1,s-i+1}\right)\left(1-\varrho_s\right)\times \\ &\left(\left(\alpha_{1,s-h+1}+\beta_{s-h+1}\right)\gamma_{y}^{(s-h)}(0)-\beta_{s-h+1}\left(1+\frac{\varrho_{s-h}}{1-\varrho_{s-h}}\mu_{y,s-h}\right)\mu_{y,s-h}\right)\end{aligned}$$

The autocovariance $\gamma_{\lambda}^{(s)}(h)$, $h \geq 1$, is given as follows :

$$\gamma_{\lambda}^{(s)}(h) = Cov\left(\lambda_s; \lambda_{s-h}\right) = \left(\alpha_{1,s}\left(1-\varrho_{s-1}\right)+\beta_s\right)\gamma_{\lambda}^{(s-1)}(h-1).$$

Iterating the equation m times and replacing m by h, we obtain

$$\gamma_{\lambda}^{(s)}(h) = \left(\prod_{i=1}^{h}\psi_{1,s-i+1}\right)\gamma_{\lambda}^{(s-h)}(0).$$

□

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