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Neutrosophic Log-Logistic Distribution: A Novel Approach for Modeling Uncertain Survival and Reliability Data

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Abstract:

- In survival and reliability engineering, log-logistic distribution is commonly employed, particularly for modeling lifetime data in electronic and human design processes. This research aims to present a modified neutrosophic log-logistic distribution (NLLD) designed to handle data indeterminacies. Unlike the approach by Rao (2023), which modeled the neutrosophic random variable as $X_N \in (X_L, X_U)$, our method defines it as $X_N \in (1 + I_N)X_L$. It is particularly useful for modeling ambiguous data with a roughly positive skew. This study examines the key statistical properties of the NLLD, including the neutrosophic mean, variance, median, quartiles, skewness, kurtosis, and reliability measures. Additionally, four modified entropy measures are derived for the NLLD, with numerical computations provided for the model. The maximum likelihood estimation method is used to estimate the neutrosophic parameters, and a simulation study is conducted to validate their accuracy. Finally, real-world applications demonstrate the effectiveness of the proposed NLLD compared to existing models.

Keywords:

- *Neutrosophic, log-logistic distribution, indeterminacy, reliability*

AMS Subject Classification:

- 49A05, 78B26.

1. INTRODUCTION

Several domains have found use for Neutrosophic theory. These include the study of the static model from a Neutrosophic perspective, the integration of renewable energy employing a range of resources, including wind turbines, and COVID-19 and its Omicron mutation. While in real issues there exist unclear facts, in conventional mathematics sharpness is the most important precondition. Mathematics based on uncertainty must be applied to tackle these problems. Many scientists and engineers are interested in uncertainty modeling because it helps them to identify and interpret the valuable information that is concealed in uncertain data. Despite being a vital instrument with real-world applications, the theory of Neutrosophic probability has not received much attention. Still, some research has been done on it. Neutrosopic statistics encompasses several domains, such as probability distributions, test methods, correlation, regression analysis, and so on. In recent years, more research has been done in these areas.

In recent years, many academics have been creating a variety of studies based on neutrosophic statistics. Smarandache (1998) launched the first studies on neutrosophic statistics. This new field of study applies fuzzy logic to uncertain environments and is a generalization of that environment. Neutronistic statistics is essential to statistics and other study domains because it can manage collections of values in an interval format. There are useful applications for the neutrosophic theory of probability, which makes it necessary. There has not been much research done in this field. In recent years, several writers have directed more of their attention toward the neutrosophic statistics method and its applications in other domains.13. Patro and Smarandache (2016) introduced the neutrosophic statistical distribution along with additional issues and their resolutions. Alhasan and Smarandache (2019) used the DUS-Weibull transformation to develop a novel neutrosophic model. Alhabib et al. (2018) investigated a few neutrosophic probability distributions by extending the Poisson, exponential, and uniform distributions from classical probability distributions to the neutrosophic class. Sherwani et al. (2021c) studied the applications and properties of neutrosophic beta distribution. Neutrosophic normal distribution was developed by Sherwani et al. (2021a). The neutrosophic exponential distribution was interoduced by Duan et al. (2021). The neutrosophic random variables were developed by Zeina and Hatip (2021). They explored several statistical examples and properties. Other applications of neutrosophic statistics in a variety of fields have been done by different researchers such as process capacity analysis, Aslam et al. (2019); sampling plans and quality control, Aslam (2019); and the indeterminacy environment in social science, Aslam and Arif (2018). Smarandache (2023) examined the evolution of plithogenic and neutrosophic probability and statistics between 1998 and 2021, emphasizing its uses in the medical domains. In order to manage data in unpredictable contexts, Sherwani et al. (2021b) extends classical statistics notions by introducing crucial measurements of central tendency and dispersion inside neutrosophic statistics. Altounji et al. (2023) investigated the use of a quadratic loss function to estimate parameters from a neutrosophic gamma distribution. The study focuses on the concepts of neutrosophic loss and risk functions, and posterior risk minimization.

The motivation for this research stems from the limitations of the neutrosophic log-logistic distribution introduced by Rao (2023). Rao's model was developed under the assumption that each observation is uncertain and falls between two specific ranges, where the uncertainty or indeterminacy lies between two values of a single observation for the random variable. For example, if the random variable represents the lifetime of a product, an observation might ambiguously suggest that the lifetime is between 20-25 hours, and this applies similarly to subsequent observations. Rao (2023) addressed this by considering the neutrosophic random variable as $X_N \in (X_L, X_U)$ follows the log-logistic distribution. In contrast, our proposed methodology addresses a different type of uncertainty or indeterminacy commonly encountered in real-life applications, where ambiguity or indeterminacy exists only in the upper limit of the recorded observation. For instance, when recording the lifetime of a product, we might know that the product lasts at least 20 hours, but we are uncertain about the exact upper limit. In such cases, the lower limit is known, while the ambiguity lies in the upper limit. To address this type of neutrosophic data, we propose a new neutrosophic log-logistic distribution, where the neutrosophic random variable is expressed as $X_N = X_L + I_N X_L = (1 + I_N)X_L$ where I_N being the indeterminate factor that determines the upper limit of the observation.

1.1. Neutrosophic Random Variable

As part of an interdisciplinary National Science Foundation project, Smarandache (2000) expands on classical theories of probability, fuzzy sets, and fuzzy logic to create neutrosophic probability, sets, and logic, highlighting their applicability in domains like artificial intelligence, neural networks, evolutionary programming, dynamic systems, and quantum mechanics. According to his, neutrosophic statistics possess a neutrosophic probability distribution as they are random variables. The values of a neutrosophic statistic behave in a certain way over an extended period of time when they are computed for several identical samples. The information used in neutrosophic statistics may be confusing, inaccurate, doubtful, partial, or even unknown. The following is the conventional form for neutrosophic numbers, which is based on classical statistics.

$$X_N = E + I.$$

Two components make up data: E represents the precise or determined portion of the data, whereas I represents the uncertain, inexact, or indeterminate portion of the data. To identify the neutrosophic random variable, use N in the subscript such as X_N. Let suppose the random variable X occurred from any probability distribution and if we consider that there is ambiguity in the values of random variable then the neutrosophic random variable $X_N \in (X_L, X_U)$ generates the neutrosophic values of data. According to this, the neutrosophic variable is defined as $X_N = X_L + I_N X_L = (1 + I_N)X_L$ indeterminate and determined parts are described by X_L and $I_N X_L$, respectively. Additionally, the expectation properties, mean and variance of the $X_N = X_L + I_N X_L = (1+I_N)X_L$ are defined as $E(X_N)^r = (1+I_N)E(X_L)^r$; mean is $E(X_N) = (1 + I_N)^r E(X_L)$, and variance is $Var(X_N) = (1 + I_N)^2 Var(X_L)$.

1.2. Log-logistic Distribution

The log-logistic distribution (LLD) also named as Fisk distribution is first presented and developed by Fisk (1961) for useful in survival analysis, economic, wealth, and networking diffusion time data applications. LLD is the special case of Burr-XII distribution. According to Gupta et al. (1999), the LLD is more useful in survival analysis because of its increasing and decreasing hazard function characteristics. The probability density function (p.d.f.) and cumulative distribution function (c.d.f.) of a continuous random variable X_i with $i = 1, 2, \ldots, n$ follows LLD with shape parameter β and scale parameter α as.

$$F(x; \alpha, \beta) = \frac{(x/\alpha)^{\beta}}{(1 + (x/\alpha)^{\beta})}, \qquad x, \alpha, \beta > 0, \tag{1.1}$$

and

$$f(x; \alpha, \beta) = \frac{(\beta/\alpha)(x/\alpha)^{\beta-1}}{[1 + (x/\alpha)^{\beta}]^2}, \qquad x, \alpha, \beta > 0. \tag{1.2}$$

The structure of the rest of the article is as follows: Overview of the neutrosophic notion and p.d.f., c.d.f. of LLD are presented in section 1. The formulation of the neutrosophic log-logistic distribution is established in Section 2. Neutrosophic reliability measures are covered in detail in section 3. In section 4, we address some neutrosophic entropies supported by statistical results and pictorial depictions. In section 5, parameter estimation like maximum likelihood estimation method is shown, section 6 represents the simualtion study in two part one is simualtion study for entorpies and secondly estimation process is validated by a simulation exercise. In Section 7, the established models and methodologies are applied to real-world neutrosophic datasets to show their usefulness. Section 7 of the publication provides an overview of the findings.

2. Formulation of Neutrosophic Log Logistic distribution.

The novel approach for neutrosophic random variable can be defined as $X_N = X_L + I_N X_L$ with $I_N \in (I_L, I_U)$, where X_L and $I_N X_L$ represent the determined and indetermined portion of the data. Assume that this neutrosophic random variable follows the neutrosophic log-logistic distribution with parameters α_N and β_N.The neutrosophic probability density function and neutrosophic cumulative distribution function of the LLD are given by

$$F_N(x_N; \alpha_N, \beta_N) = \frac{[(1 + I_N)x_L/\alpha_N]^{\beta_N}}{1 + [(1 + I_N)x_L/\alpha_N]^{\beta_N}}, \qquad x_N, \alpha_N, \beta_N > 0, \tag{2.1}$$

and

$$f_N(x_N; \alpha_N, \beta_N) = \left(\frac{\beta_N}{\alpha_N}\right) \frac{(1 + I_N)[(1 + I_N)x_L/\alpha_N]^{\beta_N - 1}}{[1 + \{(1 + I_N)x_L/\alpha_N\}^{\beta_N}]^2}, \qquad x_N, \alpha_N, \beta_N > 0 \tag{2.2}$$

Theorem 2.1. *Consider that the neutrosophic random variable $X_N \in (1+I_N)X_L$, where X_N follows the p.d.f. given in (2.2), is a valid p.d.f.*

Proof: Consider the p.d.f. given in (2.2), then integrating it over the domain of X_N as

$$\left(\frac{\beta_N}{\alpha_N}\right)\int_0^{\infty}\frac{(1+I_N)[(1+I_N)x_L/\alpha_N]^{\beta_N-1}}{[1+\{(1+I_N)x_L/\alpha_N\}^{\beta_N}]^2}dx = 1$$

Let $\left(\frac{(1+I_N)x_N}{\alpha_N}\right)^{\beta_N} = u$, substituting this into the above expression and simplifying it we get,

$$\int_0^{\infty}\frac{1}{(1+u)^2}du = 1$$

This integral is equal to one, hence it proved that the function given in (2.2) is a valid p.d.f. □

Theorem 2.2. *Let the random variable $X_N \in (1+I_N)X_L$ follows the p.d.f. given in (2.2) whose c.d.f. in (2.1) is a valid distribution function.*

Proof: A c.d.f. is a valid function if it fulfills the properties $F(-\infty)$ and $F(+\infty) = 1$. Let the c.d.f. in (2.1) and

$$\lim_{x\to 0} F_N(x_N;\alpha_N,\beta_N) = \frac{[(1+I_N)0/\alpha_N]^{\beta_N}}{1+[(1+I_N)0/\alpha_N]^{\beta_N}} = \frac{0}{1} = 0$$

$$\lim_{x\to \infty} F_N(x_N;\alpha_N,\beta_N) = \frac{1}{1+[(1+I_N)\infty/\alpha_N]^{\beta_N}} = \frac{1}{1+\frac{1}{\infty}} = 1$$

Hence, proved that (2.1) is a valid distribution function. □

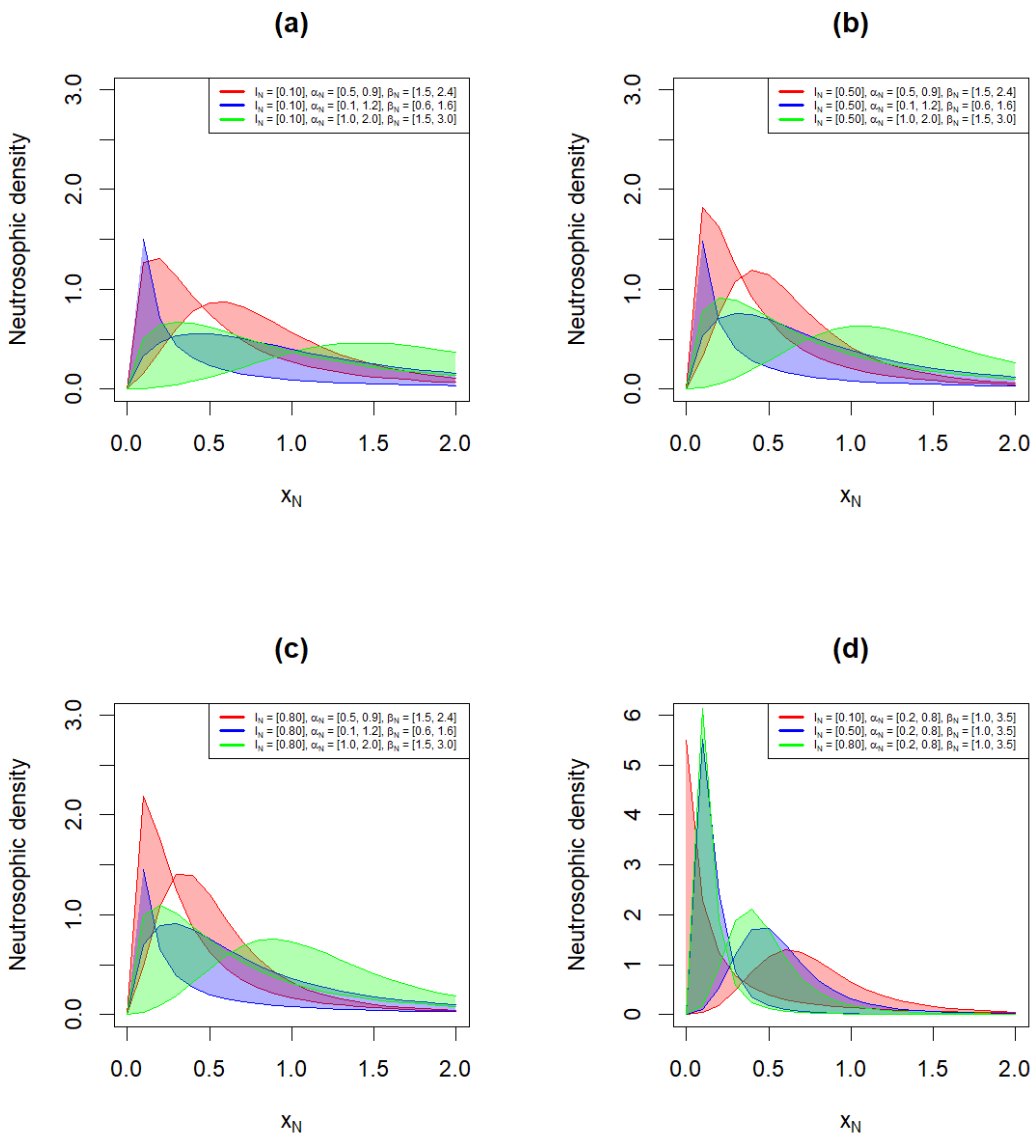

Figure 1: Density plots for the NLL distribution for different values of I_N, and parameters.

Figure 1 emphasize the impact of parameter changes on uncertainty representation by showing how parameters affect Neutrosophic density. Neutrosophic density functions for several values of I_N, α_N, and β_N, are shown. The density curves flatten as I_N, increases from 0.10 to 0.80 (figure a to c), showing less variation and less uncertainty between parameter settings. Particularly at smaller I_N, the red curve consistently displays the largest peak, with specified α_N and β_N values. All of the curves in the figure (d) have sharp peaks at lower x_N with the green curve displaying the maximum density and indicating higher density at lower x_N values.

3. Neutrosophic Reliability measures

In this section the various reliability related properties including the reliabiltiy fucniton, hazard rate fucntion (HRF), cumulaitve hazard function, reversed hazard fucntion, odd ratio, mills ratio and elascity for the NLLD has been derived. The neutrosophic moments are also discuused in this section.

The survival function of the neutrosophic LL distribution is derived as:

$$R(x) = \frac{1}{\left[1 + \left\{\frac{(1+I_N)x_L}{\alpha_N}\right\}\right]^{\beta_N}}. \tag{3.1}$$

The neutrosophic hazard function for the proposed model is:

$$h(x) = \left(\frac{\beta_N}{\alpha_N}\right) \frac{\left[\frac{(1+I_N)x_L}{\alpha_N}\right]^{\beta_N - 1}}{\left[1 + \left\{\frac{(1+I_N)x_L}{\alpha_N}\right\}^{\beta_N}\right]}. \tag{3.2}$$

The cumulative hazard function of the NLL distribution is obtained as:

$$H(x) = -\log\left[1 + \left(\frac{(1+I_N)x_L}{\alpha_N}\right)^{\beta_N}\right]. \tag{3.3}$$

The reversed hazard function is derived as:

$$r(x) = \left(\frac{\beta_N}{\alpha_N}\right) \frac{(1+I_N)}{\left[\frac{(1+I_N)x_L}{\alpha_N}\right]\left[1 + \left\{\frac{(1+I_N)x_L}{\alpha_N}\right\}^{\beta_N}\right]}. \tag{3.4}$$

The Odd ratio is obtained as given:

$$O(x) = \left[\frac{(1+I_N)x_L}{\alpha_N}\right]^{\beta_N}. \tag{3.5}$$

The Mills ratio of the model is derived as:

$$M(x) = \left(\frac{\beta_N}{\alpha_N}\right) \frac{\left[1 + \left\{\frac{(1+I_N)x_L}{\alpha_N}\right\}^{\beta_N}\right]}{(1+I_N)\left[\frac{(1+I_N)x_L}{\alpha_N}\right]^{\beta_N - 1}}. \tag{3.6}$$

The Elasticity of the NLL distribution is derived as:

$$\epsilon(x) = \left(\frac{\beta_N}{\alpha_N}\right) \frac{x_L(1+I_N)}{\left[\frac{(1+I_N)x_L}{\alpha_N}\right]\left[1 + \left\{\frac{(1+I_N)x_L}{\alpha_N}\right\}^{\beta_N}\right]}. \tag{3.7}$$

The neutrosophic rth moments for the NLLD is defined as:

$$\mu_r' = E[(1+I_N)X^r] = (1+I_N)^r \alpha^r B\left(1 - \frac{r}{\beta_N}, 1 + \frac{r}{\beta_N}\right).$$

The rth moment exist when $\frac{r}{\beta_N} < 1 \Rightarrow r < \beta_N$. So, the final expression is:

$$\mu_r' = E[(1+I_N)X^r] = \frac{(1+I_N)\alpha_N^r \pi \left(\frac{r}{\beta_N}\right)}{\sin\left(\frac{\pi r}{\beta_N}\right)}. \tag{3.8}$$

Theorem 3.1. *Consider the mean of a neutrosophic random variable is defined as $\mu_1' = E[(1+I_N)X]$. Substituting $r = 1$ in eq. (3.8) , the neutrosophic mean for the NLLD is obtained as:*

$$\mu_r' = E[(1+I_N)X].$$

$$\mu_1' = \frac{(1+I_N)\alpha_N \left(\frac{\pi}{\beta_N}\right)}{\sin\left(\frac{\pi}{\beta_N}\right)}. \tag{3.9}$$

Theorem 3.2. *The formula for the neutrosophic variance is given as:*

$$\sigma_N^2 = E[(1+I_N)X^2] - [E((1+I_N)X)]^2$$

To find $E[(1+I_N)X^2]$, take $r = 2$ in the rth moment expression as:

$$\mu_2' = E[(1+I_N)X^2] = \frac{(1+I_N)\alpha_N^2 \pi \left(\frac{2}{\beta_N}\right)}{\sin\left(\frac{2\pi}{\beta_N}\right)}$$

So, the neutrosophic variance is:

$$\sigma_N^2 = \frac{(1+I_N)\alpha_N^2 \pi \left(\frac{2}{\beta_N}\right)}{\sin\left(\frac{2\pi}{\beta_N}\right)} - \left[\frac{(1+I_N)\alpha_N \pi/\beta_N}{\sin\left(\frac{\pi}{\beta_N}\right)}\right]^2$$

or equivalenlty,

$$\sigma_N^2 = (1+I_N)\alpha_N^2 \left[\frac{\pi\left(\frac{2}{\beta_N}\right)}{\sin\left(\frac{2\pi}{\beta_N}\right)} - \frac{(1+I_N)\left(\frac{\pi}{\beta_N}\right)^2}{\sin^2\left(\frac{\pi}{\beta_N}\right)}\right], \quad \beta > 2 \tag{3.10}$$

4. Neutrosophic entropies

Neutrosophic Entropy is a measure of uncertainty that is frequently used in information theory to determine exact insights in unclear situations. It is rarely used to get accurate measurements and parameter estimations. Either there is uncertainty or confusion, or the value cannot be computed and measured exactly. In this section, we proposed four types of neutrosophic entropies of the NLL distribution that are neutrosophic Shannon Entropy, neutrosophic Rényi entropy, neutrosophic Tsallis Entropy and neutrosophic Arimoto Entropy.

4.1. Neutrosophic Shannon Entropy (NSE) of NLL distribution

The NLL distribution's Shannon entropy is suggested using the neutrosophic concept. The expression of neutrosophic Shannon entropy is obtained as:

$$H(x_N) = -\int_0^\infty f(x_N)\log(f(x_N))\,dx$$

or

$$H(X_N) = -\int_0^\infty \frac{\frac{\beta_N}{\alpha_N}(1+I_N)\left(\frac{(1+I_N)x_L}{\alpha_N}\right)^{\beta-1}}{\left[1+\left\{\frac{(1+I_N)x_L}{\alpha_N}\right\}^{\beta}\right]^2}\log\left(\frac{\frac{\beta_N}{\alpha_N}(1+I_N)\left(\frac{(1+I_N)x_L}{\alpha_N}\right)^{\beta-1}}{\left[1+\left\{\frac{(1+I_N)x_L}{\alpha_N}\right\}^{\beta}\right]^2}\right)dx \tag{4.1}$$

4.2. Neutrosophic Rényi entropy (NRE) of NLL distribution

The neutrosophic Rényi entropy is derived as:

$$R(x_N) = \frac{1}{1-v}\log\left[\int_0^\infty (f(x_N))^v\,dx\right], \quad v>0,\ v\neq 1$$

or

$$R(x_N) = \frac{1}{1-v}\log\int_0^\infty \left(\frac{\frac{\beta_N}{\alpha_N}(1+I_N)\left(\frac{(1+I_N)x_L}{\alpha_N}\right)^{\beta_N-1}}{\left[1+\left(\frac{(1+I_N)x_L}{\alpha_N}\right)^{\beta_N}\right]^2}\right)^v dx, \quad v>0,\ v\neq 1 \tag{4.2}$$

4.3. Neutrosophic Arimoto entropy (NAE) of NLL distribution

The neutrosophic arimoto entropy is derived as:

$$A(x_N) = \frac{1}{1-v} \log \left[\left\{ \int_0^{\infty} \left(f(x_N)\right)^v \, dx \right\}^{\frac{1}{v}} - 1 \right], \quad v > 0,\, v \neq 1$$

or

$$A(x_N) = \frac{1}{1-v} \log \left[\left\{ \int_0^{\infty} \left(\frac{\frac{\beta_N}{\alpha_N}(1+I_N)\left(\frac{(1+I_N)x_L}{\alpha_N}\right)^{\beta_N - 1}}{\left[1 + \left(\frac{(1+I_N)x_L}{\alpha_N}\right)^{\beta_N}\right]^2} \right)^v dx \right\}^{\frac{1}{v}} - 1 \right], \quad v > 0,\, v \neq 1 \tag{4.3}$$

4.4. Neutrosophic Tsallis entropy (NTE) of NLL distribution

The neutrosophic tsallis entropy is derived as:

$$T(x_N) = \frac{1}{v-1} \left[1 - \int_0^{\infty} \left(f(x_N)\right)^v \, dx \right], \quad v > 0,\, v \neq 1$$

or

$$T(x_N) = \frac{1}{v-1} \left[1 - \int_0^{\infty} \left(\frac{\frac{\beta_N}{\alpha_N}(1+I_N)\left(\frac{(1+I_N)x_L}{\alpha_N}\right)^{\beta_N - 1}}{\left[1 + \left\{\frac{(1+I_N)x_L}{\alpha_N}\right\}^{\beta_N}\right]^2} \right)^v dx \right], \quad v > 0,\, v \neq 1 \tag{4.4}$$

5. Parameter estimation

In this section, the maximum likelihood estimation is used to estimate the neutrosophic parameters for the NLLD and a Monte Carlo simulation study is presented to show the efficiency of the estimated parameters.

5.1. Maximum likelihood estimation method

Suppose that $(1+I_N)X_{N1}, (1+I_N)X_{N2}, \ldots, (1+I_N)X_{Nn}$, be a neutrosophic random samples of NLL distribution then log-likelihood function is derived as:

$$l(\alpha_N, \beta_N) = n\log(\beta_N) - n\log(\alpha_N) + (\beta_N - 1)\sum_{i=1}^{n}\log\left(\frac{(1+I_N)x_L}{\alpha_N}\right) \\ - 2\sum_{i=1}^{n}\log\left[1 + \left(\frac{(1+I_N)x_L}{\alpha_N}\right)^{\beta_N}\right] \tag{5.1}$$

To find the values of the parameters, taking derivative of the above expression with respect to α and β as:

$$\frac{\partial l(\theta)}{\partial \alpha_N} = -\frac{n}{\alpha_N} - (\beta_N - 1)\sum_{i=1}^{n}\left(\frac{1}{\alpha_N}\right) + 2\sum_{i=1}^{n}\frac{\beta_N\left(\frac{(1+I_N)x_L}{\alpha_N}\right)^{\beta_N}}{\alpha_N\left[1 + \left(\frac{(1+I_N)x_L}{\alpha_N}\right)^{\beta_N}\right]}, \tag{5.2}$$

and

$$\frac{\partial l(\theta)}{\partial \beta_N} = \frac{n}{\beta_N} + \sum_{i=1}^{n}\log\left(\frac{(1+I_N)x_L}{\alpha_N}\right) - 2\sum_{i=1}^{n}\frac{\left(\frac{(1+I_N)x_L}{\alpha_N}\right)^{\beta_N}\log\left(\frac{(1+I_N)x_L}{\alpha_N}\right)}{1 + \left(\frac{(1+I_N)x_L}{\alpha_N}\right)^{\beta_N}}. \tag{5.3}$$

The equations (5.2) and (5.3) are nonlinear and cannot be solved mathematically therefore RStudio and MATHEMATICA are used to get the estimated values of parameters for simulations and applications.

6. Simulation study

In order to assess the efficiency and performance of the proposed methods, we conduct a simulation study in this part.The two sub-sections of the simulation study are as follows: the first one addresses the proposed entropy measures, and the second one evaluates the efficacy of the Maximum Likelihood Estimation (MLE) technique for estimating the parameters for the NLLD.

6.1. Simulation study for entropy measures

In this section a simulation study for the proposed entropy measures is performed to check the efficacy of the proposed measures.

i. The theoretical entropy measures i.e. Shannon, Renyi, Tsallis and Arimoto are computed for selected interval values for the neutrosophic parameters of NLLD.

ii. We generated the neutrosophic random numbers of from the NLLD by using the quantile function by taking a sample of size $n = 1000$ from a size of $N = 10,000$.

iii. Then computed the estimates for the parameters α and β in the neutrosophic forms.

iv. And then putting the estimates into the equations (4.1), (4.2), (4.3), and (4.4), the proposed entropy measures are obtained.

v. This process is repeated 1000 random samples and with different parameters to assess the efficiency of the proposed measures.

vi. The I_{N_α} and I_{N_β} are calculated from the given formulas such as:

$$I_{N_\alpha} = \frac{\alpha_U - \alpha_L}{\alpha_U} \quad \text{and} \quad I_{N_\beta} = \frac{\beta_U - \beta_L}{\beta_U}$$

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Table 1: Neutrosophic Shannon Entropy

$\boldsymbol{I_{N_\alpha}, \alpha_N}$	$\boldsymbol{\beta_N}$	**NSE**
0.83, [0.50002, 2.9997]	[5.50853, 7.00636]	[-1.0055, 0.5456]
	[5.50612, 7.00816]	[-1.0051, 0.5453]
	[5.50471, 7.00992]	[-1.0048, 0.5451]
	[5.50614, 7.01071]	[-1.0051, 0.5449]
	[5.50490, 7.00769]	[-1.0049, 0.5454]
	[5.50608, 7.00962]	[-1.0051, 0.5451]
	[5.50864, 7.01162]	[-1.0050, 0.5454]
	[5.50745, 7.00727]	[-1.0053, 0.5454]
$\boldsymbol{I_{N_\beta}, \beta_N}$	$\boldsymbol{\alpha_N}$	**NSE**
0.21, [5.50471, 7.00992]	[0.50002, 2.9997]	[-0.5932, 0.9568]
	[0.50007, 3.00002]	[-0.5931, 0.9569]
	[0.50002, 2.99966]	[-0.5932, 0.9567]
	[0.50002, 3.00034]	[-0.5932, 0.9569]
	[0.50006, 3.00053]	[-0.5931, 0.9569]
	[0.50006, 3.00042]	[-0.5932, 0.9569]
	[0.50007, 2.99966]	[-0.5931, 0.9567]
	[0.49993, 3.00080]	[-0.5934, 0.9570]

Table 2: Neutrosophic Renyi, Tsallis and Arimoto entropies for $\nu = 0.5$

$\boldsymbol{I_{N_\alpha}, \alpha_N}$	$\boldsymbol{\beta_N}$	**NRE**	**NTE**	**NAE**
0.83, [0.50002, 2.9997]	[5.50853, 7.00636]	[-0.8396, 0.7007]	[-0.5431, 1.1143]	[-0.4693, 1.4247]
	[5.50612, 7.00816]	[-0.8392, 0.7005]	[-0.5427, 1.1139]	[-0.4691, 1.4240]
	[5.50471, 7.00992]	[-0.8389, 0.7002]	[-0.5425, 1.1134]	[-0.4689, 1.4233]
	[5.50614, 7.01071]	[-0.8392, 0.7001]	[-0.5427, 1.1132]	[-0.4691, 1.4230]
	[5.50490, 7.00769]	[-0.8389, 0.7005]	[-0.5425, 1.1140]	[-0.4689, 1.4242]
	[5.50608, 7.00962]	[-0.8391, 0.7002]	[-0.5427, 1.1135]	[-0.4691, 1.4235]
	[5.50864, 7.01162]	[-0.8396, 0.7000]	[-0.5431, 1.1130]	[-0.4693, 1.4227]
	[5.50745, 7.00727]	[-0.8394, 0.7006]	[-0.5429, 1.1141]	[-0.4692, 1.4244]
$\boldsymbol{I_{N_\beta}, \beta_N}$	$\boldsymbol{\alpha_N}$	**NRE**	**NTE**	**NAE**
0.21, [5.50471, 7.00992]	[0.50002, 2.9997]	[-0.4273, 1.1119]	[-0.2094, 1.8251]	[-0.1985, 2.6578]
	[0.50007, 3.00002]	[-0.4272, 1.1119]	[-0.2093, 1.8251]	[-0.1984, 2.6578]
	[0.50002, 2.99966]	[-0.4273, 1.1118]	[-0.2094, 1.8249]	[-0.1985, 2.6574]
	[0.50002, 3.00034]	[-0.4273, 1.1120]	[-0.2094, 1.8253]	[-0.1985, 2.6578]
	[0.50006, 3.00053]	[-0.4273, 1.1121]	[-0.2094, 1.8254]	[-0.1984, 2.6585]
	[0.50002, 3.00042]	[-0.4273, 1.1121]	[-0.2094, 1.8254]	[-0.1985, 2.6583]
	[0.50007, 2.99966]	[-0.4273, 1.1118]	[-0.2093, 1.8249]	[-0.1984, 2.6574]
	[0.49993, 3.00080]	[-0.4275, 1.1122]	[-0.2096, 1.8256]	[-0.1986, 2.6588]

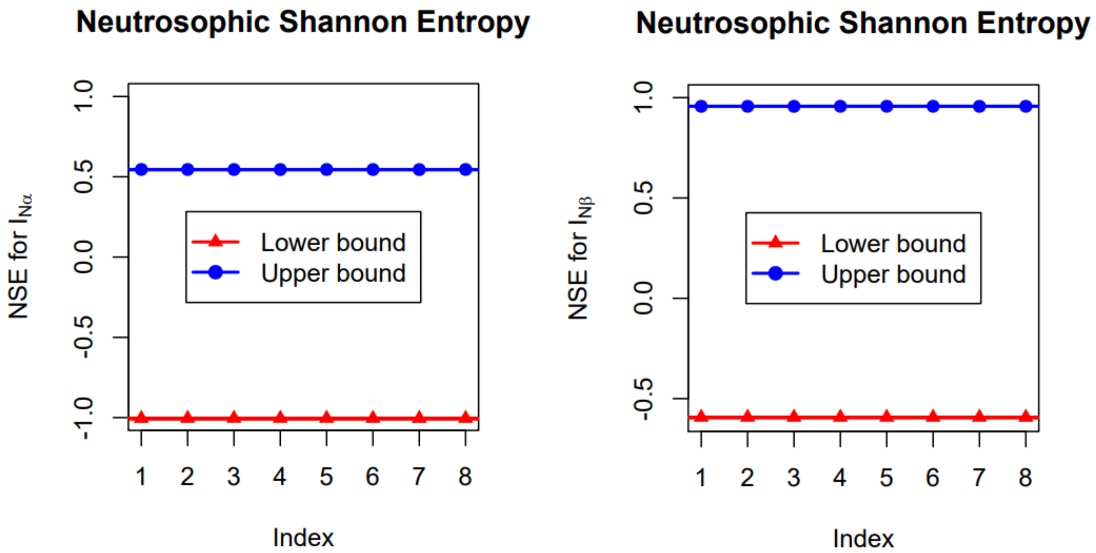

Figure 2: Plot of NSE for LLD.

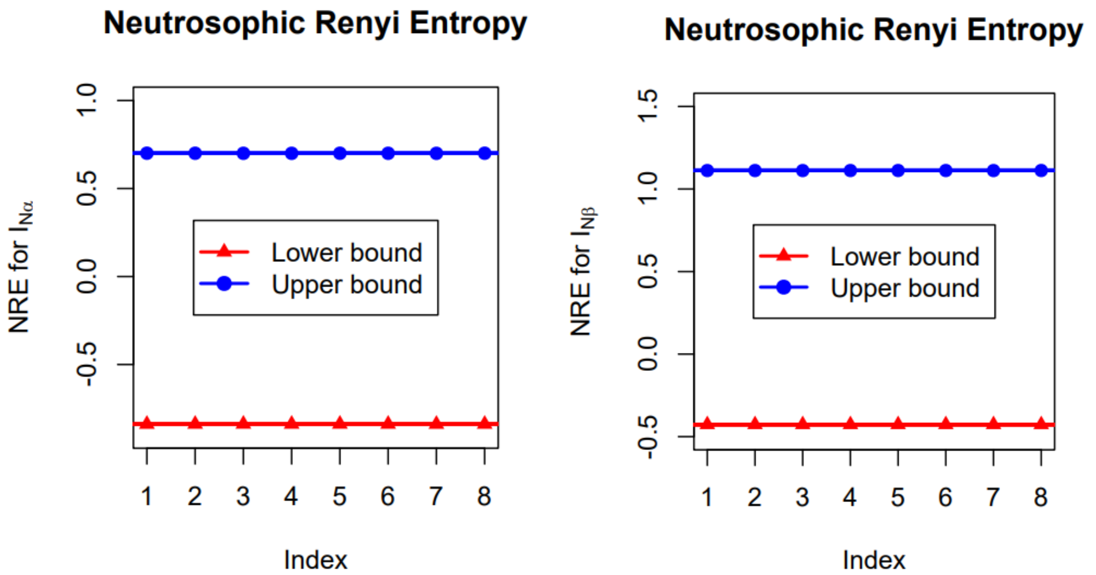

Figure 3: Plot of NRE for LLD.

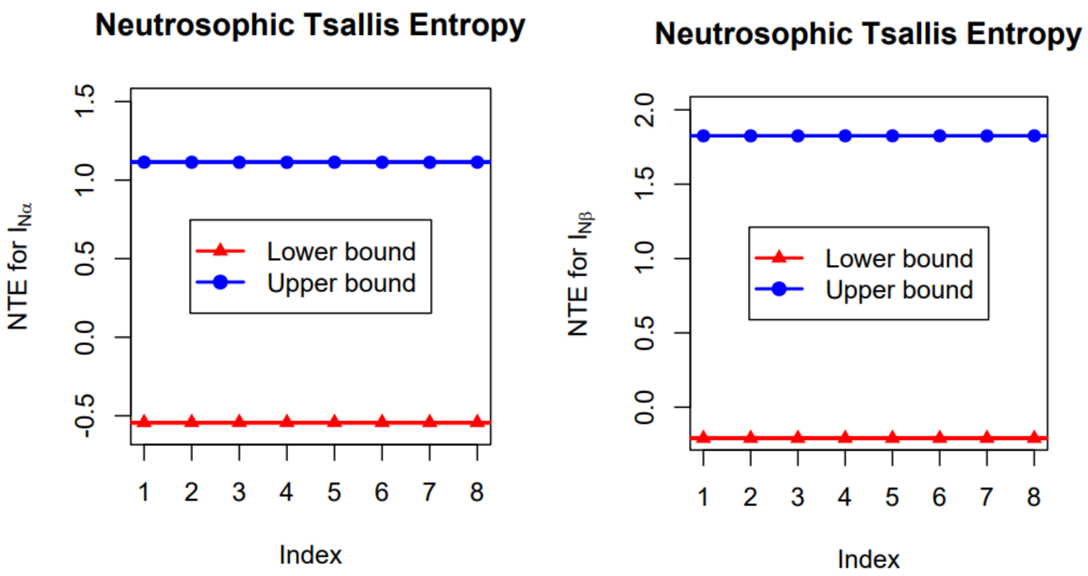

Figure 4: Plot of NTE for LLD.

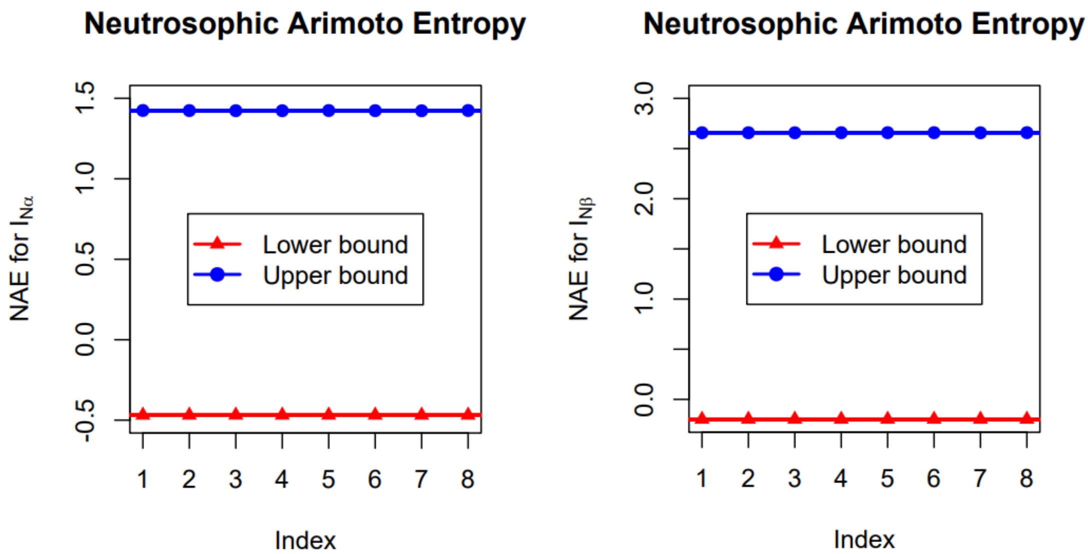

Figure 5: Plot of NAE for LLD.

Table 3: Classical Shannon Entropy (Shannon entropy for classical Lindley)

$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$	**SE**
0.10006	2.00279	-0.99653
	2.00181	-0.99604
	2.00256	-0.99644
	2.00270	-0.99648
	2.00284	-0.99653
	2.00129	-0.99578
	2.00262	-0.99641
	2.00357	-0.99692
2.00256	0.10002	-0.99681
	0.09999	-0.99711
	0.10006	-0.99641
	0.10004	-0.99651
	0.10004	-0.99661
	0.10004	-0.99631
	0.10002	-0.99651
	0.10007	-0.99631

Table 4: Classical Renyi, Tsallis and Arimoto entropies (Renyi, Tsallis and Arimoto for classical Lindley) for $\boldsymbol{\nu = 0.5}$

$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$	**RE**	**TE**	**AE**
0.10006	2.00279	-0.01611	-0.01604	-0.01598
	2.00181	-0.01485	-0.01479	-0.01474
	2.00256	-0.01581	-0.01575	-0.01569
	2.00270	-0.01599	-0.01593	-0.01586
	2.00284	-0.01617	-0.01610	-0.01604
	2.00129	-0.01418	-0.01413	-0.01408
	2.00159	-0.01582	-0.01576	-0.01570
	2.00357	-0.01710	-0.01703	-0.01696
2.00256	0.10000	-0.99681	-0.01615	-0.01607
	0.09999	-0.99711	-0.01644	-0.01637
	0.10006	-0.99641	-0.01585	-0.01578
	0.10004	-0.99651	-0.01611	-0.01605
	0.10004	-0.99661	-0.01611	-0.01607
	0.10004	-0.99631	-0.01598	-0.01597
	0.10002	-0.99651	-0.01614	-0.01608
	0.10007	-0.99631	-0.01615	-0.01609

In the Table 1 firstly for fixed value of α_N and I_N is calculated by α_N but varying the values of β_N, the NSE values are estimated, secondly for fixed values of β_N,and I_N is calculated by β_N, but varying the values of α_N, the NSE values are estimated. A similar trend is shown in the table to estimate the values of NTE, NRE and NAE. Rather in Tables 3-4, the same trend follows but taking $I_N = 0$ which means the resultant entropy are not neutrosophic.

From Tables 1-2 it is observed that for the interval values of parameters the entropies are also observed in intervals. The following pattern from Tables 1-2 is seen: as β increase NSE, NRE, NAE and NTE are decreases and as α the NSE, NRE, NAE and NTE are increases. Moreover, it is also observed as ν is from 0.5 to 0.7 the NRE, NTE and NAE are decreasing. While from Tables 3-4 it are observed that the entropies are without interval because it is simulated from the classical LLD. But in in case of interval data or ambiguous data the entropies should also be in interval form. Therefore, use of neutrosophic entropies is logical in ambiguous data sets. As a more flexible and accurate measure of disorder or uncertainty, neutrosophic entropies are better than classical distribution entropies because they take into consideration the indeterminacy and uncertainty in the data. What traditional entropies cannot manage, they successfully capture ambiguous or partial information. Since there is ambiguity and partial truth in real-world datasets, this makes them more appropriate. In graphical representation, Figures 2-5 provide upper and lower bounds that are consistent for all indices, and they depict several neutrosophic entropy measures (Shannon, Tsallis, Renyi, and Arimoto) for different indices. This consistency shows that these entropy measures reflect data uncertainty well and with little variance. The flexibility of neutrosophic models in handling data ambiguity is demonstrated by the stability of both boundaries across varying entropies.

6.2. Simulation study for MLE

The performance of the estimated parameters for the NLLD is evaluated by a Monte Carlo simulation study carried out in this section. By employing the neutrosophic root mean square error and the neutrosophic average biased, the performance of the neutrosophic maximum likelihood estimator is obtained. The mathematical expressions of the neutrosophic average bias and neutrosophic root mean square are:

$$AB_N = \frac{1}{N}\sum_{i=1}^{N}\left|\hat{\theta}_{Ni} - \theta_N\right|,$$

$$RMSE_N = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{\theta}_{Ni} - \theta_N\right)^2,$$

and

$$MRE = \frac{1}{N}\sum_{i=1}^{N}\left|\frac{\hat{\theta}_{Ni} - \theta_N}{\theta_N}\right|^2.$$

With the R language software, a Monte Carlo simulation is run with different sample sizes while keeping the neutrosophic parameters constant at $\alpha = [0.1, 0.5]$ and $\beta = [0.1, 0.5]$. An uncertain dataset with interval values of parameters is constructed using the NLL distribution. The simulation is repeated N = 10,000 times with sample sizes of n = 50, 100, 300, and 500, respectively. Next, a computation is made, and Tables 5-7 display the results of the neutrosophic maximum likelihood estimators' performance. In Table 5-6, $I_{N_\alpha}$is calculated from α_N, and I_{N_β} is calculated from β_N. While in Table 7, I_N is set as 0.

Table 5: Parameter's Bias, Average Bias, Mean Standard Error (MSE), and Mean Relative Error (MRE) for $I_{N\alpha}$ with $\alpha_N = [0.1, 0.5]$, $\beta_N = [1.0, 1.5]$, and $I_{N\alpha} = 0.88$ calculated from α_N.

Sizes	**MLE Estimates**	$I_{N\alpha}, \alpha_N = 0.88, [0.1, 0.5]$	$\beta_N = [1.0, 1.5]$
50	Bias	[0.1025, 0.5067]	[1.0260, 1.5383]
	Average Bias	[0.0198, 0.0663]	[0.0986, 0.1492]
	MSE	[0.0007, 0.0070]	[0.0163, 0.0363]
	MRE	[0.1981, 0.1326]	[0.0986, 0.0994]
100	Bias	[0.1017, 0.5020]	[1.0114, 1.5190]
	Average Bias	[0.0141, 0.0463]	[0.0679, 0.1027]
	MSE	[0.0003, 0.0034]	[0.0074, 0.0169]
	MRE	[0.1405, 0.0926]	[0.0679, 0.0685]
300	Bias	[0.1004, 0.5006]	[1.0049, 1.5063]
	Average Bias	[0.0079, 0.0267]	[0.0383, 0.0589]
	MSE	[0.0001, 0.0011]	[0.0023, 0.0054]
	MRE	[0.0792, 0.0534]	[0.0383, 0.0393]
500	Bias	[0.1002, 0.5003]	[1.0029, 1.5032]
	Average Bias	[0.0062, 0.0204]	[0.0300, 0.0454]
	MSE	[0.0001, 0.0007]	[0.0014, 0.0033]
	MRE	[0.0619, 0.0409]	[0.0300, 0.0303]

Table 6: Parameter's Bias, Average Bias, Mean Standard Error (MSE), and Mean Relative Error (MRE) for $\alpha_N = [0.1, 0.5]$, $\beta_N = [1.0, 1.5]$ and $I_{N\beta} = 0.33$ calculated from β_N.

Sizes	**MLE Estimates**	$\alpha_N = [0.1, 0.5]$	$I_{N\beta}, \beta_N = 0.33, [1.0, 1.5]$
50	Bias	[0.1030, 0.5074]	[1.0263, 1.5413]
	Average Bias	[0.0200, 0.0656]	[0.1000, 0.1473]
	MSE	[0.0007, 0.0069]	[0.0166, 0.0356]
	MRE	[0.2001, 0.1311]	[0.1000, 0.0982]
100	Bias	[0.1013, 0.5029]	[1.0131, 1.5187]
	Average Bias	[0.0139, 0.0465]	[0.0691, 0.1018]
	MSE	[0.0003, 0.0034]	[0.0077, 0.0166]
	MRE	[0.1393, 0.0931]	[0.0691, 0.0679]
300	Bias	[0.1005, 0.5000]	[1.0045, 1.5060]
	Average Bias	[0.0080, 0.0266]	[0.0383, 0.0591]
	MSE	[0.0001, 0.0011]	[0.0023, 0.0055]
	MRE	[0.0797, 0.0532]	[0.0383, 0.0394]
500	Bias	[0.1002, 0.5005]	[1.0027, 1.5041]
	Average Bias	[0.0062, 0.0205]	[0.0298, 0.0455]
	MSE	[0.0001, 0.0007]	[0.0014, 0.0033]
	MRE	[0.0620, 0.0409]	[0.0298, 0.0303]

Table 7: Classical Parameter's Bias, Average Bias, Mean Standard Error (MSE), and Mean Relative Error (MRE) for $\alpha_N = [0.1, 0.5]$, $\beta_N = [1.0, 1.5]$, and $I_N = 0$.

Sizes	**MLE Estimates**	$\alpha = 0.1$	$\beta = 1.0$
50	Bias	0.1029	1.0227
	Average Bias	0.0201	0.0985
	MSE	0.0007	0.0160
	MRE	0.2013	0.0985
100	Bias	0.1016	1.0118
	Average Bias	0.0140	0.0676
	MSE	0.0003	0.0074
	MRE	0.1401	0.0676
300	Bias	0.1006	1.0041
	Average Bias	0.0081	0.0390
	MSE	0.0001	0.0024
	MRE	0.0807	0.0390
500	Bias	0.1003	1.0027
	Average Bias	0.0063	0.0300
	MSE	0.0001	0.0014
	MRE	0.0625	0.0300

From Tables 5-6, it is observed that the resultant measures such as bias, average bias, MSE and MRE are also in the interval form due to neutrosophic parameters while in table the indeterminant factor I_N is obtained from α and in Table 6 this factor is obtained from β. It is also observed that as sample size increases the bias, MSE and MRE for parameters are reducing. From 7 demonstrates that the results are consistent with classical statistical measures and decrease in error with increasing sample size when the indeterminacy factor $I_N = 0$ indicates that there is no ambiguity in the data. This emphasizes that neutrosophic measurements are not seen when there is no uncertainty. Nonetheless, this emphasizes how crucial neutrosophic modeling is for datasets that are inherently uncertain, since it can handle uncertainty more skillfully and produce estimates that are more trustworthy. When uncertainty exists, neutrosophic technique is essential since it provides a notable benefit over conventional methods.

7. Applications

This section uses a practical application based on three real-world datasets to quantify interest in the NLL distribution when the data contains complicated or uncertain values. To determine which model is optimal, a few model selection techniques are employed to evaluate the suggested distribution's performance and contrast it with rival distributions. Presented below are the three datasets. Some model selection criteria, including the Consistent Akaike Information Criterion (CAIC), the Hannan-Quinn Information Criterion (HQIC), the Bayesian Information Criterion (BIC), and the Akaike Information Criterion (AIC). The lowest value is regarded as the best for each of these criteria. Additionally, the KS-test is employed.

It is important to clear that all three datasets are in neutrosophic form available on the referenced sources. In this research while modeling the densities for all the three datasets we used the lower values and upper values are not used rather upper values are calculated by applying the indeterminant factor (I_N) into it.

Covid-19 dataset: Dataset 1 shows a 30-day COVID-19 data from the Netherlands that was recorded between March 31, 2020, and April 30, 2020. The data consist of rough mortality rate taken from **?**.

[14.918, 15.66390]	[10.056, 11.18880]	[12.274, 12.88770]	[10.289, 10.80345]	[10.832, 11.37360]
[7.099, 7.45395]	[5.928, 6.22440]	[13.211, 13.87155]	[7.968, 8.36640]	[7.584, 7.96320]
[5.555, 5.83275]	[6.027, 6.32835]	[4.097, 4.30185]	[3.611, 3.79155]	[4.960, 5.20800]
[5.048, 5.30040]	[2.857, 2.99985]	[6.940, 7.28700]	[5.307, 5.57235]	[5.431, 5.70255]
[4.462, 4.68510]	[3.883, 4.07715]	[3.461, 3.63405]	[3.647, 3.82935]	[1.974, 2.07270]
[1.273, 1.33665]	[1.416, 1.48680]	[4.235, 4.44675]		

Alloy melting dataset: Dataset 2 represents the measurements of alloy melting points that are taken from Kacprzyk et al. (2018).

Batteries lifetime dataset: The third dataset concerns the batteries lifetime taken from Aslam (2020). Below is the lifetime of 23 batteries in 100h.

[563.3, 545.5]	[529.4, 511.6]	[523.1, 503.5]	[470.1, 449.2]	[506.7, 489.0]
[495.6, 479.1]	[495.3, 467.9]	[520.9, 495.6]	[496.9, 472.8]	[542.9, 519.1]
[505.4, 484.0]	[550.7, 525.9]	[517.7, 500.9]	[499.2, 483.0]	[500.6, 480.0]
[516.8, 499.6]	[535.0, 515.1]	[489.3, 464.4]		

[2.9, 3.99]	[12.65, 17.4]	[17.4, 23.93]	[26.07, 35.84]
[5.24, 7.2]	[13.24, 18.21]	[17.8, 24.48]	[30.29, 41.65]
[6.56, 9.02]	[13.67, 18.79]	[19.01, 26.14]	[43.97, 60.46]
[7.14, 9.82]	[13.88, 19.09]	[19.34, 26.59]	[48.09, 66.13]
[11.6, 15.96]	[15.64, 21.51]	[23.13, 31.81]	[73.48, 98.04]
[12.14, 16.69]	[17.05, 23.45]	[23.34, 32.09]	

Table 8: Neutrosophic statistics of datasets by using the proposed distribution for $I_N = 0.05$.

Descriptives	COVID-19	Alloy Melting	Lifetime Batteries
Mean	[6.5389, 6.8659]	[6.95e-06, 6.66e-05]	[21.2216, 29.1340]
Variance	[32.2161, 36.3698]	[264269.6, 243959.7]	[542.8737, 1009.091]
Median	[5.3231, 5.5962]	[512.9543, 492.6982]	[16.3864, 22.5306]
Lower Quartile	[3.6385, 3.8186]	[498.6709, 478.0587]	[10.7213, 14.7589]
Upper Quartile	[7.7876, 8.2011]	[527.6469, 507.7860]	[25.0450, 34.3947]
Skewness	[11.2136, 9.3485]	[1.0033, 1.1311]	[6.2431, 2.4635]
Kurtosis	[612.0019, 582.1537]	[-1.9912, -1.9900]	[116.4486, 68.9212]

7.1. Discussion

Table 8 represents the descriptive measures for the three data sets. The average time by mean and median are described in the interval form. Such as the average mortality rate for the COVID-19 data is between the interval [6.5389, 6.8659] by using mean or [5.3231, 5.5962] by using the median. The variance is also lying in the interval. Lower quartile shows that 25% of the patients has equal to or less than the mortality rate between the interval [3.6385, 3.8186] and 75% has equal to or less than the mortality rate between the interval [7.7876, 8.2011]. The interval values of skewness and kurtosis show that the COVID-19 data is heavily positively skewed and leptokurtic. Similarly, the descriptions can be explained from the alloy melting and lifetime of batteries data sets. The average alloy melting point lies between the interval $[6.96 \times 10^{-6}, 6.66 \times 10^{-5}]$ or [512.9543, 492.6982] by using the mean or median, respectively. The alloy melting point data is slightly positively skewed and platykurtic. The average lifetime of batteries between the interval [21.2216, 29.1340] or [16.3864, 22.5306] by using the mean or median, respectively. 25% of the batteries have lifetime less than or equal to between the interval [10.7213, 14.7589] and 75% have less than or equal to between the interval [25.0450, 34.3947]. Lifetime of batteries data is positively skewed and leptokurtic. From these findings it is clear that when the data is not clear and has uncertainty between the values then the resultant measures should also be in the uncertain forms as lying between the two ranges or bounds which can only be modeled by the neutrosophic probability distributions unlike classical probability distribution. Table 9 shows the estimated parameters from the three data sets. Tables 10-12 demonstrate the modeling of the NLL distribution on the three data sets mortality rate of COVID-19, alloy melting point, and lifetime of batteries respectively. The proposed NLLD is not only fulfilling the goodness fit criteria but also showing more compatibility and flexibility over the other neutrosophic distribution. From Table 9 it is seen that NLLD shows lesser values of goodness of fit and higher P-values than the other competitive model which shows that it performs better fit than the neutrosophic Burr-III distribution. From Table 11, even the proposed distribution fulfills the goodness of fit criteria for the alloy melting point data, but the neutrosophic Burr-III distribution does not provide good fit criteria which shows the applicability of the NLLD in such data sets. From Table 12, the NLLD is more compatible over the others. It is also seen that classical Burr-XII does not show the goodness of fit for all three neutrosophic data sets due to the reason that classical probability distributions are not appropriate to model the neutrosophic data sets.

8. Conclusion

A novel neutrosophic Log Logistic distribution is proposed in this research article due to its wide applications in the lifetime datasets. The classical LLD is widely used in survival analysis, economic, wealth and networking datasets but in many realistic scenarios the true and exact value for the observation is not possible to record and, in such cases, modeling classical LLD is not appropriate. Rao (2023) introduced the NLLD to handle the situations when the dataset is ambiguous between the two interval values. In many cases when the dataset is ambiguous, and the values are not lying between the two bounds/intervals rather the values or recorded observation has indeterminacy of appearing with lower bound only and upper bound is unknow. For example, if the room temperature is recorded in a way that the observations fluctuate, and it is recorded as 300C or more then lower bound is certain, but uncertainty or indeterminacy is with the upper bound. To handle such type of indeterminant dataset, the NLLD is proposed with various of its neutrosophic properties, neutrosophic reliability measures, neutrosophic moments, neutrosophic entropies with simulation study are presented in the article. The neutrosophic parameters of the proposed density are estimated by MLE and to assess the performance a simulation study is conducted with the results that as sample size increases the bias, MSE and MRE are reducing. Finally, the proposed NLLD is modeled on three lifetime datasets mortality rate of COVID-19 in Netherland, alloy melting point, and lifetime of batteries. NLLD is more flexible and optimal than the neutrosophic Burr-III distribution. It is also seen that the classical Burr-XII distribution depicts very low p value for KS test and illustrates that modeling of classical probability distributions are not appropriate for the neutrosophic datasets.

Table 9: ML estimates and standard errors for the three datasets (first column shows estimates and second column shows the standard errors).

Model	Par.	COVID data		Alloy melting dataset		Batteries lifetime dataset	
NLL	α	[5.5893, 5.8687]	[0.6149, 0.6456]	[538.612, 565.507]	[5.7283, 6.0084]	[17.2057, 23.6572]	[5.7283, 6.0084]
	β	[2.8875, 2.8875]	[0.4404, 0.4404]	[38.956, 38.968]	[7.5797, 7.5820]	[2.5897, 2.597]	[7.5797, 7.5820]
NBurr-III	θ	[1.033, 1.0232]	[0.0601, 0.0591]	[0.9694, 1.0595]	[0.0914, 0.0966]	[1.3736, 1.3714]	[0.0914, 0.0966]
	λ	[0.0601, 0.0591]	[4.3325, 4.5161]	[370.2229, 704.2702]	[197.1436, 403.3048]	[30.7449, 32.6433]	[197.1436, 403.3048]
Burr-XII	θ	21.4342	32.6983	1.8086	9.0505	6.3318	9.0505
	λ	0.0283	0.0432	9.0505	0.0886	0.0567	0.0886
LL	λ	5.5893	0.6149	538.612	5.7283	17.2057	5.7283

Table 10: Model Selection Criteria for the COVID-19 Dataset

Model	ℓ	AIC	BIC	CAIC	HQIC	KS-value	P-value
NLL,$I_N = [0.05]$	$[-77.318, -78.964]$	$[158.637, 161.928]$	$[161.439, 164.731]$	$[159.081, 162.373]$	$[159.533, 162.825]$	$[0.063, 0.069]$	$[0.999, 0.997]$
NBurr-III,$I_N = [0, 0.05]$	$[-80.267, -81.901]$	$[164.534, 167.803]$	$[167.337, 170.605]$	$[164.979, 168.247]$	$[165.431, 168.699]$	$[0.139, 0.139]$	$[0.563, 0.5616]$
Burr-XII	-94.298	192.595	195.397	193.039	193.492	0.362	0.000
LL	-77.318	158.637	161.439	159.081	159.533	0.063	0.999

Table 11: Model Selection Criteria for the Alloy melting dataset

Model	ℓ	AIC	BIC	CAIC	HQIC	KS-value	P-value
NLL,$I_N = [0.05]$	$[-82.143, -82.581]$	$[168.287, 169.162]$	$[170.067, 170.943]$	$[169.087, 169.962]$	$[168.532, 169.408]$	$[0.101, 0.117]$	$[0.942, 0.984]$
NBurr-III,$I_N = [0, 0.05]$	$[-131.240, -130.440]$	$[266.480, 264.881]$	$[268.260, 266.662]$	$[267.280, 265.681]$	$[266.725, 265.126]$	$[0.532, 0.538]$	$[0.000, 0.000]$
Burr-XII	-163.3192	330.6385	332.4192	331.4385	330.884	0.627	0.000
LL	-82.143	168.287	170.067	169.087	168.532	0.101	0.942

Table 12: Model Selection Criteria for the Batteries lifetime dataset

Model	ℓ	**AIC**	**BIC**	**CAIC**	**HQIC**	**KS-value**	**P-value**
NLL,$I_N = [0.05]$	$[-88.391, -95.611]$	$[180.781, 195.222]$	$[183.052, 197.493]$	$[181.381, 195.822]$	$[181.352, 195.793]$	$[0.116, 0.116]$	$[0.880, 0.881]$
NBurr-III,$I_N = [0, 0.05]$	$[-91.172, -92.307]$	$[186.344, 188.613]$	$[188.615, 190.885]$	$[186.944, 189.214]$	$[186.915, 189.185]$	$[0.203, 0.203]$	$[0.263, 0.262]$
Burr-XII	-110.5761	225.1521	227.4231	225.7521	225.7233	0.4115	0.000
LL	-88.391	180.781	183.052	181.381	181.352	0.116	0.880

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