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# PARAMETER ESTIMATION FOR INAR PROCESSES BASED ON HIGH-ORDER STATISTICS

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Abstract:

- The high-order statistics (moments and cumulants of order higher than two) have been widely applied in several fields, specially in problems where it is conjectured a lack of Gaussianity and/or non-linearity. Since the INteger-valued AutoRegressive, INAR, processes are non-Gaussian, the high-order statistics can provide additional information that allows a better characterization of these processes. Thus, an estimation method for the parameters of an INAR process, based on Least Squares for the third-order moments is proposed. The results of a Monte Carlo study to investigate the performance of the estimator are presented and the method is applied to a set of real data.

Key-Words:

- *INAR process; estimation; high-order statistics.*

AMS Subject Classification:

- 62M10, 62F10.

## 1. INTRODUCTION

In the recent past, the high-order statistics (HOS) have been widely applied in several fields. By HOS it is meant the moments and cumulants of order higher than two, in the time domain, and the corresponding multidimensional Fourier transform (polyspectrum), in the frequency domain. In this work, the time domain approach is considered. The HOS comprise information about stochastic processes such as the degree of nonlinearity and deviations from Gaussianity that is not contained in the second-order statistics.

Let $\{X_t\}$ be a $k$-th-order stationary stochastic process. The $k$-th-order joint moment of $X_t, X_{t+s_1}, ..., X_{t+s_{k-1}}$, for $s_1, ..., s_{k-1} \in \mathbb{R}$, is a function of $k-1$ variables defined by

$$\mu_X(s_1, ..., s_{k-1}) \;=\; \mathrm{E}[X_t\, X_{t+s_1} \dots X_{t+s_{k-1}}] \ ,$$

with $\mu_X = \mathrm{E}[X_t]$. For a stationary stochastic process, the moments have the following symmetry properties:

$$\mu_X(m) \;=\; \mu_X(-m) \ , \qquad m > 0 \ ,$$
$$\mu_X(m,n) \;=\; \mu_X(n,m) \;=\; \mu_X(-n, m-n) \;=\; \mu_X(n-m, -m) \ , \qquad m,n > 0 \ .$$

Then, it follows that the third-order moments over the entire plane may be obtained from the values of the third-order moments over the infinite wedge bounded by the straight lines $m = 0$ and $m = n$, $m,n > 0$.

Recently, the integer-valued autoregressive process has been proposed in the literature to model time series of counts. The $p$-th-order integer-valued autoregressive, INAR($p$), process is defined as a discrete time non-negative integer-valued stochastic process, $\{X_t\}$, that satisfies the following equation (Latour, 1998):

$$X_t \;=\; \alpha_1 \circ X_{t-1} + \alpha_2 \circ X_{t-2} + \cdots + \alpha_p \circ X_{t-p} + e_t \ , \tag{1.1}$$

where

1. $\{e_t\}$, designated the innovation process, is a sequence of independent and identically distributed (i.i.d.) non-negative integer-valued random variables with $\mathrm{E}[e_t] = \mu_e$, $\mathrm{Var}[e_t] = \sigma_e^2$ and $\mathrm{E}[e_t^3] = \gamma_e$;
2. the symbol $\circ$ represents the thinning operation (Steutel and Van Harn, 1979; Gauthier and Latour, 1994), defined by

   $$\alpha_i \circ X_{t-i} \;=\; \sum_{j=1}^{X_{t-i}} Y_{i,j}, \qquad \text{for} \;\; i = 1, ..., p \ ,$$

   where $\{Y_{i,j}\}$, designated the counting series, is a set of i.i.d. non-negative integer-valued random variables such that $\mathrm{E}[Y_{i,j}] = \alpha_i$, $\mathrm{Var}[Y_{i,j}] = \sigma_i^2$ and $\mathrm{E}[Y_{i,j}^3] = \gamma_i$. All the counting series are assumed independent of $\{e_t\}$;

3. $0 \leq \alpha_i < 1$, $i = 1, ..., p-1$, and $0 < \alpha_p < 1$. Note that the stationarity condition for the INAR($p$) process is that $\sum_{k=1}^{p} \alpha_k < 1$.

A special case is the Poisson INAR process with binomial thinning operation, where $\{e_t\}$ has a Poisson distribution with parameter $\lambda$ and the counting series, $\{Y_{i,j}\}$, are a set of Bernoulli random variables with $P(Y_{i,j}=1) = 1 - P(Y_{i,j}=0) = \alpha_i$.

Since the INAR models are non-Gaussian, the HOS can provide additional information in the characterization of these processes. Thus, an estimation method for the parameters of an INAR model that uses HOS is proposed in this work. This approach applies the Least Squares estimation method to minimize the errors between the third-order moment of the observations and of the fitted model.

This work is organized as follows: in Section 2 the third-order characterization of INAR($p$) models is provided and the proposed Least Squares Estimation method based on HOS (LS_HOS) is described. In Section 3 the results of a simulation study to assess the small sample properties of the proposed estimator are given and the method is applied to a set of observations concerning the number of plants within the industrial sector in Section 4. Finally, some remarks are presented in Section 5.

## 2. PARAMETER ESTIMATION BASED ON HOS

### 2.1. Third-order characterization of INAR($p$) models

The third-order characterization, in terms of moments and cumulants, of INAR models has been obtained by Silva and Oliveira (2004, 2005) and Silva (2005). In particular, the third-order moments of an INAR($p$) process, defined by (1.1), satisfy a set of Yule–Walker type equations similar to those satisfied by the bilinear process, that can be written as:

$$
\begin{aligned}
\mu_X(0,0) \;=\; & \sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p} \alpha_i\,\alpha_j\,\alpha_k\,\mu_X(i-j,i-k) \\
& +\; 3\sum_{i=1}^{p}\sum_{j=1}^{p} \alpha_j\,{\sigma_i}^2\,\mu_X(i-j) \;+\; 3\,\mu_X(\sigma_e^2+{\mu_e}^2)\sum_{i=1}^{p}\alpha_i \\
& +\; 3\,\mu_e\sum_{i=1}^{p}\sum_{j=1}^{p}\alpha_i\,\alpha_j\,\mu_X(i-j) \;+\; 3\,\mu_X\,\mu_e\sum_{i=1}^{p}{\sigma_i}^2 \\
& +\; \mu_X\sum_{i=1}^{p}(\gamma_i - 3\,\alpha_i\,{\sigma_i}^2 - \alpha_i^3) \;+\; \gamma_e\;,
\end{aligned}
\tag{2.1}
$$

$$\text{(2.2)} \qquad \mu_X(0,k) \;=\; \sum_{i=1}^{p} \alpha_i\, \mu_X(0,k-i) \;+\; \mu_e\, \mu_X(0) \;, \qquad k>0 \;,$$

$$\text{(2.3)} \qquad \begin{aligned} \mu_X(k,k) \;=\; & \sum_{i=1}^{p}\sum_{j=1}^{p} \alpha_i\,\alpha_j\, \mu_X(k-i,k-j) \;+\; \sum_{i=1}^{p} {\sigma_i}^2 \mu_X(k-i) \\ & +\; 2\,\mu_e\, \mu_X(k) - \mu_X({\mu_e}^2 - {\sigma_e}^2) \;, \qquad k>0 \;, \end{aligned}$$

$$\text{(2.4)} \qquad \mu_X(k,m) \;=\; \sum_{i=1}^{p} \alpha_i\, \mu_X(k,m-i) \;+\; \mu_e\, \mu_X(k) \;, \qquad m>k>0 \;,$$

where

$$\text{(2.5)} \qquad \mu_X(0) \;=\; \sum_{i=1}^{p} \alpha_i\, \mu_X(i) \;+\; \mu_e\, \mu_X + V_p \;,$$

is the second-order moment of $\{X_t\}$, with

$$V_p \;=\; {\sigma_e}^2 + \mu_X \sum_{i=1}^{p} {\sigma_i}^2 \;,$$

which represents the variance of the one-step-ahead prediction error (Silva, 2005).

These equations indicate that the INAR processes have a non-linear structure, therefore the first- and second-order moments are not sufficient to describe the dependence structure of the process. In the next section, is described an estimation method for the parameters of an INAR($p$) process that uses the additional information provided by the HOS.

## 2.2. Least squares estimation based on HOS

Let $\{x_1, x_2, ..., x_n\}$ be a realization of a non-negative integer-valued stationary stochastic process with third-order moments $\mu(0,k)$, $k>0$. The approximating model considered is an INAR($p$) process (order known) with parameters $\alpha_1, ..., \alpha_p$, $\mu_e$, $\sigma_e^2$ and third-order moments $\mu_X(0,k)$, $k>0$, satisfying (2.2), which can be represented in the following matrix form

$$\text{(2.6)} \qquad \boldsymbol{\mu}_{3,X} = \mathbf{M}_{3,X}\, \boldsymbol{\alpha} + \mu_e\, \mu_X(0)\, \mathbf{1}_p \;,$$

where $\boldsymbol{\mu}_{3,X}$ is defined as

$$\boldsymbol{\mu}_{3,X} = \big[\; \mu_X(0,1) \;\; \cdots \;\; \mu_X(0,p) \;\big]^{\mathrm{T}} \,,$$

$\mathbf{M}_{3,X}$ is the $p \times p$ non-symmetric Toeplitz matrix of the third-order moments of the INAR($p$) process

$$\mathbf{M}_{3,X} = \begin{bmatrix} \mu_X(0,0) & \mu_X(1,1) & \cdots & \mu_X(p-1,p-1) \\ \mu_X(0,1) & \mu_X(0,0) & \cdots & \mu_X(p-2,p-2) \\ \vdots & \vdots & \ddots & \vdots \\ \mu_X(0,p-1) & \mu_X(0,p-2) & \cdots & \mu_X(0,0) \end{bmatrix},$$

with $\mu_X(\cdot,\cdot)$ given in (2.1) to (2.4), $\boldsymbol{\alpha} = [\,\alpha_1 \;\cdots\; \alpha_p\,]^{\mathrm{T}}$ is the vector of coefficients, $\mu_X(0)$ is the second-order moment of the INAR($p$) process given in (2.5) and $\mathbf{1}_p$ is a $p \times 1$ vector of ones.

Defining

$$\mathbf{H} = \big[\mathbf{M}_{3,X} \;\; \mu_X(0)\mathbf{1}_p\big] \qquad \text{and} \qquad \boldsymbol{\theta} = \big[\,\alpha_1 \;\cdots\; \alpha_p \;\, \mu_e\,\big]^{\mathrm{T}},$$

equation (2.6) can be rewritten as

$$\boldsymbol{\mu}_{3,X} = \mathbf{H}\,\boldsymbol{\theta}\ ,$$

suggesting that $\boldsymbol{\theta}$ may be estimated by least squares, i.e., minimizing the squared error between the third-order moments of the fitted INAR($p$) model, $\boldsymbol{\mu}_{3,X}$, and the third-order moments of the data,

$$\boldsymbol{\mu}_3 = \big[\,\mu(0,1) \;\cdots\; \mu(0,p)\,\big]^{\mathrm{T}}.$$

Thus, $\hat{\boldsymbol{\theta}}$, the Least Squares estimator of $\boldsymbol{\theta}$ based on HOS (LS_HOS) satisfies

$$\hat{\boldsymbol{\theta}} = \min_{\boldsymbol{\theta}}\big\{L^*(\boldsymbol{\theta})\big\}$$

where

$$L^*(\boldsymbol{\theta}) = (\boldsymbol{\mu}_3 - \mathbf{H}\,\boldsymbol{\theta})^{\mathrm{T}}\,(\boldsymbol{\mu}_3 - \mathbf{H}\,\boldsymbol{\theta})\ .$$

In practice, the estimator is calculated by substituting the moments in $\boldsymbol{\mu}_3$ and $\mathbf{H}$ by their sample counterparts, using the usual estimators of the moments

$$\hat{\mu}_X(0) = \frac{1}{N}\sum_{t=1}^{N} {X_t}^2, \quad \hat{\mu}_X(0,k) = \frac{1}{N}\sum_{t=1}^{N-k} {X_t}^2 X_{t+k}\,, \quad \hat{\mu}_X(k,k) = \frac{1}{N}\sum_{t=1}^{N-k} X_t\, {X_{t+k}}^2\ .$$

Thus,

$$\hat{\boldsymbol{\theta}} = \min_{\boldsymbol{\theta}}\big\{\hat{L}^*(\boldsymbol{\theta})\big\} = \min_{\boldsymbol{\theta}}\big\{(\hat{\boldsymbol{\mu}}_3 - \hat{\mathbf{H}}\,\boldsymbol{\theta})^{\mathrm{T}}\,(\hat{\boldsymbol{\mu}}_3 - \hat{\mathbf{H}}\,\boldsymbol{\theta})\big\}\ .$$

Note that an estimator for $\sigma_e^2$ can be obtained by

$$\hat{\sigma}_e^2 = \hat{V}_p - \overline{X}\sum_{i=1}^{p}\hat{\sigma}_i^2\ ,$$

where $\overline{X}$ is the sample mean of the observations, $\hat{\sigma}_i^2$ is an estimator of the counting series variance for the $i$-th thinning operation, $\alpha_i \circ X_{t-i}$, $i=1,...,p$, and $\hat{V}_p = \hat{R}(0) - \sum_{i=1}^{p} \hat{\alpha}_i \hat{R}(i)$, with $\hat{R}(i) = \frac{1}{N} \sum_{t=1}^{N-i} (X_t - \overline{X})(X_{t+i} - \overline{X})$, representing the sample autocovariance function. The estimation of $\hat{\sigma}_i^2$ depends on the distribution of the counting series, for instance, in the case of the binomial thinning operation (when the counting series are Bernoulli distributed), $\hat{\sigma}_i^2 = \hat{\alpha}_i(1-\hat{\alpha}_i)$, for $i = 1,...,p$.

The asymptotic distribution of the LS_HOS estimator depends on the sixth-order moments and cumulants of the processes, and therefore is too complex and not useful in practice. So, the finite sample properties of the estimator are investigated by a simulation study, which results are presented in the next section.

## 3. MONTE CARLO RESULTS

The aim of the simulation study presented in this section is twofold: to examine the small sample properties of the estimator previously described and compare its performance with other estimation methods for the parameters of an INAR process.

Thus, 1000 realizations of Poisson INAR($p$) processes ($e_t \sim \mathcal{P}o(\lambda)$) with binomial thinning operation are generated, for $p = 0,...,3$. The sample size, $N$, and parameters values considered are:

- $N = 50$, 200, 500 and 1000 observations,
- $\lambda \in \{1.0, 3.0\}$,
- for $p=1$, $\alpha_1 \in \{0.1, 0.4, 0.6, 0.9\}$,
- for $p=2$, $(\alpha_1, \alpha_2) \in \{(0.1,0.6),(0.6,0.1),(0.3,0.4),(0.4,0.3),(0.1,0.1),(0.4,0.4)\}$,
- for $p=3$, $(\alpha_1, \alpha_2, \alpha_3) \in \{(0.1,0.1,0.4), (0.1,0.4,0.1), (0.4,0.1,0.1), (0.3,0.3,0.3)\}$.

For each realization, the estimation methods used to obtain $\hat{\boldsymbol{\theta}} = [\hat{\alpha}_1,...,\hat{\alpha}_p,\hat{\mu}_e]^{\mathrm{T}}$ are Yule–Walker estimation (YW), Conditional Least Squares estimation (CLS), Whittle estimation (WHT) and unconstrained and constrained Least Squares estimation based on HOS (LS_HOS and LS_HOS_C). For a detailed description of the YW, CLS and WHT estimation methods see Silva (2005). The constraints considered are $0 < \alpha_i < 1$, $i=1,...,p$, $\sum_{i=1}^{p} \alpha_i < 1$ and ${\sigma_e}^2 > 0$. The initial values of the iterative methods are the YW estimates.

The unconstrained minimizations necessary in the methods CLS, WHT and LS_HOS are performed through the MATLAB function *fminunc*, which finds a minimum of a scalar unconstrained multivariable function by using the BFGS

Quasi-Newton method with a mixed quadratic and cubic line search procedure (MathWorks (2004)). The constrained minimization of the method is accomplished by the MATLAB function *fmincon*, which finds a minimum of a scalar constrained nonlinear multivariable function by using a Sequential Quadratic Programming method (MathWorks (2004)).

For each case, the following sample statistical measures are evaluated:

- mean bias: $\overline{\mathrm{Bias}}(\hat{\theta}_i) = \dfrac{1}{N}\displaystyle\sum_{j=1}^{N}(\hat{\theta}_i^{(j)} - \theta_i)$,
- variance: $\mathrm{Var}(\hat{\theta}_i) = \dfrac{1}{N-1}\displaystyle\sum_{j=1}^{N}(\hat{\theta}_i^{(j)} - \overline{\theta}_i)^2$,
- mean square error: $\mathrm{MSE}(\hat{\theta}_i) = \dfrac{1}{N-1}\displaystyle\sum_{j=1}^{N}(\hat{\theta}_i^{(j)} - \theta_i)^2$,

where $\hat{\theta}_i^{(j)}$ represents the parameters estimates, $\hat{\alpha}_1, ..., \hat{\alpha}_p, \hat{\mu}_e$, in the $j$-th repetition, for $j=1, ..., N=1000$, and $\overline{\theta}_i = \frac{1}{N}\sum_{j=1}^{N}\hat{\theta}_i^{(j)}$ is its sample mean.

With respect to the small sample properties of the LS_HOS and LS_HOS_C estimators, the following conclusions can be drawn from the analysis of all the simulations. In general, the sample bias, variance and mean square error decrease as the sample size increases, indicating that the distribution of the estimators is consistent and symmetric. However, for a small sample size there is evidence of departure from symmetry in the marginal distributions, specially for values of the parameters near the non-stationary region.

Table 1 presents numerical results for two INAR(1) processes, with parameter values $\boldsymbol{\theta} = (\alpha_1, \lambda) = (0.1, 3.0)$ and $\boldsymbol{\theta} = (0.6, 3.0)$, respectively, and two different sample sizes: $N = 50$ and 500. As expected, the results for the unconstrained and constrained estimations only differ when the value of the coefficient is near of the non-admissible region, specially when it presents a small value ($\alpha_1 = 0.1$).

Figure 1 presents the boxplots of the sample bias for the parameter estimates of an INAR(2) processes, with parameter values $\boldsymbol{\theta} = (\alpha_1, \alpha_2, \lambda) = (0.6, 0.1, 1.0)$, obtained by LS_HOS and LS_HOS_C, for the four different sample sizes. As can be seen, the boxplots indicate that the marginal distributions of the estimators are, generally, symmetric.

When the several estimation methods are compared it is found that the LS_HOS provides similar results, in terms of the smallest values of sample bias, variance and mean square error, to the other methods. It is also verified that, in general, the proportion of non-admissible estimates of the methods is less for LS_HOS, followed by WHT and CLS. The results show that, in general, the sample mean bias of $\hat{\alpha}_i$ is negative, indicating that $\alpha_i$ is underestimated, while $\lambda$ is overestimated, since the sample mean bias of the parameter estimate is positive.

**Table 1**: Sample statistical measures for the parameters estimates of Poisson INAR(1) processes.

| Measure | $N$ | $\theta = (\alpha_1, \lambda)$ | $\hat{\theta}_1 = \hat{\alpha}_1$ | | $\hat{\theta}_2 = \hat{\lambda}$ | |
|---|---|---|---|---|---|---|
| | | | LS_HOS | LS_HOS_C | LS_HOS | LS_HOS_C |
| $\overline{\text{Bias}}(\hat{\theta}_i)$ | 50 | (0.1, 3.0)<br>(0.6, 3.0) | −0.0411<br>−0.0893 | −0.0041<br>−0.0826 | 0.0949<br>0.5208 | −0.0897<br>0.4611 |
| | 500 | (0.1, 3.0)<br>(0.6, 3.0) | −0.0019<br>−0.0094 | −0.0003<br>−0.0099 | 0.0029<br>0.0532 | −0.0050<br>0.0584 |
| $\text{Var}(\hat{\theta}_i)$ | 50 | (0.1, 3.0)<br>(0.6, 3.0) | 0.0190<br>0.0158 | 0.0099<br>0.0149 | 0.2623<br>0.9229 | 0.1566<br>0.8841 |
| | 500 | (0.1, 3.0)<br>(0.6, 3.0) | 0.0021<br>0.0014 | 0.0020<br>0.0014 | 0.0286<br>0.0787 | 0.0270<br>0.0772 |
| $\text{MSE}(\hat{\theta}_i)$ | 50 | (0.1, 3.0)<br>(0.6, 3.0) | 0.0206<br>0.0238 | 0.0100<br>0.0217 | 0.2711<br>1.1933 | 0.1645<br>1.0958 |
| | 500 | (0.1, 3.0)<br>(0.6, 3.0) | 0.0021<br>0.0015 | 0.0020<br>0.0015 | 0.0286<br>0.0815 | 0.0270<br>0.0805 |

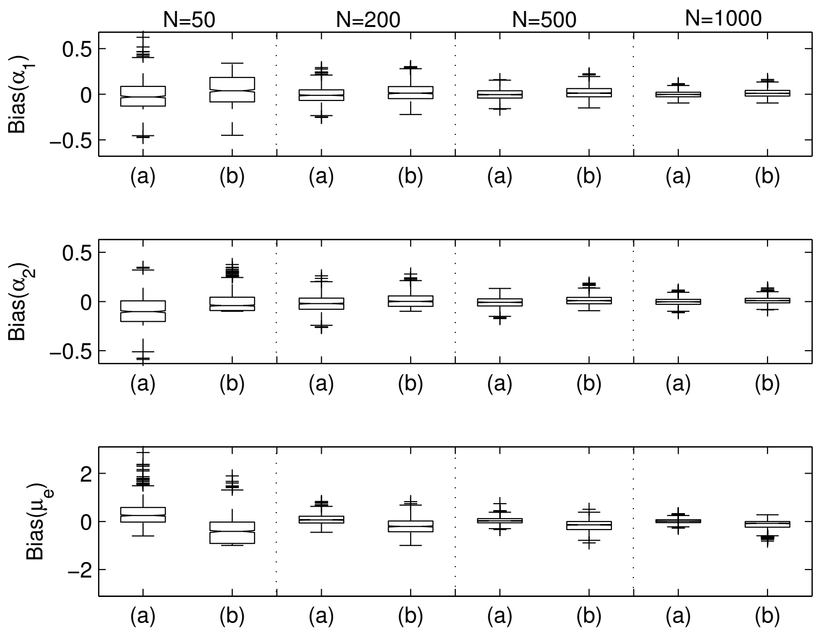


**Figure 1**: Boxplots of the sample bias for the estimates obtained by (a): LS_HOS and (b): LS_HOS_C, in 1000 realizations of the Poisson INAR(2) model: $X_t = 0.6 \circ X_{t-1} + 0.1 \circ X_{t-2} + e_t$, where $e_t \sim \mathcal{P}o(1)$.

In order to illustrate some of these conclusions, Figure 2 shows the boxplots of the sample bias for the estimates obtained from 50 and 200 observations of the INAR(1) process with parameter values $\boldsymbol{\theta} = (\alpha_1, \lambda) = (0.9, 1.0)$. Note that the value of $\alpha_1$ is near the non-stationary region, however, even for $N = 50$ observations the LS_HOS estimates presents the best results.

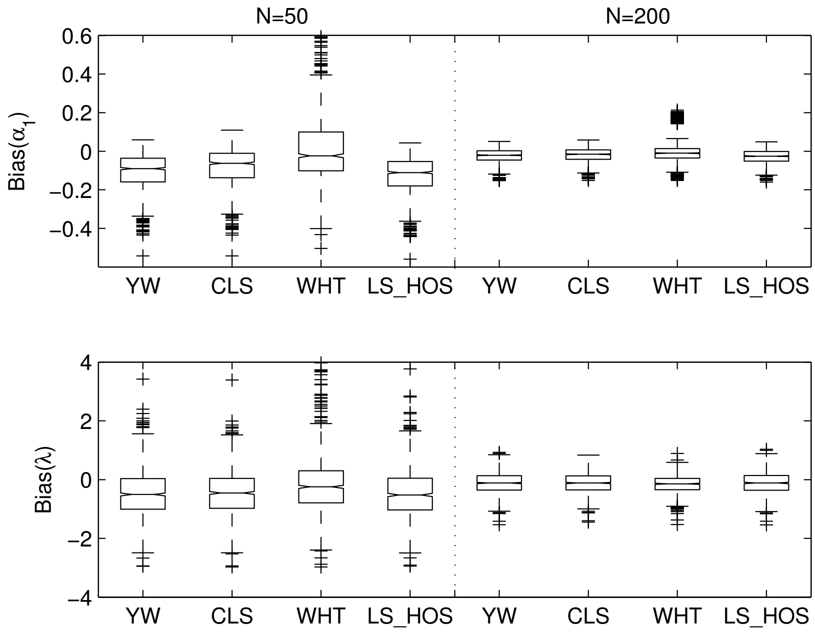


**Figure 2**: Boxplots of the sample bias for the estimates obtained in 1000 realizations of 50 and 200 observations of the Poisson INAR(1) model: $X_t = 0.9 \circ X_{t-1} + e_t$, where $e_t \sim \mathcal{P}o(1)$.

## 4. APPLICATION TO REAL DATA

In this section, the proposed estimation method is applied to a real dataset concerning the number of Swedish mechanical paper and pulp mills, from 1921 to 1981 (see Figure 3). This dataset was used by Brännäs (1995) and Brännäs and Hellström (2001), and these authors fitted an INAR(1) process to this dataset using some explanatory variables (the industrial gross profit margin and GNP). Here, an INAR(1) process, with binomial thinning operation, where the innovations are i.i.d. random variables with mean $\mu_e$ and variance $\sigma_e^2$ is considered. Since the mean of the data is 20.40 and its variance is 155.16, a Poisson innovation process is not assumed but then the method does not require that or any other assumption on the distribution of the innovations.

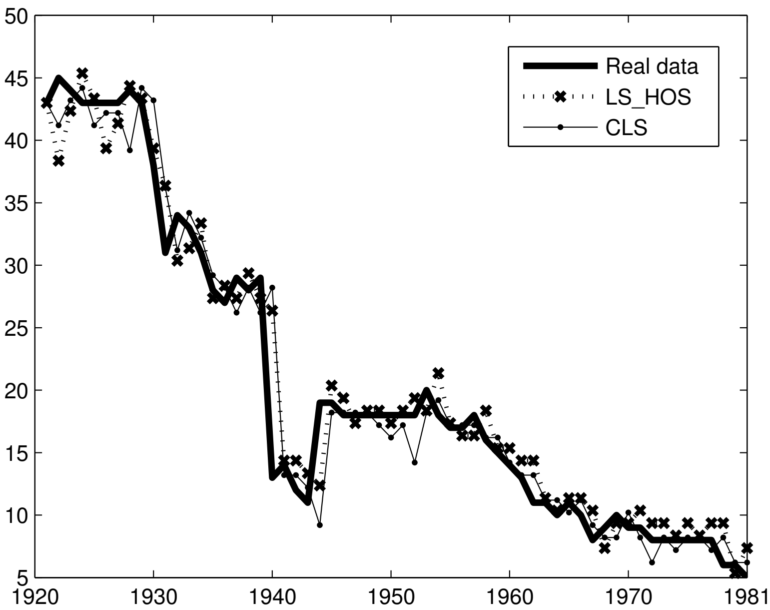


**Figure 3**: The number of Swedish mechanical paper and pulp mills, from 1921 to 1981 (Brännäs (1995) and Brännäs and Hellström (2001)), and the fitted values considering the LS_HOS and CLS estimates.

Table 2 presents the parameter estimates obtained by CLS and LS_HOS estimation methods. The fit of both models, based on LS_HOS and CLS estimates, are also shown in Figure 3. As can be observed, the two fits are very similar and similar to the dataset. The mean square errors (MSE) between the observations and the fitted values are also exhibited in Table 2. It can be seen that the MSE is slightly smaller for the LS_HOS fit than for CLS fit.

**Table 2**: The parameter estimates of the number of Swedish mechanical paper and pulp mills, from 1921 to 1981.

| Method | $\hat{\alpha}$ | $\hat{\mu}_e$ | $\hat{\sigma}_e^2$ | MSE | $\hat{\mu}_x$ | $\hat{\sigma}_x^2$ |
|---|---|---|---|---|---|---|
| CLS | 0.9591 | 0.2017 | 15.2268 | 8.5494 | 4.9315 | 192.2764 |
| LS_HOS | 0.9269 | 1.3635 | 19.2253 | 7.4465 | 18.6525 | 145.4513 |

The last two columns of the Table 2 present the mean and variance of the estimated models:

$$\hat{\mu}_x = \frac{\hat{\mu}_e}{1-\hat{\alpha}_1} \qquad \text{and} \qquad \hat{\sigma}_x^2 = \frac{\hat{\mu}_e\,\hat{\alpha}_1 + \hat{\sigma}_e^2}{1-\hat{\alpha}_1^2} \ .$$

It is noticeable that the model estimated by LS_HOS presents mean and variance closer to the sample values.

The goodness-of-fit of both fitted models is investigated by the residuals. The analysis of the sample autocorrelation and sample partial autocorrelation functions, as well as the usual tests of randomness, do not reject the hypothesis of uncorrelated random variables for the residual series from both fitted models.

## 5. FINAL REMARKS

The principal advantage of HOS is the capability to detect and characterize the deviations from Gaussianity and non-linearity of the processes. Thus in this work a new estimation method for the parameters of INAR processes based on HOS is proposed. This method uses the Least Squares estimation to minimize the errors between the third-order moment of the observations and of the fitted model. Note that this estimation method does not assume any particular discrete distribution to the counting series and to the innovation process. A Monte Carlo study indicates that this estimation method provides good results in small samples, in terms of sample bias, variance and mean square error. Moreover, when used in the context of a non-Poisson real dataset the LS_HOS estimates provide a model with mean, variance and autocorrelations closer to the sample values.

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