REVSTAT – Statistical Journal
Volume 0, Number 0, Month 0000, 000–000
https://doi.org/00.00000/revstat.v00i0.000

A novel test of fit based on Phi-divergence with real data application

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Abstract:

- The paper presents a novel approach to estimating the Phi-divergence measure, accompanied by a detailed and thorough goodness-of-fit test. The proposed test exhibits valuable qualities such as consistency and invariance, which have been rigorously established. To assess the effectiveness of the proposed test, an extensive simulation study is conducted. The study compares the performance of the proposed test with several well-known competing tests under various alternative scenarios involving exponential, Weibull, and log-normal distributions. This comparison allows for a comprehensive evaluation of the proposed test's efficacy. Additionally, to provide a practical understanding of the proposed methodology, two illustrative examples are included. These examples serve as concrete demonstrations of how the proposed procedure can be applied in real-world situations.

Keywords:

- *Goodness-of-fit test, Phi-divergence, Exponential distribution, Weibull distribution, Log-normal distribution, Monte Carlo simulation, Test power.*

AMS Subject Classification:

- 62G10, 62E15.

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1. INTRODUCTION

The Phi-divergence measures the difference between two probability distributions, as defined by Csiszar (1963). Suppose F is an unknown continuous distribution with a probability density function $f(x)$, and $G(x;\theta)$ is a parametric family of distributions with a probability density function $g(x;\theta)$ where θ is parameter and Θ is parameter space. The Phi-divergence measure between $g(x;\theta)$ and $f(x)$ is defined as follows:

$$D_\phi(g,f;\theta)=\int_{-\infty}^{\infty} f(x)\phi(\frac{g(x;\theta)}{f(x)})dx,$$

where $\phi(x):[0,\infty)\to\mathbb{R}$ is a convex function satisfying $\phi(1)=0$, $0\phi(0/0)=0$, and $0\phi(u/0)=u\lim_{t\to\infty}\phi(t)/t$ (Pardo, 2006). For a random sample $X_1,...,X_n$ from an unknown continuous distribution F with a density function f, the hypotheses

$$\begin{cases} H_0: f(x) = g(x,\theta) & \text{for some } \theta\in\Theta, \\ H_1: f(x) \neq g(x,\theta) & \text{for any } \theta\in\Theta, \end{cases}$$

are of interest. Goodness-of-fit(GOF) tests are statistical procedures for testing the null hypothesis H_0 against the alternative hypothesis H_1. GOF tests play a crucial role in diverse applications, including quality control, model selection, hypothesis testing, predictive modeling, and risk management. Through the evaluation of GOF, these tests contribute to validating and ensuring the reliability of statistical analyses, decision-making processes, and predictive models across various fields. Researchers have introduced a range of GOF tests, among them prominent ones like the Kolmogorov-Smirnov (Kolmogorov, 1933; Smirnov, 1948), Cramér-Von Mises (Correa, 1928; Von Mises, 1932), and Anderson-Darling (Anderson and Darling, 1954) tests.

Kullback-Leibler information is obtained for $\phi(x)=x\log(x)-x+1$ or $\phi(x)=x\log(x)$. Vasicek (1976), Correa (1995), Van Es (1992), Yousefzadeh and Arghami (2008), Ebrahimi et al. (1994), and Zamanzade and Arghami (2011) introduced GOF tests based on Kullback-Leibler information for various statistical distributions. Alizadeh and Balakrishnan (2016) introduced a GOF test based on Phi-divergence for exponential, uniform, normal, and Laplace distributions. Additionally, Desgagné et al. (2022) conducted a comparative analysis of the test proposed by Alizadeh and Balakrishnan (2016) specifically for the Laplace distribution, comparing it against other existing tests. Furthermore, Zamanzade and Mahdizadeh (2017) presented a GOF test for the Rayleigh distribution, utilizing Phi-divergence. They employed kernel density estimation to calculate test statistics and assessed the performance of their proposed test through simulation. Al-Omari and Zamanzade (2018) proposed test for the logistic distribution. Moreover, Tavakoli et al. (2021) introduced a test of fit for normal distribution via Phi-divergence.

In this manuscript, a new estimator for the Phi-divergence criterion is presented, aiming to evaluate goodness-of-fit with highly useful features. In Section 2, a new estimator of Phi-divergence is proposed, and the features of the test statistic are illustrated in Section 3. In Section 4, a simulation study is undertaken to assess the performance of the tests across various lifetime distributions. Section 5 presents the analysis of two real datasets. The conclusions drawn from the study are summarized in Section 6.

2. A new estimator of Phi-divergence

Phi-divergence can be represented as:

$$D_\phi(g, f; \theta) = E_f\Big(\phi(\frac{g(X;\theta)}{f(X)})\Big).$$

Let $X_1, \ldots, X_n$ be a random sample and $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(n)}$ be the order statistics of the random sample. The proposed estimator of $D_\phi(g, f; \theta)$ is given by:

$$DB_\phi = \frac{1}{n}\sum_{i=1}^{n}\phi(\frac{g(X_i;\hat{\theta})}{\hat{f}(X_i)}), \tag{2.1}$$

where $\hat{\theta}$ is the maximum likelihood estimation (MLE) of θ and a semi-parametric estimator of $g(x;\theta)$ is

$$g(X_{(i)};\hat{\theta}) \simeq \frac{G(X_{(i+m)};\hat{\theta}) - G(X_{(i-m)};\hat{\theta})}{X_{(i+m)} - X_{(i-m)}}. \tag{2.2}$$

Moreover, similar to that in Vasicek (1976), $\hat{f}$ be obtained as follows:

$$\hat{f}(X_{(i)}) = \frac{F_n(X_{(i+m)}) - F_n(X_{(i-m)})}{X_{(i+m)} - X_{(i-m)}} = \frac{2m/n}{X_{(i+m)} - X_{(i-m)}}, \tag{2.3}$$

where F_n is the empirical distribution function.

Substituting the expressions (2.2) and (2.3), we derive the following estimator of $D_\phi(g, f; \theta)$, which serves as the test statistic for goodness-of-fit:

$$DB_\phi = \frac{1}{n}\sum_{i=1}^{n}\phi\Big(\frac{G(X_{(i+m)};\hat{\theta}) - G(X_{(i-m)};\hat{\theta})}{2m/n}\Big), \tag{2.4}$$

where m is a positive integer, $m \leq n/2$, and also $X_{(i)} = X_{(1)}$ if $i < 1$, $X_{(i)} = X_{(n)}$ if $i > n$.

3. Exploring the proposed test and its features

The non-negativity of the $D_\phi(g, f; \theta)$ measure is established, and if $\phi(x)$ is strictly convex at $x = 1$, then $D_\phi(g, f; \theta) = 0$ holds if and only if the null hypothesis H_0 is confirmed (Pardo, 2006). Leveraging this characteristic, we employ the DB_ϕ measure to construct GOF tests, rejecting H_0 when the estimates exceed a sufficiently large threshold. Denoting $Q_{1-\alpha}$ as the $1-\alpha$ quantile of the DB_ϕ distribution, the critical region is defined as $DB_\phi > Q_{1-\alpha}$.

Remark 3.1. For uniformity test on $(0, 1)$, the proposed test statistic is given by

$$DB_\phi = \frac{1}{n}\sum_{i=1}^{n}\phi\Big(\frac{X_{(i+m)} - X_{(i-m)}}{2m/n}\Big),$$

where, if we choose $\phi = -log(x)$, the proposed test statistic aligns with the uniformity test introduced by Dudewicz and Van Der Meulen (1981) based on entropy. Moreover, If we use density function $g(x;\hat{\theta})$ instead of equation (2.2), then the proposed test is the same with the test introduced by Tavakoli et al. (2021) for the normal distribution.

Remark 3.2. The generating function ϕ for Phi-divergence $D_\phi(g,f;\theta)$ is not unique. For real constant c, the Phi-divergence defined by

$$\psi(x) = \phi(x) - c(x-1)$$

is equivalent to the Phi-divergence based on $\phi(x)$, i.e., $D_\phi(g,f;\theta) = D_\psi(g,f;\theta)$. Moreover, ensuring that $\phi(x) \geqslant 0$, can be achieved by setting c equal to or any subdifferential at $x = 1$ if is not differentiable (Polyanskiy and Wu, 2023, Page 118). Thus, the set of these functions is denoted by Ψ.

The subsequent theorem establishes the non-negativity of DB_ϕ, akin to $D_\phi(g,f;\theta)$.

Theorem 3.1. *Assuming $X_1,...,X_n$ is a random sample from an unknown continuous distribution F and G is known, it follows that*

$$DB_\phi \geq 0.$$

Proof: Since G is a distribution function, the value of $G(X_{(i+m)};\hat{\theta}) - G(X_{(i-m)};\hat{\theta})$, $i = 1,2,\ldots,n$, is non-negative. Additionally, for any $\phi \in \Psi$ (see Remark 3.2), we have $\phi(x) \geq 0$. Therefore,

$$\phi\Big(\frac{G(X_{(i+m)};\hat{\theta}) - G(X_{(i-m)};\hat{\theta})}{2m/n}\Big) \geqslant 0,$$

and as a result DB_ϕ is a non-negative function. □

Next, we will demonstrate the invariance property of the proposed test.

Theorem 3.2. *If the family of distributions G is assumed to be invariant under the transformations group S, and $x_1,...,x_n$ constitute a random sample, then*

$$U(\mathbf{x}) = G(x_{(i)};T(\mathbf{x}))$$

is invariant under S, where $T(x)$ is an equivariant estimator of θ.

Proof: The family of distributions $G = \{P_\theta, \theta \in \Theta\}$ with the sample space Ω under a group on transformations $S = \{s\}$ is invariant if

$$P_{\bar{s}(\theta)}(X \in A) = P_\theta(s(X) \in A), \quad \forall A \in \mathcal{A}, \forall s \in S,$$

where $\mathcal{A}$ denotes a class of measurable sets, and $\bar{S} = \{\bar{s}\}$ is the group transformation on Θ generated by S (Lehmann and Romano, 2005, Page 213). Therefore, suppose that $\mathbf{y} = (s(x_1), s(x_2),\ldots,s(x_n))$ for all $s \in S$, then

$$P_{\bar{s}(\theta)}(X \in A) = P_\theta(s(X) \in A), \quad \forall A \in \mathcal{A}.$$

Under the transformation group S, since T is equivariant, $T(s(\mathbf{X})) = \tilde{s}(T(\mathbf{X})), \forall s \in S$, where $\tilde{S} = \{\tilde{s}\}$ is the group transformation on Ω that is generated by S (Lehmann and Casella, 2006, Page 161). So, we have

$$\begin{aligned} G\big(y_{(i)};T(\mathbf{y})\big) &= G\big(y_{(i)};\tilde{s}(T(\mathbf{x}))\big) \\ &= P_{\bar{s}(T(\mathbf{x}))}(X \leq y_{(i)}) \\ &= P_{T(\mathbf{x})}(s(X) \leq y_{(i)}). \end{aligned}$$

Because $y_{(i)} = s(x_{(i)})$ and s is an increasing function, thus

$$\begin{aligned} U(\mathbf{y}) &= G\big(y_{(i)}; T(\mathbf{y})\big) \\ &= P_{T(\mathbf{x})}(s(X) \le s(x_{(i)})) \\ &= P_{T(\mathbf{x})}(X \le x_{(i)}) \\ &= U(\mathbf{x}). \end{aligned}$$

□

Corollary 3.1. *If $\bar{S}$ is transitive on Θ, and Theorem 3.2 is satisfied, the distribution of DB_ϕ does not depend on θ, i.e., the test is accurate.*

Proof: A group $\bar{S}$ on the parameter space Θ is considered transitive if, for every $\theta_1, \theta_2 \in \Theta$, there exists a $\bar{s} \in \bar{S}$ such that $\bar{s}(\theta_1) = \theta_2$ (Lehmann and Casella, 2006, Page 162). Hence, the distribution of any invariant statistic is independent of θ, and this result holds. □

Remark 3.3. When the parameters of the distribution are known as $\theta = \theta_0$ (indicating a simple null hypothesis), the test statistic is as follows:

$$DB_\phi = \frac{1}{n}\sum_{i=1}^{n} \phi\Big(\frac{G(X_{(i+m)};\theta_0) - G(X_{(i-m)};\theta_0)}{2m/n}\Big).$$

The distribution of DB_ϕ does not depend on G under H_0. As a result, the proposed test retains a constant critical value table that is applicable to all distributions.

In the subsequent theorem, we establish the consistency of the proposed test.

Theorem 3.3. *Suppose that $X_1, ..., X_n$ constitutes a random sample from an unknown continuous distribution F with probability density function $f(x)$, and let the distribution function G be known. Then*

$$DB_\phi \xrightarrow{p} D_\phi(g, f; \theta).$$

Proof: Since $\hat{\theta}$ is the MLE, we have (Newey and McFadden, 1994, Theorem 2.5):

$$\hat{\theta} \xrightarrow{p} \theta, \quad as \quad n \to \infty.$$

Utilizing Gut (2006, Theorem 10.3), one can demonstrate that

$$G(x; \hat{\theta}) \xrightarrow{p} G(x; \theta), \quad as \quad n \to \infty.$$

There exists a value $X_i^{'} \in (X_{(i-m)}, X_{(i+m)})$ such that (Vasicek, 1976, Page 56):

$$g(X_i^{'}; \hat{\theta}) = \frac{G(X_{(i+m)}; \hat{\theta}) - G(X_{(i-m)}; \hat{\theta})}{X_{(i+m)} - X_{(i-m)}}.$$

Thus, we have, $g(x,\hat{\theta}) \xrightarrow{p} g(x,\theta), \quad as \quad n,m \to \infty, m/n \to 0$. The weak law of large numbers implies (Gut, 2006, Page 276): $F_n(x) \xrightarrow{p} F(x), \quad as \quad n \to \infty$. Therefore, by Vasicek (1976, Page 56), we have:

$$\hat{f}(x) \xrightarrow{p} f(x), \quad as \quad n,m \to \infty, m/n \to 0.$$

Therefore, utilizing Gut (2006, Theorem 10.3) and the weak law of large numbers, as $n,m \to \infty, m/n \to 0$, the outcome is:

$$DB_\phi = \frac{1}{n}\sum_{i=1}^{n}\Big(\phi(\frac{g(X_i;\hat{\theta})}{\hat{f}(X_i)})\Big) \xrightarrow{p} E_f\Big(\phi(\frac{g(X;\theta)}{f(X)})\Big) = D_\phi(g,f;\theta).$$

□

4. Simulation Study

In this section, we conduct a simulation study for some special distributions. The optimal value of m will be determined through simulation. The critical values will be computed, and the power of the proposed test will be compared with that of some competing tests.

4.1. Test for specific distributions

Suppose $X_1,\ldots,X_n$ constitutes a random sample from a continuous distribution F with a density function $f(x)$ defined over a non-negative support. The hypotheses under consideration for exponential ($Exp(\lambda)$), Weibull ($W(\theta)$), and log-normal ($Ln(\mu,\sigma)$) distributions are respectively:

$$\begin{aligned}
&H_0 : f(x) = \lambda e^{-\lambda x} \quad vs. \quad H_1 : f(x) \neq \lambda e^{-\lambda x},\\
&H_0 : f(x) = \theta x^{\theta-1} exp(-x^\theta) \quad vs. \quad H_1 : f(x) \neq \theta x^{\theta-1} exp(-x^\theta),\\
&H_0 : f(x) = \frac{1}{x\sigma\sqrt{2\pi}} exp\{-\frac{(ln(x)-\mu)^2}{2\sigma^2}\} \quad vs. \quad H_1 : f(x) \neq \frac{1}{x\sigma\sqrt{2\pi}} exp\{-\frac{(ln(x)-\mu)^2}{2\sigma^2}\}.
\end{aligned}$$

where $\sigma,\lambda,\theta > 0$ and $\mu \in \mathbb{R}$.

The test statistics for each distribution are given by:

- Exponential distribution: $DB_\phi = \frac{1}{n}\sum_{i=1}^{n}\phi\Big(\frac{\exp\{-\hat{\lambda}X_{(i-m)}\}-\exp\{-\hat{\lambda}X_{(i+m)}\}}{2m/n}\Big)$,
- Weibull distribution: $DB_\phi = \frac{1}{n}\sum_{i=1}^{n}\phi\Big(\frac{\exp\{-X_{(i-m)}^{\hat{\theta}}\}-\exp\{-X_{(i+m)}^{\hat{\theta}}\}}{2m/n}\Big)$,
- Log-normal distribution: $DB_\phi = \frac{1}{n}\sum_{i=1}^{n}\phi\Big(\frac{\Phi(Z_{(i+m)})-\Phi(Z_{(i-m)})}{2m/n}\Big)$.

where $\hat{\lambda} = 1/\bar{X}$, $Y_i = \ln(X_i)$, $Z_i = (Y_i - \hat{\mu})/\hat{\sigma}$, $\Phi(.)$ is the standard normal distriution function, and $\hat{\mu} = \bar{Y} = \frac{1}{n}\sum_{i=1}^{n}Y_i$; $\hat{\sigma} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(Y_i-\bar{Y})^2}$. Also, $\hat{\theta}$(MLE of θ) does not have a closed-form solution, and we computed it through numerical calculations with the assistance of the *EnvStats* package in the *R* software.

The exponential distribution is invariant under scale transformation, and $\hat{\lambda}$ is an equivariant estimator under scale transformation. Under the group $S = \{s; s(x) = a\ln(x) + b, a > 0, b \in \mathbb{R}\}$, the log-normal distribution is invariant, and $\hat{\mu}$ and $\hat{\sigma}$ are equivariant. Under $S = \{s; s(x) = x^c, c > 0\}$, the Weibull distribution is invariant, and $\hat{\theta}$ is equivariant. Hence, according to Corollary 3.1, the critical region is an exact test. As a result, by modifying parameters in the simulation, critical values will remain unchanged.

4.2. Simulation Configuration

We apply the proposed test to evaluate the goodness-of-fit for exponential($Exp(\lambda)$), log-normal ($Ln(\mu, \sigma)$), and Weibull($W(\theta)$) distributions. All simulations were carried out using R 4.1.2 with $10,000$ replications. Critical values are determined for each test, and their power is evaluated and compared with that of several alternative tests, such as Kolmogorov (1933), Kolmogorov (1962), Correa (1928), Anderson and Darling (1954), and Jager and Wellner (2007). The test statistics for alternative tests are as follows (Alizadeh and Balakrishnan, 2016):

- CM: Cramer von Mises $\left(CM = \frac{1}{12n} + \sum_{i=1}^{n} \left(\frac{2i-1}{2n} - G(X_{(i)}; \hat{\theta})\right)^2\right)$
- KS: Kolmogorov $\left(KS = max\{D^+, D^-\}\right)$,
- KP: Kuiper $\left(KP = D^+ + D^-\right)$,
- AD: Anderson–Darling $\left(AD = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\{\ln(G(X_{(i)}; \hat{\theta})) + \ln(1 - G(X_{(n-i+1)}; \hat{\theta}))\}\right)$,
- JW: Jager–Wellner $\left(JW_s = \frac{1}{s(1-s)}(1 - (\frac{i}{n})^s(G(X_{(i)}; \hat{\theta}))^{1-s} - (1 - \frac{i}{n})^s(1 - G(X_{(i)}; \hat{\theta}))^{1-s}), s \neq 0, 1\right)$,

where $D^+ = max\{\frac{i}{n} - G(X_{(i)}; \hat{\theta})\}$ and $D^- = max\{G(X_{(i)}; \hat{\theta}) - \frac{i-1}{n}\}$. Moreover, the most notable special cases of JW_s are $s \in \{0.5, 2\}$, which we use in the simulation (Jager and Wellner, 2007).

Simulation study is conducted for specific functions of ϕ. Various established divergence measures are defined by choosing appropriate functions $\phi \in \Psi$ (refer to Remark 3.2). In Table 1, we present several significant measures of divergence that have been examined in the simulation (Pardo, 2006, Page 6).

4.3. Critical values

For the proposed tests, the critical values are determined as follows:

1- Generate a sample from $Exp(1)$, $W(1)$, $LN(0, 1)$ distributions to test Exponential, Log-normal, and Weibull distributions, respectively, with a pre-chosen size n;

Divergence	ϕ-function
KL: Kullback-Liebler (Pardo, 2006)	$\phi(x) = x\log(x) - x + 1$
PE: Pearson(Pardo, 2006)	$\phi(x) = \frac{1}{2}(x-1)^2$
BS: Balakrishnan and Sanghvi (1968)	$\phi(x) = \frac{(x-1)^2}{(x+1)^2}$
TD: Triangular divergence(Tavakoli et al., 2021)	$\phi(x) = \frac{(1-x)^2}{1+x}$
CR: Cressie and Read (1984)	$\phi(x) = \frac{x^{\frac{2}{3}+1} - x - \frac{2}{3}(x-1)}{\frac{2}{3}(\frac{2}{3}+1)}$
MD: Minimum Discrimination Information(Pardo, 2006)	$\phi(x) = -\log(x) + x - 1$
JD: Jeffreys distance(Pardo, 2006)	$\phi(x) = (x-1)\log(x)$
HE: Hellinger distance(Alizadeh and Balakrishnan, 2016)	$\phi(x) = \frac{1}{2}(\sqrt{x}-1)^2$
TV: Total variation(Pardo, 2006)	$\phi(x) = \lvert 1-x \rvert$
JS: Jensen-Shannon(Qiao and Minematsu, 2008)	$\phi(x) = \frac{1}{2}\big(x\log(x) - (x+1)\log(\frac{x+1}{2})\big)$

Table 1: Various divergence measures explored in the simulation

2- Calculate the maximum likelihood estimation (MLE) of parameters under the null hypothesis H_0;

3- Compute DB_ϕ for the sample data;

4- Repeat perform Steps 1–3 a large number of times (e.g., $10,000$ in our case) and determine the $(1-\alpha)$th quantile of DB_ϕ.

Using the aforementioned procedure, we derived the critical values for the proposed tests for sample sizes ranging from 10 to 100 at $\alpha = 0.05$. These values are presented in Table 2. The critical values in this table represent the 0.95th quantile of the test statistic values for the simulated data under the null hypothesis. Therefore, as the sample size increases, we expect these values to approach zero. The results matched our expectations, and the critical values tend to approach zero as the sample size increases.

Remark 4.1. Although distributions $Exp(1)$ and $W(1)$ are identical, and their critical values can be used interchangeably, it is important to note that, due to the reliance on parameter estimation in calculating the test statistic for these distributions, we employ distinct critical values. This distinction becomes more significant for smaller sample sizes, where parameter estimation is less accurate. As illustrated in Table 2, with an increase in sample size, the critical values for these two distributions converge towards each other.

4.4. Evaluating the robustness of proposed tests

A test that is less affected by minor deviations from H_0, especially due to a few outliers or anomalies, is termed robust (Stehlík et al., 2014). Most common tests have a significant drawback: they lack robustness and are overly sensitive to outliers. Therefore, in this section, we assess the sensitivity of the proposed tests to minor deviations from the null hypothesis. To achieve this, we simulate 5% of the data as outliers under the null hypothesis. Subsequently,

Test	n	KL	PE	BS	TD	CR	MD	JD	HE	TV	JS
Weibull	5	0.300	0.215	0.301	0.327	0.287	0.672	0.907	0.103	0.611	0.200
	10	0.147	0.109	0.108	0.161	0.139	0.230	0.349	0.043	0.400	0.083
	15	0.108	0.086	0.084	0.114	0.103	0.152	0.261	0.033	0.334	0.061
	20	0.089	0.078	0.073	0.096	0.088	0.123	0.210	0.026	0.310	0.051
	30	0.076	0.057	0.055	0.079	0.078	0.094	0.176	0.020	0.269	0.042
	50	0.059	0.053	0.042	0.064	0.058	0.079	0.131	0.017	0.242	0.032
	75	0.054	0.047	0.036	0.060	0.055	0.065	0.120	0.014	0.235	0.031
	100	0.050	0.044	0.033	0.054	0.051	0.059	0.106	0.013	0.225	0.027
Log-normal	5	0.168	0.113	0.161	0.188	0.151	0.316	0.500	0.059	0.369	0.104
	10	0.105	0.081	0.117	0.111	0.099	0.138	0.245	0.030	0.316	0.059
	15	0.082	0.064	0.074	0.086	0.078	0.115	0.194	0.024	0.288	0.047
	20	0.072	0.064	0.058	0.079	0.070	0.094	0.169	0.022	0.314	0.040
	30	0.063	0.055	0.044	0.065	0.065	0.077	0.138	0.017	0.298	0.034
	50	0.052	0.045	0.036	0.057	0.055	0.064	0.118	0.015	0.222	0.028
	75	0.050	0.044	0.033	0.054	0.047	0.058	0.108	0.013	0.212	0.027
	100	0.048	0.040	0.031	0.049	0.046	0.055	0.106	0.013	0.203	0.025
Exponential	5	0.318	0.223	0.292	0.354	0.289	0.571	0.870	0.104	0.645	0.194
	10	0.162	0.126	0.145	0.185	0.157	0.267	0.435	0.052	0.438	0.094
	15	0.116	0.091	0.105	0.127	0.110	0.165	0.280	0.034	0.382	0.066
	20	0.094	0.076	0.077	0.100	0.088	0.126	0.218	0.026	0.326	0.052
	30	0.072	0.064	0.053	0.078	0.072	0.092	0.163	0.020	0.279	0.040
	50	0.059	0.050	0.040	0.062	0.056	0.070	0.127	0.016	0.244	0.031
	75	0.050	0.044	0.034	0.053	0.049	0.060	0.109	0.014	0.225	0.027
	100	0.047	0.039	0.031	0.049	0.046	0.056	0.101	0.013	0.212	0.025

Table 2: Critical values of the test statistics.

with this setup (including 5% outlier data), we determine the true type I error rate using critical values from Table 2. We anticipate that the nominal level (e.g., 0.05) and the true Type I error rate will closely approximate each other. For this purpose, in simulating the Weibull distribution, 95% of the sample is generated with a parameter of 1, and 5% of the sample is generated with a parameter of 0.5. For the log-normal distribution, 95% of the sample was simulated from the distribution $LN(0,1)$ and 5% from the distribution $LN(0,1.5)$. In simulating the exponential distribution, 95% of the sample is generated with a parameter of 1, and 5% of the sample is generated with a parameter of 0.25. In $10,000$ repetitions of the simulation with a predetermined sample size n, the number of times the null hypothesis is rejected represents the true value of the type I error.

Table 3 shows the results of the true value of type I error for a nominal level of 0.05. The results in this table show that minor deviations from H_0 do not prevent the nominal level of the type I error from being maintained.

4.5. The optimal value of m

The optimal value of m is determined to maximize the power of the test for a specific value of n. For different values of $m (m \leq n/2)$, after obtaining critical values, we simulated alternative distributions for $n = 5, 6, ..., 100$. Afterwards, the test power was computed for each alternative distribution(introduced in a subsequent section), and the average of the

Test	n	KL	PE	BS	TD	CR	MD	JD	HE	TV	JS
Weibull	5	0.055	0.055	0.045	0.050	0.048	0.034	0.056	0.043	0.051	0.042
	10	0.057	0.058	0.076	0.051	0.049	0.050	0.054	0.051	0.058	0.062
	15	0.055	0.047	0.056	0.059	0.062	0.044	0.054	0.029	0.078	0.035
	20	0.064	0.056	0.038	0.056	0.053	0.049	0.062	0.058	0.038	0.061
	30	0.061	0.059	0.045	0.062	0.044	0.052	0.061	0.034	0.070	0.059
	50	0.057	0.055	0.043	0.059	0.051	0.045	0.048	0.053	0.054	0.065
	75	0.052	0.079	0.045	0.063	0.047	0.047	0.063	0.055	0.081	0.065
	100	0.061	0.046	0.042	0.055	0.045	0.054	0.053	0.041	0.082	0.063
Log-normal	5	0.058	0.058	0.052	0.056	0.056	0.053	0.052	0.043	0.061	0.062
	10	0.050	0.060	0.057	0.063	0.059	0.069	0.058	0.064	0.064	0.058
	15	0.055	0.055	0.056	0.072	0.064	0.052	0.052	0.050	0.051	0.059
	20	0.049	0.061	0.058	0.063	0.047	0.051	0.058	0.065	0.063	0.054
	30	0.059	0.066	0.052	0.059	0.065	0.055	0.067	0.050	0.079	0.065
	50	0.060	0.066	0.055	0.056	0.052	0.057	0.056	0.057	0.065	0.050
	75	0.055	0.055	0.051	0.055	0.057	0.063	0.054	0.053	0.057	0.060
	100	0.055	0.057	0.047	0.043	0.056	0.042	0.061	0.052	0.059	0.056
Exponential	5	0.040	0.036	0.032	0.035	0.040	0.040	0.037	0.035	0.039	0.037
	10	0.045	0.040	0.052	0.052	0.056	0.051	0.051	0.046	0.048	0.043
	15	0.045	0.041	0.050	0.047	0.048	0.052	0.049	0.046	0.054	0.049
	20	0.044	0.042	0.049	0.050	0.054	0.045	0.048	0.053	0.046	0.047
	30	0.045	0.038	0.051	0.048	0.045	0.058	0.053	0.053	0.045	0.051
	50	0.045	0.038	0.059	0.044	0.047	0.055	0.048	0.049	0.047	0.050
	75	0.049	0.041	0.063	0.050	0.042	0.056	0.052	0.053	0.044	0.053
	100	0.045	0.039	0.055	0.050	0.046	0.054	0.051	0.043	0.055	0.050

Table 3: The true value of type I error with minor deviations from H_0.

resulting powers was calculated for each n and its corresponding m. Subsequently, for each n, a value of m was was selected to maximize the average test power. Finally, following a similar idea as Crzcgorzewski and Wirczorkowski (1999), the regression model fitted is as $m = \beta_0 + \beta_1 n + \varepsilon$. For exponential, log-normal, and Weibull distributions and the ϕ functions presented in Table 1, the regression model was fitted using the Ordinary Least Squares (OLS). The model fitting indicated that, across various types of tests and values of ϕ, the regression model exhibited an exceptionally high level of fitting with $R^2 > 0.98$. The following equation obtains the approximate value of m for the specified values of $\hat{\beta}_0$ and $\hat{\beta}_1$ in Table 4:

$$m = [\hat{\beta}_0 + \hat{\beta}_1 n], \quad n > 5, m \leq n/2,$$

where the greatest integer function is denoted by [.].

4.6. Power assessment for different tests

By eliminating the distribution associated with the hypothesis H_0, we utilize competitor distributions specified in Henze and Meintanis (2005) and Krit (2014) along with their selected parameter configurations for power assessment. The competitor distributions include W(θ): Weibull, $\Gamma(\alpha,\beta)$: Gamma, LN(μ,σ): Log-normal, HN(σ): Half-normal, U(0,b): Uniform, EV(θ): Modified extreme value, LF(θ): Linear increasing failure rate law, D1(β,b): Dhillon (Dhillon, 1981), D2(λ,b): Dhillon (Dhillon, 1981), CH(λ,β): Chen (Chen, 2000),

Type of Phi	Weibull test		Log-normal test		Exponential test	
	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_0$	$\hat{\beta}_1$
KL	0.846	0.419	0.505	0.414	1.035	0.376
PE	1.298	0.416	0.652	0.419	0.789	0.381
BS	0.054	0.384	−0.310	0.394	0.137	0.354
TD	0.775	0.408	0.569	0.403	0.671	0.369
CR	1.154	0.404	0.413	0.411	0.880	0.376
MD	0.962	0.401	0.221	0.405	0.529	0.371
JD	0.952	0.407	0.813	0.402	0.969	0.370
HE	0.977	0.401	0.219	0.413	0.667	0.373
TV	0.582	0.358	1.762	0.329	0.793	0.324
JS	0.988	0.403	0.314	0.410	0.793	0.372

Table 4: Estimation of regression model parameters.

EW(θ,η,β): Exponentiated Weibull(Mudholkar and Srivastava, 1993), AW (θ,η,β): Additive Weibull (Xie and C.D., 1995), IG(α,β): Inverse-Gamma, HJ(θ,η,β): Hjorth (Hjorth, 1980), GG(α,η,β): Generalized Gamma (Stacy, 1962), GB(α,η,β): Generalized Burr (Mudholkar et al., 1996), PW(α,η,β): Power Generalized Weibull (Nikulin and Haghighi, 2006), MO(α,η,β): Extended Marshall-Olkin (Marshall and Olkin, 1997), and PA(1,β): Pareto. The details of the competitor distributions are presented in Table 5.

The parameters of the competitor distributions are selected to achieve distinct shapes of the hazard rate, as follows:

- Increasing hazard rate(IHR):
  U(0, 1), Γ(2,1), EV(0.5), HN($\sqrt{\pi/2}$), LF(2).
- Decreasing hazard rate(DHR):
  W(0.8), D2(1,0), AW(2,3,0.9), PA(1,0.4), MO(0.5,1,0.5).
- Bathtub-shaped hazard rate(BT):
  D1(1,0.8), CH(2,0.4), GG(0.1,1,4), HJ(2,1,1), PW(0.9,2,0.9).
- Upside-down bathtub-shaped hazard rate(UBT):
  LN(0,0.8), EW(4,12,0.6), IG(3,1), IS(1,4), GB(1,1,2).

The power of both Phi-divergence based tests and classic tests for alternative distributions is illustrated in the Appendix (refer to Tables 9 to 11), with tests categorized by Weibull, log-normal, and exponential distributions. The average power for each category, namely IHR, DHR, BT, and UBT, was computed, and the findings are detailed in Table 6. Also, to draw clearer conclusions about the best tests, we define the AP (Average Power) index, which represents the average power across all alternative distributions. The average power results suggest that:

For $n = 10$, the BS test has demonstrated superior performance for Weibull distribution. For Weibull distribution, as the sample size increases, particularly for categories IHR and DHR, the Kullback-Leibler divergence (KL) demonstrates excellent performance. The Anderson–Darling test has demonstrated good performance for exponential distribution in the categories of DHR and BT. In the categories of IHR and UBT for the exponential

Distribution	density/distribution function
W(θ)	$f(x) = \theta x^{\theta-1} exp(-x^{\theta}), \quad \theta > 0, x > 0$
Γ(α,β)	$f(x) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} exp(-\beta x), \quad \alpha, \beta > 0, x > 0$
LN(μ,σ)	$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} exp\{-\frac{(ln(x)-\mu)^2}{2\sigma^2}\}, \quad \mu \in \mathbb{R}, \sigma > 0, x > 0$
HN(σ)	$f(x) = \frac{\sqrt{2}}{\sigma\sqrt{\pi}} exp(-\frac{x^2}{2\sigma^2}), \quad \sigma > 0, x > 0$
U(0,b)	$f(x) = \frac{1}{b}, \quad 0 < x < b,$
EV(θ)	$F(x) = 1 - exp(\frac{1-e^{x}}{\theta}), \quad \theta > 0, x > 0$
LF(θ)	$f(x) = (1 + \theta x) exp(-x - \theta x^2/2), \quad \theta > 0, x > 0$
D1(β,b)	$F(x) = 1 - exp(-(e^{(\beta x)^b} - 1)), \quad b > 0, \beta > 0, x > 0$
D2(λ,b)	$F(x) = 1 - exp(-(\ln(\lambda x + 1))^{b+1}), \quad b > 0, \beta > 0, x > 0$
CH(λ,β)	$F(x) = 1 - exp(\lambda(1 - e^{x^{\beta}})), \quad \lambda, \beta > 0, x > 0$
EW(θ,η,β)	$F(x) = \left(1 - exp(-(x/\eta)^{\beta})\right)^{\theta}, \quad \theta, \eta, \beta > 0, x > 0$
AW(θ,η,β)	$F(x) = 1 - exp(-\theta x - \eta x^{\beta}), \quad \theta, \eta, \beta > 0, x > 0,$
IG(α,β)	$f(x) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} (1/x)^{\alpha+1} exp(-\beta/x), \quad \alpha, \beta > 0, x > 0$
HJ(θ,η,β)	$F(x) = 1 - \frac{\exp(-\eta x^2/2)}{(1+\beta x)^{\theta/\beta}}, \quad \theta, \eta, \beta > 0, x > 0$
GG(α,η,β)	$F(x) = \frac{1}{\Gamma(\alpha)} \gamma(\alpha, (x/\eta)^{\beta}), \quad \gamma(s, x) = \int_0^x \nu^{s-1} e^{-\nu} d\nu, \quad x, \alpha, \eta, \beta > 0$
GB(α,η,β)	$F(x) = 1 - \left(1 + \alpha(x/\eta)^{\beta}\right)^{-1/\alpha}, \quad \alpha, \beta, \eta > 0, x > 0$
PW(α,η,β)	$F(x) = 1 - \exp\left(1 - \left(1 + (x/\eta)^{\beta}\right)^{1/\alpha}\right), \quad \alpha, \beta, \eta > 0, x > 0$
MO(α,η,β)	$F(x) = 1 - \frac{\alpha \exp(-(x/\eta)^{\beta})}{1-(1-\alpha)\exp(-(x/\eta)^{\beta})}, \quad \alpha, \beta, \eta > 0, x > 0$
PA(1,β)	$f(x) = \frac{\beta}{x^{\beta+1}}, \quad \beta > 0, x > 1$

Table 5: The competitor distributions explored in the simulation

distribution, the Kullback-Leibler (KL) test has exhibited superior performance for sample sizes $n = 10$ and $n = 20$. For the log-normal distribution, the tests via Phi-divergence have exhibited superior performance compared to classical tests. The HE test for $n = 10$ and the TD test for $n = 20$ have shown good performance for the log-normal distribution. However, with an increased sample size of 50, the KL test has demonstrated higher power, in the categories of IHR and BT. TD test exhibits very high power and shows the best performance for the category DHR for the log-normal distribution with a sample size of 50. Based on the AP index, it can be concluded that for the Weibull distribution with a small sample size ($n = 10$), the BS test exhibited the highest power. For other sample sizes, both for this distribution and the exponential distribution, the AD test performed better. However, the results were different for the log-normal distribution. The highest AP was observed for sample size 10 in the HE test, for sample size 20 in the TD test, and for sample size 50 in the KL test.

Table 6: Comparative power analysis of tests at level $\alpha = 0.05$.

		$n=10$					$n=20$					$n=50$				
Dist.	*Test*	*IHR*	*DHR*	*BT*	*UBT*	*AP*	*IHR*	*DHR*	*BT*	*UBT*	*AP*	*IHR*	*DHR*	*BT*	*UBT*	*AP*
Weibull	KL	0.618	0.598	0.559	0.446	0.556	**0.781**	**0.705**	0.745	0.593	0.706	**0.845**	**0.901**	0.925	**0.839**	0.878
	PE	0.641	0.614	**0.593**	0.467	0.579	0.775	0.697	0.736	0.588	0.699	0.843	0.885	0.876	0.816	0.855
	BS	**0.655**	**0.619**	0.590	**0.479**	**0.586**	0.732	0.645	0.683	0.537	0.649	0.823	0.838	0.923	0.743	0.832
	TD	0.622	0.602	0.563	0.454	0.560	0.771	0.688	0.740	0.577	0.694	0.844	0.894	0.927	0.828	0.873
	CR	0.626	0.601	0.568	0.453	0.562	0.777	0.701	0.740	0.586	0.701	0.840	0.890	0.919	0.828	0.869
	MD	0.552	0.581	0.474	0.446	0.513	0.758	0.676	0.714	0.555	0.676	0.840	0.888	0.905	0.810	0.861
	JD	0.627	0.604	0.562	0.458	0.563	0.770	0.691	0.734	0.576	0.693	0.835	0.876	0.933	0.805	0.862
	HE	0.622	0.600	0.560	0.454	0.559	0.774	0.701	0.744	0.585	0.701	0.836	0.883	0.910	0.809	0.860
	TV	0.627	0.605	0.591	0.446	0.567	0.777	**0.705**	0.750	0.602	0.708	0.840	0.879	0.924	0.828	0.868
	JS	0.629	0.601	0.550	0.463	0.561	0.771	0.696	0.735	0.576	0.695	0.839	0.885	0.934	0.818	0.869
	CM	0.583	0.594	0.515	0.448	0.535	0.769	0.692	0.753	0.582	0.699	0.828	0.893	0.957	0.780	0.865
	KS	0.525	0.580	0.460	0.434	0.500	0.741	0.663	0.696	0.546	0.662	0.817	0.854	0.928	0.729	0.832
	KP	0.378	0.547	0.325	0.413	0.416	0.671	0.602	0.577	0.485	0.584	0.794	0.756	0.833	0.614	0.749
	AD	0.544	0.597	0.469	0.464	0.518	0.766	0.701	**0.758**	**0.631**	**0.714**	0.830	0.898	**0.967**	0.834	**0.882**
	$JW_{0.5}$	0.541	0.582	0.527	0.459	0.524	0.719	0.681	0.714	0.575	0.672	0.836	0.830	0.922	0.812	0.851
	JW_2	0.548	0.570	0.537	0.464	0.528	0.733	0.692	0.717	0.557	0.673	0.838	0.830	0.924	0.820	0.854
		$n=10$					$n=20$					$n=50$				
Dist.	*Test*	*IHR*	*DHR*	*BT*	*UBT*	*AP*	*IHR*	*DHR*	*BT*	*UBT*	*AP*	*IHR*	*DHR*	*BT*	*UBT*	*AP*
Log-normal	KL	0.243	0.181	0.240	0.065	0.182	0.509	0.351	0.517	0.080	0.364	**0.825**	0.580	**0.875**	0.131	**0.603**
	PE	0.246	0.183	0.241	0.066	0.184	0.471	0.337	0.472	0.095	0.344	0.810	0.575	0.853	0.143	0.595
	BS	0.206	0.160	0.200	0.060	0.156	0.457	0.313	0.460	0.073	0.326	0.800	0.542	0.849	0.116	0.577
	TD	0.269	0.203	0.267	0.076	0.204	**0.519**	**0.365**	**0.523**	**0.098**	**0.376**	0.821	**0.583**	0.865	0.139	0.602
	CR	0.260	0.192	0.255	0.066	0.193	0.495	0.342	0.505	0.075	0.354	0.791	0.557	0.829	0.132	0.577
	MD	0.266	0.202	0.261	0.076	0.201	0.513	0.353	**0.523**	0.082	0.368	0.816	0.569	0.868	0.130	0.596
	JD	0.255	0.195	0.253	0.070	0.193	0.504	0.353	0.508	0.086	0.363	0.819	0.582	0.866	0.137	0.601
	HE	**0.278**	**0.209**	**0.277**	**0.077**	**0.211**	0.489	0.351	0.494	0.095	0.357	0.817	0.578	0.864	0.133	0.598
	TV	0.230	0.179	0.232	0.072	0.178	0.482	0.355	0.487	0.106	0.357	0.792	0.555	0.843	0.133	0.581
	JS	0.252	0.188	0.250	0.069	0.190	0.516	0.352	0.521	0.085	0.369	0.812	0.566	0.863	0.120	0.590
	CM	0.201	0.151	0.196	0.059	0.152	0.419	0.298	0.420	0.083	0.305	0.742	0.519	0.767	0.126	0.539
	KS	0.170	0.129	0.168	0.058	0.131	0.336	0.241	0.334	0.073	0.246	0.655	0.452	0.667	0.108	0.471
	KP	0.167	0.128	0.162	0.055	0.128	0.349	0.251	0.343	0.071	0.254	0.656	0.436	0.671	0.100	0.466
	AD	0.215	0.162	0.211	0.061	0.162	0.451	0.320	0.453	0.086	0.328	0.780	0.559	0.814	**0.144**	0.574
	$JW_{0.5}$	0.190	0.183	0.161	0.065	0.154	0.409	0.326	0.341	0.083	0.301	0.731	0.546	0.762	0.126	0.563
	JW_2	0.195	0.179	0.165	0.069	0.156	0.405	0.331	0.342	0.080	0.300	0.743	0.551	0.763	0.122	0.567
		$n=10$					$n=20$					$n=50$				
Dist.	*Test*	*IHR*	*DHR*	*BT*	*UBT*	*AP*	*IHR*	*DHR*	*BT*	*UBT*	*AP*	*IHR*	*DHR*	*BT*	*UBT*	*AP*
Exponential	KL	**0.267**	0.254	0.058	**0.472**	0.263	**0.461**	0.453	0.118	0.639	0.418	**0.741**	0.602	0.256	0.829	0.607
	PE	0.246	0.212	0.040	0.461	0.240	0.451	0.426	0.097	**0.650**	0.406	0.729	0.592	0.232	0.837	0.598
	BS	0.220	0.350	0.121	0.365	0.264	0.340	0.532	0.239	0.493	0.401	0.627	0.646	0.386	0.685	0.586
	TD	0.236	0.355	0.124	0.398	0.278	0.437	0.511	0.188	0.601	0.434	0.726	0.616	0.304	0.800	0.611
	CR	0.235	0.362	0.130	0.412	0.285	0.448	0.510	0.187	0.615	0.440	0.738	0.613	0.299	0.814	0.616
	MD	0.210	0.367	0.139	0.344	0.265	0.401	0.520	0.200	0.561	0.421	0.724	0.627	0.323	0.783	0.614
	JD	0.217	0.367	0.134	0.366	0.271	0.452	0.476	0.137	0.629	0.424	0.723	0.622	0.314	0.790	0.612
	HE	0.223	0.361	0.130	0.372	0.272	0.432	0.517	0.203	0.600	0.438	0.729	0.620	0.314	0.796	0.615
	TV	0.233	0.354	0.129	0.465	0.295	0.400	0.539	0.262	0.621	0.456	0.679	0.649	0.417	**0.839**	0.646
	JS	0.262	0.273	0.064	0.461	0.265	0.427	0.514	0.192	0.590	0.431	0.697	0.633	0.356	0.764	0.612
	CM	0.190	0.468	0.197	0.361	0.304	0.359	0.605	0.301	0.528	0.448	0.667	0.699	0.419	0.729	0.629
	KS	0.157	0.447	0.183	0.336	0.281	0.287	0.588	0.283	0.500	0.415	0.592	0.685	0.406	0.697	0.595
	KP	0.169	0.400	0.169	0.332	0.268	0.306	0.555	0.281	0.518	0.415	0.580	0.664	0.421	0.743	0.602
	AD	0.139	**0.510**	**0.287**	0.305	**0.310**	0.325	**0.630**	**0.376**	0.527	**0.465**	0.654	**0.725**	**0.457**	0.773	**0.652**
	$JW_{0.5}$	0.175	0.480	0.182	0.340	0.294	0.247	0.601	0.289	0.461	0.400	0.369	0.721	0.445	0.645	0.545
	JW_2	0.151	0.483	0.181	0.342	0.289	0.217	0.602	0.272	0.464	0.389	0.331	0.710	0.436	0.641	0.529

5. Practical demonstration using real data

The implementation of GOF tests using Phi-divergence is exemplified through examples for the Weibull, log-normal, and exponential distributions in this section.

Example 5.1. In this illustration, we have analyzed data related to the duration (in hours) required to repair Automated Teller Machines (ATMs) at one of the branches of a bank in Asia. The dataset includes the repair duration for 32 ATMs, presented as follows:

29.8, 5.4, 469.2, 39.5, 26.7, 614.6, 60.2, 2.2, 131.1, 528.7, 638.0, 609.5, 126.0, 404.2, 98.4, 554.9, 108.7, 113.4 23.2, 4.8, 635.0, 1.1, 23.3, 98.6, 21.3, 374.7, 610.7, 619.5, 20.3, 105.4, 117.8, 253.5.

The Weibull, log-normal, and exponential distributions are frequently employed for modeling the time required to repair a maintainable system (ÓConnor and Kleyner, 2012). Figure 1a presents the histogram of the given data, along with the fitted Weibull, log-normal, and exponential density functions. The fit of the exponential distribution to the data seems to be superior, as observed in Figure 1a. To further investigate this, GOF tests based on Phi-divergence, as discussed in the previous section, are conducted for this dataset. The determination of the value of m is referenced to Table 4. The results are displayed in Table 7. The results, including the test statistic (DB_ϕ), critical value ($Q_{0.95}$), and the test outcome for each distribution, are displayed in Table 7. It is apparent that all tests relying on Phi-divergence rejected the suitability of Weibull and log-normal distributions for the data. However, with the exception of the TV test, the goodness-of-fit tests based on Phi-divergence confirmed the adequacy of the exponential distribution for the provided data.

	Test for Weibull distribution									
	KL	PE	BS	TD	CR	MD	JD	HE	TV	JS
DB_ϕ	0.618	0.345	0.621	0.643	0.494	10.131	10.760	0.285	0.805	0.421
$Q_{0.95}$	0.072	0.059	0.053	0.075	0.071	0.091	0.163	0.019	0.267	0.039
Result	*REJ.*	*REJ.*	*REJ.*	*REJ.*	*REJ.*	*REJ.*	*REJ.*	*REJ.*	*REJ.*	*REJ.*
	Test for log-normal distribution									
	KL	PE	BS	TD	CR	MD	JD	HE	TV	JS
DB_ϕ	0.075	0.063	0.052	0.082	0.073	0.097	0.173	0.021	0.281	0.042
$Q_{0.95}$	0.060	0.052	0.044	0.064	0.058	0.076	0.133	0.016	0.248	0.033
Result	*REJ.*	*REJ.*	*REJ.*	*REJ.*	*REJ.*	*REJ.*	*REJ.*	*REJ.*	*REJ.*	*REJ.*
	Test for exponential distribution									
	KL	PE	BS	TD	CR	MD	JD	HE	TV	JS
DB_ϕ	0.053	0.044	0.042	0.056	0.051	0.063	0.116	0.014	0.284	0.028
$Q_{0.95}$	0.072	0.057	0.051	0.076	0.067	0.089	0.159	0.020	0.279	0.038
Result	*ACC.*	*ACC.*	*ACC.*	*ACC.*	*ACC.*	*ACC.*	*ACC.*	*ACC.*	*REJ.*	*ACC.*

Table 7: GOF Test Results for repair duration data.

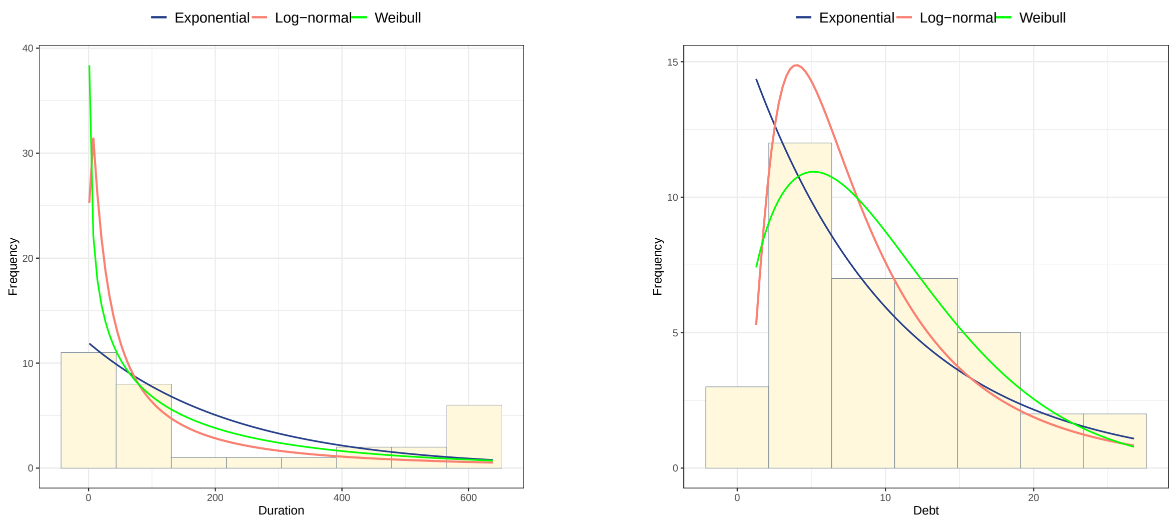
Example 5.2. In this instance, the examination revolves around the amount of debt owed by 38 government institutions to banks in one of the Asian countries, quantified in the local currency, as detailed in the following dataset:

4.839, 16.000, 10.595, 2.990, 16.774, 2.070, 16.600, 7.532, 2.124, 8.966, 6.951, 18.573, 4.171, 26.771, 7.228, 22.329, 21.683, 17.749, 24.705, 13.500, 2.522, 11.377, 3.057, 5.258, 2.219, 10.836, 4.741, 11.325, 13.037, 7.531, 3.632, 1.274, 9.232, 3.574, 5.236, 3.728, 11.395, 13.325.

An application of lifetime distribution involves its utilization in the examination of the survival of bank loans (Chimedza and Marimo, 2017). In a similar context, we entertain the idea that one of the Weibull, log-normal, and exponential distributions might be a suitable fit for the data. The histogram of this data, accompanied by the fitted Weibull, log-normal, and exponential density functions, is illustrated in Figure 1b. It is apparent from Figure 1b that the fit of the log-normal distribution to the data appears to be superior. GOF tests via Phi-divergence are conducted for this dataset after determining the critical value and the m value(refer to Table 4). DB_ϕ, critical value($Q_{0.95}$), and the test result are presented in Table 8. All tests indicated the lack of fit for the Weibull and exponential distributions to the data. In contrast, with the exception of the BS test, tests based on Phi-divergence confirmed the suitability of the log-normal distribution for the data.

	Test for Weibull distribution									
	KL	PE	BS	TD	CR	MD	JD	HE	TV	JS
DB_ϕ	0.691	0.386	0.686	0.716	0.556	6.405	7.095	0.297	0.866	0.459
$Q_{0.95}$	0.065	0.056	0.048	0.072	0.064	0.085	0.147	0.018	0.258	0.036
Result	$REJ.$	$REJ.$	$REJ.$	$REJ.$	$REJ.$	$REJ.$	$REJ.$	$REJ.$	$REJ.$	$REJ.$
	Test for log-normal distribution									
	KL	PE	BS	TD	CR	MD	JD	HE	TV	JS
DB_ϕ	0.051	0.042	0.041	0.057	0.050	0.066	0.116	0.015	0.218	0.029
$Q_{0.95}$	0.057	0.048	0.041	0.060	0.057	0.070	0.128	0.016	0.239	0.031
Result	$ACC.$	$ACC.$	$REJ.$	$ACC.$	$ACC.$	$ACC.$	$ACC.$	$ACC.$	$ACC.$	$ACC.$
	Test for exponential distribution									
	KL	PE	BS	TD	CR	MD	JD	HE	TV	JS
DB_ϕ	0.082	0.068	0.057	0.088	0.081	0.103	0.186	0.023	0.319	0.045
$Q_{0.95}$	0.065	0.054	0.046	0.069	0.065	0.080	0.148	0.018	0.267	0.035
Result	$REJ.$	$REJ.$	$REJ.$	$REJ.$	$REJ.$	$REJ.$	$REJ.$	$REJ.$	$REJ.$	$REJ.$

Table 8: GOF Test Results for bank debt data.



(a) Repair duration. (b) Bank debt

Figure 1: Data histograms for Examples 5.1 and 5.2.

6. Conclusion

In this paper, we introduced a general goodness-of-fit(GOF) test based on Phi-divergence. The non-negativity of the proposed test statistic, akin to Phi-divergence, was illustrated. Furthermore, we established the invariant and consistency of the proposed test. The application of the test to lifetime distributions, such as exponential, log-normal, and Weibull distributions, was demonstrated. The test's performance was assessed for specific functions of ϕ corresponding to each distribution.

The tests' performance was notably influenced by the shape of the hazard rate. The average power of the competitor distributions for categories IHR, DHR, BT, and UBT was computed in a simulation study. The simulation results showed that the Balakrishnan-Sanghvi test exhibits the most power with a sample size of 10 for testing the Weibull distribution. With an increase in the sample size, especially in the categories of IHR and DHR, the Kullback-Leibler test consistently demonstrates excellent performance. The Anderson–Darling test has shown good performancein assessing the fit of the exponential distribution, particularly in the categories of DHR and BT. In the categories of IHR and UBT for assessing the fit of the exponential distribution, the Kullback-Leibler test has demonstrated superior performance for sample sizes $n = 10$ and $n = 20$. The tests via Phi-divergence have shown superior performance compared to classical tests for assessing the fit of the log-normal distribution. For testing this distribution, the Hellinger test for $n = 10$ and the Triangular divergence for $n = 20$ have demonstrated good performance. With an increased sample size of 50, the Kullback-Leibler test demonstrated higher power, particularly in the categories of IHR and BT. Triangular divergence exhibits very high power and demonstrates the best performance for the category DHR in testing the log-normal distribution with a sample size of 50. In general, based on the average power of all competing distributions, the BS test exhibited the highest power for the Weibull distribution with a small sample size ($n = 10$). For larger sample sizes, the AD test performed better for both the Weibull and exponential distributions. The HE test showed the highest AP for a sample size of 10, the TD test for a sample size of 20, and the KL test for a sample size of 50.

Acknowledgements

The authors are grateful to the anonymous referee and the associate editor for providing useful comments on an earlier version of this manuscript.

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Appendix: Power comparison results

	$n = 10$															
Alt.	KL	PE	BS	TD	CR	MD	JD	HE	TV	JS	CM	KS	KP	AD	$JW_{0.5}$	JW_2
U	0.982	0.999	0.985	0.982	0.992	0.794	0.974	0.974	1.000	0.984	0.736	0.630	0.348	0.642	0.612	0.622
Γ	0.599	0.576	0.689	0.601	0.588	0.713	0.655	0.639	0.505	0.726	0.811	0.768	0.622	0.873	0.448	0.451
EV	0.944	0.969	0.957	0.948	0.954	0.841	0.944	0.943	0.974	0.918	0.902	0.827	0.657	0.847	0.512	0.531
HN	0.078	0.094	0.101	0.080	0.083	0.058	0.079	0.074	0.087	0.075	0.055	0.051	0.044	0.041	0.612	0.622
LF	0.490	0.566	0.541	0.498	0.514	0.352	0.482	0.483	0.570	0.444	0.412	0.347	0.220	0.318	0.520	0.514
$D2$	0.171	0.205	0.208	0.179	0.185	0.148	0.183	0.170	0.199	0.183	0.191	0.166	0.107	0.259	0.814	0.821
AW	0.998	0.999	0.998	0.998	0.999	0.991	0.998	0.998	0.999	0.998	0.997	0.989	0.954	0.992	0.121	0.132
PA	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.999	0.981
MO	0.223	0.251	0.269	0.229	0.220	0.183	0.234	0.231	0.223	0.224	0.187	0.166	0.128	0.135	0.392	0.345
$D1$	0.335	0.397	0.380	0.341	0.348	0.225	0.332	0.335	0.405	0.299	0.253	0.211	0.127	0.187	0.345	0.351
CH	0.944	0.969	0.957	0.948	0.954	0.841	0.944	0.943	0.974	0.918	0.902	0.827	0.657	0.847	0.512	0.525
GG	0.956	0.986	0.964	0.960	0.971	0.771	0.949	0.949	0.997	0.924	0.779	0.687	0.422	0.693	0.691	0.703
HJ	0.447	0.504	0.491	0.448	0.460	0.338	0.443	0.448	0.487	0.414	0.390	0.335	0.224	0.302	0.581	0.592
PW	0.116	0.107	0.160	0.118	0.107	0.192	0.141	0.125	0.090	0.194	0.252	0.239	0.195	0.317	0.508	0.512
LN	0.142	0.161	0.180	0.144	0.139	0.148	0.149	0.143	0.152	0.169	0.195	0.164	0.107	0.270	0.571	0.584
EW	0.984	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.983	1.000	1.000	1.000	1.000	0.581	0.591
IG	0.852	0.865	0.889	0.862	0.856	0.845	0.871	0.861	0.795	0.878	0.774	0.764	0.776	0.685	0.411	0.425
IS	0.082	0.106	0.118	0.084	0.082	0.088	0.089	0.094	0.082	0.101	0.080	0.075	0.074	0.106	0.431	0.421
GB	0.171	0.205	0.209	0.179	0.185	0.148	0.183	0.170	0.199	0.183	0.191	0.166	0.107	0.259	0.299	0.301
	$n = 20$															
Alt.	KL	PE	BS	TD	CR	MD	JD	HE	TV	JS	CM	KS	KP	AD	$JW_{0.5}$	JW_2
U	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.731	0.742
Γ	0.936	0.910	0.952	0.958	0.929	0.931	0.940	0.940	0.858	0.935	0.987	0.971	0.900	0.993	0.521	0.525
EV	1.000	1.000	0.996	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.999	0.998	0.964	1.000	0.841	0.893
HN	0.123	0.111	0.070	0.099	0.115	0.089	0.102	0.107	0.141	0.105	0.077	0.063	0.045	0.067	0.751	0.761
LF	0.848	0.852	0.640	0.799	0.839	0.771	0.807	0.824	0.889	0.813	0.783	0.675	0.443	0.770	0.751	0.742
$D2$	0.406	0.399	0.283	0.366	0.396	0.342	0.380	0.398	0.433	0.387	0.430	0.354	0.197	0.518	0.931	0.944
AW	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.251	0.261
PA	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
MO	0.415	0.388	0.296	0.385	0.409	0.361	0.384	0.405	0.386	0.397	0.340	0.296	0.210	0.286	0.541	0.561
$D1$	0.691	0.689	0.434	0.631	0.683	0.585	0.638	0.667	0.772	0.647	0.561	0.439	0.236	0.549	0.431	0.454
CH	1.000	1.000	0.996	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.999	0.998	0.964	1.000	0.741	0.751
GG	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.999	0.999	0.941	1.000	0.851	0.842
HJ	0.788	0.792	0.613	0.751	0.790	0.721	0.762	0.776	0.824	0.770	0.743	0.631	0.430	0.715	0.831	0.829
PW	0.248	0.201	0.375	0.318	0.228	0.265	0.272	0.279	0.154	0.261	0.464	0.415	0.312	0.529	0.714	0.708
LN	0.365	0.345	0.285	0.352	0.339	0.293	0.328	0.341	0.359	0.326	0.403	0.322	0.166	0.516	0.631	0.624
EW	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.682	0.692
IG	0.995	0.994	0.990	0.996	0.995	0.994	0.995	0.995	0.986	0.994	0.971	0.966	0.980	0.954	0.561	0.554
IS	0.201	0.200	0.129	0.173	0.199	0.147	0.177	0.191	0.231	0.176	0.107	0.087	0.083	0.165	0.591	0.505
GB	0.406	0.399	0.283	0.366	0.396	0.342	0.380	0.398	0.433	0.387	0.430	0.354	0.197	0.518	0.412	0.408
	$n = 50$															
Alt.	KL	PE	BS	TD	CR	MD	JD	HE	TV	JS	CM	KS	KP	AD	$JW_{0.5}$	JW_2
U	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.861	0.863
Γ	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.752	0.742
EV	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.921	0.925
HN	0.228	0.219	0.129	0.220	0.205	0.202	0.181	0.187	0.204	0.201	0.144	0.102	0.054	0.151	0.821	0.832
LF	0.998	0.997	0.986	0.997	0.996	0.997	0.995	0.995	0.998	0.996	0.997	0.986	0.917	0.997	0.824	0.830
$D2$	0.831	0.807	0.693	0.818	0.816	0.802	0.788	0.804	0.815	0.795	0.853	0.773	0.528	0.905	1.000	1.000
AW	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.491	0.488
PA	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
MO	0.774	0.733	0.658	0.759	0.745	0.750	0.714	0.728	0.701	0.744	0.719	0.643	0.495	0.686	0.829	0.831
$D1$	0.981	0.980	0.930	0.982	0.981	0.974	0.968	0.973	0.983	0.976	0.960	0.897	0.663	0.976	0.732	0.735
CH	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.951	0.961
GG	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
HJ	0.995	0.992	0.979	0.994	0.993	0.992	0.990	0.992	0.993	0.992	0.992	0.969	0.880	0.993	1.000	1.000
PW	0.649	0.407	0.705	0.658	0.624	0.560	0.704	0.585	0.645	0.701	0.832	0.772	0.620	0.864	0.925	0.924
LN	0.823	0.756	0.707	0.813	0.807	0.777	0.792	0.781	0.798	0.812	0.820	0.705	0.405	0.900	0.785	0.791
EW	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.913	0.929
IG	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.825	0.818
IS	0.541	0.518	0.318	0.509	0.514	0.470	0.445	0.462	0.525	0.482	0.228	0.169	0.138	0.363	0.742	0.751
GB	0.831	0.807	0.693	0.818	0.816	0.802	0.788	0.804	0.815	0.795	0.853	0.773	0.528	0.905	0.795	0.812

Table 9: Comparison of power for the Weibull distribution tests at level $\alpha = 0.05$.

Alt.	KL	PE	BS	TD	CR	MD	JD	HE	TV	JS	CM	KS	KP	AD	$JW_{0.5}$	JW_2
	$n = 10$															
U	0.423	0.419	0.442	0.458	0.482	0.465	0.450	0.511	0.392	0.445	0.366	0.288	0.315	0.393	0.252	0.253
Γ	0.099	0.108	0.075	0.112	0.099	0.115	0.103	0.110	0.097	0.103	0.087	0.080	0.072	0.090	0.161	0.162
EV	0.223	0.223	0.172	0.250	0.231	0.237	0.234	0.252	0.217	0.232	0.176	0.154	0.147	0.190	0.264	0.261
HN	0.235	0.239	0.167	0.260	0.244	0.252	0.243	0.255	0.217	0.235	0.187	0.161	0.147	0.199	0.151	0.158
LF	0.235	0.241	0.176	0.268	0.245	0.260	0.245	0.265	0.226	0.246	0.189	0.166	0.154	0.204	0.122	0.139
W	0.148	0.154	0.112	0.174	0.143	0.166	0.159	0.167	0.153	0.155	0.116	0.103	0.095	0.125	0.091	0.061
$D2$	0.078	0.080	0.056	0.089	0.074	0.084	0.079	0.081	0.092	0.082	0.066	0.068	0.059	0.070	0.324	0.325
AW	0.142	0.152	0.113	0.163	0.146	0.168	0.161	0.158	0.140	0.146	0.123	0.109	0.101	0.129	0.111	0.119
PA	0.421	0.414	0.433	0.460	0.486	0.469	0.453	0.515	0.392	0.441	0.359	0.281	0.310	0.386	0.197	0.194
MO	0.114	0.117	0.085	0.130	0.113	0.124	0.121	0.126	0.121	0.115	0.091	0.086	0.075	0.098	0.194	0.197
$D1$	0.266	0.265	0.208	0.298	0.276	0.281	0.278	0.299	0.259	0.274	0.212	0.182	0.173	0.232	0.158	0.157
CH	0.223	0.223	0.172	0.250	0.231	0.237	0.234	0.252	0.217	0.232	0.176	0.154	0.147	0.190	0.154	0.157
GG	0.393	0.389	0.375	0.426	0.438	0.429	0.413	0.464	0.362	0.410	0.336	0.268	0.280	0.357	0.163	0.159
HJ	0.158	0.163	0.127	0.179	0.171	0.183	0.168	0.192	0.158	0.169	0.129	0.120	0.105	0.139	0.178	0.189
PW	0.161	0.165	0.117	0.181	0.160	0.173	0.171	0.180	0.165	0.166	0.129	0.115	0.104	0.138	0.151	0.162
EW	0.060	0.060	0.057	0.072	0.059	0.073	0.067	0.071	0.064	0.065	0.054	0.053	0.053	0.059	0.062	0.063
IG	0.083	0.082	0.073	0.094	0.083	0.094	0.088	0.097	0.084	0.083	0.068	0.064	0.063	0.070	0.063	0.074
IS	0.040	0.043	0.054	0.049	0.048	0.052	0.047	0.060	0.047	0.046	0.046	0.046	0.046	0.044	0.089	0.079
GB	0.078	0.080	0.056	0.089	0.074	0.084	0.079	0.081	0.092	0.082	0.066	0.068	0.059	0.070	0.044	0.058
	$n = 20$															
Alt.	KL	PE	BS	TD	CR	MD	JD	HE	TV	JS	CM	KS	KP	AD	$JW_{0.5}$	JW_2
U	0.880	0.771	0.865	0.854	0.868	0.890	0.843	0.805	0.790	0.880	0.730	0.585	0.683	0.778	0.511	0.501
Γ	0.171	0.186	0.139	0.198	0.166	0.173	0.178	0.186	0.190	0.168	0.144	0.123	0.108	0.154	0.361	0.359
EV	0.491	0.453	0.425	0.504	0.481	0.495	0.488	0.476	0.470	0.498	0.396	0.314	0.307	0.431	0.561	0.552
HN	0.501	0.471	0.425	0.517	0.475	0.494	0.506	0.477	0.477	0.512	0.403	0.319	0.314	0.439	0.331	0.329
LF	0.504	0.475	0.432	0.523	0.488	0.511	0.505	0.499	0.481	0.524	0.422	0.339	0.332	0.455	0.281	0.285
W	0.301	0.295	0.236	0.330	0.285	0.304	0.311	0.305	0.313	0.298	0.248	0.205	0.190	0.269	0.181	0.182
$D2$	0.071	0.112	0.054	0.100	0.071	0.070	0.082	0.109	0.141	0.077	0.096	0.084	0.086	0.104	0.612	0.621
AW	0.297	0.286	0.239	0.322	0.290	0.297	0.313	0.308	0.303	0.298	0.238	0.194	0.173	0.261	0.174	0.168
PA	0.881	0.780	0.871	0.847	0.867	0.880	0.846	0.807	0.783	0.877	0.728	0.579	0.674	0.773	0.251	0.261
MO	0.207	0.212	0.164	0.224	0.198	0.213	0.214	0.223	0.232	0.213	0.177	0.144	0.134	0.193	0.412	0.422
$D1$	0.585	0.531	0.520	0.588	0.570	0.591	0.576	0.556	0.548	0.590	0.480	0.377	0.385	0.516	0.311	0.312
CH	0.491	0.453	0.425	0.504	0.481	0.495	0.488	0.476	0.470	0.498	0.396	0.314	0.307	0.431	0.401	0.411
GG	0.830	0.724	0.802	0.805	0.817	0.839	0.796	0.756	0.743	0.831	0.686	0.551	0.618	0.726	0.312	0.332
HJ	0.358	0.331	0.297	0.370	0.344	0.358	0.352	0.348	0.333	0.361	0.269	0.211	0.200	0.295	0.332	0.321
PW	0.322	0.320	0.258	0.346	0.313	0.331	0.330	0.334	0.340	0.326	0.272	0.215	0.205	0.297	0.351	0.332
EW	0.076	0.091	0.072	0.094	0.071	0.080	0.083	0.085	0.090	0.079	0.075	0.067	0.062	0.074	0.092	0.089
IG	0.129	0.135	0.110	0.149	0.118	0.124	0.135	0.142	0.145	0.133	0.113	0.096	0.087	0.122	0.083	0.081
IS	0.044	0.044	0.058	0.049	0.040	0.054	0.043	0.045	0.046	0.051	0.046	0.046	0.050	0.044	0.091	0.089
GB	0.071	0.112	0.054	0.100	0.071	0.070	0.082	0.109	0.141	0.077	0.096	0.084	0.086	0.104	0.066	0.061
	$n = 50$															
Alt.	KL	PE	BS	TD	CR	MD	JD	HE	TV	JS	CM	KS	KP	AD	$JW_{0.5}$	JW_2
U	1.000	1.000	1.000	1.000	0.998	1.000	1.000	0.999	1.000	1.000	0.991	0.964	0.991	0.997	0.811	0.821
Γ	0.400	0.393	0.339	0.402	0.360	0.373	0.388	0.389	0.343	0.368	0.309	0.253	0.211	0.353	0.821	0.836
EV	0.914	0.886	0.895	0.904	0.857	0.906	0.909	0.900	0.876	0.904	0.792	0.668	0.678	0.844	0.812	0.817
HN	0.915	0.889	0.888	0.901	0.867	0.907	0.900	0.897	0.871	0.896	0.803	0.690	0.696	0.850	0.622	0.654
LF	0.897	0.883	0.879	0.898	0.871	0.894	0.898	0.899	0.870	0.894	0.812	0.702	0.705	0.855	0.591	0.585
W	0.671	0.651	0.618	0.674	0.616	0.655	0.676	0.656	0.617	0.659	0.545	0.451	0.412	0.605	0.401	0.414
$D2$	0.067	0.104	0.041	0.088	0.105	0.060	0.078	0.082	0.095	0.059	0.133	0.107	0.119	0.156	0.891	0.886
AW	0.661	0.633	0.609	0.661	0.603	0.649	0.660	0.656	0.604	0.636	0.552	0.444	0.407	0.609	0.331	0.326
PA	1.000	1.000	1.000	0.999	0.998	1.000	1.000	1.000	1.000	1.000	0.991	0.961	0.989	0.997	0.502	0.514
MO	0.501	0.486	0.441	0.493	0.461	0.482	0.496	0.497	0.457	0.475	0.375	0.297	0.254	0.429	0.603	0.613
$D1$	0.957	0.944	0.947	0.954	0.922	0.958	0.956	0.949	0.940	0.955	0.875	0.769	0.794	0.914	0.823	0.819
CH	0.914	0.886	0.895	0.904	0.857	0.906	0.909	0.900	0.876	0.904	0.792	0.668	0.678	0.844	0.821	0.815
GG	0.999	0.997	0.999	0.999	0.995	0.998	0.998	0.998	0.999	0.999	0.984	0.940	0.972	0.993	0.662	0.674
HJ	0.782	0.751	0.744	0.767	0.708	0.772	0.761	0.765	0.737	0.761	0.601	0.479	0.463	0.670	0.751	0.759
PW	0.723	0.688	0.662	0.701	0.663	0.704	0.705	0.706	0.663	0.694	0.586	0.481	0.448	0.650	0.752	0.749
EW	0.136	0.138	0.119	0.136	0.125	0.127	0.140	0.125	0.126	0.115	0.111	0.095	0.081	0.121	0.115	0.116
IG	0.265	0.275	0.237	0.278	0.253	0.273	0.279	0.277	0.242	0.252	0.211	0.176	0.146	0.247	0.119	0.118
IS	0.058	0.056	0.068	0.055	0.044	0.059	0.053	0.050	0.068	0.054	0.051	0.053	0.053	0.050	0.125	0.119
GB	0.067	0.104	0.041	0.088	0.105	0.060	0.078	0.082	0.095	0.059	0.133	0.107	0.119	0.156	0.145	0.135

Table 10: Comparison of power for the log-normal distribution tests at level $\alpha = 0.05$.

	$n = 10$															
Alt.	KL	PE	BS	TD	CR	MD	JD	HE	TV	JS	CM	KS	KP	AD	$JW_{0.5}$	JW_2
U	0.470	0.410	0.472	0.479	0.460	0.463	0.467	0.474	0.357	0.476	0.353	0.266	0.338	0.276	0.121	0.114
Γ	0.347	0.345	0.231	0.273	0.274	0.211	0.229	0.240	0.349	0.348	0.236	0.201	0.191	0.167	0.451	0.330
EV	0.133	0.116	0.106	0.114	0.114	0.107	0.106	0.106	0.116	0.117	0.089	0.082	0.081	0.063	0.102	0.091
HN	0.172	0.155	0.131	0.139	0.147	0.124	0.132	0.133	0.152	0.157	0.120	0.104	0.110	0.085	0.121	0.127
LF	0.214	0.203	0.162	0.175	0.179	0.147	0.153	0.164	0.194	0.213	0.153	0.131	0.127	0.102	0.078	0.092
W	0.018	0.013	0.046	0.047	0.052	0.055	0.058	0.054	0.048	0.019	0.114	0.101	0.081	0.170	0.106	0.112
$D2$	0.217	0.166	0.322	0.330	0.342	0.332	0.337	0.331	0.327	0.240	0.524	0.488	0.405	0.561	0.831	0.829
AW	0.036	0.034	0.044	0.043	0.044	0.046	0.047	0.046	0.041	0.039	0.058	0.055	0.050	0.069	0.071	0.053
PA	0.721	0.655	0.827	0.826	0.832	0.846	0.843	0.835	0.809	0.750	0.869	0.858	0.825	0.871	1.000	1.000
MO	0.280	0.191	0.513	0.528	0.541	0.554	0.550	0.540	0.545	0.317	0.775	0.734	0.640	0.881	0.391	0.421
$D1$	0.078	0.068	0.076	0.077	0.079	0.078	0.074	0.076	0.074	0.073	0.059	0.057	0.063	0.057	0.092	0.081
CH	0.110	0.057	0.316	0.323	0.345	0.363	0.358	0.346	0.352	0.138	0.601	0.552	0.461	0.790	0.077	0.089
GG	0.016	0.008	0.117	0.122	0.126	0.149	0.139	0.134	0.136	0.023	0.226	0.209	0.227	0.475	0.111	0.141
HJ	0.052	0.043	0.048	0.056	0.057	0.055	0.057	0.053	0.046	0.053	0.044	0.044	0.046	0.044	0.115	0.119
PW	0.033	0.027	0.047	0.042	0.043	0.049	0.043	0.044	0.039	0.033	0.053	0.053	0.049	0.069	0.513	0.474
LN	0.297	0.288	0.194	0.222	0.240	0.181	0.201	0.209	0.295	0.293	0.187	0.171	0.174	0.136	0.148	0.144
EW	0.136	0.120	0.097	0.103	0.111	0.096	0.096	0.101	0.130	0.127	0.084	0.075	0.078	0.057	1.000	1.000
IG	0.715	0.706	0.534	0.584	0.604	0.490	0.530	0.539	0.695	0.706	0.489	0.464	0.464	0.398	0.083	0.095
IS	0.956	0.962	0.818	0.880	0.898	0.778	0.823	0.832	0.968	0.947	0.853	0.795	0.768	0.786	0.181	0.171
GB	0.257	0.229	0.182	0.203	0.210	0.177	0.181	0.181	0.235	0.234	0.194	0.177	0.176	0.150	0.290	0.301
	$n = 20$															
Alt.	KL	PE	BS	TD	CR	MD	JD	HE	TV	JS	CM	KS	KP	AD	$JW_{0.5}$	JW_2
U	0.860	0.813	0.832	0.876	0.873	0.865	0.855	0.871	0.748	0.870	0.674	0.520	0.666	0.637	0.213	0.211
Γ	0.610	0.631	0.337	0.543	0.566	0.460	0.586	0.521	0.583	0.514	0.476	0.389	0.362	0.448	0.508	0.375
EV	0.204	0.201	0.143	0.192	0.194	0.165	0.205	0.194	0.166	0.183	0.148	0.128	0.121	0.120	0.194	0.191
HN	0.269	0.268	0.164	0.250	0.263	0.221	0.266	0.253	0.209	0.247	0.207	0.168	0.161	0.173	0.162	0.151
LF	0.359	0.341	0.224	0.326	0.346	0.293	0.350	0.322	0.296	0.319	0.290	0.233	0.219	0.244	0.159	0.157
W	0.018	0.013	0.087	0.052	0.054	0.064	0.028	0.061	0.104	0.058	0.197	0.168	0.133	0.264	0.289	0.290
$D2$	0.521	0.481	0.636	0.621	0.618	0.634	0.568	0.630	0.647	0.620	0.804	0.767	0.684	0.822	0.962	0.960
AW	0.028	0.026	0.049	0.040	0.033	0.037	0.032	0.042	0.049	0.039	0.062	0.057	0.053	0.084	0.157	0.148
PA	0.977	0.968	0.992	0.991	0.989	0.992	0.984	0.992	0.989	0.990	0.989	0.987	0.981	0.990	1.000	1.000
MO	0.724	0.642	0.897	0.853	0.858	0.874	0.766	0.863	0.908	0.863	0.975	0.962	0.924	0.991	0.598	0.612
$D1$	0.101	0.097	0.098	0.105	0.099	0.101	0.106	0.115	0.092	0.103	0.084	0.080	0.084	0.086	0.171	0.167
CH	0.378	0.302	0.707	0.598	0.602	0.636	0.453	0.626	0.742	0.607	0.892	0.855	0.784	0.965	0.141	0.153
GG	0.030	0.016	0.289	0.145	0.145	0.176	0.051	0.173	0.378	0.161	0.417	0.378	0.435	0.700	0.222	0.211
HJ	0.054	0.046	0.055	0.055	0.058	0.051	0.050	0.057	0.048	0.055	0.044	0.043	0.046	0.044	0.239	0.225
PW	0.030	0.023	0.047	0.036	0.031	0.037	0.026	0.042	0.050	0.035	0.067	0.060	0.056	0.086	0.672	0.606
LN	0.574	0.607	0.289	0.492	0.530	0.415	0.555	0.488	0.555	0.475	0.346	0.297	0.323	0.346	0.242	0.246
EW	0.201	0.211	0.098	0.177	0.188	0.142	0.206	0.180	0.175	0.167	0.118	0.098	0.115	0.106	1.000	1.000
IG	0.984	0.986	0.851	0.966	0.973	0.936	0.980	0.963	0.976	0.959	0.852	0.828	0.848	0.867	0.270	0.292
IS	1.000	1.000	0.989	1.000	1.000	0.999	1.000	1.000	1.000	1.000	0.999	0.996	0.991	0.999	0.352	0.343
GB	0.437	0.444	0.239	0.370	0.385	0.313	0.407	0.368	0.400	0.351	0.325	0.280	0.314	0.317	0.440	0.437
	$n = 50$															
Alt.	KL	PE	BS	TD	CR	MD	JD	HE	TV	JS	CM	KS	KP	AD	$JW_{0.5}$	JW_2
U	1.000	0.999	1.000	1.000	1.000	1.000	1.000	1.000	0.998	1.000	0.984	0.928	0.988	0.987	0.382	0.371
Γ	0.937	0.947	0.750	0.914	0.928	0.899	0.903	0.908	0.953	0.876	0.903	0.833	0.799	0.922	0.581	0.425
EV	0.449	0.432	0.362	0.447	0.458	0.450	0.443	0.460	0.347	0.418	0.331	0.266	0.239	0.303	0.291	0.281
HN	0.597	0.567	0.459	0.569	0.593	0.576	0.569	0.572	0.476	0.534	0.478	0.394	0.365	0.446	0.290	0.254
LF	0.723	0.702	0.566	0.703	0.709	0.694	0.700	0.708	0.621	0.656	0.642	0.538	0.507	0.611	0.302	0.322
W	0.058	0.034	0.213	0.104	0.091	0.142	0.120	0.117	0.237	0.162	0.423	0.367	0.284	0.513	0.435	0.402
$D2$	0.933	0.914	0.960	0.948	0.949	0.961	0.953	0.953	0.958	0.960	0.991	0.985	0.967	0.992	0.999	0.999
AW	0.024	0.020	0.059	0.027	0.027	0.036	0.036	0.031	0.050	0.046	0.082	0.075	0.071	0.121	0.301	0.281
PA	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
MO	0.995	0.993	0.999	0.999	0.998	0.999	0.998	0.999	1.000	0.999	1.000	1.000	1.000	1.000	0.869	0.868
$D1$	0.211	0.182	0.224	0.222	0.216	0.235	0.226	0.234	0.160	0.224	0.146	0.129	0.137	0.148	0.351	0.330
CH	0.896	0.857	0.979	0.955	0.953	0.964	0.957	0.957	0.993	0.977	0.999	0.999	0.996	1.000	0.281	0.272
GG	0.092	0.050	0.594	0.251	0.239	0.318	0.287	0.283	0.816	0.469	0.815	0.772	0.837	0.962	0.291	0.351
HJ	0.059	0.054	0.075	0.063	0.064	0.067	0.068	0.065	0.057	0.072	0.044	0.045	0.055	0.054	0.451	0.442
PW	0.021	0.017	0.058	0.027	0.026	0.032	0.032	0.030	0.059	0.040	0.092	0.085	0.080	0.121	0.851	0.784
LN	0.954	0.961	0.736	0.928	0.943	0.903	0.919	0.917	0.966	0.881	0.753	0.715	0.753	0.851	0.460	0.459
EW	0.438	0.454	0.218	0.379	0.407	0.352	0.362	0.378	0.430	0.326	0.219	0.192	0.244	0.270	1.000	1.000
IG	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.999	1.000	1.000	0.532	0.532
IS	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.551	0.542
GB	0.754	0.770	0.473	0.693	0.722	0.659	0.668	0.683	0.801	0.612	0.673	0.579	0.719	0.745	0.682	0.671

Table 11: Comparison of power for the exponential distribution tests at level $\alpha = 0.05$.