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Hypothesis Testing and Interval Estimation On Common Hazard Rate of Two Exponential Populations

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Abstract:

- This paper discusses the hypothesis testing and interval estimation on the common hazard rate parameter of two exponential populations with different locations. In the case of hypothesis testing, we derived test procedures using a generalized p-value approach, parametric bootstrap likelihood ratio approach, and computational approach. In the case of interval estimation, we derived generalized, bootstrap, and highest posterior density (HPD) intervals. The test procedures are evaluated using size and power values, whereas the interval estimators are evaluated using coverage probability and average length. Finally, we demonstrate the potential applicability of the current model problem using real-life datasets.

Keywords:

- *Average length; Common hazard rate; Coverage probability; Generalized p-value; Numerical comparison; Powers; Sizes.*

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1. INTRODUCTION

Inference on a function of the parameter(s), such as quantiles, hazard rates, reliability function, and stress-strength in the case of the two-parameter exponential model, is quite challenging and interesting due to its real-world applications. Particularly, the hazard rate (failure rate) function is helpful in reliability engineering and quality control since it provides information about the failure time distribution. Basically, hazard rate defines the probability that a system will fail before time $(t + \triangle t)$ if it survives up to time t. In reliability and life testing, inference about the hazard rate parameter of an exponential distribution is quite necessary. The simplest and the most widely used model in life testing situations is the (one or two-parameter) exponential distribution.

Suppose we have samples from two two-parameter exponential distributions with a common scale parameter σ and different location parameters μ_1 and μ_2. The objective is to obtain test procedures and interval estimators for the common hazard rate parameter $\lambda = 1/\sigma$. Mathematically, the testing problem is stated as, test the hypothesis

$$H_0 : \lambda = \lambda_0 \ \text{ against } \ H_a : \lambda \neq \lambda_0, \tag{1.1}$$

where λ_0 is some known specified value.

Inference on the common parameter is not new in the literature and has a long history in the area of statistical inference. The problem has been considered under various probabilistic model assumptions in the past. For example, the problem of estimating the common mean of several normal populations is quite old, and several researchers have done a lot of studies from classical and decision-theoretic viewpoints. We refer to Fairweather (1972) [9], Mitra and Sinha (2007) [24], Chang and Pal (2008) [3], Tripathy and Kumar (2015) [30] and the references cited therein for a detailed review on inference on the common mean of several normal populations. The problem of estimating the common standard deviation of several normal populations with ordered means has been considered by Tripathy et al. (2013) [32]. The estimation of ordered means with common variance has been considered by Gupta and Singh (1992) [14].

Some studies have also been done to estimate common parameters in the case of exponential distribution. The problem of estimating the common location parameter of several exponential distributions has been considered by Ghosh and Razmpour (1984) [10]. Some further results on inference on common location parameter of several exponential populations can be seen in Jin and Pal (1991) [19], Tripathy et al. (2014) [31], Gunasekera (2019) [12] and the references cited therein. The problem of inferences on the common scale parameter for several exponential populations has been considered by Madi and Tsui (1990) [21], Madi and Leonard (1996) [22], Gunasekera (2013) [11], Jena and Tripathy (2019) [18]. From a decision-theoretic point of view, Rukhin and Zidek (1985) [28] considered the estimation of quantiles of several exponential populations with a common scale parameter. The problem of estimating the common hazard rate parameter under ordered location for several exponential populations has been considered by Patra and Kumar (2018) [27]. The authors have derived certain inadmissibility results in the classes of affine equivariant estimators. Mahapatra et al. (2012) [23] considered simultaneous estimation of hazard rates of several exponential populations with a common location parameter. The authors adopted a differential inequality approach to obtain improved estimators. Jana et al. (2019) [17] considered estimating the stress strength reliability parameter of two exponential distributions with a common scale parameter. Moreover, the authors proposed several confidence intervals, such as generalized confidence interval, percentile bootstrap interval, and Bayesian interval for the stress-strength reliability parameter.

The problem of estimating the hazard rate (reciprocal of scale parameter) of an exponential population was first considered by Sharma (1977) [29]. The author proved some inadmissibility results in the class of affine equivariant estimators and consequently obtained improved estimators dominating the best affine equivariant estimator. Based on a doubly censored sample, Elfessi (1997) [8] derived improved estimators for the scale parameter, the reciprocal of the scale parameter, and the location parameter of a two-parameter exponential distribution. For an exponential family, Berger (1980) [1] and Gupta (1986) [13] investigated simultaneous estimation of the scale parameters and the reciprocals of the scale parameters. The authors used the differential inequality approach to obtain a class of dominating estimators.

The current article contributes in two main directions: one is related to hypothesis testing, where specific test procedures have been developed, and the other is related to interval estimation. The following are the main contributions of this study: Hypothesis testing of the common hazard rate parameter λ is addressed in Section 2. In particular, in Subsection 2.1, we explain the concept of the generalized variable approach, and several test statistics have been constructed using it. The computational approach test (CAT) and its modified version are utilized to produce specific test statistics in Subsection 2.2. In Subsection 2.3, we derive the test statistics using the popular parametric bootstrap method. In Subsection 2.4, a detailed simulation study has been conducted to numerically compare the performances of all the test procedures in terms of powers and sizes. Section 3 considers the interval estimation of the common hazard rate parameter under the same model assumption. In Subsection 3.1, we derive the generalized confidence intervals using the generalized pivot variable. In Subsection 3.2, we propose the bootstrap confidence intervals using bootstrap samples. In Subsection 3.3, we consider the Bayesian interval estimation using the Markov Chain Monte Carlo (MCMC) method and the Metropolis-Hastings algorithm. In Subsection 3.4, we compare the performances of all the interval estimators in terms of coverage probability (CP) and average length (AL). In Section 4, we discuss the application of the current model problem using real-life data sets and conclude the remarks.

2. HYPOTHESIS TESTING ON COMMON HAZARD RATE

In this section, we derive various test procedures to test the null hypothesis $H_0 : \lambda = \lambda_0$ against the alternative $H_a : \lambda \neq \lambda_0$.

2.1. Hypothesis Testing Using the Generalized Variable Approach

In this subsection, several generalized test statistics are constructed to test the hypothesis H_0 against H_a. Tsui and Weerahandi (1989) [33] and Weerahandi (1993) [34] proposed this generalized variable method, which is used to construct the test statistics and confidence intervals for a function of the parameter(s). First, we will discuss the hypothesis testing procedure that uses the generalized variable and the generalized p-value. The generalized p-value approach test procedure gives a closed-form expression for the test variable, though the power and size can be obtained numerically. The following two definitions are essential for constructing generalized pivot variables and the corresponding p- values.

Definition 2.1. Let X be a random variable with the probability density function $f_X(x, \omega, \kappa)$, where ω is the parameter of interest, and κ is a nuisance parameter. Suppose the problem is to test the hypothesis:

$$H_0 : \omega \leq \omega_0 \quad \text{against} \quad H_a : \omega > \omega_0, \tag{2.1}$$

where ω_0 is a particular value of ω. A variable $U(X; x, \omega, \kappa)$ is called a generalized pivot variable for testing the hypothesis given in (2.1) if it satisfies the following criteria.

- For a fixed value of x, the distribution of $U(X; x, \omega, \kappa)$ is free from the nuisance parameter κ.
- When $X = x$, is fixed the distribution of $U(X; x, \omega, \kappa)$ is free from all the unknown parameters.
- The distribution of $U(X; x, \omega, \kappa)$ is either stochastically increasing or stochastically decreasing in ω for fixed x and κ. That is, for every fixed real number a, $P\{U(X; x, \omega, \kappa) \geq a\}$ is either an increasing or decreasing function of ω.

Definition 2.2. Let $u = U(X; x, \omega, \kappa)$ be the observed value of $U(X; x, \omega, \kappa)$ at $X = x$. If $U(X; x, \omega, \kappa)$ is stochastically increasing in ω, the generalized p-value for testing the hypothesis defined in (2.1) is given by

$$\sup_{H_0} P\{U(X; x, \omega, \kappa) \geq u\} = P\{U(X; x, \omega_0, \kappa) \geq u\} \tag{2.2}$$

and if $U(X; x, \omega, \kappa)$ is stochastically decreasing in ω, then the generalized p-value for testing the hypothesis given in (2.1) can be obtain as

$$\sup_{H_0} P\{U(X; x, \omega, \kappa) \leq u\} = P\{U(X; x, \omega_0, \kappa) \leq u\}. \tag{2.3}$$

2.1.1. Generalized Variable Method for Testing the Common Hazard Rate

Let $\underset{\sim}{X} = (X_1, X_2, \ldots, X_m)$ and $\underset{\sim}{Y} = (Y_1, Y_2, \ldots, Y_n)$ be independent random samples obtained from two exponential populations $\text{Exp}(\mu_1, \sigma)$ and $\text{Exp}(\mu_2, \sigma)$, respectively. For the proposed model, the complete and sufficient statistics exist and are given by $X_{(1)}$, $Y_{(1)}$, and S, where the variables are defined as

$$X_{(1)} = \min_{1 \leq i \leq m} X_i, \quad Y_{(1)} = \min_{1 \leq j \leq n} Y_j, \text{ and } S = \sum_{i=1}^{m}(X_i - X_{(1)}) + \sum_{j=1}^{n}(Y_j - Y_{(1)}).$$

It is easy to observe that $X_{(1)} \sim \text{Exp}(\mu_1, \sigma/m)$, $Y_{(1)} \sim \text{Exp}(\mu_2, \sigma/n)$, and $S \sim \text{Gamma}(m + n - 2, \sigma)$, a Gamma distribution with shape parameter $m + n - 2$ and scale parameter σ. Moreover the random variables $X_{(1)}$, $Y_{(1)}$, and S are independently distributed.

The Maximum likelihood estimators (MLEs) of the parameters μ_1, μ_2, and σ are, respectively, obtained as $X_{(1)}$, $Y_{(1)}$, and $S/(m + n)$. Further, the uniformly minimum variance unbiased estimator (UMVUE) of the parameter σ is obtained as $S/(m + n - 2)$. Moreover, the best affine equivariant estimator (BAE) of σ is obtained as $S/(m + n - 1)$. Using the invariant property of MLE, the MLE of the common hazard rate parameter λ is obtained as $\hat{\lambda}_{ml} = (m + n)/S$.

In order to compute the generalized pivot variable and the generalized p-value, we introduce a new random variable $W_1 = 2S/\sigma$. It is easy to verify that $W_1 \sim \chi^2_{2(m+n-2)}$, a chi-square distribution with $2(m + n - 2)$ number of degrees of freedom. Utilizing the random variable W_1, the generalized pivot variable for λ is obtained as

$$U_s = \frac{W_1}{2\bar{s}}, \tag{2.4}$$

where $\bar{s}$ is the observed value of the sufficient statistic S. The generalized test variable to test the null hypothesis H_0 against the alternative H_a is thus obtained as $T_s = U_s - \lambda$. Easily, it can be shown that T_s meets the definition's requirements given in Definition 2.1 and Definition 2.2. As a consequence, the generalized p-value for testing the hypothesis $H_0: \ \lambda = \lambda_0$ against $H_a: \ \lambda \neq \lambda_0$ is calculated as

$$2\min[(P(U_s \geq \lambda_0), P(U_s \leq \lambda_0)]. \tag{2.5}$$

Next, utilizing the estimators of common scale parameter σ, we construct some more generalized test statistics for testing the underlying hypothesis. The

MLE of the parameter σ is given by $\hat{\sigma}_{ml} = S/(m+n)$. Using the MLE $\hat{\sigma}_{ml}$, the generalized pivot variable for λ is given by

$$U_{ml} = \frac{W_1}{2(m+n)\bar{\sigma}_{ml}}, \tag{2.6}$$

where $\bar{\sigma}_{ml}$ is the observed value of MLE $\hat{\sigma}_{ml}$. Thus the generalized test variable to test the hypothesis H_0 against H_a is obtained as $T_{ml} = U_{ml} - \lambda$, and the corresponding p-value is computed as

$$2\min[P(U_{ml} \geq \lambda_0), P(U_{ml} \leq \lambda_0)]. \tag{2.7}$$

In a similar manner, using the UMVUE of σ, that is $\hat{\sigma}_{mv} = S/(m+n-2)$ the generalized pivot statistic for λ is obtained as

$$U_{mv} = \frac{W_1}{2(m+n-2)\bar{\sigma}_{mv}}, \tag{2.8}$$

where $\bar{\sigma}_{mv}$ is the observed value of UMVUE $\hat{\sigma}_{mv}$. Thus the generalized test variable to test the hypothesis H_0 against H_a is given by $T_{mv} = U_{mv} - \lambda$, and the corresponding p-value is computed as

$$2\min[P(U_{mv} \geq \lambda_0), P(U_{mv} \leq \lambda_0)]. \tag{2.9}$$

Finally, considering the BAE of σ, that is $\hat{\sigma}_{ba} = \frac{S}{m+n-1}$ the generalized pivotal statistic for λ is obtained as

$$U_{ba} = \frac{W_1}{2(m+n-1)\bar{\sigma}_{ba}}, \tag{2.10}$$

where $\bar{\sigma}_{ba}$ is the observed value of $\hat{\sigma}_{ba}$. Thus the generalized test statistic to test the hypothesis defined in (1.1) is obtained as $T_{ba} = U_{ba} - \lambda$, and the corresponding p-value is determined as

$$2\min[P(U_{ba} \geq \lambda_0), P(U_{ba} \leq \lambda_0)]. \tag{2.11}$$

2.2. The Computational Approach Test (CAT)

In this subsection, we employ a computational approach method known as the computational approach test(CAT) method to test the null hypothesis H_0 against the alternative hypothesis H_a. Pal et al. (2007) [26] proposed this computational method and effectively used by Chang and Pal (2008) [3] to test the common mean of several normal populations. This method is a step-by-step procedure implemented through computer programming without knowing the exact distribution of sufficient statistics involved. Generally, one can use the MLEs of the parameters to construct the test statistics so that the size value and power of the test can be determined numerically. The following are the detailed algorithmic steps for determining the test procedure size and power in the proposed problem.

- Step-1: Generate samples $(X_1, X_2, \dots, X_m)$ and $(Y_1, Y_2, \dots, Y_n)$ from two exponential populations Exp(μ_1, σ) and Exp(μ_2, σ), respectively, and obtain the MLEs of the parameters μ_1, μ_2 and σ, say $\hat{\mu}_{1ml}$, $\hat{\mu}_{2ml}$, and $\hat{\sigma}_{ml}$. Using the invariance property of MLE, the MLE of the common hazard rate parameter λ is given as $\hat{\lambda}_{ml} = 1/\hat{\sigma}_{ml}$, and is denoted as Δ.

- Step-2: It is important to mention that the MLEs of the parameters μ_1, and μ_2 do not depend on the parameter σ, when the null hypothesis H_0 is true. Therefore, generate artificial samples $(X_1^*, X_2^*, \dots, X_m^*)$ and $(Y_1^*, Y_2^*, \dots, Y_n^*)$ from Exp$(\hat{\mu}_{1ml}, 1/\lambda_0)$ and Exp$(\hat{\mu}_{2ml}, 1/\lambda_0)$, respectively, and calculate the MLE of parameter λ, say $\Delta_0 = \hat{\lambda}_{0ml}$.

- Step-3: Repeat the Step-2 for a large number of times, say N times to obtain the estimates of λ, say $\Delta_{01}, \Delta_{02}, \dots, \Delta_{0N}$. Arrange all these estimates in increasing order as $\Delta_{0(1)} \leq \Delta_{0(2)} \leq \dots \leq \Delta_{0(N)}$.

- Step-4: Define the lower and upper cut-off points as $\Delta_L = \Delta_{0(\frac{\alpha}{2}N)}$ and $\Delta_U = \Delta_{0((1-\frac{\alpha}{2})N)}$, respectively.

- Step-5: Reject the null hypothesis H_0 if $\Delta \leq \Delta_L$ or $\Delta \geq \Delta_U$, otherwise accept the hypothesis H_0.

- Step-6: Let us call this test as T_{ct}. The power of the test T_{ct} is computed as

$$\Upsilon_{ml} = P\left(\Delta \leq \Delta_L \ \cup \ \Delta \geq \Delta_U\right). \tag{2.12}$$

One can modify the two sided test problem defined in (1.1) to a one sided test by rewriting it as,

$$H_0^* : h(\lambda) = (\lambda - \lambda_0)^2 = 0 \text{ against } H_a^* : h(\lambda) > 0. \tag{2.13}$$

The algorithmic steps to implement this modified CAT are as follows:

1. Generate the artificial samples as discussed above and calculate $\hat{h}_0 = (\hat{\lambda}_{0ml} - \lambda_0)^2$ for a large number of times, say N times and determine the estimates $\hat{h}_{01}, \hat{h}_{02}, \dots, \hat{h}_{0N}$. Arrange all these estimates in ascending order as $\hat{h}_{0(1)} \leq \hat{h}_{0(2)} \leq \dots \leq \hat{h}_{0(N)}$.

2. Obtain the statistic $\delta = (\hat{\lambda}_{ml} - \lambda_0)^2$. Reject the null hypothesis H_0^* if $\delta > \hat{h}_{0((1-\alpha)N)}$, otherwise accept the hypothesis H_0^*.

3. Let us name this test as T_{mt}, and thus the power of this modified CAT (T_{mt}) procedure can be determined as

$$\Upsilon_{mt} = P(\delta > \hat{h}_{0((1-\alpha)N)}). \tag{2.14}$$

2.3. Parametric Bootstrap Method

In this subsection, we propose a likelihood ratio test procedure based on the artificial bootstrap samples obtained using the MLEs of the parameters under the null hypothesis H_0. Chang et al. (2010) [4] suggested this well-known parametric bootstrap likelihood ratio test (PBLRT) method, which has certain advantages over the conventional likelihood ratio test (LRT) procedure. To use this method, we first derive the likelihood function in our model from computing the test statistic and the corresponding power.

Based on the complete and sufficient statistic $(X_{(1)}, Y_{(1)}, S)$, the likelihood function for the proposed problem is given by

$$\begin{aligned} L(\mu_1, \mu_2, \sigma) &= \frac{mns^{m+n-3}}{\Gamma(m+n-2)\sigma^{m+n}} \\ &\times \exp\left\{-\frac{(mx+ny+s-m\mu_1-n\mu_2)}{\sigma}\right\}, \quad s>0, x>\mu_1, y>\mu_2. \end{aligned} \tag{2.15}$$

Here the random variables $X_{(1)}$, $Y_{(1)}$, and S are denoted by X, Y, and S, respectively. In order to obtain a test statistic using the LRT approach, one needs to maximize the likelihood function L with respect to the null space $\Theta_0 = \big((\mu_1, \mu_2, \sigma), -\infty < \mu_1, \mu_2 < \infty, \sigma = 1/\lambda_0\big)$ and over the whole parameter space $\Theta = \big((\mu_1, \mu_2, \sigma), -\infty < \mu_1, \mu_2 < \infty, \sigma > 0\big)$, then take the ratio. Thus, the LRT statistic is given by

$$\Lambda = \frac{\sup_{\Theta_0} L}{\sup_{\Theta} L}, \tag{2.16}$$

and after some calculations, one gets

$$\Lambda = \left(\frac{\lambda_0}{\hat{\lambda}_{ml}}\right)^{m+n} \left[\exp\left\{1 - \frac{\lambda_0}{\hat{\lambda}_{ml}}\right\}\right]^{m+n}. \tag{2.17}$$

Taking logarithms on both the sides of (2.17), the statistic becomes

$$\Lambda^* = \log(\Lambda) = (m+n)\left[\log(\lambda_0) - \log(\hat{\lambda}_{ml}) + \left\{1 - \frac{\lambda_0}{\hat{\lambda}_{ml}}\right\}\right]. \tag{2.18}$$

Due to the complicated structure, it is quite challenging to determine the exact distribution of the likelihood ratio statistic Λ^*, under the null hypothesis H_0. Furthermore, under H_0, the chi-square distribution can not be used for $(-2\Lambda^*)$ (see Chang et al. (2013) [2]). Because of this and inspired by the results of Chang et al. (2010) [4], we adopt the parametric bootstrap method to test the hypothesis H_0 against H_a. The detailed algorithmic steps necessary for computing the size and power of the PBLRT procedure are described below.

- Step-1: Using the original samples $(X_1, X_2, \ldots, X_m)$ and $(Y_1, Y_2, \ldots, Y_n)$ from two exponential populations $\text{Exp}(\mu_1, \sigma)$ and $\text{Exp}(\mu_2, \sigma)$, compute the statistic Λ^*.

- Step-2: Under the hypothesis H_0, generate artificial samples $(X_1^*, X_2^*, \ldots, X_m^*)$ and $(Y_1^*, Y_2^*, \ldots, Y_n^*)$ from $\text{Exp}(\hat{\mu}_{1ml}, 1/\lambda_0)$, and $\text{Exp}(\hat{\mu}_{2ml}, 1/\lambda_0)$, respectively. Based on these sample values, calculate the statistic Λ^* and denote it as Λ_0^*.

- Step-3: Repeat the Step-2 a large number of times, say Q times to determine the estimates $\Lambda_{01}^*, \Lambda_{02}^*, \ldots, \Lambda_{0Q}^*$ of Λ^*. Arrange all these Λ^* values in an ascending order as $\Lambda_{01}^* \leq \Lambda_{02}^* \leq \ldots \leq \Lambda_{0Q}^*$. Define the lower and upper cut-off points as $\Lambda_L^* = \Lambda_{0(\frac{\alpha}{2}Q)}^*$ and $\Lambda_U^* = \Lambda_{0((1-\frac{\alpha}{2})Q)}^*$, respectively.

- Step-4: Reject H_0 if $\Lambda^* \leq \Lambda_L^*$ or $\Lambda^* \geq \Lambda_U^*$; else, accept it. Let us call this test as T_{pb}. Moreover, the power of the test T_{pb} is computed as

  $$\Upsilon_{pb} = P(\Lambda^* \leq \Lambda_L^* \ \cup \ \Lambda^* \geq \Lambda_U^*). \tag{2.19}$$

2.4. Numerical Comparison

This section conducts a detailed simulation study to evaluate and compare the performances of all suggested test methods in terms of size values and powers. It is essential to mention that no other test procedures have closed-form expressions except for the generalized test. Therefore, it is quite challenging to analyze the test procedures behavior analytically. So, taking advantage of advanced computational facilities available nowadays, we attempt to numerically compare the performances of all the proposed test procedures in terms of their powers and size values.

To compute the powers and sizes of all the test procedures, we have generated $20,000$ random samples each from the two exponential populations $\text{Exp}(\mu_1,\sigma)$ and $\text{Exp}(\mu_2, \sigma)$ of sample sizes m and n, respectively. In the case of the generalized variable approach test, the number of replications in the inner loop is taken as $10,000$ times. The same number of replications, $10,000$ times, is used for the computational approach test procedure (both CAT and modified CAT) and for the PBLRT procedure. In our simulation investigation, we used $\alpha = 0.05$ as the significance level for computing the tests' sizes and powers. Note that the sizes of all the test procedures are independent of μ_1, μ_2, and only depend on the sample values. Therefore, the sizes of all the tests are computed by considering several combinations of $\lambda = \lambda_0$ values and varying the sample sizes. The powers of the selected tests (whose size values lie within 15% of the significance level α) are computed by keeping fixed as $\lambda_0 = 1$ and varying the value of λ from 1.10 to 5.0 for several combinations of sample sizes. In computing the size/power, a high level of accuracy has been reached in the sense that the simulation's standard error is bounded above by 0.005.

The simulation study has been conducted for several combinations of sample sizes and parameter values; however, for illustration purposes, we have reported the sizes/powers only for a few choices of sample sizes and parameters. The sizes of all the recommended tests for equal and unequal sample sizes are reported in Tables 1 − 2. The powers of all test procedures have been presented in Table 3 − 4. In Tables 1 − 4, corresponding to one choice λ (in the first column), there correspond six values that represent the sizes/powers of the tests for six different combinations of sample sizes. The values given in the tables should be read vertically downward in the given order of sample sizes. The results of our extensive simulation analysis and the Tables 1 − 4 are summarized below.

- It is observed that all the test procedures are very conservative and attain the size value more frequently within 15% of nominal level $\alpha = 0.05$.
- It is noticed that as the sample sizes increase, the powers of all the test procedures increase. For fixed sample sizes, as the values of λ go away from λ_0, the powers of all the test procedures increase and become 1.
- The sizes and powers of all the generalized tests T_s, T_{ml} $T_{mv,}$ and T_{ba} are almost equal for all choices of sample sizes and parameter values. Therefore, we presented the sizes and powers of the test T_{ml} only in the tables.
- In terms of powers, the modified CAT (T_{mt}) has the best performance with the highest power for almost all sample sizes and parameter values. The CAT (T_{ct}) comes in second position, followed by the generalized test (T_{ml}), and in the end the PBLRT (T_{pb}).
- Similar trends have been seen for other combinations of sample sizes and the parameter choices used in our simulation study in terms of the sizes and powers of all the proposed test procedures.

Table 1: **Comparing the Sizes of All the Proposed Tests of λ for Equal Sample Sizes** $(m, n) = (5,5), (8,8), (12,12), (20,20), (30,30), (40,40)$ **with** $\alpha = 0.05$.

λ	T_{ml}	T_{ct}	T_{mt}	T_{pb}	λ	T_{ml}	T_{ct}	T_{mt}	T_{pb}
0.10	0.0480	0.0476	0.0482	0.0486	2.00	0.0488	0.0498	0.0518	0.0460
	0.0504	0.0488	0.0506	0.0484		0.0494	0.0482	0.0472	0.0476
	0.0506	0.0504	0.0530	0.0498		0.0504	0.0498	0.0494	0.0498
	0.0468	0.0472	0.0474	0.0472		0.0514	0.0528	0.0520	0.0502
	0.0480	0.0500	0.0484	0.0500		0.0488	0.0496	0.0464	0.0440
	0.0478	0.0480	0.0496	0.0514		0.0508	0.0522	0.0534	0.0518
0.50	0.0536	0.0544	0.0550	0.0534	2.50	0.0422	0.0424	0.0452	0.0468
	0.0452	0.0462	0.0486	0.0530		0.0526	0.0534	0.0512	0.0492
	0.0510	0.0514	0.0492	0.0490		0.0504	0.0488	0.0458	0.0500
	0.0508	0.0510	0.0538	0.0488		0.0544	0.0526	0.0462	0.0506
	0.0518	0.0524	0.0486	0.0462		0.0498	0.0526	0.0532	0.0488
	0.0476	0.0492	0.0518	0.0532		0.0490	0.0488	0.0470	0.0516
1.00	0.0500	0.0486	0.0528	0.0494	3.00	0.0486	0.0486	0.0494	0.0484
	0.0512	0.0534	0.0498	0.0506		0.0482	0.0480	0.0478	0.0522
	0.0494	0.0490	0.0484	0.0468		0.0454	0.0456	0.0484	0.0464
	0.0556	0.0562	0.0558	0.0524		0.0484	0.0464	0.0392	0.0438
	0.0524	0.0518	0.0516	0.0484		0.0458	0.0448	0.0482	0.0448
	0.0526	0.0538	0.0544	0.0518		0.0482	0.0482	0.0474	0.0450
1.50	0.0500	0.0500	0.0544	0.0500	4.00	0.0486	0.0476	0.0460	0.0476
	0.0504	0.0520	0.0482	0.0508		0.0484	0.0492	0.0532	0.0506
	0.0522	0.0526	0.0496	0.0458		0.0504	0.0500	0.0484	0.0534
	0.0490	0.0500	0.0518	0.0508		0.0522	0.0508	0.0468	0.0496
	0.0502	0.0498	0.0480	0.0564		0.0474	0.0486	0.0492	0.0440
	0.0532	0.0540	0.0524	0.0530		0.0454	0.0456	0.0450	0.0488

Table 2: **Comparing the Sizes of All the Proposed Tests of** λ **for Unequal Sample Sizes** $(m, n) = (5, 10), (5, 15), (12, 20), (10, 5), (15, 5), (20, 12)$ **with** $\alpha = 0.05$.

λ	T_{ml}	T_{ct}	T_{mt}	T_{pb}	λ	T_{ml}	T_{ct}	T_{mt}	T_{pb}
0.10	0.0468	0.0448	0.0528	0.0472	2.00	0.0492	0.0498	0.0550	0.0558
	0.0520	0.0528	0.0514	0.0520		0.0502	0.0516	0.0498	0.0486
	0.0446	0.0464	0.0510	0.0510		0.0524	0.0514	0.0476	0.0524
	0.0490	0.0482	0.0514	0.0544		0.0496	0.0486	0.0564	0.0486
	0.0530	0.0540	0.0528	0.0542		0.0488	0.0498	0.0450	0.0452
	0.0448	0.0456	0.0476	0.0518		0.0488	0.0494	0.0482	0.0466
0.50	0.0480	0.0472	0.0480	0.0498	2.50	0.0526	0.0526	0.0534	0.0534
	0.0504	0.0522	0.0500	0.0516		0.0506	0.0508	0.0538	0.0502
	0.0510	0.0492	0.0500	0.0466		0.0532	0.0546	0.0540	0.0550
	0.0466	0.0464	0.0504	0.0494		0.0568	0.0552	0.0528	0.0540
	0.0512	0.0514	0.0504	0.0492		0.0524	0.0522	0.0542	0.0568
	0.0508	0.0518	0.0482	0.0452		0.0502	0.0506	0.0534	0.0506
1.00	0.0460	0.0456	0.0426	0.0478	3.00	0.0470	0.0480	0.0476	0.0492
	0.0478	0.0476	0.0436	0.0476		0.0508	0.0520	0.0522	0.0526
	0.0472	0.0474	0.0456	0.0476		0.0490	0.0490	0.0484	0.0492
	0.0478	0.0484	0.0474	0.0485		0.0484	0.0476	0.0502	0.0506
	0.0486	0.0486	0.0504	0.0524		0.0508	0.0522	0.0556	0.0500
	0.0498	0.0498	0.0450	0.0478		0.0510	0.0508	0.0506	0.0514
1.50	0.0488	0.0496	0.0506	0.0476	4.00	0.0548	0.0564	0.0520	0.0504
	0.0486	0.0500	0.0492	0.0520		0.0514	0.0502	0.0530	0.0490
	0.0518	0.0518	0.0518	0.0504		0.0458	0.0478	0.0536	0.0478
	0.0480	0.0474	0.0526	0.0520		0.0570	0.0574	0.0504	0.0516
	0.0500	0.0500	0.0518	0.0518		0.0568	0.0558	0.0550	0.0512
	0.0496	0.0508	0.0516	0.0506		0.0466	0.0484	0.0544	0.0500

Table 3: **Powers of the Proposed Test Methods for Testing $\lambda = \lambda_0$ against $\lambda \neq \lambda_0$ with Equal Sample Sizes** $(m, n) = (5,5), (8,8), (12,12), (20,20), (30,30), (40,40)$.

λ	T_{ml}	T_{ct}	T_{mt}	T_{pb}	λ	T_{ml}	T_{ct}	T_{mt}	T_{pb}
1.10	0.0512	0.0526	0.0738	0.0616	3.00	0.8128	0.8098	0.9078	0.8070
	0.0564	0.0562	0.0858	0.0650		0.9802	0.9808	0.9952	0.9798
	0.0662	0.0680	0.1096	0.0764		0.9996	0.9996	1.0000	0.9996
	0.0884	0.0890	0.1354	0.0910		1.0000	1.0000	1.0000	1.0000
	0.0988	0.0964	0.1652	0.1012		1.0000	1.0000	1.0000	1.0000
	0.1320	0.1305	0.2095	0.1235		1.0000	1.0000	1.0000	1.0000
1.50	0.1542	0.1574	0.2558	0.1596	3.50	0.9148	0.9164	0.9702	0.9142
	0.2532	0.2536	0.3828	0.2506		0.9982	0.9980	0.9994	0.9980
	0.4154	0.4162	0.5506	0.4052		1.0000	1.0000	1.0000	1.0000
	0.6618	0.6608	0.7822	0.6420		1.0000	1.0000	1.0000	1.0000
	0.8478	0.8486	0.9216	0.8236		1.0000	1.0000	1.0000	1.0000
	0.9495	0.9495	0.9765	0.9360		1.0000	1.0000	1.0000	1.0000
2.00	0.3918	0.3946	0.5520	0.3924	4.00	0.9654	0.9646	0.9898	0.9632
	0.6706	0.6712	0.7994	0.6642		0.9990	0.9990	0.9998	0.9990
	0.8838	0.8864	0.9402	0.8796		1.0000	1.0000	1.0000	1.0000
	0.9884	0.9888	0.9956	0.9872		1.0000	1.0000	1.0000	1.0000
	0.9998	0.9996	1.0000	0.9996		1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000		1.0000	1.0000	1.0000	1.0000
2.50	0.6380	0.6398	0.7762	0.6358	5.00	0.9962	0.9964	0.9994	0.9962
	0.9028	0.9056	0.9570	0.9012		1.0000	1.0000	1.0000	1.0000
	0.9870	0.9884	0.9976	0.9872		1.0000	1.0000	1.0000	1.0000
	1.0000	0.9998	1.0000	0.9998		1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000		1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000		1.0000	1.0000	1.0000	1.0000

Table 4: **Powers of the Proposed Test Methods for Testing** $\lambda = \lambda_0$ **against** $\lambda \neq \lambda_0$ **with Unequal Sample Sizes** $(m, n) = (5, 10), (5, 15), (12, 20), (10, 5), (15, 5), (20, 12)$.

λ	T_{ml}	T_{ct}	T_{mt}	T_{pb}	λ	T_{ml}	T_{ct}	T_{mt}	T_{pb}
1.10	0.0530	0.0538	0.0892	0.0642	3.00	0.9738	0.9722	0.9912	0.9714
	0.0596	0.0590	0.0956	0.0672		0.9972	0.9976	0.9998	0.9972
	0.0674	0.0676	0.1206	0.0754		1.0000	1.0000	1.0000	1.0000
	0.0532	0.0530	0.0834	0.0648		0.9744	0.9742	0.9912	0.9730
	0.0618	0.0606	0.1004	0.0680		0.9976	0.9976	0.9998	0.9974
	0.0728	0.0710	0.1244	0.0808		1.0000	1.0000	1.0000	1.0000
1.50	0.2466	0.2470	0.3696	0.2466	3.50	0.9952	0.9948	0.9986	0.9944
	0.3378	0.3382	0.4806	0.3296		0.9990	0.9990	0.9998	0.9990
	0.5496	0.5510	0.6916	0.5320		1.0000	1.0000	1.0000	1.0000
	0.2446	0.2466	0.3612	0.2456		0.9948	0.9946	0.9988	0.9940
	0.3380	0.3422	0.4834	0.3334		0.9990	0.9988	1.0000	0.9988
	0.5612	0.5634	0.7042	0.5448		1.0000	1.0000	1.0000	1.0000
2.00	0.6148	0.6160	0.7596	0.6086	4.00	0.9992	0.9990	1.0000	0.9990
	0.7962	0.7930	0.8880	0.7852		1.0000	1.0000	1.0000	1.0000
	0.9642	0.9640	0.9872	0.9598		1.0000	1.0000	1.0000	1.0000
	0.6196	0.6212	0.7586	0.6166		0.9994	0.9992	1.0000	0.9992
	0.7916	0.7900	0.8886	0.7820		1.0000	1.0000	1.0000	1.0000
	0.9656	0.9670	0.9880	0.9616		1.0000	1.0000	1.0000	1.0000
2.50	0.8740	0.8728	0.9396	0.8696	5.00	1.0000	1.0000	1.0000	1.0000
	0.9684	0.9686	0.9882	0.9670		1.0000	1.0000	1.0000	1.0000
	0.9988	0.9992	0.9998	0.9984		1.0000	1.0000	1.0000	1.0000
	0.8784	0.8786	0.9436	0.8742		1.0000	1.0000	1.0000	1.0000
	0.9692	0.9680	0.9882	0.9648		1.0000	1.0000	1.0000	1.0000
	0.9988	0.9990	1.0000	0.9986		1.0000	1.0000	1.0000	1.0000

3. CONFIDENCE INTERVAL FOR THE COMMON HAZARD RATE

In this section, we derive several confidence intervals for the common hazard rate parameter λ utilizing some of the existing methodologies.

3.1. Generalized Confidence Intervals

The generalized confidence interval is a method for approximately obtaining the confidence interval of a function of the parameter(s). To employ the technique, we introduce the concept of generalized pivot variable proposed by Weerahandi (1993) [34] for constructing the generalized confidence intervals in this subsection. The following definition is helpful in constructing the generalized confidence intervals for λ.

Definition 3.1. Let X be a random variable with density $f_X(x, \omega, \kappa)$, where ω is the parameter of interest, and κ is the nuisance parameter. Moreover, let x be the observed value of the random variable X. If the statistic $U = U(X; x, \omega, \kappa)$ satisfies the following two conditions, then it is considered a generalized pivotal variable for constructing generalized confidence intervals of parameter ω.

- The statistic $U(X; x, \omega, \kappa)$ has a probability distribution which is free of all unknown parameters when $X = x$ is fixed.
- At $X = x$, the value of the statistic U is ω, that is $U(X; x, \omega, \kappa) = \omega$, the parameter of interest.

Observe that the generalized pivotal statistic U_s defined in (2.4) satisfies the conditions of Definition 3.1. The generalized pivot variable U_s for the parameter λ is $U_s = W_1/2\bar{s}$, where W_1 is a chi-square random variable with degrees of freedom $2(m + n - 2)$ and $\bar{s}$ is the observed value of S. Thus $(1 - \alpha)100\%$ generalized confidence interval for the parameter λ is obtained as

$$(U_s(\alpha/2),\ U_s(1 - \alpha/2)). \tag{3.1}$$

Similarly, utilizing the pivot variables U_{ml}, U_{mv}, and U_{ba}, we propose $(1 - \alpha)100\%$ generalized confidence intervals for the common hazard rate parameter λ, and are respectively, defined as

$$(U_{ml}(\alpha/2),\ U_{ml}(1 - \alpha/2)), \tag{3.2}$$

$$(U_{mv}(\alpha/2),\ U_{mv}(1-\alpha/2)), \tag{3.3}$$

and

$$(U_{ba}(\alpha/2),\ U_{ba}(1-\alpha/2)). \tag{3.4}$$

3.2. Parametric Bootstrap Confidence Intervals

In this subsection, we propose two parametric bootstrap methods, namely bootstrap-p, and bootstrap-t, to obtain approximate confidence intervals for the common hazard rate parameter λ. This method is purely based on computer programming and provides intervals as good as other exact intervals. The bootstrap-p (Boot-p) confidence interval was proposed by Efron (1982) [7], whereas Hall and Martin (1988) [15] proposed the bootstrap-t (Boot-t) confidence interval. The following are the step-by-step procedures for obtaining these two approximate confidence intervals for the common hazard rate parameter λ.

3.2.1. Bootstrap-p Method

The algorithmic steps for applying the bootstrap-p method to construct the estimated confidence interval of the parameter λ are as follows.

- Step-1: Using the original samples $(X_1, X_2, \ldots, X_m)$ and $(Y_1, Y_2, \ldots, Y_n)$ from two exponential populations $\text{Exp}(\mu_1, \sigma)$ and $\text{Exp}(\mu_2, \sigma)$, compute the MLEs $\hat{\mu}_{1ml}$, $\hat{\mu}_{2ml}$, and $\hat{\sigma}_{ml}$ for parameters μ_1, μ_2 and σ, respectively. Considering the MLE's invariance property, the MLE of the common hazard rate parameter λ is obtained as $\hat{\lambda}_{ml} = 1/\hat{\sigma}_{ml}$.
- Step-2: Generate the artificial bootstrap samples $(X_1^*, X_2^*, \ldots, X_m^*)$ and $(Y_1^*, Y_2^*, \ldots, Y_n^*)$ from $\text{Exp}(\hat{\mu}_{1ml}, \hat{\sigma}_{ml})$, and $\text{Exp}(\hat{\mu}_{2ml}, \hat{\sigma}_{ml})$, respectively, and obtain the bootstrap MLEs $\hat{\mu}_{1bml}$, $\hat{\mu}_{2bml}$, and $\hat{\sigma}_{bml}$. Consequently, compute the bootstrap MLE for parameter λ, as $\hat{\lambda}_{bml} = 1/\hat{\sigma}_{bml}$.
- Step-3: Repeat the Step-2 for a large number of times, say B times to determine the bootstrap MLEs $\hat{\lambda}_{1bml}, \hat{\lambda}_{2bml}, \ldots, \hat{\lambda}_{Bbml}$ of λ. Arrange all these MLE values in an ascending order.
- Step-4: The $(1-\alpha)100\%$ bootstrap-p confidence interval for λ is obtained using the bootstrap MLEs and is given by

  $$\left(\hat{\lambda}_{Boot-p}(\alpha/2), \hat{\lambda}_{Boot-p}(1-\alpha/2)\right),$$

  where $\hat{\lambda}_{Boot-p}(x) = F^{-1}(x),\ F(x) = P(\hat{\lambda}_{bml} \leq x)$.

3.2.2. Bootstrap-t Method

The algorithmic steps for applying the bootstrap-t method to construct the approximate confidence interval of the parameter λ are discussed below.

- Step-1: Compute the MLE $\hat{\lambda}_{ml}$ for parameter λ, as discussed in Step-1 of bootstrap-p method.
- Step-2: This step is also the same as Step-2 of the bootstrap-p method. Obtain $\hat{\lambda}_{bml}= 1/\hat{\sigma}_{bml}$.
- Step-3: Compute the new statistic $T = \dfrac{\hat{\lambda}_{bml} - \hat{\lambda}_{ml}}{\hat{se}(\hat{\lambda}_{bml})}$, where $\hat{se}(\hat{\lambda}_{bml})$ is the standard error estimate of the bootstrap MLE $\hat{\lambda}_{bml}$.
- Step-4: Repeat the Steps 2 and 3, a large number of times, say B times and collect the several values of the bootstrap statistic T.
- Step-5: The approximate $(1-\alpha)100\%$ bootstrap-t confidence interval of the parameter λ is computed as
$$\Big(\hat{\lambda}_{Boot-t}(\alpha/2), \hat{\lambda}_{Boot-t}(1-\alpha/2)\Big),$$
where $\hat{\lambda}_{Boot-t}(x) = \hat{\lambda}_{ml} + G^{-1}(x)\hat{se}(\hat{\lambda}_{ml}),\ G(x) = P(T \leq x)$.

In the following subsection, we will propose another approximate confidence interval based on the Markov Chain Monte Carlo (MCMC) method. This method is very effective when some prior information regarding the parameter is available.

3.3. Highest Posterior Density (HPD) Interval Using MCMC Method

This section uses the Markov Chain Monte Carlo (MCMC) process and the Metropolis-Hastings algorithm to compute the HPD confidence interval for the common hazard rate parameter λ.

Suppose independent random samples $\underset{\sim}{X} = (X_1, X_2, \dots, X_m)$ and $\underset{\sim}{Y} = (Y_1, Y_2, \dots, Y_n)$ are available from two exponential populations $\text{Exp}(\mu_1, \sigma)$ and $\text{Exp}(\mu_2, \sigma)$, respectively. In order to compute the HPD interval for the parameter λ, some appropriate prior information about the parameters μ_1, μ_2, and σ are required. In this connection, we consider the joint prior for the parameters σ, μ_1 and μ_2 as given by (Madi and Leonard (1996) [22]).

The prior distribution of μ_1 given σ is taken as exponential and is given by

$$\pi(\mu_1|\sigma) = \frac{c}{\sigma}\ \exp\left\{\frac{-c(\mu-\mu_1)}{\sigma}\right\};\ \mu \geq \mu_1. \tag{3.5}$$

Similarly the prior distribution of μ_2 given σ is also exponential and is given by

$$\pi(\mu_2|\sigma) = \frac{c}{\sigma}\ \exp\left\{\frac{-c(\mu-\mu_2)}{\sigma}\right\};\ \mu \geq \mu_2. \tag{3.6}$$

The distribution of σ is taken as inverse gamma (IGamma) distribution with shape parameter k and scale parameter θ and is given by,

$$\pi(\sigma) = \frac{\theta^k}{\Gamma(k)}\Big(\frac{1}{\sigma}\Big)^{k+1}\exp\{-\theta/\sigma\};\ \sigma > 0,\ \theta > 0,\ k > 0. \tag{3.7}$$

It is further assumed that c, k, and θ are specified positive constants and μ has a uniform distribution which is independent of σ. The density of μ is given by

$$\pi(\mu) \propto 1, \quad -\infty < \mu < \infty. \tag{3.8}$$

Now, combining all the above information, and the likelihood function given in (2.15), one gets the joint posterior density of μ_1, μ_2 and σ, given μ, as

$$\begin{aligned}\pi\Big((\mu_1,\mu_2,\sigma)\mid(\underset{\sim}{x},\underset{\sim}{y},\mu)\Big) \;=\; & \frac{mnc^2\theta^k s^{m+n-3}}{\Gamma(k)\Gamma(m+n-2)}\Big(\frac{1}{\sigma}\Big)^{m+n+k+3} \\ & \times\ \exp\left\{-\frac{(mx+ny+s+2c\mu-(m+c)\mu_1-(n+c)\mu_2)}{\sigma}\right\},\end{aligned} \tag{3.9}$$

where $-\infty < \mu_1 < \delta_1$, $-\infty < \mu_2 < \delta_2$; and $\delta_1 = \min(x,\mu)$, and $\delta_2 = \min(y,\mu)$.

Observe that, from the joint posterior density of (μ_1,μ_2,σ), it is easy to obtain the marginal posterior density of the parameter σ, and is obtained as

$$\pi\Big(\sigma \mid (\underset{\sim}{x},\underset{\sim}{y},\mu_1,\mu_2,\mu)\Big) \propto \text{IGamma}(k_1,\theta_1)$$

where $k_1 = m+n+k$ and $\theta_1 = mx+ny+s+2c\mu-(m+c)\delta_1-(n+c)\delta_2$.

Thus the marginal posterior distribution of the common hazard rate parameter λ, given μ_1, μ_2, μ and the data is proportional to Gamma distribution with the shape parameter k_1 and the rate parameter θ_1, that is,

$$\pi\Big(\lambda \mid (\underset{\sim}{x},\underset{\sim}{y},\mu_1,\mu_2,\mu)\Big) \propto \text{Gamma}(k_1,\theta_1)$$

Since the marginal posterior density of the required parameter λ is proportional to the well-known gamma distribution; we use the MCMC method along with the Random walk Metropolis-Hastings algorithm proposed by Hastings (1970) [16] to generate posterior samples for the parameter λ. The details of the algorithmic steps of the Random-Walk-Metropolis method are described below.

Random Walk Metropolis-Hastings Algorithm:

- Step-1: Let the k^{th} iteration step of the Markov chain consists of $\lambda^{(k-1)}$ with a suitable initial value $\lambda^{(0)}$.
- Step-1: Generate a sample ε from $N(0, \sigma_\lambda^2)$ and put $\lambda^* = \lambda^{(k-1)} + \varepsilon$.
- Step-3: Determine the term $\varphi = \min\left(1, \dfrac{\pi\Big(\lambda^* \mid (\underset{\sim}{x}, \underset{\sim}{y}, \mu_1, \mu_2, \mu)\Big)}{\pi\Big(\lambda^{(k-1)} \mid (\underset{\sim}{x}, \underset{\sim}{y}, \mu_1, \mu_2, \mu)\Big)}\right)$
- Step-4: Generate a random sample u from $U(0,1)$. If $u \leqslant \varphi$, accept λ^* and update the parameter as $\lambda^{(k)} = \lambda^*$ and otherwise, reject λ^* and do $\lambda^{(k)} = \lambda^{(k-1)}$. This completes the transition from k^{th} iteration step to $(k+1)^{th}$ iteration step.
- Step-5: Repeat the above Random walk Metropolis algorithm for $k = 1, 2, \ldots, M$ where M is a suitable large number. Collect these M independent MCMC samples as $\lambda_1, \lambda_2, \ldots, \lambda_M$.

In order to compute the $(1-\alpha)100\%$ HPD confidence interval for λ, arrange all the MCMC samples in the ascending order as $\Big(\lambda_{(1)}, \lambda_{(2)}, \ldots, \lambda_{(M)}\Big)$. Then using the technique of Chen and Shao (1999) [5], we obtain the $(1-\alpha)100\%$ HPD confidence interval for λ as

$$\text{HPD} = (\lambda_{(s^*)}, \lambda_{(s^*+[(1-\alpha)M])}) \tag{3.10}$$

where s^* is chosen in such a way that

$$\lambda_{(s^*+[(1-\alpha)M])} - \lambda_{(s^*)} = \min_{1 \leq s \leq M-[(1-\alpha)M]} (\lambda_{(s+[(1-\alpha)M])} - \lambda_{(s)}).$$

Remark 3.1. In order to start the MCMC chain, we have used the suitable initial guess $\lambda^{(0)}$ as $\lambda^{(0)} = \hat{\lambda}_{ml}$, the MLE of λ.

Remark 3.2. The choice of the value of σ_λ^2 is quite crucial in the Random walk Metropolis method. Chib and Greenberg (1995) [6] discussed this issue and reported that for small choices of σ_λ^2 there is a high chance of accepting the sample. Hence to make the process more fair, we choose the value of σ_λ^2 in such a way that the acceptance rate is in between 20% and 30%, that is neither too high nor too low.

Remark 3.3. It is important to note that, we have formed test procedures using the confidence intervals, such as bootstrap-p, bootstrap-t and HPD. The null hypothesis H_0 (given in (1.1) is rejected, if the intervals do not contain the true value of parameter λ_0 with probability 0.95. We have computed the sizes for all these three test procedures, and seen that none of these attain the size value 0.05 within 15% of the nominal level. Thus we have not considered these test procedures for power comparison, and omitted in our simulation study.

3.4. Simulation Study: Comparing the Confidence Intervals

In this section, we compare all the proposed confidence intervals (CIs) in terms of coverage probability (CP) and average length (AL) by considering many combinations of sample sizes and parameters.

It is important to mention that the CPs and ALs of all the proposed CIs for λ could not be compared analytically, so utilizing the superior computational facilities available nowadays, we attempt to evaluate and compare the CPs and ALs of all the intervals, numerically.

The sample generation and the parameter choices for computing the CPs and ALs remain the same as in the previous section (Subsection 2.4). Moreover, in computing the bootstrap intervals, the number of replications for generating the bootstrap samples is set to $B = 5000$. The same number of replications is used in the inner loops for computing the HPD and the generalized intervals. The confidence level for computing all the intervals is fixed at $1 - \alpha = 0.95$. Note that the CPs and ALs of all the intervals are independent of μ_1 and μ_2, and only depend upon the sample sizes. Therefore, the CPs and ALs of all the CIs of λ are computed by keeping $\mu_1 = \mu_2 = 1$ fixed and considering various combinations of sample sizes. However, we have reported for selected combinations of sample sizes and parameters for illustration purposes. Table 5 presents the ALs and CPs of all the CIs of λ for some equal and unequal sample sizes. Note that all the generalized CIs have equal ALs and CPs; that is, they perform equally well in terms of ALs and CPs. Therefore, we only consider the generalized interval U_{ml} in the tables for presentation purposes.

The following observation can be drawn from our simulation study as well as from Table 5.

Remark 3.4. (i) The ALs of all the CIs decrease as the sample sizes increase; however; whereas, their CPs lie in the range 60% and 98%. Notably, the CPs of all generalized intervals are always 0.95, that is, they are exact.

(ii) All the generalized CIs perform equally well in terms of AL and CP for both equal and unequal sample sizes. Thus, we only present the AL and CP of the interval U_{ml} in the Table 5.

(iii) If we compare the performances of all the CIs in terms of CPs, considering 0.95 as the nominal level, then all the generalized CIs and Boot-t interval attain it. Among these CIs, the Boot-t CI performs better with higher CP, followed by the generalized CIs. However, for small values of λ (say $\lambda < 0.50$) the HPD interval attains the CP, for almost all the combinations of samples sizes that we have considered.

(iv) In terms of AL, and for small values of λ (say $\lambda < 0.50$) the generalized

CI U_{ml} performs better with the shortest AL; followed by the Boot-t, Boot-p and finally the HPD interval. For other choices of λ the HPD interval performs better with shortest AL, however its CP falls short from attaining the nominal level of 0.95.

(v) While ranking all the proposed CIs in terms of shortest AL and highest CP, it is seen that all the generalized CIs performs the best among all the proposed CIs with the shortest AL and also attain the confidence level of 0.95.

(v) A similar type of trend has been observed for other combinations of sample sizes and parameters choices. We have also considered the simulation study considering $1 - \alpha = 0.90, 0.99$, which have produced similar performance in terms of CPs and ALs.

Table 5: Average Lengths and Coverage Probabilities of Various Confidence Intervals of λ for Sample Sizes $(m,n) = (5,5)$, $(8,8)$, $(12,12)$, $(20,20)$, $(30,30)$, $(40,40)$, $(5,10)$, $(5,15)$, $(12,20)$, $(10,5)$, $(15,5)$, $(20,12)$ with $\alpha = 0.05$.

λ	Boot-p	Boot-t	HPD	U_{ml}
	0.314 (0.60)	0.187 (0.97)	0.408 (0.94)	0.157 (0.95)
	0.166 (0.73)	0.124 (0.97)	0.276 (0.96)	0.112 (0.95)
	0.113 (0.80)	0.095 (0.97)	0.177 (0.98)	0.087 (0.95)
	0.076 (0.85)	0.070 (0.96)	0.109 (0.98)	0.065 (0.95)
	0.057 (0.88)	0.055 (0.96)	0.083 (0.99)	0.052 (0.95)
$\lambda = 0.10$	0.048 (0.90)	0.069 (0.99)	0.045 (0.94)	0.045 (0.95)
	0.181 (0.70)	0.131 (0.97)	0.283 (0.97)	0.116 (0.95)
	0.133 (0.76)	0.107 (0.97)	0.217 (0.98)	0.097 (0.95)
	0.089 (0.84)	0.079 (0.96)	0.133 (0.98)	0.074 (0.95)
	0.178 (0.72)	0.129 (0.97)	0.292 (0.97)	0.117 (0.95)
	0.133 (0.77)	0.106 (0.97)	0.207 (0.98)	0.097 (0.95)
	0.089 (0.84)	0.079 (0.97)	0.131 (0.97)	0.074 (0.95)
	1.543 (0.60)	0.92 (0.98)	0.646 (0.89)	0.787 (0.95)
	0.837 (0.73)	0.624 (0.97)	0.567 (0.92)	0.558 (0.95)
	0.563 (0.80)	0.472 (0.96)	0.496 (0.92)	0.435 (0.95)
	0.377 (0.85)	0.347 (0.96)	0.397 (0.94)	0.326 (0.95)
	0.287 (0.89)	0.277 (0.95)	0.319 (0.96)	0.261 (0.95)
$\lambda = 0.50$	0.241 (0.90)	0.270 (0.98)	0.238 (0.96)	0.224 (0.95)
	0.891 (0.73)	0.648 (0.97)	0.565 (0.95)	0.588 (0.95)
	0.665 (0.76)	0.532 (0.96)	0.520 (0.94)	0.488 (0.95)
	0.445 (0.84)	0.396 (0.97)	0.432 (0.98)	0.372 (0.95)
	0.899 (0.70)	0.654 (0.96)	0.576 (0.96)	0.585 (0.95)
	0.666 (0.76)	0.533 (0.97)	0.523 (0.94)	0.486 (0.95)
	0.445 (0.84)	0.396 (0.96)	0.430 (0.95)	0.370 (0.95)
	3.093 (0.60)	1.844 (0.98)	0.673 (0.87)	1.584 (0.95)
	1.657 (0.74)	1.234 (0.97)	0.621 (0.89)	1.119 (0.95)
	1.125 (0.80)	0.945 (0.97)	0.569 (0.89)	0.874 (0.95)
	0.756 (0.85)	0.696 (0.96)	0.493 (0.91)	0.650 (0.95)
	0.574 (0.89)	0.554 (0.96)	0.434 (0.92)	0.523 (0.95)
$\lambda = 1.00$	0.479 (0.90)	0.388 (0.93)	0.472 (0.96)	0.448 (0.95)
	1.800 (0.70)	1.309 (0.97)	0.632 (0.89)	1.163 (0.95)
	1.324 (0.77)	1.060 (0.96)	0.588 (0.91)	0.968 (0.95)
	0.890 (0.84)	0.791 (0.96)	0.524 (0.91)	0.741 (0.95)
	1.801 (0.70)	1.311 (0.96)	0.628 (0.98)	1.164 (0.95)
	1.328 (0.77)	1.063 (0.97)	0.594 (0.98)	0.970 (0.95)
	0.892 (0.83)	0.794 (0.96)	0.528 (0.90)	0.739 (0.95)

4. APPLICATION USING REAL-LIFE DATA SETS

Example 1: In this section, we consider real data sets to demonstrate the test procedures and interval estimation methods. The data sets have been taken from Xia et al. (2009) [35] and are presented in Table 6. These data sets represent the breaking strength of jute fiber at gauge lengths 10 mm and 20 mm, respectively. Using the Kolmogorov-Smirnov test, it is seen the two-parameter exponential distribution fits well to these two datasets, with p-values 0.2831 and 0.5891, respectively. The equality of the scale parameters holds true at 5% significance level, using the test proposed by Nagarsenker and Nagarsenker (1986) [25]. Suppose X and Y denote the breaking strengths of jute fiber with 10 mm and 20 mm, respectively. Using the data X and Y, we computed the p-values and the confidence intervals for the common hazard rate parameter λ. In Table 7, we have reported all the intervals with their corresponding lengths for λ with confidence level $(1-\alpha) = 95\%$. From Table 7, it is clear that the generalized confidence interval U_{ml} has the shortest length, which also validates our simulation results.

Suppose we are interested in testing the hypothesis $H_0 : \lambda = 0.003$ against the alternative $H_a : \lambda \neq 0.003$ at a level of significance $\alpha = 0.05$ using the given data sets. The p-values for all the proposed test procedures are computed and reported in Table 8. The p-values indicate that all the test procedures accept the null hypothesis, except the PBLRT T_{pb}, at a level of significance $\alpha = 0.05$.

Table 6: **Breaking Strength of Jute Fiber of Gauge Length 10mm and 20 mm.**

10 mm(X)	693.73	704.66	323.83	778.17	123.06	637.66
	383.43	151.48	108.94	50.16	671.49	183.16
	257.44	727.23	291.27	101.15	376.42	163.40
	141.38	700.74	262.90	353.24	422.11	43.93
	590.48	212.13	303.90	506.60	530.55	177.25
20 mm(Y)	71.46	419.02	284.64	585.57	456.60	113.85
	187.85	688.16	662.66	45.58	578.62	756.70
	594.29	166.49	99.72	707.36	765.14	187.13
	145.96	350.70	547.44	116.99	375.81	581.60
	119.86	48.01	200.16	36.75	244.53	83.55

Table 7: **Computing the Lower Limits, Upper Limits and Lengths of Confidence Intervals with Confidence Level** $(1-\alpha) = 0.95$.

Method	Boot-p	Boot-t	HPD	U_{ml}
Lower	0.00263	0.00204	0.00193	0.00233
Upper	0.00434	0.00375	0.93563	0.00394
Length	0.00171	0.00171	0.09163	0.00160

Table 8: **Computing the p-values of All the Proposed Tests with** $\alpha = 0.05$.

Method	T_{ml}	T_{ct}	T_{mt}	T_{pb}
p-value	0.7572	0.7668	0.9988	0.0024

Example 2: Let us consider the data sets given in Lawless (2011) [20], and the data sets are tabulated in Table 9. The data sets represent the failure times of two types of electrical insulators in an experiment when the insulators were subjected to a continuously increasing voltage stress. Using the Kolmogorov-Smirnov test, the two-parameter exponential distribution is fitted to these two data sets, whose p-values are obtained as 0.2831 and 0.5891, respectively. Equality of the scale

parameters has been tested at 5% significance level, using the method proposed Nagarsenker and Nagarsenker (1986) [25]. Suppose X and Y denote the failure times (in minutes) of the two types of electrical insulators, say Type-A and Type-B, respectively.

Using the data X and Y, we computed the p-values and the confidence intervals for the common hazard rate parameter λ. In Table 10, we have reported all the intervals with their corresponding lengths for λ with confidence level $(1-\alpha) = 95\%$. From Table 10, it is clear that the generalized confidence interval U_{ml} has the shortest length.

Further, we are interested in testing the hypothesis $H_0 : \lambda = 0.0166$ against the alternative $H_a : \lambda \neq 0.0166$ at the level of significance $\alpha = 0.05$. The p-values for all the proposed test procedures are computed and given in Table 11. The p-values indicate that all the test procedures accept the null hypothesis, except the PBLRT T_{pb}.

Table 9: **Failure Times (in Minutes) for Two Types of Electrical Insulators.**

Type-A (X)	219.3	79.4	86.0	150.2	21.7	18.5
	121.9	40.5	147.1	35.1	42.3	48.7
Type-B (Y)	21.8	70.7	24.4	138.6	151.9	75.3
	12.3	95.3	98.1	43.2	28.6	46.9

Table 10: **Computing the Lower Limits, Upper Limits and Lengths of Confidence Intervals with Confidence Level** $(1-\alpha) = 0.95$.

Method	Boot-p	Boot-t	HPD	U_{ml}
Lower	0.01250	0.00529	0.00785	0.00972
Upper	0.02785	0.02063	0.02596	0.02224
Length	0.01534	0.01534	0.01811	0.01252

Table 11: **Computing the p-values of All the Proposed Tests with** $\alpha = 0.05$.

Method	T_{ml}	T_{ct}	T_{mt}	T_{pb}
p-value	0.6048	0.6428	1.0000	0.0000

5. CONCLUDING REMARKS

The problems of point estimation of the common scale parameter σ and the hazard function $1/\sigma$ of several exponential distributions have been considered by researchers in the past from classical and decision-theoretic viewpoints. In this article, we have investigated the problems of hypothesis testing and confidence interval of the common hazard rate parameter when samples are available from two two-parameter exponential distributions with different location parameters. The problem is quite important from an application point of view.

Several test procedures, namely the generalized tests, tests based on the computational approach we call CAT and modified CAT, and the parametric bootstrap likelihood ratio test, have been derived for testing the common hazard rate. The powers and sizes of all the test procedures have been computed numerically. It has been seen that all the generalized tests perform quite well as compared to other tests. Several confidence intervals, such as generalized confidence intervals, bootstrap-p and bootstrap-t, and highest posterior density intervals, have been proposed for the common hazard rate. The performances of all the intervals in terms of coverage probability and average length have been evaluated. It is seen that the generalized confidence intervals perform quite well. The current problem can be considered from a Bayesian point of view, such as Bayesian hypothesis testing.

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