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The γ-order generalized Chi-γ distribution ${}_{\gamma}\mathcal{X}_n^{\gamma}$

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Abstract:

- The various attempts to present a generalized Normal distribution came to a conclusion by trial and error, rather than a theoretical framework. It is the γ-order Generalized Normal distribution with a strong theoretical background due to the Euclidean Logarithm Sobolev Inequality (LSI), adopting an extra, shape parameter. The introduced Generalized Gaussian offers extensions to the Information Theory oriented results among other applications. Based on this distribution we go a step forward to generalize the Chi-square distribution and we present in this paper the γ-order Generalized Chi-γ. Moreover, both Rayleigh and Maxwell–Boltzmann distributions have been generalized.

Keywords:

- *Normal Distribution; γ-order Generalized Normal Distribution; Chi-square distribution; Rayleigh distribution; Maxwell-Boltzmann distribution.*

AMS Subject Classification:

- 60E05, 62P35, 44A10, 62B10.

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1. INTRODUCTION

The Binomial distribution gave rise for the development of the Normal and the Chi-square distributions. The first one who worked with the Normal distribution was De Moivre, in 1733, Daw and Pearson (1972), while the Weak Law of Large Numbers (WLLN) provided by Bernoulli, on 1713 was the first step towards the Probability Theory. The Statistical Theory on the Normal distribution was strengthened and enlarged with Gauss' Least Square Method, where his "bell-shaped" curve, the "Gaussian", in current terminology, was the backbone of his research Seal (1967), as Gauss worked with the distribution of the involved errors.

Through the Binomial distribution and the Normal (the idea to sum up the square of Normals was there) Pearson worked to provide his cumulative test statistic Pearson (1900), which is approaching the Chi-square distribution. For the development of the Chi-square see Pearson and Plackett (1983).

Both, Normal and Chi-square, are very popular to applications: It was Adolph Quetelet who adopted the Gaussian distribution to Astronomy, being fundamental Airy (1861) for astronomers, while Francis Galton applied the Normal distribution to all of his research fields, working also with the logarithms of the observations. He also moved a step forward working with pairs of measurements, while Karl Pearson and others developed the "correlation coefficients", so the Bivariate Normal was "ante portas" and was approached eventually by Laplace (1811), Gauss (1823). Recently the Normal distribution is so essential to the Information Theory, Cover and Thomas (2006), while the Chi-square tests, Schervish (1997) for a theoretical approach, are widely used in Applied Statistics. Moreover recall that the Chi-square $\mathcal{X}_n^2$ distribution comes from the sum of squares of n given independent standard Normal random variables, $Z_i, i = 1, 2, \ldots, n$. We denote by $Y = \sum_{i=1}^{n} Z_i^2 \sim \mathcal{X}_n^2$. This result has been generalized in Section 3. Recall that for the expected value and the variance of Y it holds that $\mathbb{E}(Y) = n = \frac{1}{2}Var(Y)$. This result has been generalized in Section 3, Corollary 3.3.

The multivariate Normal distribution, see Anderson (2003) among others, attracts more interest in applications than the multivariate Chi-square Royen (1991). One explanation might be that the multivariate Normal offers very useful results to many fields of interest, the Information Theory being one of them, Cover and Thomas (2006). Even as an inverse Gaussian, as introduced by Schrödinger (1915), Normal distribution is really very attractive, especially to the experimentalists. Therefore a number of attempts have been made to offer a generalized form for the Normal distribution.

Defining a Generalized Normal, Nadarajah (2005), Domínguez-Molina et al. (2003), is referred on a shape parameter of a generalized Gaussian, with no theoretical framework, see also Gómez et al. (1998), similar to those from Kitsos and Toulias (2010). But neither in Gómez et al. (1998) nor in Nadarajah (2005) the presentation of the "new" Gaussian is based on theoretical background. A hard attempt of trials and errors it might be behind, to form coefficients, while Yu et al. (2012), Dytso et al. (2018) are reviewing the exponential power distribution. Therefore, with an extra shape parameter γ and the strong theoretical background of the Logarithm Sobolev Inequalities (LSI), Kitsos and Tavoularis (2009) proposed the γ-order Generalized Normal, $N_\gamma(\mu, \Sigma)$ as an *extremal* function to Euclidean LSI.

This distribution, introduced in Kitsos and Tavoularis (2009), has been discussed in detail in Kitsos and Toulias (2011), Kitsos et al. (2014) while it was adopted to Economical problems by Halkos and Kitsos (2018), Halkos and Kitsou (2018) and to Transfer Entropy problems by Hlavackova-Schindler (2011), Hlavackova-Schindler et al. (2016).

The $N_\gamma(\mu, \Sigma)$ distribution is presented in a compact form in Section 2 and the generalized γ-order standard Normal $N_\gamma(0, 1)$ is obtained. Now we are moving in this paper one step forward: Due to the $N_\gamma(0, 1)$, we introduce the γ-order Generalized Chi-γ. Indeed: for independent variables $Z_i, i = 1, 2, \ldots, n$ from $N_\gamma(0, 1)$ i.e. $Z_i \sim N_\gamma(0, 1), i = 1, 2, \ldots, n$ the sum

$$Y_n = \sum_{i=1}^{n} Z_i^{\gamma_1}, \quad \gamma_0 = \frac{\gamma - 1}{\gamma}, \quad \gamma_1 = \gamma_0^{-1} = \frac{\gamma}{\gamma - 1}, \gamma \in \mathbb{R} - [0, 1] \tag{1.1}$$

is obtained and the γ-order generalized Chi-γ, ${}_{\gamma}\mathcal{X}_n^{\gamma}$ is defined, while for $\gamma = 2$ the Chi-square can be achieved. The distribution of the square root of Y_n with $n = 2$ and $n = 3$ (and $\gamma = 2$) are essential both in Statistics and Physics: with $n = 2$ the Rayleigh distribution, Rayleigh (1919), is obtained and with $n = 3$ the Maxwell-Boltzmann distribution, Mandl (2008), see Section 3. A small number of different graphs is presented and the extensions are discussed.

2. THE γ-ORDER GENERALIZED NORMAL $N_\gamma(\mu, \Sigma)$

For a given function g such that $1 = \|g\|_2 \in L^2(\mathbb{R}^p, dm)$ let us define

$$\begin{aligned} I_p(\kappa, \lambda, dm) &= \int_{\mathbb{R}^p} \|g\|^\kappa \log \|g\|^\lambda dm, \\ J_p(\kappa, dm) &= \int_{\mathbb{R}^p} \|g\|^\kappa dm. \end{aligned}$$

Then, following Kitsos and Tavoularis (2009), the Gross logarithm inequality with respect to a Gaussian weight $d\mu = \exp\{-\pi|x|^2\}dx$, see Gross (1975), states that

$$I_p(2, 2, d\mu) \leq \frac{1}{\pi} J_p(2, d\mu). \tag{2.1}$$

Inequality (2.1) is equivalent to the Euclidean LSI of the form

$$I_p(2, 2, dx) \leq \frac{p}{2} \log \left[\frac{2}{\pi p e} J_p(2, dx) \right] \tag{2.2}$$

with e the well-known mathematical constant (Euler's number), $g \in W^{1,2}(\mathbb{R}^p)$ and $\|g\|_2 = 1$. Moreover inequality (2.2) is optimal, in the sense of Weissler (1978), and can be extended, Pino et al. (2004), to a, γ-order say, LSI of the form

$$I_p(\gamma, 1, dx) \leq \frac{p}{\gamma^2} \log \left[A(\gamma, p) I_p(\gamma, dx) \right], \tag{2.3}$$

where

$$A(\gamma, p) = \frac{\gamma}{p} \left(\frac{\gamma - 1}{e} \right)^{\gamma - 1} \pi^{-\gamma/2} \left[\frac{\Gamma(\frac{p}{2} + 1)}{\Gamma(p\frac{\gamma - 1}{\gamma} + 1)} \right]^{\gamma/p}$$

with $\Gamma(\cdot)$ being the Gamma function. Inequality (2.3) is optimal and the equation holds when, Kitsos and Tavoularis (2009), the function $g(x)$ equals to

$$g(x) = \phi_\gamma(x) = C \exp\left\{-\frac{\gamma-1}{\gamma}[Q(x)]^{\frac{\gamma}{2(\gamma-1)}}\right\} \tag{2.4}$$

with

$$C = C_p(\mu, \Sigma; \gamma) = \frac{1}{\pi^{p/2}} \frac{1}{|\Sigma|^{1/2}} \frac{\Gamma(\frac{p}{2}+1)}{\Gamma(p\frac{\gamma-1}{\gamma}+1)} \left(\frac{\gamma-1}{\gamma}\right)^{p\frac{\gamma-1}{\gamma}} \tag{2.5}$$

and

$$Q(x) = (x-\mu)^T \Sigma^{-1}(x-\mu), \quad \mu \in \mathbb{R}^p, \quad \Sigma \in \mathbb{R}^{p\times p}.$$

For the random variable X following (2.4) we shall denote $X \sim N_\gamma(\mu, \Sigma)$ and we shall refer to it as the γ-order Generalized Normal ($\gamma - GN$) distribution. It is essential to notice that the defined probability density function (pdf) $\phi_\gamma(x)$ works as an extremal function, see Kitsos and Tavoularis (2009) for details. For the probability function (2.4) a strong theoretical background was developed, see Kitsos and Toulias (2011) and Kitsos et al. (2014), while for some Information Theory results see Kitsos (2023). Moreover it holds:

Theorem 2.1. *In case where $X \sim N_\gamma(\mu, \sigma^2 I_p)$ the Shannon entropy is*

$$H(X) = p\frac{\gamma-1}{\gamma} + \ln \sigma^p - \ln \Lambda(p, \gamma),$$

where $\Lambda(p,\gamma) = \sigma^p C$, with C as in (2.5).Therefore for the (spherically) contoured multivariate distributions holds

$$H(X) = \begin{cases} \ln \frac{(\pi\sigma^2)^{p/2}}{\Gamma(\frac{p}{2}+1)} & X \sim N_1(\mu, \sigma^2 I) \quad \textit{Uniform}, \quad \gamma = 1 \\ p\ln(\sqrt{2\pi e}\sigma) & X \sim N_2(\mu, \sigma^2 I) \quad \textit{Normal}, \quad \gamma = 2 \\ \ln \frac{p! e^p (\pi\sigma^2)^{p/2}}{\Gamma(\frac{p}{2}+1)} & X \sim N_{\pm\infty}(\mu, \sigma^2 I) \quad \textit{Laplace}, \quad \gamma \to \pm\infty \\ \infty & X \sim N_0(\mu, \sigma^2 I) \quad \textit{Dirac}, \quad \gamma \uparrow 0,\ p = 1, 2 \end{cases}$$

Consider the $N_\gamma(\mu, \sigma^2)$, see Kitsos and Toulias (2011), with position (mean) μ, positive scale parameter σ^2, extra shape parameter γ ($\gamma \in \mathbb{R} - [0, 1]$) and density function $\phi_\gamma(x; \mu, \sigma^2)$ given by

$$\phi_\gamma(x; \mu, \sigma^2) = \frac{\lambda_\gamma}{\sqrt{\pi\sigma^2}} \exp\left\{-\frac{\gamma-1}{\gamma}\left(\frac{|x-\mu|}{\sqrt{\sigma^2}}\right)^{\frac{\gamma}{\gamma-1}}\right\} \tag{2.6}$$

where

$$\lambda_\gamma = \frac{\Gamma(\frac{1}{2}+1)}{\Gamma(\frac{\gamma-1}{\gamma}+1)} \left(\frac{\gamma-1}{\gamma}\right)^{\frac{\gamma-1}{\gamma}}. \tag{2.7}$$

Figure 1 represents (2.6), the family of γ-order standard Normal with different values of γ corresponding to $\gamma \uparrow 0$ (Dirac), $\gamma \downarrow 1$ (Uniform), $\gamma = 2$ (Normal), $\gamma \to \infty$ (Dirac).

For the Laplace transform of $N_\gamma(\mu, \sigma^2)$ the following theorem has been stated and proved.

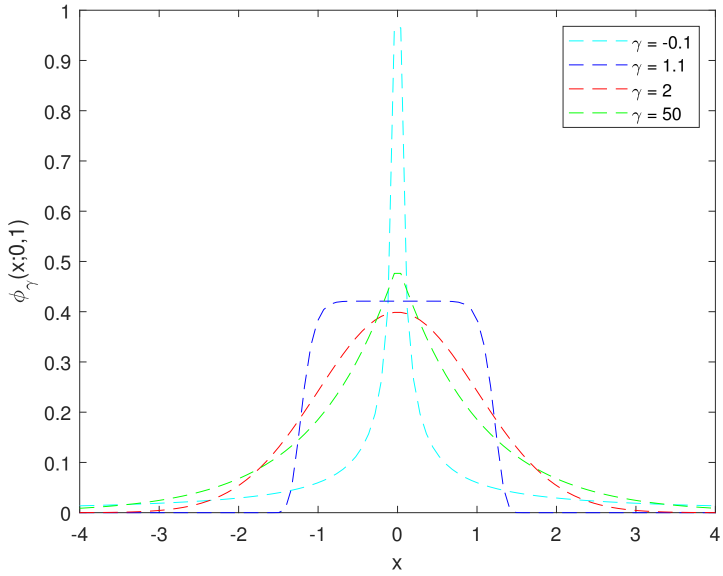

Figure 1: Plots of the univariate $\phi_\gamma(x;0,1)$ for different γ.

Theorem 2.2. *The Laplace transform of $\phi_\gamma(x;\mu,\sigma^2)$, recall (2.6), is*

$$\mathcal{L}\phi_\gamma(\xi) = \frac{e^{\xi\mu}}{\Gamma(\gamma_0)} \sum_{j=0}^{\infty} \frac{1}{(2j)!} \left(\xi\sigma(\gamma_1)^{\gamma_0}\right)^{2j} \Gamma\left((2j+1)\gamma_0\right).$$

Proof: See Kitsos and Stamatiou (2024). □

Therefore due to Theorem 2.2 the Moment Generating Function is well defined, as well as the cumulative distribution function (cdf) of $N_\gamma(\mu,\sigma^2)$. Furthermore, the well known for applications in Physics Heat Equation (Karlin and Taylor, 1975, Chapter 7) can be generalized through the $\phi_\gamma(x)$ distribution, see Kitsos (2023) for details.

3. THE γ-ORDER GENERALIZED CHI-γ, ${}_\gamma\mathcal{X}_n^\gamma$

In cases where "fat-tailed" distributions are needed, the Normal distribution is inadequate and the $N_\gamma(\mu,\sigma^2)$ can be applied with an appropriate choice of γ. Therefore for independent $Z_i, i = 1,2,\ldots,n$ as in (1.1) an extension is needed. The basis of our extension from Chi-square to γ-order Chi-γ is the well defined γ-order Generalized Normal distribution discussed in Section 2. We shall proceed as in (1.1). Thinking in terms of "shape parameter" we can say that in $\mathcal{X}_n^2$ the degrees of freedom (df) n are acting as a shape parameter. In principle all the distributions are associated with a shape parameter, to which we traditionally refer as the "degrees of freedom", for example $\mathcal{X}_n^2, t_n$, the Gamma distribution e.t.c. With the γ-order GN, the Normal distribution is associated with an extra shape parameter γ with the value $\gamma = 2$. This is certainly a new approach.

Practically, the consideration of the shape parameter influences the Chi-square distribution emerged from $N_\gamma(\mu,\sigma^2)$, and not only, to be associated with the same extra shape parameter. Notice that in classical situations the standard Normal distribution is unique, an ideal model, that a collected data set never reaches exactly. Thus, to be realistic, we need $N_\gamma(\mu,\sigma^2)$ which can be "intuitively" perceived, as we now work in three dimensions (position (mean), scale (variance), shape) to approach the collected data set. In addition, for all the distributions according to Central Limit Theorem for "large" n, theoretically $n \to \infty$, the shape is "stabilized" so the shape parameter is meaningless, as the "ideal" standard Normal distribution appears. But the shape exists in a *real data set* and we don't perceive it, because finally, even now (with powerful computers) n is getting large, but does not tend to infinity! So n acts as a shape parameter for small n to $\mathcal{X}^2_n$. Mainly for the approach of a fat-tailed distribution we can adopt an "almost Normal distribution" with a shape parameter γ. The following theorem is useful to our development.

Theorem 3.1. *Let $Z \sim N_\gamma(0,1), \gamma \in \mathbb{R} - [0,1], \gamma_0 = \frac{\gamma-1}{\gamma}, \quad \gamma_1 = 1/\gamma_0$. Consider the transformation*

$$W = 2\gamma_0 |Z|^{\gamma_1} \quad \text{with} \quad \gamma = \frac{1}{1-k},\ k \in \mathbb{N}, k > 1. \tag{3.1}$$

Then $W \sim \mathcal{X}^2_{2\gamma_0}$, that is W follows a Chi-square distribution with $2\gamma_0$ degrees of freedom. Moreover $\sum_{i=1}^n W_i \sim \mathcal{X}^2_{2n\gamma_0}$ for $\gamma = \frac{2n}{2n-k},\ k \in \mathbb{N}, k \neq 2n$.

Proof: As γ_0 needs to be integer this requires $\gamma = 1/(1-k)$ for $k \in \mathbb{N}, k > 1$. By (3.1) it is $|Z| = \left(\frac{W}{2\gamma_0}\right)^{1/\gamma_1}$ so the pdf of W can be obtained using the Jacobian of the transformation. The absolute value of Z and the symmetry of $N_\gamma(0,1)$, see (2.6), give rise to the coefficient 2 below.

$$\begin{aligned}
f_W(w) &= 2\phi_\gamma(w;0,1)\frac{dz}{dw} \\
&= 2\frac{\lambda_\gamma}{\sqrt{\pi}} e^{-w/2}\frac{1}{\gamma_1}\left(\frac{w}{2\gamma_0}\right)^{\gamma_0-1} \qquad \text{(recall (2.7))} \\
&= \frac{\gamma_0^{\gamma_0}\gamma_0}{\Gamma(\gamma_0+1)}\left(\frac{1}{2\gamma_0}\right)^{\gamma_0} 2^{\frac{2\gamma_0}{2}}\Gamma\left(\frac{2\gamma_0}{2}\right)\frac{w^{\frac{2\gamma_0}{2}-1}e^{-w/2}}{2^{\frac{2\gamma_0}{2}}\Gamma(\frac{2\gamma_0}{2})} \qquad \text{(pdf of Chi-square)} \\
&= \frac{1}{2^{\frac{2\gamma_0}{2}}\Gamma(\frac{2\gamma_0}{2})} w^{\frac{2\gamma_0}{2}-1}e^{-w/2} = f_{\mathcal{X}^2_{2\gamma_0}}(w).
\end{aligned}$$

Consequently we come across the result that

$$W_{n,\gamma_1} = \sum_{i=1}^n 2\gamma_0 |Z_i|^{\gamma_1} \sim \mathcal{X}^2_{2n\gamma_0}. \tag{3.2}$$

□

With Theorem 3.1 we are working on Chi-square line of thought due to the proposed transformation. The idea behind is to restrict n to "small" values and Theorem 3.1 offers food for thought for the "small" values of n, as the sum of $n = 3, 2$ Chi-square distributions is essential in applied Physics, as Maxwell-Boltzmann and Rayleigh distributions are emerged respectively with such n.

Corollary 3.1. *With $n = 3$ and $\gamma = 2$ thus $\gamma_0 = 1/2, \gamma_1 = 2$ we get from (3.2) that*

$$W_{3,2} = \sum_{i=1}^{3} |Z_i|^2 \sim \mathcal{X}_3^2. \tag{3.3}$$

Notice that $W_{3,2} = Z_1^2 + Z_2^2 + Z_3^2$ with $Z_i \sim N_2(0,1) \equiv N(0,1)$. That is in principle the Maxwell-Boltzmann distribution depends on a $\mathcal{X}_3^2$ with given scale parameter. Notice that we consider for convenience $\sigma = 1$. The scale parameter has a physical meaning considering (3.4) for the chosen values of γ. Then $\sqrt{W_{3,2}}$ is known to follow a Maxwell-Boltzmann distribution, Mandl (2008). Therefore we say that

$$MB_\gamma = \sqrt{\sum_{i=1}^{3} 2\gamma_0 |Z_i|^{\gamma_1}} = \sqrt{W_{3,\gamma_1}} \tag{3.4}$$

follows a γ-order generalized Maxwell-Boltzmann distribution with $\gamma = \frac{6}{6-k}$ and $k \in \mathbb{N}, k \neq 6$, see Theorem 3.2 and Figure 2.

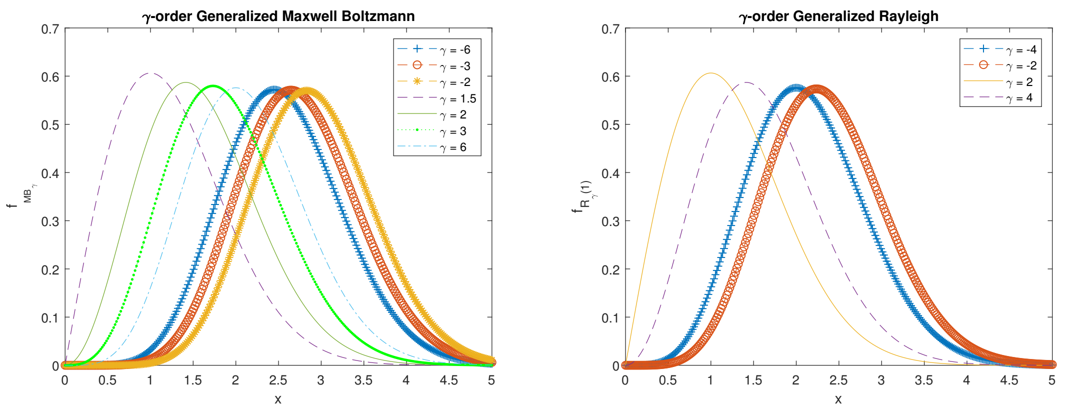

Figure 2: Plots of pdf of MB_γ and $R_\gamma(1)$ for different values of γ.

Example 3.1. Let the independent random variables v_x, v_y and v_z be distributed according to the γ-order generalized standard Normal distribution and assume a particle with velocity $v = (v_x, v_y, v_z)$ and velocity measure equal to $|v| = \sqrt{v_x^2 + v_y^2 + v_z^2}$. Then according to the above discussion $|v| \sim MB_\gamma$. If we let γ take the value 2 the usual Maxwell-Boltzmann distribution is implied, $|v| \sim MB_2$ while when we suppose that $v_x, v_y, v_z \sim N_{1.5}(0,1)$ or $N_3(0,1)$ respectively then $|v| \sim MB_{1.5}$ and $|v| \sim MB_3$ respectively. In particular according to (3.3), (3.4) and (3.2) the distribution of the velocity for $\gamma = 2, 1.5$ and 3 is

$$MB_2 = \sqrt{(v_x)^2 + (v_y)^2 + (v_z)^2} \sim \sqrt{\mathcal{X}_3^2} \quad \text{when} \quad v_x, v_y, v_z \sim N_2(0,1)$$

$$MB_{1.5} = \sqrt{\frac{2}{3}\left(|v_x|^{2+1} + |v_y|^{2+1} + |v_z|^{2+1}\right)} \sim \sqrt{\mathcal{X}_2^2} \quad \text{when} \quad v_x, v_y, v_z \sim N_{1.5}(0,1),$$

where $\mathcal{X}_2^2$ appears due to (3.2) with $2n\gamma_0 = 2 \cdot 3 \cdot \frac{1.5-1}{1.5} = 2$ df, while $\frac{2}{3} = 2\gamma_0$ and the exponent $\gamma_1 = 3 = 2 + 1$ and

$$MB_3 = \sqrt{\frac{4}{3}\left(|v_x|^{2-\frac{1}{2}} + |v_y|^{2-\frac{1}{2}} + |v_z|^{2-\frac{1}{2}}\right)} \sim \sqrt{\mathcal{X}_4^2} \quad \text{when} \quad v_x, v_y, v_z \sim N_3(0,1),$$

where $\mathcal{X}_4^2$ appears due to (3.2) with $2n\gamma_0 = 2 \cdot 3 \cdot \frac{3-1}{3} = 4$ df, while $\frac{4}{3} = 2\gamma_0$ and the exponent $\gamma_1 = \frac{3}{2} = 2 - \frac{1}{2}$, respectively, see Figure 3. The exponents in $MB_{1.5}$ and MB_3 are presented in relation with the value $\gamma = 2$ as in MB_2.

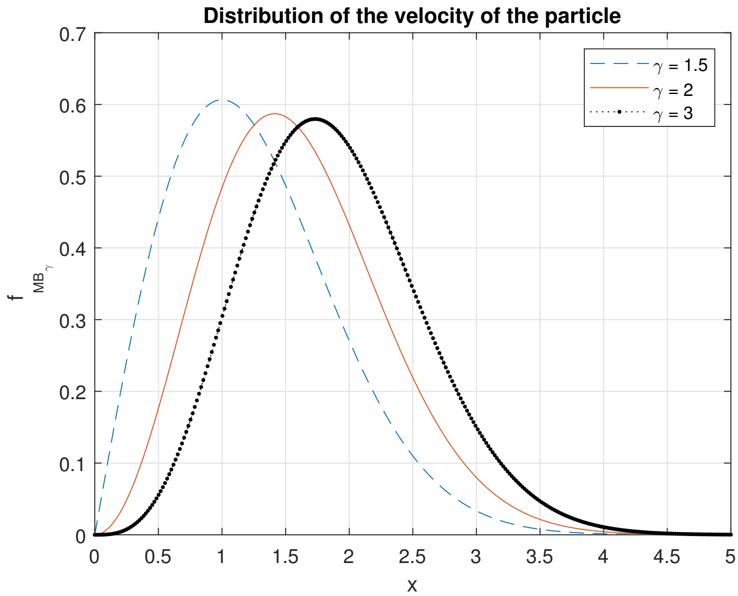

Figure 3: The Maxwell-Boltzmann distribution of the velocity of the particle for different values of γ, see Example 3.1.

Now, recall the definition of Rayleigh distribution, Meyer (1970), Rayleigh (1919). From (3.2) for $n = 2$ and $\gamma = 2$ thus $\gamma_0 = 1/2, \gamma_1 = 2$ as above, the $\sqrt{W_{2,2}}$ follows a Rayleigh distribution. Therefore we define the γ-order Generalized Rayleigh distribution and denote $R_\gamma(1)$ to be

$$R_\gamma(1) = \sqrt{\sum_{i=1}^{2} 2\gamma_0 |Z_i|^{\gamma_1}} = \sqrt{W_{2,\gamma_1}} \tag{3.5}$$

with $\gamma = \frac{4}{4-k}$ and $k \in \mathbb{N}, k \neq 4$, see Theorem 3.3 and Figure 2.

In Theorem 3.2 and Theorem 3.3 we obtain the pdf to these generalizations we worked with and with $\gamma = 2$ the classical case is obtained.

Theorem 3.2. *Consider the γ-order generalized Maxwell-Boltzmann distribution. Let $Y \sim MB_\gamma$ as in (3.4) with $\gamma = \frac{6}{6-k}$ and $k \in \mathbb{N}, k \neq 6$. Then its pdf is equal to*

$$f_{MB_\gamma}(y) = \frac{2^{1-3\gamma_0}}{\Gamma(3\gamma_0)} y^{6\gamma_0 - 1} e^{-y^2/2}. \tag{3.6}$$

Proof: Due to the Jacobian of the transformation of (3.4) and to the result in (3.2) it is $MB_{\gamma} = \sqrt{W_{3,\gamma_1}} = \sqrt{\mathcal{X}^2_{6\gamma_0}}$. Thus

$$f_{MB_{\gamma}}(y) = 2y f_{\mathcal{X}^2_{6\gamma_0}}(y^2) = 2y \frac{2^{-6\gamma_0/2}}{\Gamma(6\gamma_0/2)} (y^2)^{\frac{6\gamma_0}{2}-1} e^{-y^2/2} = \frac{2^{1-3\gamma_0}}{\Gamma(3\gamma_0)} y^{6\gamma_0-1} e^{-y^2/2},$$

which is exactly (3.6). □

Notice that (3.6) has a Gamma function flavor, see Theorem 3.7. The following theorem refers to Rayleigh distribution, $R_{\gamma}(1)$.

Theorem 3.3. *Consider the γ-order generalized Rayleigh distribution. Let $Y \sim R_{\gamma}(1)$ as in (3.5) with $\gamma = \frac{4}{4-k}$ and $k \in \mathbb{N}, k \neq 4$. Then its pdf is equal to*

$$f_{R_{\gamma}(1)}(y) = \frac{2^{1-2\gamma_0}}{\Gamma(2\gamma_0)} y^{4\gamma_0-1} e^{-y^2/2}. \tag{3.7}$$

Proof: Due to the Jacobian of the transformation of (3.5) and to the result in (3.2) it is $R_{\gamma}(1) = \sqrt{W_{2,\gamma_1}} = \sqrt{\mathcal{X}^2_{4\gamma_0}}$. Thus

$$f_{R_{\gamma}(1)}(y) = 2y f_{\mathcal{X}^2_{4\gamma_0}}(y^2) = 2y \frac{2^{-4\gamma_0/2}}{\Gamma(4\gamma_0/2)} (y^2)^{\frac{4\gamma_0}{2}-1} e^{-y^2/2} = \frac{2^{1-2\gamma_0}}{\Gamma(2\gamma_0)} y^{4\gamma_0-1} e^{-y^2/2},$$

which is exactly (3.7). □

Let the independent random variables $Z_i \sim N_{\gamma}(0,1)$, where $i = 1, 2, \ldots, n$. Denote the sum of their power to γ_1, Y_n, recall (1.1), i.e. $Y_n = \sum_{i=1}^{n} Z_i^{\gamma_1}$. We define Y_n as the γ-order generalized Chi-γ distribution with n degrees of freedom and we write $Y_n \sim {}_{\gamma}\mathcal{X}_n^{\gamma}$. The pdf of Y_n is evaluated in the following theorem.

Theorem 3.4. *The pdf of Y_n as in (1.1) equals to*

$$f_{Y_n}(y) = \left(\sqrt{\pi} \frac{\gamma_0^{\gamma_0}}{2\Gamma(\gamma_0+1)} \right)^n \frac{2\gamma_0}{\Gamma(n/2)} y^{n\gamma_0-1} \exp\{-\gamma_0 y\}, \quad \gamma_0 = \frac{\gamma-1}{\gamma}. \tag{3.8}$$

Proof: Consider n samples z_i from the γ-order standard generalized normal, i.e. from (2.6) with $\mu = 0$ and $\sigma = 1$. We let

$$y = \sum_{i=1}^{n} z_i^{\gamma_1}, \tag{3.9}$$

and V be the elemental shell volume at y, proportional to the $(n-1)$-dimensional surface in n-space for which (3.9) holds. Then due to (2.6) and then to (3.9)

$$\begin{aligned} f_{Y_n}(y)dy &= \int_V \prod_{i=1}^{n} N_\gamma(z_i;0,1)dz_i \\ &= \int_V \left(\frac{\lambda_\gamma}{\sqrt{\pi}}\right)^n \exp\{-\gamma_0(z_1^{\gamma_1}+z_2^{\gamma_1}+\ldots+z_n^{\gamma_1})\}d\mathbf{z} \\ &= \frac{\lambda_\gamma^n}{\pi^{n/2}}\exp\{-\gamma_0 y\}\int_V dz_1dz_2\ldots dz_n, \end{aligned} \tag{3.10}$$

which remains constant within the set V. As $N_\gamma(\mu,\Sigma)$ is a Kotz-type elliptical distribution, see Kitsos and Tavoularis (2009), the considered $N_\gamma(0,1)$ is a spherical contoured distribution. The integral in (3.10) is equal to the sphere $V = S_{n-1}$ with radius $r = y^{1/\gamma_1}$ times the infinitesimal thickness of the sphere $dr = \frac{1}{\gamma_1}y^{\frac{1}{\gamma_1}-1}dy$. Equivalently

$$\begin{aligned} \int_V dz_1dz_2\ldots dz_n &= \frac{2(y^{-\gamma_1})^{n-1}\pi^{n/2}}{\Gamma(n/2)}\cdot\frac{1}{\gamma_1}y^{\frac{1}{\gamma_1}-1}dy \\ &= \frac{2}{\gamma_1}y^{\frac{n}{\gamma_1}-1}\frac{\pi^{n/2}}{\Gamma(n/2)}dy \end{aligned}$$

Therefore,

$$f_{Y_n}(y)dy = \frac{\lambda_\gamma^n}{\Gamma(n/2)}2\gamma_0 y^{n\gamma_0-1}\exp\{-\gamma_0 y\}dy,$$

which by (2.7) reduces to (3.8). □

Notice that also (3.8) has a Gamma function flavor, see Theorem 3.7. The trivial case $\gamma = 2$ is discussed below.

Corollary 3.2. *When $\gamma = 2$ the pdf of the Chi-square distribution with n degrees of freedom is obtained*

$$f_{{}_2\mathcal{X}_n^2}(y) = \frac{1}{2^{n/2}\Gamma(n/2)}y^{\frac{n}{2}-1}\exp\{-y/2\},$$

for $y > 0$.

Proof of Corollary 3.2: See the Appendix. □

The central moments of ${}_\gamma\mathcal{X}_n^\gamma$ can be evaluated due to the following theorem, which we state and prove.

Theorem 3.5. *Let $Y_n \sim {}_\gamma\mathcal{X}_n^\gamma$ and $m \in \mathbb{N}$. Then*

$$\mathbb{E}(Y_n^m) = \frac{\pi^{n/2}}{2^{n-1}}\frac{\Gamma(m+n\gamma_0)}{\Gamma^n(\gamma_0+1)\Gamma(n/2)}\gamma_0^{1-m}. \tag{3.11}$$

Proof: Recall the definition of the moment of a random variable and (3.8) we have

$$\begin{aligned}\mathbb{E}(Y_n^m) &= \int_0^\infty y^m f_Y(y)dy \\ &= \left(\sqrt{\pi}\frac{\gamma_0^{\gamma_0}}{2\Gamma(\gamma_0+1)}\right)^n \frac{2\gamma_0}{\Gamma(n/2)} \int_0^\infty y^{m+n\gamma_0-1}e^{-\gamma_0 y}dy \\ &= \left(\sqrt{\pi}\frac{\gamma_0^{\gamma_0}}{2\Gamma(\gamma_0+1)}\right)^n \frac{2\gamma_0}{\Gamma(n/2)} \left(\frac{1}{\gamma_0}\right)^{m+n\gamma_0} \Gamma\left(m+n\gamma_0\right)\end{aligned}$$

where the following result was applied from the second to the third step

$$\int_0^\infty x^{s-1}e^{-\gamma_0 x}dx = \left(\frac{1}{\gamma_0}\right)^s \Gamma(s) \tag{3.12}$$

with $s = m + n\gamma_0$. Thus (3.11) is proved. □

In Table 1 the central moments of $Y_n \sim {}_\gamma\mathcal{X}_n^\gamma$ are presented for various choices of γ, n and m according to (3.11). Notice that with $\gamma = 2$ and $m = 1$ it is clear that $\mathbb{E}(Y_n) = n$ while for $\gamma > 2, \mathbb{E}(Y_n) > n$. In Table 2 the central moments of $Y_n \sim {}_\gamma\mathcal{X}_n^\gamma$ are presented with "small" and "large" positive values of γ, ($\gamma = 1.1$ and $\gamma = 50$), which correspond to $\gamma \to 1$ and $\gamma \to \pm\infty$, i.e. Uniform and Laplace distributions, Kitsos and Toulias (2011), for the defined γ-order Generalized Normal (2.4). It is clear that there is no need for m to be greater than 3, while n was considered "small".

	$\mathbb{E}(Y_n^m)$								
n	$\gamma = 2$			$\gamma = 2.5$			$\gamma = 3$		
1	1	3	15	1	2.667	11.556	1	2.5	10
2	2	8	48	2.168	7.949	42.393	2.295	8.032	40.161
3	3	15	105	3.692	17.228	109.107	4.270	19.216	115.297
4	4	24	192	4.264	32.698	239.785	7.453	40.992	286.941
5	5	35	315	8.665	57.766	481.379	12.704	82.574	660.589
m	1	2	3	1	2	3	1	2	3

Table 1: Central moments for $Y_n \sim {}_\gamma\mathcal{X}_n^\gamma$ for different values of γ, n, m.

	$\mathbb{E}(Y_n^m)$					
n	$\gamma = 1.1$			$\gamma = 50$		
1	1	102	276	1	2.020	6.144
2	1.590	20.667	496.012	3.080	9.301	37.586
3	1.626	22.769	569.218	8.966	36.048	181.711
4	1.319	19.791	514.560	27.156	136.334	823.571
5	0.916	14.665	395.963	86.832	522.764	3680.689
m	1	2	3	1	2	3

Table 2: Central moments of $Y_n \sim {}_\gamma\mathcal{X}_n^\gamma$ for "small" and "big" γ.

Corollary 3.3. *Let $Y_n \sim {}_\gamma\mathcal{X}_n^\gamma$. Then the variance of Y_n is equal to*

$$Var(Y_n) = Q(n,\gamma_0)\,(n+\gamma_1 - Q(n,\gamma_0))\,, \tag{3.13}$$

where

$$Q(n,\gamma_0) = \frac{\pi^{n/2}}{2^{n-1}}\frac{\Gamma(1+n\gamma_0)}{\Gamma^n(\gamma_0+1)\Gamma(n/2)}. \tag{3.14}$$

In particular when $\gamma = 2$, i.e. $Y_n \sim {}_2\mathcal{X}_n^2$, then the variance of the Chi-square distribution with n degrees of freedom is obtained That is

$$Var(Y_n) = Var({}_2\mathcal{X}_n^2) = Var(\mathcal{X}_n^2) = 2n.$$

Proof of Corollary 3.3: See the Appendix. □

From (3.13) we can easily evaluate the values of the variances of $Y_n \sim {}_\gamma\mathcal{X}_n^\gamma$.

The Laplace transform of the defined generalized γ-order Chi-γ distribution, ${}_\gamma\mathcal{X}_n^\gamma$, can be evaluated due to the following theorem.

Theorem 3.6. *The Laplace transform of $Y_n \sim {}_\gamma\mathcal{X}_n^\gamma$ is*

$$\mathcal{L}\{f_{Y_n}(\xi)\} = \Xi_n^\gamma \left(\frac{1}{\gamma_0-\xi}\right)^{n\gamma_0}$$

for $\xi < \gamma_0$ where, see also (3.14), with

$$\Xi_n^\gamma = Q(n,\gamma_0)\frac{\gamma_0^{n\gamma_0}}{n}. \tag{3.15}$$

Proof: By the definition of the Laplace transform and denoting c_n^γ the normalizing constant in (3.8), it holds

$$\begin{aligned}\mathcal{L}\{f_{Y_n}(\xi)\} = \int_0^\infty e^{\xi x} c_n^\gamma x^{n\gamma_0-1} e^{-\gamma_0 x}dx &= c_n^\gamma \int_0^\infty x^{n\gamma_0-1}e^{-(\gamma_0-\xi)x}dx \\ &= c_n^\gamma \Gamma(n\gamma_0)\left(\frac{1}{\gamma_0-\xi}\right)^{n\gamma_0},\end{aligned}$$

where in the last step (3.12) is appropriately used. By (3.14) it holds that

$$\begin{aligned} c_n^\gamma\Gamma(n\gamma_0) &= \left(\sqrt{\pi}\frac{\gamma_0^{\gamma_0}}{2\Gamma(\gamma_0+1)}\right)^n \frac{2\gamma_0}{\Gamma(n/2)}\Gamma(n\gamma_0) \\ &= Q(n,\gamma_0)\frac{\gamma_0^{n\gamma_0}}{n} =: \Xi_n^\gamma \end{aligned}$$

as in (3.15). □

Corollary 3.4. *Let us consider the case where $Y_n \sim {}_2\mathcal{X}_n^2$. Then the Laplace transform of Y_n is*

$$\mathcal{L}\{f_{Y_n}(\xi)\} = (1-2\xi)^{-n/2} = \mathcal{L}\mathcal{X}_n^2(\xi) \tag{3.16}$$

for $\xi < 1/2$.

Proof of Corollary 3.3: See the Appendix. □

Therefore due to the theoretical results produced in this section the introduced generalized γ-order distributions are well-defined, their moments have been evaluated and the comparison with the "classical cases" is easily performed. We were thinking that there exists a Gamma distribution flavor, see (3.6) and (3.8), to these results and therefore we stated and proved Theorem 3.7. The following theorem proves that indeed there is a such line of thought.

Theorem 3.7. *Let the random variable Z be from the γ-order generalized standard normal, $Z \sim N_\gamma(0,1)$. Consider the transformation*

$$Y = \gamma_0 |Z|^{\gamma_1}, \quad \gamma_0 = \frac{\gamma - 1}{\gamma}, \quad \gamma_1 = \gamma_0^{-1} = \frac{\gamma}{\gamma - 1}, \quad \gamma \in \mathbb{R} - [0,1].$$

Then Y follows a Gamma distribution with shape parameter γ_0, scale parameter 1, that is

$$f_Y(y) = \frac{1}{\Gamma(\gamma_0)} y^{\gamma_0 - 1} \exp\{-y\}$$

for any $y > 0$.

Proof: Let $F_Y(y)$ be the cdf of Y

$$\begin{aligned} F_Y(y) = \mathbb{P}(Y \le y) &= \mathbb{P}(\gamma_0|Z|^{\gamma_1} \le y) \\ &= \mathbb{P}(|Z| \le (\gamma_1 y)^{1/\gamma_1}) \\ &= \mathbb{P}(Z \le (\gamma_1 y)^{1/\gamma_1}) - \mathbb{P}(Z \le -(\gamma_1 y)^{1/\gamma_1}) \\ &= 2\mathbb{P}(Z \le (\gamma_1 y)^{1/\gamma_1}) - 1 \end{aligned} \tag{3.17}$$

for any $y > 0$. Differentiating (3.17) and considering (2.7) with $\mu = 0, \sigma = 1$ and proceeding as in the proof of Theorem 3.1, the pdf of Y, say f_Y, is

$$\begin{aligned} f_Y(y) &= 2\frac{1}{\gamma_1}(\gamma_1 y)^{\frac{1}{\gamma_1}-1}\gamma_1 \frac{\gamma_0^{\gamma_0}}{2\Gamma(\gamma_0+1)} \exp\{-\gamma_0((\gamma_1 y)^{1/\gamma_1})^{\gamma_1}\} \\ &= (\gamma_1 y)^{\frac{1}{\gamma_1}-1} \frac{\gamma_1^{-1/\gamma_1}}{\gamma_0 \Gamma(\gamma_0)} \exp\{-\gamma_0\gamma_1 y\} \\ &= \frac{1}{\Gamma(1/\gamma_1)} y^{\frac{1}{\gamma_1}-1} \exp\{-y\}, \end{aligned}$$

which is the Gamma distribution with shape parameter $1/\gamma_1 = \gamma_0$. □

Corollary 3.5. *Assume the setting of Theorem 3.7, where $Y = \gamma_0|Z|^{\gamma_1}$, with $Z \sim N_\gamma(0,1)$. Then*

$$\mathbb{E}(Y) = Var(Y) = \gamma_0 = \frac{\gamma - 1}{\gamma}.$$

Proof of Corollary 3.5: By Theorem 3.7 Y follows a Gamma distribution with shape parameter γ_0 and scale parameter 1 therefore it is known that the mean of Y is the product of these parameters, that is $\mathbb{E}(Y) = \gamma_0 \cdot 1 = \gamma_0$, and the variance of Y is the square of the scale parameter times the shape parameter, that is $Var(Y) = \gamma_0 \cdot 1^2 = \gamma_0$. □

4. CONCLUSION

In this paper it was considered that all the distributions under investigation are associated with a shape parameter γ, even the "ideal" standard normal distribution, the case $\gamma = 2$. Every deviation from $\gamma = 2$ creates another Normal distribution within the family of γ-order Generalized Normal distributions in which well known distribution are included, see also Theorem 2.1. Therefore, the generalized "Chi-square" distribution was introduced due to the γ-order Generalized Normal, as ${}_\gamma\mathcal{X}_n^\gamma$ and the pdf, as well as the moments, were evaluated. For $\gamma = 2$ the $\mathcal{X}_n^2$ distribution is obtained, while for fat tailed $N_\gamma(0,1)$ distributions, for values of $\gamma = 2.5, 3$ the corresponding ${}_\gamma\mathcal{X}_n^\gamma$ for different values of n are generated, see Figure 4. In Figure 5 we keep n constant to 3 and 5 respectively and vary between different values of γ. Recall that values of γ near to 1 or "large" correspond to Uniform or Laplace. Finally with $\gamma = 2$, $N_\gamma(0,1)$ becomes a Normal distribution and $\mathbb{E}(Y_{50})$ is equal to 50 as expected. In Tables 1 and 2 the values of $\mathbb{E}(Y_n^m)$ with $n \geq 10$ are very large. With Theorem 3.7 there is a transformation of the $N_\gamma(0,1)$ to a Gamma distribution with shape parameter γ_0, scale parameter 1. For any request on software providing calculations communicate with the authors.

Moreover the well-known in Physics distributions, Maxwell-Boltzmann and Rayleigh, have been generalized, due to the existent extra shape-parameter γ.

The international constant $(\frac{1}{1-\gamma})^{\frac{1}{1-\gamma}}$, Takashi and Takasi (2001), see for example (3.15), plays an important role to our development from Normal distribution to γ-order Generalized Normal, then to γ-order Generalized Chi-γ and γ-order Rayleigh, γ-order Maxwell-Boltmzann Generalized distributions.

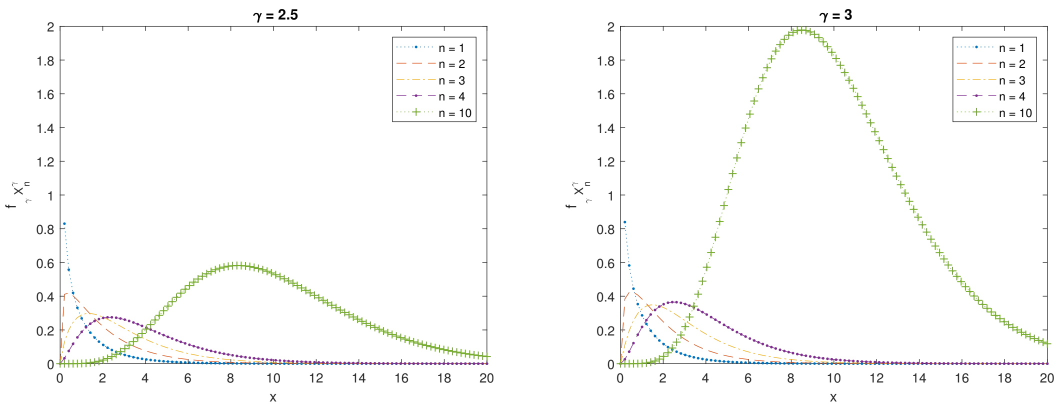

Figure 4: Plots of $f_{{}_\gamma\mathcal{X}_n^\gamma}$ for different values of n with $\gamma = 2.5$ and 3 respectively.

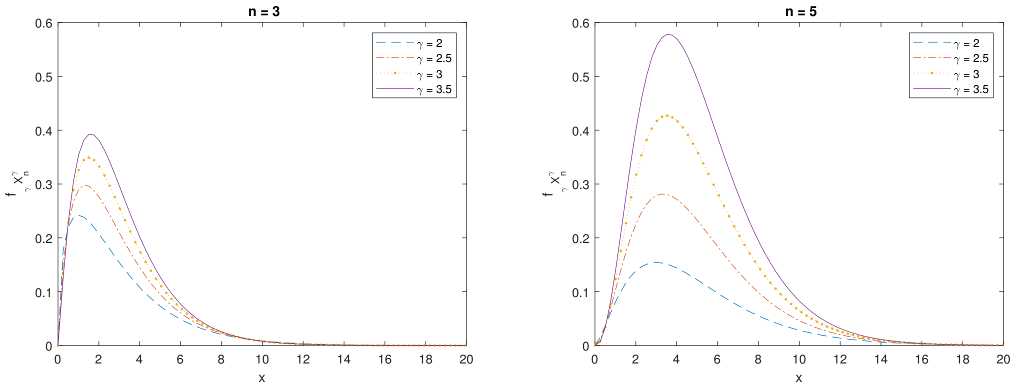

Figure 5: Plots of $f_{{}_\gamma\mathcal{X}_n^\gamma}$ for different values of γ with $n = 3$ and 5 respectively.

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APPENDIX. PROOFS

Proof of Corollary 3.2: When $\gamma = 2$ then $\gamma_0 = 1/2$. Using this value for γ_0 in (3.8), implies that

$$\begin{aligned} f_{{}_2\mathcal{X}_n^2}(y) &= \left(\sqrt{\pi}\frac{\sqrt{1/2}}{2\Gamma(\frac{1}{2}+1)}\right)^n \frac{1}{\Gamma(n/2)} y^{\frac{n}{2}-1}\exp\{-y/2\} \\ &= \frac{1}{2^{n/2}\Gamma(n/2)} y^{\frac{n}{2}-1}\exp\{-y/2\} \end{aligned}$$

which is the pdf of the Chi-square distribution, Schervish (1997), with n degrees of freedom. □

Proof of Corollary 3.3: Since $Var(Y_n) = \mathbb{E}(Y_n)^2 - \mathbb{E}^2(Y_n)$, Theorem 3.5 implies that

$$\begin{aligned} Var(Y_n) &= \frac{\pi^{n/2}}{2^{n-1}}\frac{\Gamma(2+n\gamma_0)}{\Gamma^n(\gamma_0+1)\Gamma(n/2)}\gamma_0^{-1} - \left(\frac{\pi^{n/2}}{2^{n-1}}\frac{\Gamma(1+n\gamma_0)}{\Gamma^n(\gamma_0+1)\Gamma(n/2)}\right)^2 \\ &= \frac{\pi^{n/2}}{2^{n-1}}\frac{\Gamma(1+n\gamma_0)}{\Gamma^n(\gamma_0+1)\Gamma(n/2)}\frac{n\gamma_0+1}{\gamma_0} - \left(\frac{\pi^{n/2}}{2^{n-1}}\frac{\Gamma(1+n\gamma_0)}{\Gamma^n(\gamma_0+1)\Gamma(n/2)}\right)^2 \\ &= Q(n,\gamma_0)(n+\gamma_1 - Q(n,\gamma_0)). \end{aligned}$$

Therefore (3.13) is true. When $\gamma = 2$ then $\gamma_0 = 1/2$ and $\gamma_1 = 2$ so that

$$Q(n,1/2) = \frac{\pi^{n/2}}{2^{n-1}}\frac{\Gamma(1+\frac{n}{2})}{\Gamma^n(\frac{1}{2}+1)\Gamma(n/2)} = \frac{\pi^{n/2}}{2^{n-1}}\frac{n/2}{(\sqrt{\pi}/2)^n} = n. \tag{A1}$$

By (3.13) the variance for ${}_2\mathcal{X}_n^2$ equals

$$Var({}_2\mathcal{X}_n^2) = n(n+2-n) = 2n$$

which is the variance of the Chi-square distribution with n degrees of freedom. □

Proof of Corollary 3.4: Application of Theorem 3.6 with $\gamma = 2$ implies

$$\mathcal{L}\{f_{Y_n}(\xi)\} = \Xi_n^{1/2}\left(\frac{1}{\frac{1}{2}-\xi}\right)^{n/2} = Q(n,1/2)\frac{(1/2)^{n/2}}{n}\left(\frac{2}{1-2\xi}\right)^{n/2} = \left(\frac{1}{1-2\xi}\right)^{n/2}$$

for $\xi < 1/2$ where (3.15) and (A1) are used or equivalently (3.16). □