Proof of Proposition 2.1 for the Hall class

It holds that

$$\begin{aligned}
g_{\boldsymbol{\gamma}_k}(x) =&\frac{1}{a_k}\left(1+\gamma\frac{x-b_k}{a_k}\right)^{-\alpha-1}\exp\left(-\left(1+\gamma\frac{x-b_k}{a_k}\right)^{-\alpha}\right)\\
=&\alpha(Ak)^{-\gamma}(Akx^{-\alpha})^{1+\gamma}\exp\left(-Akx^{-\alpha}\right)\\
\ln f_{(m)}(x) =&-M_n\left\{1+Bx^{-\beta}+\frac{1}{2}Ax^{-\alpha}+o(x^{-\alpha}+x^{-\beta})\right\}\\
&+\log\left(x^{-1}M_n\left\{\alpha+B(\alpha+\beta)x^{-\beta}+A\alpha x^{-\alpha}+o(x^{-\alpha}+x^{-\beta})\right\}\right)\\
f_{(m)}(x) =&\exp(-M_n)\exp\left[-M_n\left\{Bx^{-\beta}+\frac{1}{2}Ax^{-\alpha}+o(x^{-\alpha}+x^{-\beta})\right\}\right]\\
&\left(x^{-1}M_n\left\{\alpha+B(\alpha+\beta)x^{-\beta}+A\alpha x^{-\alpha}+o(x^{-\alpha}+x^{-\beta})\right\}\right).
\end{aligned}$$

When $(M_n \vee K_n) = o(1)$ and $(M_n \wedge K_n) \to \infty$,

$$\begin{aligned}
\widetilde{\tau_n} :=&f_{(m)}(x) - g_{\boldsymbol{\gamma}_k}(x) = \alpha x^{-1}(M_n - K_n + o(M_n \vee K_n)) \to 0,\\
\widetilde{\tau_n} =&\alpha x^{-1}\left(M_n\exp\left(-M_n\right) - K_n\exp\left(-K_n\right) + o(M_n\exp\left(-M_n\right) - K_n\exp\left(-K_n\right))\right) \to 0,
\end{aligned}$$

respectively. When there exists some positive constant δ s.t. $M_n - \delta =: \epsilon_n \to 0$, $\widetilde{\tau_n}$ converges only if $K_n - \delta =: \widetilde{\epsilon_n} \to 0$. Then,

$$\begin{aligned}
\widetilde{\tau_n} =\exp(-\delta)\Bigg\{&\exp\left[-\epsilon_n-\delta\left\{Bm^{-\beta\gamma}+\frac{1}{2}Am^{-1}+o(m^{-\beta\gamma}+m^{-1})\right\}\right]\\
&x^{-1}(\delta+\epsilon_n)\left(\alpha+B(\alpha+\beta)x^{-\beta}+A\alpha x^{-\alpha}+o(x^{-\alpha}+x^{-\beta})\right)-\exp\left(-\widetilde{\epsilon_n}\right)\Bigg\},
\end{aligned}$$

and the convergence rate of $\widetilde{\tau_n}$ is x^{-1} times $(m^{-\beta\gamma} \vee m^{-1} \vee \epsilon_n \vee \widetilde{\epsilon_n})$.

Proof of Proposition 2.1 for the Weibull class

It holds that

$$\begin{aligned}
g_{\boldsymbol{\gamma}_k}(x) =&\exp\left(-\frac{x-b_k}{a_k}\right)\exp\left(-\exp\left(-\frac{x-b_k}{a_k}\right)\right)\\
\sim&k^{\kappa}\exp(-\kappa C^{1/\kappa}(\ln k)^{\theta}x)\exp[-k^{\kappa}\exp(-\kappa C^{1/\kappa}(\ln k)^{\theta}x)].
\end{aligned}$$

Set $K_n := k^{\kappa}\exp(-\kappa C^{1/\kappa}(\ln k)^{\theta}x)$; then, we have

$$\begin{aligned}
f_{(m)}(x) =&\kappa Cmx^{\kappa-1}\left(1-\exp\{-Cx^{\kappa}\}\right)^{m-1}\\
\sim&\kappa Cmx^{\kappa-1}\exp\left[(m-1)\ln\left(1-\exp\{-Cx^{\kappa}\}\right)\right]\\
\sim&\kappa Cmx^{\kappa-1}\exp\left[(m-1)\left(-\exp\{-Cx^{\kappa}\}-2^{-1}\exp\{-2Cx^{\kappa}\}\right)\right]\\
\sim&\begin{cases}\kappa Cmx^{\kappa-1}\{1+(-1+m^{-1})M_n-2^{-1}M_n^2\} & \text{for} \quad M_n\to 0\\ \kappa Cmx^{\kappa-1}\exp\left(-M_n\right) & \text{for} \quad M_n\to\infty,\end{cases}
\end{aligned}$$

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where $M_n := m\exp(-Cx^\kappa)$. It follows that the convergence rate is

$$\begin{cases}\kappa Cmx^{\kappa-1} - M_n - K_n \quad \text{for} \quad M_n \to 0 \\ \kappa Cmx^{\kappa-1}\exp(-M_n) - K_n\exp(-K_n) \quad \text{for} \quad M_n \to \infty.\end{cases}$$

$\widetilde{\tau}_n$ for $M_n \to \delta$ converges only if both $K_n \to \delta$ and $K_n \sim \kappa Cmx^{\kappa-1}$, i.e., $k \sim m$.

Proof of Proposition 2.1 for the bounded class

It holds that

$$\begin{aligned} g_{\boldsymbol{\gamma}_k}(x) =& \frac{1}{a_k}\left(1+\gamma\frac{x-b_k}{a_k}\right)^{-\mu-1}\exp\left(-\left(1+\gamma\frac{x-b_k}{a_k}\right)^{-\mu}\right) \\ =& \frac{1}{-\gamma(x^*-x)}Dk(x^*-x)^{-\mu}\exp\left(-Dk(x^*-x)^{-\mu}\right) \\ \ln f_{(m)}(x) =& -(m-1)\{(1-F(x))+\frac{1}{2}(1-F(x))^2+O((1-F(x))^3\} \\ &+\log\left(\frac{m}{(x^*-x)}[\frac{1}{\gamma}(1-F(x))-(x^*-x)^{-\mu}E\frac{\sigma}{\gamma}(x^*-x)^{\mu\sigma}+o((x^*-x)^{-\mu+\mu\sigma})]\right) \\ f_{(m)}(x) =& \exp(-(m-1)\{(1-F(x))+\frac{1}{2}(1-F(x))^2\}) \\ & \frac{m}{(x^*-x)}[\frac{1}{-\gamma}(1-F(x))+(x^*-x)^{-\mu}E\frac{\sigma}{\gamma}(x^*-x)^{\mu\sigma}+o((x^*-x)^{-\mu+\mu\sigma})]. \end{aligned}$$

When $(M_n \vee K_n) = o(1)$ and $(M_n \wedge K_n) \to \infty$,

$$\begin{aligned} \widetilde{\tau}_n :=& f_{(m)}(x) - g_{\boldsymbol{\gamma}_k}(x) = \frac{1}{-\gamma(x^*-x)}(M_n - K_n + o(M_n \vee K_n)) \to 0, \\ \widetilde{\tau}_n =& \frac{1}{-\gamma(x^*-x)}\left(M_n\exp(-M_n+o(M_n)) - K_n\exp(-K_n+o(K_n))\right) \to 0, \end{aligned}$$

respectively. When there exists some positive constant δ s.t. $M_n - \delta =: \epsilon_n \to 0$, $\widetilde{\tau}_n$ converges only if $K_n - \delta =: \widetilde{\epsilon}_n \to 0$. Then, the convergence rate of $\widetilde{\tau}_n$ is $(x^*-x)^{-1}$ times $(m^{-\sigma} \vee m^{-1} \vee \epsilon_n \vee \widetilde{\epsilon}_n)$.

Proof of Theorem 2.1 for the Hall class and the bounded class

First, we decompose the difference as follows:

$$\begin{aligned} f_{(m)}(x) - g_{\widehat{\boldsymbol{\gamma}}_k}(x) =& \left[f_{(m)}(x) - g_{\boldsymbol{\gamma}_k}(x)\right] - \left[g_{\boldsymbol{\gamma}_k}(x) - g_{\widehat{\boldsymbol{\gamma}}_k}(x)\right] \\ =& \widetilde{\tau}_n + \widetilde{\zeta}_n \qquad \text{(say)}. \end{aligned}$$

It holds that

$$\widetilde{\zeta}_n = g_{\boldsymbol{\gamma}_k}(x) - g_{\widehat{\boldsymbol{\gamma}}_k}(x) = -\frac{\partial}{\partial\boldsymbol{\gamma}}g_{\boldsymbol{\gamma}}(x)\bigg|_{\boldsymbol{\gamma}=\widetilde{\boldsymbol{\gamma}}_k}(\widehat{\boldsymbol{\gamma}}_k - \boldsymbol{\gamma}_k)$$

and $\widetilde{\boldsymbol{\gamma}}_k = (\widetilde{\gamma}_k, \widetilde{a}_k, \widetilde{b}_k)^\mathsf{T}$ is between $\widehat{\boldsymbol{\gamma}}_k$ and $\boldsymbol{\gamma}_k$ with probability 1. By calculating the derivative, we obtain

$$\begin{aligned}\frac{\partial}{\partial \alpha} g_{\boldsymbol{\gamma}}(x)\bigg|_{\boldsymbol{\gamma}=\boldsymbol{\gamma}_k} =& a_k^{-1}(z_{k,n})^{-\alpha-1}\exp(-(z_{k,n})^{-\alpha}) \\ & \{(1-z_{k,n}^{-1})(1+\gamma) - \ln(z_{k,n}) - (z_{k,n})^{-\alpha}((1-z_{k,n}^{-1}) - \ln(z_{k,n}))\},\end{aligned}$$

where $z_{k,n} := 1 + (\alpha a_k)^{-1}(x - b_k) = \gamma a_k^{-1} x$. Dombry and Ferreira (2019) proved that under the assumption of Theorem 2.1,

$$\sqrt{N}\begin{pmatrix}1 & & \\ & a_k^{-1} & \\ & & a_k^{-1}\end{pmatrix}(\widehat{\boldsymbol{\gamma}}_k - \boldsymbol{\gamma}_k) \xrightarrow{d} \boldsymbol{N} := \mathcal{N}(\lambda I_0^{-1}\boldsymbol{b}, I_0^{-1}),$$

where the bias term $\boldsymbol{b}$ and the Fisher information matrix I_0 are given in Dombry and Ferreira (2019). It follows from $(1 + (\widehat{\alpha}\widehat{a}_k)^{-1}(x - \widehat{b}_k))/z_{k,n} \xrightarrow{p} 1$ that

$$\frac{\partial}{\partial \alpha} g_{\boldsymbol{\gamma}}(x)\bigg|_{\boldsymbol{\gamma}=\widetilde{\boldsymbol{\gamma}}_k} - \widetilde{s}_n \xrightarrow{p} 0,$$

where

$$\widetilde{s}_n := a_k^{-1} K_n^{1+\gamma}\exp(-K_n)\left\{(1-K_n^{\gamma})(1+\gamma) + \gamma\ln K_n - K_n\left((1-K_n^{\gamma}) + \gamma \ln K_n\right)\right\}.$$

Similarly,

$$\frac{\partial}{\partial a_k} g_{\boldsymbol{\gamma}}(x)\bigg|_{\boldsymbol{\gamma}=\widetilde{\boldsymbol{\gamma}}_k} - \widetilde{t}_n^* \xrightarrow{p} 0, \quad \frac{\partial}{\partial b_k} g_{\boldsymbol{\gamma}}(x)\bigg|_{\boldsymbol{\gamma}=\widetilde{\boldsymbol{\gamma}}_k} - \widetilde{u}_n^* \xrightarrow{p} 0,$$

where

$$\begin{aligned}\widetilde{t}_n^* :=& -\frac{K_n^{-1}(1+\alpha) - \alpha K_n^{-\gamma-1} + \alpha K_n^{-\gamma} - \alpha}{K_n^{-2(1+\gamma)} a_k^2}\exp(-K_n)\\ \widetilde{u}_n^* :=& \frac{\left(K_n^{-1}(\alpha+1) - \alpha\right)}{\alpha K_n^{-2(1+\gamma)} a_k^2}\exp(-K_n).\end{aligned}$$

As seen from Dombry and Ferreira (2019), $\widetilde{\zeta}_n$ is asymptotically equivalent in distribution to $-N^{-1/2}\widetilde{\boldsymbol{\eta}}_n^\mathsf{T}\boldsymbol{N} = O_P(N^{-1/2}a_k^{-1})$, where $\widetilde{\boldsymbol{\eta}}_n = (\widetilde{s}_n, \widetilde{t}_n, \widetilde{u}_n)^\mathsf{T}$, $\widetilde{t}_n := a_k \widetilde{t}_n^*$ and $\widetilde{u}_n := a_k \widetilde{u}_n^*$. Combining Proposition 2.1 and the above results, Theorem 2.1 for the Hall class is proved. In the same manner, Theorem 2.1 is proved for the bounded class.

Proof of Theorem 2.1 for the Weibull class

It holds that

$$\frac{\partial}{\partial \alpha} g_{\boldsymbol{\gamma}}(x)\bigg|_{\boldsymbol{\gamma}=\widetilde{\boldsymbol{\gamma}}_k} \xrightarrow{p} 0, \quad \frac{\partial}{\partial a_k} g_{\boldsymbol{\gamma}}(x)\bigg|_{\boldsymbol{\gamma}=\widetilde{\boldsymbol{\gamma}}_k} - \widetilde{t}_n^* \xrightarrow{p} 0,$$

where

$$\widetilde{t}_n^* := (1-k^{-\phi_0 a_k^{1/(\kappa-1)} x_{k,n}^*})(x_{k,n}^* a_k^{-2} k^{-\phi_0 a_k^{1/(\kappa-1)} x_{k,n}^*}) \exp\left[-k^{-\phi_0 a_k^{1/(\kappa-1)} x_{k,n}^*}\right],$$

$x_{k,n}^* := x - \{C^{-1}\ln k\}^{\frac{1}{\kappa}}$ and $\phi_0 := \kappa^{1/\theta} C^{1/(\kappa-1)}$. It follows from $\phi_0 a_k^{1/(\kappa-1)} x_{k,n}^* = \kappa C^{1/\kappa}(\ln k)^{-1/\kappa} x - \kappa$ that

$$\begin{aligned} k^{-\phi_0 a_k^{1/(\kappa-1)} x_{k,n}^*} =& k^{\kappa} \times k^{-\kappa C^{1/\kappa}(\ln k)^{-1/\kappa} x} \\ =& k^{\kappa} \exp\left[\ln(k^{-\kappa C^{1/\kappa}(\ln k)^{-1/\kappa} x})\right] =: K_n. \end{aligned}$$

Combining these results and $x_{k,n}^* a_k^{-1} = -\ln K_n$, we obtain

$$\widetilde{t}_n^* = a_k^{-1} \exp(-K_n) K_n (K_n - 1) \ln K_n.$$

Similarly, we have

$$\frac{\partial}{\partial b_k} g_{\boldsymbol{\gamma}}(x)\bigg|_{\boldsymbol{\gamma}=\widetilde{\boldsymbol{\gamma}}_k} - \widetilde{u}_n^* \xrightarrow{p} 0,$$

where $\widetilde{u}_n^* := a_k^{-1} K_n (K_n - 1)\exp(-K_n)$.

It follows from the same asymptotic normality of $\widehat{\boldsymbol{\gamma}}_k$ used in the proof for the Hall class that $\widetilde{\zeta}_n$ is asymptotically equivalent in distribution to $-N^{-1/2}\widetilde{\boldsymbol{\eta}}_n^{\mathsf{T}}\boldsymbol{N}$, where $\widetilde{\boldsymbol{\eta}}_n = (\widetilde{s}_n, \widetilde{t}_n, \widetilde{u}_n)^{\mathsf{T}}$, $\widetilde{s}_n \equiv 0$, $\widetilde{t}_n := a_k \widetilde{t}_n^*$, $\widetilde{u}_n := a_k \widetilde{u}_n^*$. Theorem 2.1 for the Weibull class has now been proved.

Proof of Theorem 3.1 for the Hall class

It follows from $F(x - h_2 z)$ equals

$$\begin{aligned} & 1 - A(x-h_2 z)^{-\alpha}\left(1 + B(x-h_2 z)^{-\beta} + o(x^{-\beta})\right) \\ =& 1 - A\left\{x^{-\alpha} - \alpha h_2 x^{-\alpha-1} z + \frac{\alpha(\alpha+1)}{2} h_2^2 x^{-\alpha-2} z^2 + O(h_2^3 x^{-\alpha-3})\right\} \\ & \left(1 + B\left\{x^{-\beta} - \beta h_2 x^{-\beta-1} z + \frac{\beta(\beta+1)}{2} h_2^2 x^{-\beta-2} z^2 + O(h_2^3 x^{-\beta-3})\right\} + o(x^{-\beta})\right) \end{aligned}$$

that

$$\begin{aligned} \mathbb{E}[\widehat{F}(x;h_2)] - F(x) =& \int w(z)\{F(x-h_2 z) - F(x)\} \mathrm{d}z \\ =& -\frac{A\alpha(\alpha+1)}{2} h_2^2 x^{-\alpha-2} \int z^2 w(z)\,\mathrm{d}z + o(h_2^2 x^{-\alpha-2}), \end{aligned}$$

$$\begin{aligned} n\mathbb{V}[\widehat{F}(x;h_2)] =& \mathbb{E}\left[W^2\left(\frac{x-X_i}{h_2}\right)\right] - \left(\mathbb{E}\left[W\left(\frac{x-X_i}{h_2}\right)\right]\right)^2 \\ =& Ax^{-\alpha}\left\{1 + Bx^{-\beta} - Ax^{-\alpha} - 2\alpha h_2 x^{-1}\int zW(z)w(z)\mathrm{d}z + o(x^{-\alpha} + x^{-\beta} + h_2 x^{-1})\right\}. \end{aligned}$$

In the same manner, we have

$$\begin{aligned}\mathbb{E}[\widehat{f}(x;h_1)]-f(x)=&\frac{A\alpha(\alpha+1)(\alpha+2)}{2}h_1^2x^{-\alpha-3}\int z^2w(z)\mathrm{d}z+o(h_1^2x^{-\alpha-3})\\ n\mathbb{V}[\widehat{f}(x;h_1)]=&h_1^{-1}f(x)\int w^2(z)\mathrm{d}z+o(h_1^{-1}f(x)).\end{aligned}$$

Set $Z_n := m(\widehat{F}(x;h_2)-\mathbb{E}[\widehat{F}(x;h_2)])$, $y_n := m(\mathbb{E}[\widehat{F}(x;h_2)]-F(x))$, $\widetilde{W}_n := \widehat{f}(x;h_1)-\mathbb{E}[\widehat{f}(x;h_1)]$, and $\widetilde{u}_n := (\mathbb{E}[\widehat{f}(x;h_1)]-f(x))$. $\widehat{f}_{(m)}(x;h_1,h_2)$ equals

$$\begin{aligned}&m\frac{\widehat{f}(x;h_1)}{\widehat{F}(x;h_2)}\exp\left(m\ln(\widehat{F}(x;h_2))\right)\\ =&m\{f(x)+\widetilde{u}_n+\widetilde{W}_n\}\exp\left(m\left\{\widehat{F}(x;h_2)-1+o_P(\widehat{F}(x;h_2)-1)\right\}\right)\\ =&m\{f(x)+\widetilde{u}_n+\widetilde{W}_n\}\exp\left(-M_n+Z_n+y_n+o_P(m(\widehat{F}(x;h_2)-1))\right)\\ =&m\exp(-M_n)\{f(x)+\widetilde{u}_n+\widetilde{W}_n\}\left(1+Z_n+y_n+O_P(Z_n^2)+o_P(m(\widehat{F}(x;h_2)-1))+O(y_n^2)\right).\end{aligned}$$

Since $\widetilde{u}_n = o(1)$ and $\widetilde{W}_n = o_P(1)$ as seen from the above, $\widehat{f}_{(m)}(x;h_1,h_2)$ is asymptotically given by

$$m\exp(-M_n)\{f(x)+\widetilde{u}_n+\widetilde{W}_n+f(x)(Z_n+y_n)\}.$$

As a consequence $f_{(m)}(x)-\widehat{f}_{(m)}(x;h_1,h_2)$ is asymptotically equivalent in distribution to $m\exp(-M_n)\{-\widetilde{u}_n-\widetilde{W}_n\}+m\exp(-M_n)\ f(x)(-Z_n-y_n)$. This is the sum of i.i.d. sequence, hence the asymptotic bias is found to be

$$\frac{h_1^2}{2}\exp(-M_n)M_n^{1+3\gamma}m^{-3\gamma}\psi_n\int z^2w(z)\mathrm{d}z+\frac{h_2^2}{2}\exp(-M_n)M_nm\xi_nf(x)\int z^2w(z)\mathrm{d}z.$$

The variance of $\widetilde{W}_n+f(x)Z_n$ is asymptotically given by

$$\begin{aligned}&\frac{1}{n}[\frac{f(x)}{h_1}\int w^2(z)\mathrm{d}z+\{f'(x)\int zw^2(z)\mathrm{d}z-f^2(x)\}]+\frac{2}{n}f(x)\{f(x)\int W(z)w(z)\mathrm{d}z-F(x)f(x)\}\\ &+\frac{1}{n}M_n^2f^2(x)\{M_n^{-1}m-2h_2\omega_n\int zW(z)w(z)\mathrm{d}z\}.\end{aligned}$$

Combining these results with the asymptotic normality of Z_n and $\widetilde{W}_n$, Theorem 3.1 for the Hall class is proved.

Proof of Theorem 3.1 for the Weibull class

$$\begin{aligned}f(x)-&\kappa Cx^{\kappa-1}\exp(-Cx^{\kappa})\to 0,\\ f'(x)-&\kappa Cx^{\kappa-2}(-\kappa Cx^{\kappa}+\kappa-1)\exp(-Cx^{\kappa})\to 0\\ f''(x)-&\kappa Cx^{\kappa-3}(\kappa^2C^2x^{2\kappa}-\kappa C(3\kappa-3)x^{\kappa}+(\kappa-1)(\kappa-2))\exp(-Cx^{\kappa})\to 0\end{aligned}$$

as $x \to \infty$. There exist some $c_0 > 0$ such that

$$\left|\mathbb{E}[\widehat{F}(x;h_2)] - F(x) - h_2^2 \frac{f'(x)}{2} \int z^2 w(z)\mathrm{d}z\right| < c_0 h_2^3 \exp(-Cx^\kappa)$$

and

$$\left|\mathbb{E}[\widehat{f}(x;h_1)] - f(x) - h_1^2 \frac{f''(x)}{2} \int z^2 w(z)\mathrm{d}z\right| < c_0 h_1^3 \exp(-Cx^\kappa).$$

For $h_2 x^{\kappa-1} \to 0$, the asymptotic variances are given by

$$\begin{aligned}\mathbb{V}[\widehat{F}(x;h_2)] &\sim n^{-1}\exp(-Cx^\kappa)\{1 - 2\kappa C h_2 x^{\kappa-1}\int zW(z)w(z)\mathrm{d}z\},\\ \mathbb{V}[\widehat{f}(x;h_1)] &\sim (nh_1)^{-1}\kappa C x^{\kappa-1}\exp(-Cx^\kappa)\int w^2(z)\mathrm{d}z.\end{aligned}$$

Set $Z_n := m(\widehat{F}(x;h_2) - \mathbb{E}[\widehat{F}(x;h_2)])$, $y_n := m(\mathbb{E}[\widehat{F}(x;h_2)] - F(x))$, $\widetilde{W}_n := \widehat{f}(x;h_1) - \mathbb{E}[\widehat{f}(x;h_1)]$, and $\widetilde{u}_n := (\mathbb{E}[\widehat{f}(x;h_1)] - f(x))$. It follows from

$$\widehat{f}_{(m)}(x;h_1,h_2) = m\{f(x) + \widetilde{u}_n + \widetilde{W}_n\}\exp\left(-M_n + Z_n + y_n + o_P(m(\widehat{F}(x) - 1))\right)$$

that $f_{(m)}(x) - \widehat{f}_{(m)}(x;h_1,h_2)$ is asymptotically equivalent in distribution to

$$m\exp(-M_n)\{-\widetilde{u}_n - \widetilde{W}_n\} + m\exp(-M_n)\kappa C x^{\kappa-1}\exp(-Cx^\kappa)(-Z_n - y_n).$$

Combining these results with the asymptotic normality of Z_n and $\widetilde{W}_n$, Theorem 3.1 for the Weibull class is proved.

Proof of Theorem 3.1 for the bounded class

$F(x - h_2 z)$ asymptotically equals

$$1 - \left\{(x^* - x)^{-\mu} - \mu(x^* - x)^{-\mu-1}h_2 z + \frac{\mu(\mu+1)}{2}(x^* - x)^{-\mu-2}(h_2 z)^2\right\}$$
$$\left[\ D + E\left\{(x^* - x)^{\mu\sigma} + \mu\sigma(x^* - x)^{\mu\sigma-1}h_2 z + \frac{\mu\sigma(\mu\sigma-1)}{2}(x^* - x)^{\mu\sigma-2}(h_2 z)^2\right\}\ \right]$$

for $h_2(x^* - x)^{-1} \to 0$ and asymptotically equals

$$1 - \left\{(h_2 z)^{-\mu} - \mu(h_2 z)^{-\mu-1}(x^* - x) + \frac{\mu(\mu+1)}{2}(h_2 z)^{-\mu-2}(x^* - x)^2\right\}$$
$$\left[\ D + E\left\{(h_2 z)^{\mu\sigma} + \mu\sigma(h_2 z)^{\mu\sigma-1}(x^* - x) + \frac{\mu\sigma(\mu\sigma-1)}{2}(h_2 z)^{\mu\sigma-2}(x^* - x)^2\right\}\ \right]$$

for $h_2(x^* - x)^{-1} \to \infty$. $h_2(x^* - x)^{-1} \to 0$ means $\mathrm{supp}(w) \subset [h_2^{-1}(x - x^*), h_2^{-1}(x - x_*)]$ for a large enough n, where x_* is the lower endpoint of f. $h_2(x^* - x)^{-1} \to \infty$ means $(x^* - x)^{-\mu} = o(h_2^{-\mu})$. Therefore, $\mathbb{E}[\widehat{F}(x;h_2)] - F(x)$ is asymptotically

$$\begin{cases} -\dfrac{D\mu(\mu+1)}{2}(x^* - x)^{-\mu-2}h_2^2\displaystyle\int z^2 w(z)\mathrm{d}z & \text{for} \quad h_2(x^* - x)^{-1} \to 0 \\ -Dh_2^{-\mu}\displaystyle\int_0^\infty z^{-\mu}w(z)\mathrm{d}z & \text{for} \quad h_2(x^* - x)^{-1} \to \infty, \end{cases}$$

where x_* is the lower endpoint of f if the support of w is bounded. Similarly, $n\mathbb{V}[\widehat{F}(x;h_2)]$ is asymptotically

$$\begin{cases} D(x^*-x)^{-\mu}\left(1-2\mu(x^*-x)^{-1}h_2\int W(z)w(z)z\mathrm{d}z\right) & \text{for} \quad h_2(x^*-x)^{-1}\to 0 \\ 2Dh_2^{-\mu}\int_0^\infty z^{-\mu}w(z)\{1-W(z)\}\mathrm{d}z \quad \text{for} \quad h_2(x^*-x)^{-1}\to\infty. \end{cases}$$

It holds that $\mathbb{E}[\widehat{f}(x;h_1)]-f(x)$ is asymptotically

$$\begin{cases} -\frac{D\mu(\mu+1)(\mu+2)}{2}(x^*-x)^{-\mu-3}h_1^2\int z^2w(z)\mathrm{d}z, \quad h_1(x^*-x)^{-1}\to 0 \\ -D\mu h_1^{-\mu-1}\int_0^\infty z^{-\mu-1}w(z)\mathrm{d}z \quad h_1(x^*-x)^{-1}\to\infty \end{cases}$$

and

$$n\mathbb{V}[\widehat{f}(x)]\sim\begin{cases} h_1^{-1}f(x)\int w^2(z)\mathrm{d}z \quad h_1(x^*-x)^{-1}\to 0 \\ -D\mu h_1^{-\mu-2}\int_0^\infty z^{-\mu-1}w^2(z)\mathrm{d}z \quad h_1(x^*-x)^{-1}\to\infty. \end{cases}$$

Proof of Theorem 4.1

$$\begin{aligned} &\mathbb{E}[\bar{f}_{(m)}(x)]-f_{(m)}(x)=\int w\left(z\right)\{f_{(m)}(x-hz)-f_{(m)}(x)\}\mathrm{d}z \\ \sim&\frac{mh^2}{2}\left[f''(x)F^2(x)+f'(x)F(x)\{f(x)+(2m-3)\}+(m-1)(m-2)f(x)\right] \\ &F^{m-3}(x)\int z^2w\left(z\right)\mathrm{d}z. \end{aligned}$$

It follows for the Hall class from

$$f(x)\sim A\alpha x^{-\alpha-1},\ f'(x)\sim-\frac{A\alpha(\alpha+1)}{2}x^{-\alpha-2},\ f''(x)\sim\frac{A\alpha(\alpha+1)(\alpha+2)}{2}x^{-\alpha-3}$$

that $(\mathbb{E}[\bar{f}_{(m)}(x)]-f_{(m)}(x))$ is asymptotically equivalent to

$$\frac{mh^2}{2}f(x)\left[\frac{(\alpha+1)(\alpha+2)}{2}x^{-2}-2m\frac{\alpha+1}{2}x^{-1}+m^2\right]\int z^2w\left(z\right)\mathrm{d}z.$$

Since for the Weibull class

$$\begin{aligned} f(x)\sim&\kappa Cx^{\kappa-1}\exp(-Cx^\kappa)\quad\left(<C((1-\kappa)/C)^\theta\exp(\kappa-1)\right) \\ f'(x)\sim&\kappa Cx^{\kappa-2}(-\kappa Cx^\kappa+\kappa-1)\exp(-Cx^\kappa) \\ f''(x)\sim&\kappa Cx^{\kappa-3}(\kappa^2C^2x^{2\kappa}-\kappa C(3\kappa-3)x^\kappa+(\kappa-1)(\kappa-2))\exp(-Cx^\kappa), \end{aligned}$$

$$\mathbb{E}[\bar{f}_{(m)}(x)]-f_{(m)}(x)\sim\frac{mh^2}{2}f(x)\left[(\kappa C)^2x^{2(\kappa-1)}-2m\kappa Cx^{\kappa-1}+m^2\right]\int z^2w\left(z\right)\mathrm{d}z.$$

It follows from

$$\bar{n}\mathbb{V}[\bar{f}(x)] = h^{-1}\mathbb{E}\left[w^2\left(\frac{x-Y_i}{h}\right)\right] - \left(\mathbb{E}\left[w\left(\frac{x-Y_i}{h}\right)\right]\right)^2 \sim h^{-1}f_{(m)}(x)\int w^2(z)\,\mathrm{d}z$$

and the asymptotic normality of the kernel density estimator that Theorem 4.1 holds for the Hall class and the Weibull class. Theorem 4.1 for the bounded class for the case of $h_2(x^*-x)^{-1} \to 0$ is proved in the same manner. For the case of $h_2(x^*-x)^{-1} \to \infty$, $(\mathbb{E}[\bar{f}_{(m)}(x)] - f_{(m)}(x))$ is asymptotically equivalent to

$$-D\mu m h_1^{-\mu-1}\int_0^\infty z^{-\mu-1}w(z)\mathrm{d}z,$$

and

$$\bar{n}\mathbb{V}[\bar{f}(x)] \sim -D\mu m h_1^{-\mu-2}\int_0^\infty z^{-\mu-1}w^2(z)\mathrm{d}z \quad h_1(x^*-x)^{-1} \to \infty.$$

Proof of Theorem 6.1

From Charpentier and Flachaire (2015) we see

$$\begin{aligned}\mathbb{E}[\widehat{f}^{\ln}(x;h_1)] - f(x) \sim& \frac{h_2^2}{2}(\int z^2 w(z)\mathrm{d}z)\{f(x)+3xf'(x)+x^2f''(x)\}\\ n\mathbb{V}[\widehat{f}^{\ln}(x;h_1)] \sim& h_1^{-1}x^{-1}f(x)\int w^2(z)\mathrm{d}z.\end{aligned}$$

In the same manner, we have

$$\begin{aligned}\mathbb{E}[\widehat{F}^{\ln}(x;h_2)] - F(x) \sim& \frac{h_2^2}{2}(\int z^2 w(z)\mathrm{d}z)\{xf(x)+x^2f'(x)\}\\ n\mathbb{V}[\widehat{F}^{\ln}(x;h_2)] \sim& 1-F(x)+2h_2xf(x)\int zW(z)w(z)\mathrm{d}z.\end{aligned}$$

Set $Z_n^{\ln} := m(\widehat{F}^{\ln}(x;h_2) - \mathbb{E}[\widehat{F}^{\ln}(x;h_2)])$, $y_n^{\ln} := m(\mathbb{E}[\widehat{F}^{\ln}(x;h_2)] - F(x))$, $\widetilde{W}_n^{\ln} := \widehat{f}^{\ln}(x;h_1) - \mathbb{E}[\widehat{f}^{\ln}(x;h_1)]$, and $\widetilde{u}_n^{\ln} := (\mathbb{E}[\widehat{f}^{\ln}(x;h_1)] - f(x))$. It follows from the proof of Theorem 3.1 that $f_{(m)}(x) - \widehat{f}_{(m)}^{\ln}(x;h_1,h_2)$ is asymptotically equivalent in distribution to $m\exp(-M_n)\{-\widetilde{u}_n^{\ln} - \widetilde{W}_n^{\ln}\} + m\exp(-M_n)\, f(x)(-Z_n^{\ln} - y_n^{\ln})$. Hence, the asymptotic bias of $\widehat{f}_{(m)}^{\ln}(x;h_1,h_2)$ is given by

$$\frac{h_1^2}{2}\exp(-M_n)M_n^{1+3\gamma}m^{-3\gamma}x^2\psi_n'\int z^2w(z)\mathrm{d}z + \frac{h_2^2}{2}\exp(-M_n)M_nm\xi_n'x^2f(x)\int z^2w(z)\mathrm{d}z.$$

The asymptotic variance is

$$\begin{aligned}&\frac{m^2}{n}\exp(-2M_n)\left[h_1^{-1}x^{-1}f(x)\int w^2(z)\mathrm{d}z + f'(x)\int zw^2(z)\mathrm{d}z - f^2(x)\right.\\ &\left.+2f(x)\{f(x)\int W(z)w(z)\mathrm{d}z - F(x)f(x)\} + M_n^2f^2(x)\{M_n^{-1}m - 2h_2x\omega_n\int zW(z)w(z)\mathrm{d}z\}\right].\end{aligned}$$

The asymptotic normality of $Z_n^{\ln}$ and $\widetilde{W}_n^{\ln}$ yields Theorem 6.1.

Proof of Theorem 6.2

From Charpentier and Flachaire (2015) we see

$$\mathbb{E}[\bar{f}^{\ln}_{(m)}(x)] - f_{(m)}(x) \sim \frac{h_2^2}{2}(\int z^2 w(z)\mathrm{d}z)\{f_{(m)}(x) + 3xf'_{(m)}(x) + x^2 f''_{(m)}(x)\}$$

$$\sim \frac{mh^2}{2} x \left[f''(x)F^2(x) + f'(x)F(x)\{f(x) + (2m-3)\} + (m-1)(m-2)f(x)\right]$$

$$F^{m-3}(x)\int z^2 w\left(z\right)\mathrm{d}z$$

$$\sim \frac{m^3h^2}{2} x^2 \exp(-M_n)f(x)\int z^2 w\left(z\right)\mathrm{d}z$$

It follows from

$$\bar{n}\mathbb{V}[\bar{f}^{\ln}(x)] \sim h^{-1}x^{-1}f_{(m)}(x)\int w^2\left(z\right)\mathrm{d}z$$

and the asymptotic normality of the kernel density estimator that Theorem 6.2 holds.