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# COMPARISON AND EQUALITY OF BAJRAKTAREVIĆ-TYPE $\psi$-ESTIMATORS

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Abstract:

- We solve the comparison problem for Bajraktarević-type $\psi$-estimators introduced by Barczy and Páles in 2022. Namely, we derive several necessary and sufficient conditions under which a Bajraktarević-type $\psi$-estimator is less than or equal to another Bajraktarević-type $\psi$-estimator for any sample. We also solve the corresponding equality problem. As an important particular case, we obtain the solutions of the two problems in question for quasi-arithmetic-type $\psi$-estimators.

Keywords:

- *$\psi$-estimator; $Z$-estimator; comparison of estimators; equality of estimators; Bajraktarević-type estimator; quasi-arithmetic-type estimator.*

AMS Subject Classification:

- 62F10, 62D99, 26E60.

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## 1. INTRODUCTION

The $\psi$-estimators (also called $Z$-estimators) have been playing an important role in statistics since the 1960's. In what follows, let $(X, \mathcal{X})$ be a measurable space, $\Theta$ be a Borel subset of $\mathbb{R}$, and $\psi : X \times \Theta \to \mathbb{R}$ be a function which is measurable in its second variable with respect to the sigma-algebra $\mathcal{X}$. Let $(\xi_n)_{n\geqslant 1}$ be a sequence of independent identically distributed (i.i.d.) $X$-valued random variables such that the distribution of $\xi_1$ depends on an unknown parameter $\vartheta \in \Theta$. For each $n \geqslant 1$, Huber [5, 6] among others introduced an important estimator of $\vartheta$ based on the observations $\xi_1, \dots, \xi_n$ as a solution $\widehat{\vartheta}_{n,\psi}(\xi_1, \dots, \xi_n)$ of the equation (with respect to the unknown parameter):

$$\sum_{i=1}^{n} \psi(\xi_i, t) = 0, \qquad t \in \Theta,$$

provided that such a solution exists. In the statistical literature, one calls the random variable $\widehat{\vartheta}_{n,\psi}(\xi_1, \dots, \xi_n)$ a $\psi$-estimator of the unknown parameter $\vartheta \in \Theta$ based on the i.i.d. observations $\xi_1, \dots, \xi_n$, while other authors call it a $Z$-estimator (the letter $Z$ refers to "zero"). In fact, $\psi$-estimators are special $M$-estimators (where the letter $M$ refers to "maximum likelihood-type") that were also introduced by Huber [5, 6]. For a detailed exposition of $M$-estimators and $\psi$-estimators, see, e.g., Kosorok [7, Sections 2.2.5 and 13] or van der Vaart [9, Section 5].

Results on the comparison and the equality of $\psi$-estimators are quite rare in the literature. These two problems can be formulated as follows: given $\psi, \varphi : X \times \Theta \to \mathbb{R}$ (with the properties described above), we are interested in finding necessary as well as sufficient conditions for the inequality $\widehat{\vartheta}_{n,\psi} \leqslant \widehat{\vartheta}_{n,\varphi}$ and for the equality $\widehat{\vartheta}_{n,\psi} = \widehat{\vartheta}_{n,\varphi}$ to be valid for all $n \geqslant 1$, respectively. In Barczy and Páles [2, Section 3], we derived such conditions in case of generalized $\psi$-estimators, introduced in Barczy and Páles [1] (see also Definition 1.2), which are generalizations of $\psi$-estimators recalled above. In general linear models, many authors investigated the equality of ordinary least squares estimator (OLSE) and best linear unbiased estimator (BLUE) of the regression parameters, for a detailed review of the literature, see Section 2 in our arXiv version [2].

In this paper, we study the comparison and equality problems for a subclass of generalized $\psi$-estimators, namely for Bajraktarević-type $\psi$-estimators introduced in Barczy and Páles [1] (see also (2.2)). The statistical applications of Bajraktarević-type $\psi$-estimators has not been explored yet, it can be a topic of a future research. Here we point out an important field in practice, where these types of estimators may be indeed useful. Mukhopadhyay et al. [8] found that in the presence of outliers in the data, more precisely, when the data are generated by a mixture population involving a major (dominating) component and a minor (outlying) component, the power mean (also called generalized mean) estimates the mean of the dominating population more accurately compared to the usual maximum likelihood estimator. Thus the class of power means offers an alternative way for estimating the target mean parameter without invoking the complications of sophisticated robust techniques. Power means are special Bajraktarević means, that can be considered as special Bajraktarević-type $\psi$-estimators. This can indicate some potential of Bajraktarević-type $\psi$-estimators as well in estimation of parameters for date coming from a mixture population.

By applying the general results from Section 3 of our arXiv version [2] (recalled below as well), we solve the two problems in question. As one will see, our results are not easy and not immediate consequences of the general theory developed in Section 3 of [2].

In Section 2, we recall the notion of Bajraktarević-type $\psi$-estimators and then we present the solution of the comparison problem for them (cf. Theorems 2.1 and 2.2). In Section 3, we characterize the equality of Bajraktarević-type $\psi$-estimators (cf. Theorems 3.1 and 3.2). We note that, surprisingly, in the heart of the proof of the equality problem for Bajraktarević-type estimators a result about Schwarzian derivative and rational functions come into play, see Lemma 3.1. We can also characterize the equality of quasiarithmetic-type $\psi$-estimators, see Corollary 3.1. In Proposition 3.1, we derive a necessary and sufficient condition in order that two strictly increasing functions defined on a nondegenerate open interval be the Möbius transforms of each other. The results of the present paper are those contained in Section 4 in our arXiv version [2].

In the rest of this section, we introduce the basic notations and concepts that are used throughout the paper, and we recall some results from Section 3 of our arXiv version [2] that we need to use through the proofs.

Throughout this paper, we fix the following notations: the symbols $\mathbb{N}$, $\mathbb{Z}_+$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{R}_{++}$, and $\mathbb{R}_{--}$ will stand for the sets of positive integers, non-negative integers, rational numbers, real numbers, non-negative real numbers, positive real numbers, and negative real numbers, respectively. For a subset $S \subseteq \mathbb{R}$, the convex hull of $S$ (which is the smallest interval containing $S$) is denoted by $\operatorname{conv}(S)$. A real interval will be called nondegenerate if it contains at least two distinct points. For each $n \in \mathbb{N}$, let us also introduce the set $\Lambda_n := \mathbb{R}_+^n \setminus \{(0, \dots, 0)\}$. All the random variables are defined on an appropriate probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

Throughout this paper, let $X$ be a nonempty set, $\Theta$ be a nondegenerate open interval of $\mathbb{R}$ and let $\Psi(X, \Theta)$ denote the class of real-valued functions $\psi : X \times \Theta \to \mathbb{R}$ such that, for all $x \in X$, there exist $t_+, t_- \in \Theta$ such that $t_+ < t_-$ and $\psi(x, t_+) > 0 > \psi(x, t_-)$. Roughly speaking, a function $\psi \in \Psi(X, \Theta)$ satisfies the following property: for all $x \in X$, the function $t \ni \Theta \mapsto \psi(x, t)$ changes sign on the interval $\Theta$ at least once.

**Definition 1.1.** For a function $f : \Theta \to \mathbb{R}$, consider the following three level sets

$$\Theta_{f>0} := \{t \in \Theta : f(t) > 0\},$$
$$\Theta_{f=0} := \{t \in \Theta : f(t) = 0\},$$
$$\Theta_{f<0} := \{t \in \Theta : f(t) < 0\}.$$

We say that $\vartheta \in \Theta$ is a *point of sign change (of decreasing-type) for $f$* if

$$f(t) > 0 \quad \text{for } t < \vartheta, \qquad \text{and} \qquad f(t) < 0 \quad \text{for } t > \vartheta.$$

Note that there can exist at most one element $\vartheta \in \Theta$ which is a point of sign change for $f$. Further, if $f$ is continuous at a point $\vartheta$ of sign change, then $f(\vartheta) = 0$.

**Definition 1.2.** We say that a function $\psi \in \Psi(X, \Theta)$

(i) *possesses the property $[C]$ (briefly, $\psi$ is a $C$-function) if it is continuous in its second variable, i.e., if, for all $x \in X$, the mapping $\Theta \ni t \mapsto \psi(x,t)$ is continuous.*

(ii) *possesses the property $[T_n]$ (briefly, $\psi$ is a $T_n$-function) for some $n \in \mathbb{N}$* if there exists a mapping $\vartheta_{n,\psi} : X^n \to \Theta$ such that, for all $\boldsymbol{x} = (x_1, \dots, x_n) \in X^n$ and $t \in \Theta$,

$$\psi_{\boldsymbol{x}}(t) = \sum_{i=1}^{n} \psi(x_i, t) \begin{cases} > 0 & \text{if } t < \vartheta_{n,\psi}(\boldsymbol{x}), \\ < 0 & \text{if } t > \vartheta_{n,\psi}(\boldsymbol{x}), \end{cases}$$

that is, for all $\boldsymbol{x} \in X^n$, the value $\vartheta_{n,\psi}(\boldsymbol{x})$ is a point of sign change for the function $\psi_{\boldsymbol{x}}$. If there is no confusion, instead of $\vartheta_{n,\psi}$ we simply write $\vartheta_n$. We may call $\vartheta_{n,\psi}(\boldsymbol{x})$ as a generalized $\psi$-estimator for some unknown parameter in $\Theta$ based on the realization $\boldsymbol{x} = (x_1, \dots, x_n) \in X^n$. If, for each $n \in \mathbb{N}$, $\psi$ is a $T_n$-function, then we say that $\psi$ *possesses the property $[T]$ (briefly, $\psi$ is a $T$-function).*

(iii) *possesses the property $[Z_n]$ (briefly, $\psi$ is a $Z_n$-function) for some $n \in \mathbb{N}$* if it is a $T_n$-function and

$$\psi_{\boldsymbol{x}}(\vartheta_{n,\psi}(\boldsymbol{x})) = \sum_{i=1}^{n} \psi(x_i, \vartheta_{n,\psi}(\boldsymbol{x})) = 0 \qquad \text{for all} \quad \boldsymbol{x} = (x_1, \dots, x_n) \in X^n.$$

If, for each $n \in \mathbb{N}$, $\psi$ is a $Z_n$-function, then we say that $\psi$ *possesses the property $[Z]$ (briefly, $\psi$ is a $Z$-function).*

(iv) *possesses the property $[T_n^{\boldsymbol{\lambda}}]$ for some $n \in \mathbb{N}$ and $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n) \in \Lambda_n$ (briefly, $\psi$ is a $T_n^{\boldsymbol{\lambda}}$-function)* if there exists a mapping $\vartheta_{n,\psi}^{\boldsymbol{\lambda}} : X^n \to \Theta$ such that, for all $\boldsymbol{x} = (x_1, \dots, x_n) \in X^n$ and $t \in \Theta$,

$$\psi_{\boldsymbol{x},\boldsymbol{\lambda}}(t) = \sum_{i=1}^{n} \lambda_i \psi(x_i, t) \begin{cases} > 0 & \text{if } t < \vartheta_{n,\psi}^{\boldsymbol{\lambda}}(\boldsymbol{x}), \\ < 0 & \text{if } t > \vartheta_{n,\psi}^{\boldsymbol{\lambda}}(\boldsymbol{x}), \end{cases}$$

that is, for all $\boldsymbol{x} \in X^n$, the value $\vartheta_{n,\psi}^{\boldsymbol{\lambda}}(\boldsymbol{x})$ is a point of sign change for the function $\psi_{\boldsymbol{x},\boldsymbol{\lambda}}$. If there is no confusion, instead of $\vartheta_{n,\psi}^{\boldsymbol{\lambda}}$ we simply write $\vartheta_n^{\boldsymbol{\lambda}}$. We may call $\vartheta_{n,\psi}^{\boldsymbol{\lambda}}(\boldsymbol{x})$ as a weighted generalized $\psi$-estimator for some unknown parameter in $\Theta$ based on the realization $\boldsymbol{x} = (x_1, \ldots, x_n) \in X^n$ and weights $(\lambda_1, \ldots, \lambda_n) \in \Lambda_n$.

It can be seen that if $\psi$ is continuous in its second variable and, for some $n \in \mathbb{N}$, it is a $T_n$-function, then it also a $Z_n$-function.

Given $q \in \mathbb{N}$ and properties

$$[P_1], \ldots, [P_q] \in \big\{[C], [T], [Z]\big\} \cup \big\{[T_n], [Z_n] \colon n \in \mathbb{N}\big\} \cup \big\{[T_n^{\boldsymbol{\lambda}}] \colon n \in \mathbb{N},\, \boldsymbol{\lambda} \in \Lambda_n\big\},$$

the subclass of $\Psi(X, \Theta)$ consisting of elements possessing the properties $[P_1], \ldots,$ $[P_q]$ will be denoted by $\Psi[P_1, \ldots, P_q](X, \Theta)$, i.e.,

$$\Psi[P_1, \ldots, P_q](X, \Theta) := \bigcap_{i=1}^{q} \Psi[P_i](X, \Theta).$$

For a function $\psi \in \Psi[T_1](X, \Theta)$, we introduce the notation

$$\Theta_\psi := \big\{t \in \Theta \mid \exists\, x, y \in X : \vartheta_{1,\psi}(x) < t < \vartheta_{1,\psi}(y)\big\}. \tag{1.1}$$

Then $\Theta_\psi$ is open, and it coincides with the interior of the convex hull of $\vartheta_{1,\psi}(X)$, see Section 3 of our arXiv version [2]. Consequently, $\Theta_\psi$ is an open (possibly degenerate) subinterval of $\Theta$.

Next, we recall a result about the comparison of generalized $\psi$-estimators due to Barczy and Páles [2, Theorem 3.1].

**Theorem 1.1.** *Let $\psi \in \Psi[T, Z_1](X, \Theta)$ and $\varphi \in \Psi[Z](X, \Theta)$. Then the following assertions are equivalent to each other:*

*(i)* *The inequality*

$$\vartheta_{n,\psi}(x_1, \ldots, x_n) \leqslant \vartheta_{n,\varphi}(x_1, \ldots, x_n) \tag{1.2}$$

*holds for each $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$.*

*(ii)* *The inequality*

$$\vartheta_{k+m,\psi}(\underbrace{x, \ldots, x}_{k}, \underbrace{y, \ldots, y}_{m}) \leqslant \vartheta_{k+m,\varphi}(\underbrace{x, \ldots, x}_{k}, \underbrace{y, \ldots, y}_{m})$$

*holds for each $k, m \in \mathbb{N}$ and $x, y \in X$.*

*(iii)* *For all $x \in X$, we have $\vartheta_{1,\psi}(x) \leqslant \vartheta_{1,\varphi}(x)$, and the inequality*

$$\psi(x, t)\varphi(y, t) \leqslant \psi(y, t)\varphi(x, t)$$

*is valid for all $t \in \Theta$ and for all $x, y \in X$ with $\vartheta_{1,\varphi}(x) < t < \vartheta_{1,\varphi}(y)$.*

(iv) *For all $x \in X$, we have $\vartheta_{1,\psi}(x) \leqslant \vartheta_{1,\varphi}(x)$, and there exists a nonnegative function $p : \Theta_\varphi \to \mathbb{R}_+$ such that*

$$\psi(z,t) \leqslant p(t)\varphi(z,t), \qquad z \in X,\, t \in \Theta_\varphi.$$

Under some additional regularity assumptions on $\psi$ and $\varphi$, Barczy and Páles [2, Theorem 3.4] derived another set of conditions that is equivalent to (1.2). We also recall this result below.

**Theorem 1.2.** *Let $\psi, \varphi \in \Psi[C,Z](X,\Theta)$. Assume that $\vartheta_{1,\psi} = \vartheta_{1,\varphi} =: \vartheta_1$ on $X$, $\vartheta_1(X) = \Theta$, and, for all $x \in X$, the maps*

$$\Theta \ni t \mapsto \psi(x,t) \qquad \text{and} \qquad \Theta \ni t \mapsto \varphi(x,t)$$

*are differentiable at $\vartheta_1(x)$ with a non-vanishing derivative. Then any of the equivalent assertions (i), (ii), (iii) and (iv) of Theorem 1.1 is equivalent to the following one:*

(v) *For all $x, y \in X$, we have*

$$-\frac{\psi(y,\vartheta_1(x))}{\partial_2\psi(x,\vartheta_1(x))} \leqslant -\frac{\varphi(y,\vartheta_1(x))}{\partial_2\varphi(x,\vartheta_1(x))}.$$

Finally, we recall a result about the equality of generalized $\psi$-estimators due to Barczy and Páles [2, Theorem 3.5].

**Theorem 1.3.** *Let $\psi \in \Psi[T,Z_1](X,\Theta)$ and $\varphi \in \Psi[Z](X,\Theta)$. Assume that $\vartheta_{1,\psi} = \vartheta_{1,\varphi}$ on $X$. Then $\Theta_\psi = \Theta_\varphi$ and the following assertions are equivalent:*

(i) *The equality*

$$\vartheta_{n,\psi}(x_1,\ldots,x_n) = \vartheta_{n,\varphi}(x_1,\ldots,x_n)$$

*holds for each $n \in \mathbb{N}$ and $x_1,\ldots,x_n \in X$.*

(ii) *The equality*

$$\vartheta_{k+m,\psi}(\underbrace{x,\ldots,x}_{k},\underbrace{y,\ldots,y}_{m}) = \vartheta_{k+m,\varphi}(\underbrace{x,\ldots,x}_{k},\underbrace{y,\ldots,y}_{m})$$

*holds for each $k, m \in \mathbb{N}$ and $x, y \in X$.*

(iii) *There exists a positive function $p : \Theta_\varphi \to (0,\infty)$ such that*

$$\psi(z,t) = p(t)\varphi(z,t), \qquad z \in X,\, t \in \Theta_\varphi.$$

## 2. COMPARISON OF BAJRAKTAREVIĆ-TYPE $\psi$-ESTIMATORS

In this section we apply Theorems 1.1 and 1.2 for solving the comparison problem of Bajraktarević-type estimators that are special generalized $\psi$-estimators.

First, we recall the notions of Bajraktarević-type functions and then Bajraktarević-type estimators. Let $f : \Theta \to \mathbb{R}$ be strictly increasing, $p : X \to \mathbb{R}_{++}$ and $F : X \to \operatorname{conv}(f(\Theta))$. In terms of these functions, define $\psi : X \times \Theta \to \mathbb{R}$ by

$$\psi(x,t) := p(x)(F(x) - f(t)), \qquad x \in X,\ t \in \Theta. \tag{2.1}$$

By Lemma 1 in Grünwald and Páles [3], there exists a uniquely determined monotone function $g : \operatorname{conv}(f(\Theta)) \to \Theta$ such that $g$ is the left inverse of $f$, i.e.,

$$(g \circ f)(t) = t, \qquad t \in \Theta.$$

Furthermore, $g$ is monotone in the same sense as $f$ (i.e, $f$ is monotone increasing), is continuous, and the following relation holds:

$$(f \circ g)(y) = y, \qquad y \in f(\Theta).$$

The function $g : \operatorname{conv}(f(\Theta)) \to \Theta$ is called the *generalized left inverse of the strictly increasing function* $f : \Theta \to \mathbb{R}$ and is denoted by $f^{(-1)}$. It is clear that the restriction of $f^{(-1)}$ to $f(\Theta)$ is the inverse of $f$ in the standard sense, which is also strictly increasing. Therefore, $f^{(-1)}$ is the continuous and monotone extension of the inverse of $f$ to the smallest interval containing the range of $f$, that is, to the convex hull of $f(\Theta)$.

Recall also that, by Proposition 2.19 in Barczy and Páles [1], under the above assumptions, $\psi$ is a $T_n^{\boldsymbol{\lambda}}$-function for each $n \in \mathbb{N}$ and $\boldsymbol{\lambda} \in \Lambda_n$, and

$$\vartheta_{n,\psi}^{\boldsymbol{\lambda}}(\boldsymbol{x}) = f^{(-1)}\left(\frac{\lambda_1 p(x_1)F(x_1) + \cdots + \lambda_n p(x_n)F(x_n)}{\lambda_1 p(x_1) + \cdots + \lambda_n p(x_n)}\right) \tag{2.2}$$

for each $n \in \mathbb{N}$, $\boldsymbol{x} = (x_1, \ldots, x_n) \in X^n$ and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \Lambda_n$. In particular, $\vartheta_{1,\psi} = f^{(-1)} \circ F$ holds. The value $\vartheta_{n,\psi}^{\boldsymbol{\lambda}}(\boldsymbol{x})$ given by (2.2) can be called as a Bajraktarević-type $\psi$-estimator of some unknown parameter in $\Theta$ based on the realization $\boldsymbol{x} = (x_1, \ldots, x_n) \in X^n$ and weights $(\lambda_1, \ldots, \lambda_n) \in \Lambda_n$ corresponding to the Bajraktarević-type function given by (2.1). In particular, if $p = 1$ is a constant function in (2.1), then we speak about a quasi-arithmetic-type $\psi$-estimator.

As a first result, we give a necessary and sufficient condition in order that $\Theta_\psi = \emptyset$ hold, where $\Theta_\psi$ is given by (1.1).

**Lemma 2.1.** *Let $f : \Theta \to \mathbb{R}$ be a strictly increasing function, $p : X \to \mathbb{R}_{++}$, $F : X \to \operatorname{conv}(f(\Theta))$, and define $\psi : X \times \Theta \to \mathbb{R}$ by (2.1). Then $\Theta_\psi = \emptyset$*

*holds if and only if there exists $t_0 \in \Theta$ such that the range $F(X)$ of $F$ is contained in $[f(t_0-0), f(t_0+0)]$, where $f(t_0-0)$ and $f(t_0+0)$ denote the left and right hand limits of $f$ at $t_0$, respectively.*

**Proof:** First, let us suppose that there exists $t_0 \in \Theta$ such that $F(X) \subseteq J_f(t_0) := [f(t_0-0), f(t_0+0)]$. Then, using that $f$ is strictly increasing, for all $x \in X$ and $t', t'' \in \Theta$ with $t' < t_0 < t''$, we have that $f(t') < f(t_0-0) \leqslant F(x) \leqslant f(t_0+0) < f(t'')$, and therefore, taking into account that $p$ is positive, for all $x \in X$, we get

$$p(x)(F(x)-f(t)) \begin{cases} > 0 & \text{if } t < t_0,\ t \in \Theta, \\ < 0 & \text{if } t > t_0,\ t \in \Theta. \end{cases}$$

Hence, $\vartheta_{1,\psi}(x) = t_0$ for all $x \in X$. This yields that $\Theta_\psi = \emptyset$.

To prove the converse statement, we check that if there does not exist $t_0 \in \Theta$ such that $F(X) \subseteq J_f(t_0)$, then $\Theta_\psi \neq \emptyset$. Since $f$ is strictly increasing, we have

$$\operatorname{conv}(f(\Theta)) = \bigcup_{t\in\Theta} J_f(t),$$

and $J_f(t') \cap J_f(t'') = \emptyset$ for all $t', t'' \in \Theta$ with $t' \neq t''$. Using also that $F(X) \subseteq \operatorname{conv}(f(\Theta))$, there exist $t_1, t_2 \in \Theta$ with $t_1 < t_2$ such that $F(X) \cap J_f(t_i) \neq \emptyset$, $i = 1, 2$. Hence there exist $x_1, x_2 \in X$ such that $F(x_i) \in J_f(t_i)$, $i = 1, 2$. Consequently, similarly as before, we have

$$p(x_i)(F(x_i)-f(t)) \begin{cases} > 0 & \text{if } t < t_i,\ t \in \Theta, \\ < 0 & \text{if } t > t_i,\ t \in \Theta, \end{cases} \qquad i = 1, 2,$$

yielding that $\vartheta_{1,\psi}(x_i) = t_i$, $i = 1, 2$. Therefore, $(t_1, t_2) \subseteq \Theta_\psi$, showing that $\Theta_\psi$ is not empty, as expected. □

The next lemma is connected to Theorem 2.10 in Barczy and Páles [1], where we derived several implications between the property $[T]$ of a function $\Psi(X, \Theta)$ and the monotonicity properties of the map (2.3) defined in the next lemma.

**Lemma 2.2.** *Let $f : \Theta \to \mathbb{R}$ be a strictly increasing function, $p : X \to \mathbb{R}_{++}$, $F : X \to \operatorname{conv}(f(\Theta))$, and define $\psi : X \times \Theta \to \mathbb{R}$ by (2.1). Then, for all $x, y \in X$ with $\vartheta_{1,\psi}(x) < \vartheta_{1,\psi}(y)$, the map*

$$(\vartheta_{1,\psi}(x), \vartheta_{1,\psi}(y)) \ni u \mapsto -\frac{\psi(x,u)}{\psi(y,u)} \tag{2.3}$$

*is positive and strictly increasing.*

**Proof:** Let $x, y \in X$ with $\vartheta_{1,\psi}(x) < \vartheta_{1,\psi}(y)$ and let $u \in (\vartheta_{1,\psi}(x), \vartheta_{1,\psi}(y))$ be arbitrary. Then $\psi(x, u) < 0 < \psi(y, u)$, which proves that the map (2.3) is positive valued. To see the strict monotonicity property, let additionally $v \in (\vartheta_{1,\psi}(x), \vartheta_{1,\psi}(y))$ be arbitrary with $u < v$. Then $\psi(x,v) < 0 < \psi(y,v)$, which implies that $F(x) < f(v) < F(y)$. Thus $F(x) < F(y)$ and, by the strict monotonicity of $f$, we also have $f(u) < f(v)$. Therefore

$$(F(y) - F(x))(f(v) - f(u)) > 0,$$

which is equivalent to the inequality

$$(F(x) - f(u))(F(y) - f(v)) > (F(y) - f(u))(F(x) - f(v)).$$

Multiplying this inequality by $\frac{-p(x)}{p(y)(F(y)-f(u))(F(y)-f(v))} < 0$ side by side, it follows that

$$-\frac{\psi(x,u)}{\psi(y,u)} = -\frac{p(x)(F(x) - f(u))}{p(y)(F(y) - f(u))} < -\frac{p(x)(F(x) - f(v))}{p(y)(F(y) - f(v))} = -\frac{\psi(x,v)}{\psi(y,v)}.$$

This completes the proof of the strict increasingness of the map (2.3). We note that the statement also follows from the proof of Proposition 2.19 in Barczy and Páles [1]. □

In the following result, we describe sufficient conditions which imply that the function $\psi$ defined by (2.1) possesses the property $[Z_1]$ and $[Z]$, respectively.

**Lemma 2.3.** *Let $f : \Theta \to \mathbb{R}$ be a strictly increasing function, $p : X \to \mathbb{R}_{++}$, $F : X \to f(\Theta)$ and define $\psi : X \times \Theta \to \mathbb{R}$ by (2.1). Then $\psi$ has the property $[Z_1]$ and*

$$\Theta_\psi = \big\{t \in \Theta \,|\, \exists\, x, y \in X : F(x) < f(t) < F(y)\big\}. \tag{2.4}$$

*If, in addition, $\operatorname{conv}(F(X)) \subseteq f(\Theta)$, then $\psi$ has the property $[Z]$ as well.*

**Proof:** Since $F(X) \subseteq f(\Theta)$, the restriction of $f^{(-1)}$ onto $F(X)$ is the strictly increasing inverse of $f$ in the standard sense restricted to $F(X)$. Thus, for all $x \in X$, we have

$$\begin{aligned}\psi(x, \vartheta_{1,\psi}(x)) &= p(x)(F(x) - f(\vartheta_{1,\psi}(x))) = p(x)\big(F(x) - f(f^{(-1)}(F(x)))\big)\\ &= p(x)(F(x) - F(x)) = 0,\end{aligned}$$

yielding that $\psi$ has the property $[Z_1]$. Furthermore, using also that $(f^{(-1)} \circ f)(t) = t$, $t \in \Theta$, we get that

$$\begin{aligned}\Theta_\psi &= \big\{t \in \Theta \,|\, \exists\, x, y \in X : \vartheta_{1,\psi}(x) < t < \vartheta_{1,\psi}(y)\big\}\\ &= \big\{t \in \Theta \,|\, \exists\, x, y \in X : f^{(-1)}(F(x)) < t < f^{(-1)}(F(y))\big\}\\ &= \big\{t \in \Theta \,|\, \exists\, x, y \in X : (f \circ f^{(-1)})(F(x)) < f(t) < (f \circ f^{(-1)})(F(y))\big\}\\ &= \big\{t \in \Theta \,|\, \exists\, x, y \in X : F(x) < f(t) < F(y)\big\},\end{aligned}$$

as desired.

To prove the last assertion, let us assume that $\operatorname{conv}(F(X)) \subseteq f(\Theta)$. Then, by (2.2), for each $n \in \mathbb{N}$ and $\boldsymbol{x} = (x_1, \dots, x_n) \in X^n$, we have

$$\begin{aligned}
&\sum_{i=1}^{n} \psi(x_i, \vartheta_{n,\psi}(\boldsymbol{x})) \\
&\quad = \sum_{i=1}^{n} p(x_i)\big(F(x_i) - f(\vartheta_{n,\psi}(\boldsymbol{x}))\big) \\
&\quad = \sum_{i=1}^{n} p(x_i)F(x_i) - \sum_{i=1}^{n} p(x_i) f\left(f^{(-1)}\left(\sum_{j=1}^{n} \frac{p(x_j)}{p(x_1)+\cdots+p(x_n)} F(x_j)\right)\right) \\
&\quad = \sum_{i=1}^{n} p(x_i)F(x_i) - \sum_{i=1}^{n} p(x_i) \sum_{j=1}^{n} \frac{p(x_j)}{p(x_1)+\cdots+p(x_n)} F(x_j) = 0,
\end{aligned}$$

where we used that $\sum_{j=1}^{n} \frac{p(x_j)}{p(x_1)+\cdots+p(x_n)} F(x_j) \in \operatorname{conv}(F(X)) \subseteq f(\Theta)$. □

In the next remark, we point out the fact that (2.4) does not hold in general, showing that the assumption $F(X) \subseteq f(\Theta)$ in Lemma 2.3 is indispensable.

**Remark 2.1.** Let $X := \{x_1, x_2\}$, $\Theta := \mathbb{R}$, let $p : X \to \mathbb{R}_{++}$, $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(t) := \begin{cases} t & \text{if } t \leqslant 1, \\ t+1 & \text{if } 1 < t \leqslant 2, \\ t+2 & \text{if } t > 2, \end{cases}$$

and $F : X \to \mathbb{R}$ be such that $F(x_1) := 1$ and $F(x_2) := 3{,}5$. Then $\operatorname{conv}(f(\Theta)) = \mathbb{R}$, and

$$p(x_1)(F(x_1) - f(t)) \begin{cases} > 0 & \text{if } t < 1, \\ = 0 & \text{if } t = 1, \\ < 0 & \text{if } t > 1, \end{cases}$$

and

$$p(x_2)(F(x_2) - f(t)) \begin{cases} > 0 & \text{if } t \leqslant 2, \\ < 0 & \text{if } t > 2, \end{cases}$$

yielding that $\vartheta_{1,\psi}(x_j) = j$, $j = 1, 2$. Hence $\Theta_\psi = (1, 2)$. However,

$$\big\{t \in \Theta \,|\, \exists\, x, y \in X : F(x) < f(t) < F(y)\big\} = (1, 2],$$

which does not coincide with $\Theta_\psi = (1, 2)$. Note that $F(X) \subseteq f(\Theta)$ is also not valid, and $\psi$ does not have the property $[Z_1]$, since $\psi(x_2, \vartheta_{1,\psi}(x_2)) = \psi(x_2, 2) = p(x_2)(F(x_2) - f(2)) = \frac{p(x_2)}{2} > 0$. □

Below, we solve the comparison problem for Bajraktarević-type estimators.

**Theorem 2.1.** *Let $f, g : \Theta \to \mathbb{R}$ be strictly increasing functions, $p, q : X \to \mathbb{R}_{++}$, $F : X \to f(\Theta)$, $G : X \to g(\Theta)$, and suppose that $\operatorname{conv}(G(X)) \subseteq g(\Theta)$. Let $\psi : X \times \Theta \to \mathbb{R}$ and $\varphi : X \times \Theta \to \mathbb{R}$ be given by*

$$\begin{aligned} \psi(x,t) &:= p(x)(F(x) - f(t)), \qquad x \in X,\ t \in \Theta, \\ \varphi(x,t) &:= q(x)(G(x) - g(t)), \qquad x \in X,\ t \in \Theta. \end{aligned} \tag{2.5}$$

*Assume that $(f^{(-1)} \circ F)(x) \leqslant (g^{(-1)} \circ G)(x)$, $x \in X$. Then $\vartheta_{n,\psi}(\boldsymbol{x}) \leqslant \vartheta_{n,\varphi}(\boldsymbol{x})$ holds for each $n \in \mathbb{N}$ and $\boldsymbol{x} \in X^n$ if and only if the inequality*

$$\frac{q(y)}{q(x)} \cdot \frac{G(y) - g(t)}{G(x) - g(t)} \leqslant \frac{p(y)}{p(x)} \cdot \frac{F(y) - f(t)}{F(x) - f(t)} \tag{2.6}$$

*is valid for all $t \in \Theta$ and for all $x, y \in X$ with $G(x) < g(t) < G(y)$.*

**Proof:** By Proposition 2.19 in Barczy and Páles [1] and Lemma 2.3, $\psi$ has the properties $[T]$ and $[Z_1]$, and $\varphi$ has the property $[Z]$. Using Theorem 1.1, we have that $\vartheta_{n,\psi}(\boldsymbol{x}) \leqslant \vartheta_{n,\varphi}(\boldsymbol{x})$ holds for each $n \in \mathbb{N}$ and $\boldsymbol{x} \in X^n$ if and only if the inequality

$$\psi(x,t)\varphi(y,t) \leqslant \psi(y,t)\varphi(x,t)$$

is valid for all $t \in \Theta$ and for all $x, y \in X$ with $\vartheta_{1,\varphi}(x) < t < \vartheta_{1,\varphi}(y)$.

Using $G(X) \subseteq g(\Theta)$, Lemma 2.3 yields that

$$\Theta_\varphi = \big\{ t \in \Theta \,|\, \exists\, x, y \in X : G(x) < g(t) < G(y) \big\},$$

and for all $t \in \Theta$ and $x, y \in X$, the inequality $\vartheta_{1,\varphi}(x) < t < \vartheta_{1,\varphi}(y)$ holds if and only if $G(x) < g(t) < G(y)$. Consequently, $\vartheta_{n,\psi}(\boldsymbol{x}) \leqslant \vartheta_{n,\varphi}(\boldsymbol{x})$ holds for each $n \in \mathbb{N}$ and $\boldsymbol{x} \in X^n$ if and only if

$$p(x)(F(x) - f(t))q(y)(G(y) - g(t)) \leqslant p(y)(F(y) - f(t))q(x)(G(x) - g(t)) \tag{2.7}$$

is valid for all $t \in \Theta$ and for all $x, y \in X$ with $G(x) < g(t) < G(y)$. Using that $f$ is strictly increasing, $F(X) \subseteq f(\Theta)$, and $g^{(-1)}$ restricted to $g(\Theta)$ is strictly increasing, for all $t \in \Theta$ and for all $x \in X$ with $G(x) < g(t)$, we have that

$$\begin{aligned} F(x) &= (f \circ f^{(-1)})(F(x)) = f(f^{(-1)}(F(x))) \\ &\leqslant f(g^{(-1)}(G(x))) < f(g^{(-1)}(g(t))) = f(t). \end{aligned}$$

Consequently, $G(x) - g(t) < 0$ and $F(x) - f(t) < 0$ in the inequality (2.7), and hence by rearranging it, the assertion follows. □

In the next result, under some additional regularity assumptions on $f$ and $g$, we derive another set of conditions that is equivalent to (2.6).

**Theorem 2.2.** *Let $f, g : \Theta \to \mathbb{R}$ be strictly increasing functions, $p, q : X \to \mathbb{R}_{++}$, $F : X \to f(\Theta)$, $G : X \to g(\Theta)$, and suppose that $F(X) = f(\Theta)$ and $\operatorname{conv}(G(X)) \subseteq g(\Theta)$. Let $\psi : X \times \Theta \to \mathbb{R}$ and $\varphi : X \times \Theta \to \mathbb{R}$ be given by (2.5). Assume that $(f^{(-1)} \circ F)(x) = (g^{(-1)} \circ G)(x) =: \vartheta_1(x)$, $x \in X$, and that $f$ and $g$ are differentiable at $\vartheta_1(x)$ for all $x \in X$ with non-vanishing (and hence positive) derivatives. Then $\vartheta_{n,\psi}(\boldsymbol{x}) \leqslant \vartheta_{n,\varphi}(\boldsymbol{x})$ holds for each $n \in \mathbb{N}$ and $\boldsymbol{x} \in X^n$ if and only if the inequality*

$$\frac{p(y)}{p(x)} \cdot \frac{F(y) - F(x)}{f'\big(f^{(-1)}(F(x))\big)} \leqslant \frac{q(y)}{q(x)} \cdot \frac{G(y) - G(x)}{g'\big(g^{(-1)}(G(x))\big)}$$

*is valid for all $x, y \in X$.*

**Proof:** Since $F(X) = f(\Theta)$, we have that

$$\vartheta_1(X) = (f^{(-1)} \circ F)(X) = f^{(-1)}(F(X)) = f^{(-1)}(f(\Theta)) = \Theta.$$

Further, for all $x \in X$, we have

$$\partial_2\psi(x, \vartheta_1(x)) = -p(x)f'(\vartheta_1(x)) = -p(x)f'\big(f^{(-1)}(F(x))\big),$$
$$\partial_2\varphi(x, \vartheta_1(x)) = -q(x)g'(\vartheta_1(x)) = -q(x)g'\big(g^{(-1)}(G(x))\big).$$

Consequently, using Theorem 1.2, we have that $\vartheta_{n,\psi}(\boldsymbol{x}) \leqslant \vartheta_{n,\varphi}(\boldsymbol{x})$ holds for each $n \in \mathbb{N}$ and $\boldsymbol{x} \in X^n$ if and only if

$$\frac{p(y)}{p(x)} \cdot \frac{F(y) - f(\vartheta_1(x))}{f'(\vartheta_1(x))} \leqslant \frac{q(y)}{q(x)} \cdot \frac{G(y) - g(\vartheta_1(x))}{g'(\vartheta_1(x))}$$

is valid for all $x, y \in X$, which yields the statement, since

$$f(\vartheta_1(x)) = f(f^{(-1)}(F(x))) = (f \circ f^{(-1)})(F(x)) = F(x), \qquad x \in X,$$

due to the condition $F(X) \subseteq f(\Theta)$, and similarly, we also have $g(\vartheta_1(x)) = G(x)$, $x \in X$. □

In the next result, among others, we point out that, under the assumptions of Theorem 2.2, the function $g$ is continuous.

**Lemma 2.4.** *Let $f, g : \Theta \to \mathbb{R}$ be strictly increasing functions, $F : X \to f(\Theta)$, $G : X \to g(\Theta)$, and suppose that $F(X) = f(\Theta)$ and $\operatorname{conv}(G(X)) \subseteq g(\Theta)$. Assume that $(f^{(-1)} \circ F)(x) = (g^{(-1)} \circ G)(x)$, $x \in X$. Then*

*(i)* $G(X) = g(\Theta)$ *and this set is convex.*

*(ii)* $g$ *is continuous, and* $G(x) = (g \circ f^{(-1)})(F(x))$, $x \in X$.

**Proof:** (i). Since $F(X) = f(\Theta)$, we have that

$$(f^{(-1)} \circ F)(X) = f^{(-1)}(F(X)) = f^{(-1)}(f(\Theta)) = \Theta,$$

which implies that $(g^{(-1)} \circ G)(X) = (f^{(-1)} \circ F)(X) = \Theta$ holds as well. Hence, using that $G(X) \subseteq g(\Theta)$, we have that

$$g(\Theta) = g((g^{(-1)} \circ G)(X)) = (g \circ g^{(-1)})(G(X)) = G(X).$$

Consequently, $\operatorname{conv}(g(\Theta)) = \operatorname{conv}(G(X))$, and, since $\operatorname{conv}(G(X)) \subseteq g(\Theta)$, we obtain that $g(\Theta) \subseteq \operatorname{conv}(g(\Theta)) = \operatorname{conv}(G(X)) \subseteq g(\Theta)$, yielding that $g(\Theta) = \operatorname{conv}(G(X))$ is a convex set.

(ii). Since $g$ is strictly increasing and its range $g(\Theta)$ is convex (see part (i)), we can see that $g$ is continuous as well. Finally, note that the equality $(f^{(-1)} \circ F)(x) = (g^{(-1)} \circ G)(x)$, $x \in X$, implies that $G(x) = (g \circ f^{(-1)})(F(x))$, $x \in X$, since $G(X) \subseteq g(\Theta)$. □

## 3. EQUALITY OF BAJRAKTAREVIĆ-TYPE $\psi$-ESTIMATORS

In this section, we apply Theorem 1.3 for solving the equality problem for Bajraktarević-type estimators. In the proof, we will use a result on the Schwarzian derivative of a function. Given a nondegenerate open interval $I \subseteq \mathbb{R}$, for a three times differentiable function $h : I \to \mathbb{R}$ with a nonvanishing first derivative, its Schwarzian derivative $S_h : I \to \mathbb{R}$ is defined by

$$S_h(x) := \frac{h'''(x)}{h'(x)} - \frac{3}{2}\left(\frac{h''(x)}{h'(x)}\right)^2, \qquad x \in I.$$

The following result is well-known, see, e.g., Grünwald and Páles [4, Corollary 3].

**Lemma 3.1.** *Let $I \subseteq \mathbb{R}$ be a nondegenerate open interval, and $h : I \to \mathbb{R}$ be a three times differentiable function such that $h'$ does not vanish on $I$. Then $S_h(x) = 0$, $x \in I$, holds if and only if there exist four constants $a, b, c, d \in \mathbb{R}$ with $ad \neq bc$ and $0 \notin cI + d$ such that*

$$h(x) = \frac{ax + b}{cx + d}, \qquad x \in I.$$

In the proof of the subsequent theorems, the following auxiliary result plays an important role.

**Lemma 3.2.** *Let $I$ be a nondegenerate open interval of $\mathbb{R}$. Let $f, g : I \to \mathbb{R}$ be strictly increasing functions such that there exist four constants $a, b, c, d \in \mathbb{R}$ with $0 \notin cf(I) + d$ and*

$$g(t) = \frac{af(t) + b}{cf(t) + d}, \qquad t \in I.$$

*Then $ad > bc$ and $cf + d$ is either everywhere positive or everywhere negative on $I$.*

**Proof:** The condition $0 \notin cf(I)+d$ yields that $(c,d) \neq (0,0)$. If $ad = bc$, then there exists $\lambda \in \mathbb{R}$ such that $(a,b) = \lambda(c,d)$. In this case, we get that $g(t) = \lambda$ for all $t \in I$, which contradicts the strict monotonicity of $g$. Thus $ad \neq bc$ must hold.

Next, we check that $cf + d$ is either everywhere positive or everywhere negative on $I$. On the contrary, if $cf + d$ changes sign in $I$, then $c$ cannot be zero and hence $cf + d$ is also strictly monotone. Therefore, using also that $I$ is a nondegenerate open interval, there exists a unique point $\tau \in I$ such that $cf(t)+d > (<)\, 0$ for all $t < \tau$, $t \in I$, and $cf(t)+d < (>)\, 0$ for all $t > \tau$, $t \in I$. Let $t < \tau < r < s$ be arbitrarily fixed elements of $I$. Then $(cf(t)+d)(cf(r)+d) < 0$ and $(cf(r)+d)(cf(s)+d) > 0$. Consequently, using the strict increasingness of $g$, the inequalities

$$\frac{af(t)+b}{cf(t)+d} = g(t) < g(r) = \frac{af(r)+b}{cf(r)+d}$$

and

$$\frac{af(r)+b}{cf(r)+d} = g(r) < g(s) = \frac{af(s)+b}{cf(s)+d}$$

imply that

$$\begin{aligned}(af(t)+b)(cf(r)+d) &> (af(r)+b)(cf(t)+d),\\ (af(r)+b)(cf(s)+d) &< (af(s)+b)(cf(r)+d),\end{aligned}$$

or equivalently,

$$0 > (ad-bc)(f(r)-f(t)) \qquad \text{and} \qquad 0 < (ad-bc)(f(s)-f(r)). \tag{3.1}$$

Since $f$ is also strictly increasing, we have that $f(t) < f(r) < f(s)$. Therefore, $(ad-bc)(f(r)-f(t))$ and $(ad-bc)(f(s)-f(r))$ should have the same signs. This together with the inequalities (3.1) lead us to a contradiction.

Finally, to show that $ad > bc$, let $r,s \in I$ with $r < s$ be fixed. Then, using that $cf + d$ does not change sign in $I$, we have $(cf(r)+d)(cf(s)+d) > 0$, and therefore the inequality $g(r) < g(s)$, in the same way as above, implies $0 < (ad-bc)(f(s)-f(r))$. This, in view of the strict increasingness of $f$ yields that $ad - bc > 0$. □

**Theorem 3.1.** *Let $f, g : \Theta \to \mathbb{R}$ be strictly increasing functions such that $f$ is continuous, $p, q : X \to \mathbb{R}_{++}$, $F : X \to f(\Theta)$, $G : X \to g(\Theta)$, and suppose that $F(X) = f(\Theta)$ and $\operatorname{conv}(G(X)) \subseteq g(\Theta)$. Let $\psi : X \times \Theta \to \mathbb{R}$ and $\varphi : X \times \Theta \to \mathbb{R}$ be given by (2.5). If $\vartheta_{n,\psi}(\boldsymbol{x}) = \vartheta_{n,\varphi}(\boldsymbol{x})$ holds for each $n \in \mathbb{N}$*

*and $\boldsymbol{x} \in X^n$, then there exist four constants $a, b, c, d \in \mathbb{R}$ with $ad \neq bc$ and $0 \notin cf(\Theta) + d$ such that*

$$
\begin{aligned}
g(t) &= \frac{af(t)+b}{cf(t)+d}, \qquad t \in \Theta, \\
G(x) &= \frac{aF(x)+b}{cF(x)+d}, \qquad x \in X, \\
q(x) &= (cF(x)+d)p(x), \qquad x \in X.
\end{aligned} \tag{3.2}
$$

**Proof:** Since $f$ is strictly increasing and continuous and $\Theta$ is a non-degenerate open interval, we have $f(\Theta)$ is a non-degenerate open interval. Hence $\operatorname{conv}(f(\Theta)) = f(\Theta) = F(X)$, and then, as a consequence of Lemma 2.3, we get $\psi \in \Psi[Z](X, \Theta)$. Further, since $\operatorname{conv}(G(X)) \subseteq g(\Theta)$, Lemma 2.3 also yields that $\varphi \in \Psi[Z](X, \Theta)$.

We first verify that $\Theta_\psi = \Theta$. The inclusion $\Theta_\psi \subseteq \Theta$ is trivial. To prove the reversed one, let $t \in \Theta$ be arbitrary. Now choose $r, s \in \Theta$ such that $r < t < s$. Using that $F(X) = f(\Theta)$, we can find $x, y \in X$ such that $f(r) = F(x)$ and $f(s) = F(y)$. Since $f$ is strictly increasing, it follows that $F(x) = f(r) < f(t) < f(s) = F(y)$, showing that $t$ belongs to the set

$$\big\{t \in \Theta \mid \exists\, x, y \in X : F(x) < f(t) < F(y)\big\},$$

which, according to (2.4), equals $\Theta_\psi$.

Assume that $\vartheta_{n,\psi}(\boldsymbol{x}) = \vartheta_{n,\varphi}(\boldsymbol{x})$ holds for each $n \in \mathbb{N}$ and $\boldsymbol{x} \in X^n$. Then, in the case $n = 1$, this equality and (2.2) imply that $(f^{(-1)} \circ F)(x) = \vartheta_{1,\psi}(x) = \vartheta_{1,\varphi}(x) = (g^{(-1)} \circ G)(x)$, $x \in X$. Hence, according to Lemma 2.4, we get that $G(X) = \operatorname{conv}(G(X)) = g(\Theta)$ is a convex set and $g$ is continuous. Then, similarly as we derived $\Theta_\psi = \Theta$, we have that $\Theta_\varphi = \Theta$ holds as well.

For all $x, y \in X$ with $G(x) < G(y)$, let us introduce the notation

$$\Theta_{x,y} := \{t \in \Theta \mid G(x) < g(t) < G(y)\}.$$

Using that $F(X) \subseteq f(\Theta)$, $G(X) \subseteq g(\Theta)$, and that the restrictions of $f^{(-1)}$ and $g^{(-1)}$ to $f(\Theta)$ and $g(\Theta)$ are the strictly increasing inverses of $f$ and $g$ in the standard sense, respectively, for all $x, y \in X$ with $G(x) < G(y)$, we have

$$
\begin{aligned}
\Theta_{x,y} &= \{t \in \Theta \mid g^{(-1)}(G(x)) < t < g^{(-1)}(G(y))\} \\
&= \{t \in \Theta \mid f^{(-1)}(F(x)) < t < f^{(-1)}(F(y))\} \\
&= \{t \in \Theta \mid F(x) < f(t) < F(y)\}.
\end{aligned} \tag{3.3}
$$

The previous argument also shows that for all $x, y \in X$, we get $G(x) < G(y)$ if and only if $F(x) < F(y)$, and the set $\Theta_{x,y}$ is a nonempty open interval for all $x, y \in X$ with $G(x) < G(y)$, since it is the intersection of the open intervals $(g^{(-1)}(G(x)), g^{(-1)}(G(y)))$ and $\Theta$.

In view of (3.3), Lemma 2.3 and the equality $\Theta_\psi = \Theta$ we can see that

$$
\begin{aligned}
\Theta_\varphi &= \bigcup_{\{x,y\in X : G(x)<G(y)\}} \Theta_{x,y} \\
&= \bigcup_{\{x,y\in X : F(x)<F(y)\}} \{t\in\Theta \mid F(x)<f(t)<F(y)\} = \Theta_\psi = \Theta.
\end{aligned}
\tag{3.4}
$$

Using that the image under $f$ of a union of subsets is the union of the images under $f$ of the given subsets, (3.4) immediately yields that

$$
\bigcup_{\{x,y\in X : G(x)<G(y)\}} f(\Theta_{x,y}) = f(\Theta), \tag{3.5}
$$

which is an open interval, since $\Theta$ is a nonempty open interval and $f$ is strictly increasing and continuous.

Using Theorem 2.1, we have that

$$
\frac{q(y)}{q(x)} \cdot \frac{G(y)-g(t)}{G(x)-g(t)} = \frac{p(y)}{p(x)} \cdot \frac{F(y)-f(t)}{F(x)-f(t)} \tag{3.6}
$$

holds for all $t\in\Theta$ and for all $x,y\in X$ with $G(x)<g(t)<G(y)$, or equivalently, (3.6) holds for all $x,y\in X$ with $G(x)<G(y)$ and for all $t\in\Theta_{x,y}$.

One can readily check that for all $x,y\in X$ with $G(x)<G(y)$ and for all $t\in\Theta_{x,y}$, the equality (3.6) is equivalent to any of the following three equalities:

$$
\begin{aligned}
&p(x)(F(x)-f(t))q(y)(G(y)-g(t)) = p(y)(F(y)-f(t))q(x)(G(x)-g(t)),\\
&\Big(p(y)(F(y)-f(t))q(x) - p(x)(F(x)-f(t))q(y)\Big)g(t)\\
&\qquad = p(y)(F(y)-f(t))q(x)G(x) - p(x)(F(x)-f(t))q(y)G(y),\\
&(c_{x,y}f(t)+d_{x,y})g(t) = a_{x,y}f(t)+b_{x,y},
\end{aligned}
\tag{3.7}
$$

where

$$
\begin{aligned}
a_{x,y} &:= p(x)q(y)G(y) - p(y)q(x)G(x),\\
b_{x,y} &:= p(y)q(x)F(y)G(x) - p(x)q(y)F(x)G(y),\\
c_{x,y} &:= p(x)q(y) - p(y)q(x),\\
d_{x,y} &:= p(y)q(x)F(y) - p(x)q(y)F(x).
\end{aligned}
$$

Here, due to $G(x)\neq G(y)$, we have that $(a_{x,y},c_{x,y})\neq(0,0)$ and $(b_{x,y},d_{x,y})\neq(0,0)$ hold. Substituting $s:=f(t)$ (i.e., $t=f^{(-1)}(s)$) in the third equality in (3.7), it follows that

$$
(c_{x,y}s+d_{x,y})(g\circ f^{(-1)})(s) = a_{x,y}s+b_{x,y} \tag{3.8}
$$

for all $x,y\in X$ with $G(x)<G(y)$ and for all $s\in f(\Theta_{x,y})$.

Next, we check that $c_{x,y}s + d_{x,y} \neq 0$ for all $x, y \in X$ with $G(x) < G(y)$ and for all $s \in f(\Theta_{x,y})$. If $c_{x,y}s + d_{x,y} = 0$ and $c_{x,y} = 0$ were true, then $d_{x,y} = 0$, $a_{x,y} \neq 0$, $b_{x,y} \neq 0$ and $a_{x,y}s + b_{x,y} = 0$ (following from (3.8)). This leads us to a contradiction, since $c_{x,y} = d_{x,y} = 0$ implies that $p(x)q(y) = p(y)q(x)$ and $F(x) = F(y)$, which cannot happen due to $F(x) < F(y)$. If $c_{x,y}s + d_{x,y} = 0$ and $c_{x,y} \neq 0$ were true, then $s = -\frac{d_{x,y}}{c_{x,y}}$ and $a_{x,y}s + b_{x,y} = 0$, yielding that $c_{x,y}b_{x,y} - d_{x,y}a_{x,y} = 0$. This leads us to a contradiction, since an easy calculation shows that

$$c_{x,y}b_{x,y} - d_{x,y}a_{x,y} = p(x)p(y)q(x)q(y)(F(x) - F(y))(G(y) - G(x)), \tag{3.9}$$

which cannot be 0 for any $x, y \in X$ with $G(x) < G(y)$.

Consequently,

$$(g \circ f^{(-1)})(s) = \frac{a_{x,y}s + b_{x,y}}{c_{x,y}s + d_{x,y}} \tag{3.10}$$

for all $x, y \in X$ with $G(x) < G(y)$ and for all $s \in f(\Theta_{x,y})$.

We can apply Lemma 3.1 to the function $h := g \circ f^{(-1)}$ defined on the nondegenerate open interval $I := f(\Theta)$, since (3.10) implies that $h$ is three times differentiable on $I$ such that $h'$ does not vanish on $I$. Indeed, using (3.10), we have that

$$h'(s) = \frac{a_{x,y}d_{x,y} - b_{x,y}c_{x,y}}{(c_{x,y}s + d_{x,y})^2} \neq 0, \qquad s \in f(\Theta_{x,y})$$

for all $x, y \in X$ with $G(x) < G(y)$, where we used (3.9). Hence, as a consequence of (3.5), we have $h'(s) \neq 0$, $s \in f(\Theta)$. Taking into account that $f(\Theta_{x,y})$ is a nondegenerate open interval for all $x, y \in X$ with $G(x) < G(y)$, Lemma 3.1 and (3.10) imply that $S_h(s) = 0$, $s \in f(\Theta)$. Consequently, using again Lemma 3.1, there exist four constants $a^*, b^*, c^*, d^* \in \mathbb{R}$ with $a^*d^* \neq b^*c^*$ and $0 \notin c^*f(\Theta) + d^*$ such that

$$h(s) = (g \circ f^{(-1)})(s) = \frac{a^*s + b^*}{c^*s + d^*}, \qquad s \in f(\Theta). \tag{3.11}$$

By substituting $s := f(t)$, where $t \in \Theta$, it follows that

$$g(t) = \frac{a^*f(t) + b^*}{c^*f(t) + d^*}, \qquad t \in \Theta,$$

as desired. Using (3.11), the assumptions $f^{(-1)} \circ F = g^{(-1)} \circ G$ and $G(X) \subseteq g(\Theta)$, we get that

$$G(x) = g((f^{(-1)} \circ F)(x)) = (g \circ f^{(-1)})(F(x)) = \frac{a^*F(x) + b^*}{c^*F(x) + d^*}, \qquad x \in X,$$

where $0 \notin c^*F(X) + d^*$, since $F(X) = f(\Theta)$ and $0 \notin c^*f(\Theta) + d^*$.

By (3.6) and taking into account the forms of $G$ and $g$, we get

$$\frac{q(y)}{q(x)}\cdot\frac{\dfrac{a^*F(y)+b^*}{c^*F(y)+d^*}-\dfrac{a^*f(t)+b^*}{c^*f(t)+d^*}}{\dfrac{a^*F(x)+b^*}{c^*F(x)+d^*}-\dfrac{a^*f(t)+b^*}{c^*f(t)+d^*}}=\frac{p(y)}{p(x)}\cdot\frac{F(y)-f(t)}{F(x)-f(t)}$$

holds for all $x,y\in X$ with $G(x)<G(y)$ and for all $t\in\Theta_{x,y}$. Using that $a^*d^*-b^*c^*\neq 0$, after some algebraic calculations, we obtain that

$$\frac{q(y)}{p(y)}=\frac{c^*F(y)+d^*}{c^*F(x)+d^*}\cdot\frac{q(x)}{p(x)}$$

holds for all $x,y\in X$ with $G(x)<G(y)$, or equivalently,

$$\frac{\left(\frac{q}{p}\right)(y)}{c^*F(y)+d^*}=\frac{\left(\frac{q}{p}\right)(x)}{c^*F(x)+d^*}$$

holds for all $x,y\in X$ with $G(x)<G(y)$. Since $q/p$ is positive, it follows that there exists a constant $k\in\mathbb{R}\setminus\{0\}$ such that

$$q(x)=k(c^*F(x)+d^*)p(x),\qquad x\in X.$$

The statement of the proposition now holds with the choices $a:=ka^*$, $b:=kb^*$, $c:=kc^*$ and $d:=kd^*$. □

We note that in the proof of Theorem 3.1, the assumption that $f$ is continuous is used for deriving that $f(\Theta)$ is an open interval, which is essential when we apply Lemma 3.1. Note also that in the proof of Theorem 3.1 it turned out that $g$ is continuous as well, however, we did not utilize this property in the proof. Nonetheless, the continuity of $g$ also follows from the result itself, since $f$ is continuous and $g(t)=(af(t)+b)/(cf(t)+d)$, $t\in\Theta$.

Next, we will provide a set of sufficient conditions on $f$, $g$, $F$ and $G$ in order that $\vartheta_{n,\psi}(\boldsymbol{x})=\vartheta_{n,\varphi}(\boldsymbol{x})$ hold for each $n\in\mathbb{N}$ and $\boldsymbol{x}\in X^n$.

**Theorem 3.2.** *Let $f,g:\Theta\to\mathbb{R}$ be strictly increasing functions, $p,q: X\to\mathbb{R}_{++}$, $F:X\to\operatorname{conv}(f(\Theta))$, and $G:X\to\operatorname{conv}(g(\Theta))$. Let $\psi:X\times\Theta\to\mathbb{R}$ and $\varphi:X\times\Theta\to\mathbb{R}$ be given by (2.5). If there exist four constants $a,b,c,d\in\mathbb{R}$ with $0\notin cf(\Theta)+d$ such that (3.2) holds, then $\vartheta_{n,\psi}(\boldsymbol{x})=\vartheta_{n,\varphi}(\boldsymbol{x})$ holds for each $n\in\mathbb{N}$ and $\boldsymbol{x}\in X^n$.*

**Proof:** Since $p$ and $q$ are strictly positive functions, as a consequence of the equality $q=(cF+d)p$, we get that $cF+d$ is positive on $X$. Further, for all $x\in X$ and $t\in\Theta$, we obtain

$$\begin{aligned}\varphi(x,t)&=q(x)(G(x)-g(t))=(cF(x)+d)p(x)\left(\frac{aF(x)+b}{cF(x)+d}-\frac{af(t)+b}{cf(t)+d}\right)\\&=p(x)\frac{(ad-bc)(F(x)-f(t))}{cf(t)+d}=\frac{ad-bc}{cf(t)+d}\psi(x,t).\end{aligned}\tag{3.12}$$

Using Lemma 3.2, we have that $ad > bc$, and $cf + d$ is either everywhere positive or everywhere negative on $\Theta$. We show that the latter property cannot hold. To the contrary, assume that $cf + d$ is everywhere negative on $\Theta$, i.e., $cf(\Theta) + d \subseteq \mathbb{R}_{--}$. Then $c \cdot \operatorname{conv}(f(\Theta)) + d = \operatorname{conv}(cf(\Theta) + d) \subseteq \mathbb{R}_{--}$. Using that $F(X) \subseteq \operatorname{conv}(f(\Theta))$, this implies that

$$c \cdot F(X) + d \subseteq c \cdot \operatorname{conv}(f(\Theta)) + d \subseteq \mathbb{R}_{--},$$

which contradicts the positivity of $cF + d$ on $X$. Consequently, $cf + d$ must be everywhere positive on $\Theta$.

To prove the equality $\vartheta_{n,\psi} = \vartheta_{n,\varphi}$ on $X^n$, let $n \in \mathbb{N}$ and $\boldsymbol{x} = (x_1, \dots, x_n) \in X^n$ be arbitrary. Then, by (3.12), we have

$$\sum_{i=1}^{n} \varphi(x_i, t) = \frac{ad - bc}{cf(t) + d} \sum_{i=1}^{n} \psi(x_i, t), \qquad t \in \Theta.$$

Since $(ad - bc)/(cf + d)$ is positive everywhere on $\Theta$. This implies that

$$\operatorname{sign}\left(\sum_{i=1}^{n} \varphi(x_i, t)\right) = \operatorname{sign}\left(\sum_{i=1}^{n} \psi(x_i, t)\right), \qquad t \in \Theta.$$

Hence the unique points of sign change of the functions

$$\Theta \ni t \mapsto \sum_{i=1}^{n} \varphi(x_i, t) \qquad \text{and} \qquad \Theta \ni t \mapsto \sum_{i=1}^{n} \psi(x_i, t)$$

are equal to each other, which implies the equality $\vartheta_{n,\psi}(\boldsymbol{x}) = \vartheta_{n,\varphi}(\boldsymbol{x})$, as desired. □

Next, we give an equivalent form of the first equality in (3.2). Roughly speaking, we derive a necessary and sufficient condition in order that two strictly increasing functions defined on a nondegenerate open interval be the Möbius transforms of each other.

**Proposition 3.1.** *Let $I$ be a nondegenerate open interval of $\mathbb{R}$. Let $f, g : I \to \mathbb{R}$ be strictly increasing functions. The following two statements are equivalent:*

*(i)* *There exist four constants $a, b, c, d \in \mathbb{R}$ with $0 \notin cf(I) + d$ and*

$$g(t) = \frac{af(t) + b}{cf(t) + d}, \qquad t \in I. \tag{3.13}$$

*(ii)* *For all $t_1, t_2, t_3, t_4 \in I$, we have*

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ f(t_1) & f(t_2) & f(t_3) & f(t_4) \\ g(t_1) & g(t_2) & g(t_3) & g(t_4) \\ f(t_1)g(t_1) & f(t_2)g(t_2) & f(t_3)g(t_3) & f(t_4)g(t_4) \end{vmatrix} = 0. \tag{3.14}$$

**Proof:** (i)$\Longrightarrow$(ii). Let us suppose that there exist four real constants $a, b, c, d \in \mathbb{R}$ such that $0 \notin cf(I) + d$ and (3.13) hold. By Lemma 3.2, we have $ad > bc$. Further, $(cf(t) + d)g(t) = af(t) + b$, $t \in I$, yielding that

$$1 \cdot b + f(t) \cdot a + g(t) \cdot (-d) + f(t)g(t) \cdot (-c) = 0, \qquad t \in I.$$

In particular, for all $t_1, t_2, t_3, t_4 \in I$, we have that

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ f(t_1) & f(t_2) & f(t_3) & f(t_4) \\ g(t_1) & g(t_2) & g(t_3) & g(t_4) \\ f(t_1)g(t_1) & f(t_2)g(t_2) & f(t_3)g(t_3) & f(t_4)g(t_4) \end{bmatrix}^\top \cdot \begin{bmatrix} b \\ a \\ -d \\ -c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

As a consequence of the inequality $ad > bc$, we have that $(b, a, -d, -c) \neq (0, 0, 0, 0)$, which shows that $(b, a, -d, -c)$ is a nontrivial solution to the above homogeneous system of linear equations. Hence we obtain that (3.14) must hold for all $t_1, t_2, t_3, t_4 \in I$.

(ii)$\Longrightarrow$(i). Let $t_3 < t_4$ be fixed elements of $I$. By the strict monotonicity of $f$, the vectors $(1, f(t_3))$ and $(1, f(t_4))$ are linearly independent. Assume first that, for all $t \in I$,

$$\begin{vmatrix} 1 & 1 & 1 \\ f(t) & f(t_3) & f(t_4) \\ g(t) & g(t_3) & g(t_4) \end{vmatrix} = 0 \tag{3.15}$$

holds. Expanding the determinant along its first column, for all $t \in I$, we get

$$b + af(t) - dg(t) = 0$$

where

$$a := -\begin{vmatrix} 1 & 1 \\ g(t_3) & g(t_4) \end{vmatrix}, \qquad b := \begin{vmatrix} f(t_3) & f(t_4) \\ g(t_3) & g(t_4) \end{vmatrix}, \qquad d := -\begin{vmatrix} 1 & 1 \\ f(t_3) & f(t_4) \end{vmatrix} \neq 0.$$

Therefore, (3.13) holds with $c = 0$ and we also have that $0 \notin cf(I) + d = \{d\}$.

Now we consider the case when (3.15) is not valid for all $t \in I$, that is, there exists $t_2 \in I$ such that (3.15) does not hold for $t = t_2$. Then, by (3.14), for all $t \in I$,

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ f(t) & f(t_2) & f(t_3) & f(t_4) \\ g(t) & g(t_2) & g(t_3) & g(t_4) \\ f(t)g(t) & f(t_2)g(t_2) & f(t_3)g(t_3) & f(t_4)g(t_4) \end{vmatrix} = 0.$$

Expanding the determinant along its first column, for all $t \in I$, we get

$$b + af(t) - dg(t) - cf(t)g(t) = 0, \tag{3.16}$$

where

$$a := -\begin{vmatrix} 1 & 1 & 1 \\ g(t_2) & g(t_3) & g(t_4) \\ f(t_2)g(t_2) & f(t_3)g(t_3) & f(t_4)g(t_4) \end{vmatrix},$$

$$b := \begin{vmatrix} f(t_2) & f(t_3) & f(t_4) \\ g(t_2) & g(t_3) & g(t_4) \\ f(t_2)g(t_2) & f(t_3)g(t_3) & f(t_4)g(t_4) \end{vmatrix},$$

$$c := \begin{vmatrix} 1 & 1 & 1 \\ f(t_2) & f(t_3) & f(t_4) \\ g(t_2) & g(t_3) & g(t_4) \end{vmatrix} \neq 0,$$

$$d := -\begin{vmatrix} 1 & 1 & 1 \\ f(t_2) & f(t_3) & f(t_4) \\ f(t_2)g(t_2) & f(t_3)g(t_3) & f(t_4)g(t_4) \end{vmatrix}.$$

Since $c \neq 0$, we have that $cf + d$ is strictly monotone. We now prove that $cf + d$ does not vanish on $I$. Assume, on the contrary, that for some $t_1 \in I$, we have that $cf(t_1) + d = 0$. Then, $cf(t) + d \neq 0$ for $t \in I \setminus \{t_0\}$, and, by (3.16), we get that $af(t_1) + b = 0$. This implies that $ad = bc$. Therefore, applying (3.16) for $t \in I \setminus \{t_1\}$, we obtain

$$g(t) = \frac{af(t) + b}{cf(t) + d} = \frac{af(t) + \frac{ad}{c}}{cf(t) + d} = \frac{a}{c},$$

which contradicts the strict monotonicity of $g$. □

As a consequence of Theorem 3.1, we can characterize the equality of quasiarithmetic-type $\psi$-estimators.

**Corollary 3.1.** *Let $f, g : \Theta \to \mathbb{R}$ be strictly increasing functions, $F : X \to \operatorname{conv}(f(\Theta))$, and $G : X \to \operatorname{conv}(g(\Theta))$. Let $\psi : X \times \Theta \to \mathbb{R}$ and $\varphi : X \times \Theta \to \mathbb{R}$ be given by*

$$\psi(x,t) := F(x) - f(t), \qquad \varphi(x,t) := G(x) - g(t), \qquad x \in X,\ t \in \Theta.$$

*The following two assertions hold:*

*(i)* *If there exist two constants $a, b \in \mathbb{R}$ with $a \neq 0$ such that*

$$(3.17)\quad g(t) = af(t) + b, \quad t \in \Theta, \qquad \text{and} \qquad G(x) = aF(x) + b, \quad x \in X,$$

*then $\vartheta_{n,\psi}(\boldsymbol{x}) = \vartheta_{n,\varphi}(\boldsymbol{x})$ holds for each $n \in \mathbb{N}$ and $\boldsymbol{x} \in X^n$.*

*(ii)* *In addition, suppose that $f$ is continuous, $F(X) = f(\Theta)$ and $\operatorname{conv}(G(X)) \subseteq g(\Theta)$. If $\vartheta_{n,\psi}(\boldsymbol{x}) = \vartheta_{n,\varphi}(\boldsymbol{x})$ holds for each $n \in \mathbb{N}$ and $\boldsymbol{x} \in X^n$, then there exist two constants $a, b \in \mathbb{R}$ with $a \neq 0$ such that (3.17) holds.*

**Proof:** (i). Let us assume that there exist two constants $a, b \in \mathbb{R}$ with $a \neq 0$ such that (3.17) holds. By choosing $c := 0$, $d := 1$ and $p(x) := q(x) := 1$, $x \in X$, we have $cf(\Theta) + d = 1$, and hence $0 \notin cf(\Theta) + d$. Further, (3.2) is satisfied as well. Consequently, Theorem 3.2 yields that $\vartheta_{n,\psi}(\boldsymbol{x}) = \vartheta_{n,\varphi}(\boldsymbol{x})$ for each $n \in \mathbb{N}$ and $\boldsymbol{x} \in X^n$.

(ii). One can apply Theorem 3.1 with the given functions $f, g, F$ and $G$ and by choosing $p(x) := q(x) := 1$, $x \in X$. Then we obtain that there exist four constants $a, b, c, d \in \mathbb{R}$ with $ad \neq bc$ and $0 \notin cf(\Theta) + d$ such that (3.2) holds, i.e.,

$$\begin{aligned} g(t) &= \frac{af(t) + b}{cf(t) + d}, \qquad t \in \Theta, \\ G(x) &= \frac{aF(x) + b}{cF(x) + d}, \qquad x \in X, \\ q(x) &= (cF(x) + d)p(x), \qquad x \in X. \end{aligned}$$

Since $p = q = 1$, we get $cF(x) + d = 1$, $x \in X$, and hence $G(x) = aF(x) + b$, $x \in X$. Consequently, in order to prove the statement, it is enough to verify that $cf(t) + d = 1$, $t \in \Theta$. We check that $c = 0$ and hence $d = 1$. Since $\Theta$ is a nondegenerate open interval of $\mathbb{R}$ and $f$ is strictly increasing and continuous, we have that $f(\Theta)$ is a nondegenerate interval of $\mathbb{R}$. Hence, using that $F(X) = f(\Theta)$, the range $F(X)$ of $F$ contains at least two distinct elements, and consequently, there exist $x_1, x_2 \in X$ such that $F(x_1) \neq F(x_2)$. Since $cF(x_1) + d = 1$ and $cF(x_2) + d = 1$, we have $c(F(x_1) - F(x_2)) = 0$, yielding that $c = 0$, as desired. □

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