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On Progressively Type-II Censored Inverse Lomax Distribution: Characterizations, Estimation and Application to Cancer Data

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Abstract:

- This paper discusses some additional distributional characteristics of inverse Lomax distribution and explores the classical and Bayesian estimation procedures of its survival characteristics. Moreover, the estimation techniques are developed under the progressive type-II censoring scheme. In classical setup, maximum likelihood estimators for the distribution parameters and the considered survival characteristics are obtained. The Bayesian estimation techniques have been discussed with informative priors under a generalized asymmetric loss function. Further, different interval estimators such as asymptotic confidence interval, bootstrap confidence interval, and the Bayes credible interval have also been constructed for the parameters and survival characteristics of the inverse Lomax distribution under progressive type-II censoring. A Monte Carlo simulation has been carried out to evaluate the performances of the point and interval estimators of the considered survival characteristics. To illustrate the practical applicability of the proposed procedures, two data sets pertaining to cancer patients have been explored.

Keywords:

- *Bayes estimation; different characterization; inverse Lomax distribution; survival properties.*

AMS Subject Classification:

- 60E05, 62F10, 62F15.

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1. INTRODUCTION

In survival analysis, the estimation of survival characteristics plays an important role in gaining deeper insights about the observed time-to-event data. Analyzing such data using appropriate probability models are of special interest for several researchers. However, choosing a suitable lifetime model out of thousands is a mammoth task, but can be tackled skillfully if one decides to select the models based on the hazard nature of the data. Apart from the constant hazard rate, there exist monotone and non-monotone hazard rate which can occur naturally in many survival studies. The Weibull, gamma, generalized exponential distribution, etc. are usually the primary choices to analyze any survival data with monotone hazard rate. But in many cancer or tumor recurrence-related studies, the hazard associated with the events of interest can initially increase, and then can gradually decrease after a certain time point (see Boucher and Kerber, 2001, for instance). Such types of hazard pattern is known as hump type or non-monotone hazard rate and arises naturally in many demographic or pharmaceutical studies where the initial failures are very high, reaching a peak and then decline slowly. Child mortality data can be taken as another example. The inverse exponential, inverse Weibull, inverse Gaussian, generalized inverse exponential distributions, etc. can be chosen as suitable models for these types of situations.

The inverse Lomax distribution (ILD) was initially explored by Kleiber and Kotz (2003) in the context of stochastic modelling in economics and actuarial science. Later, it was studied by Rahman et al. (2013) for estimation and prediction purposes through a Bayesian approach. Having both decreasing and non-monotone hazard rates, ILD can be used to model lifetime data that exhibits a non-monotone hazard nature. Classical and the Bayesian estimation procedures for the parameters and reliability functions of ILD using type-II censored observations have been discussed by Singh et al. (2016). Yadav et al. (2016) explored the parametric estimation procedure for ILD parameters under type-I and type-II hybrid censoring setup. Kumar et al. (2021) proposed the transmuted version of ILD and discussed its distributional properties. The probability density function (PDF) and cumulative distribution function (CDF) of ILD are given as

$$f(x;\zeta,\vartheta)=\frac{\zeta\vartheta}{x^2}\left(1+\frac{\vartheta}{x}\right)^{-(1+\zeta)} \quad |_{x,\,\zeta,\,\vartheta\in\mathcal{R}^+} \tag{1.1}$$

and

$$F(x;\zeta,\vartheta)=\left(1+\frac{\vartheta}{x}\right)^{-\zeta} \quad |_{x,\,\zeta,\,\vartheta\in\mathcal{R}^+}, \tag{1.2}$$

respectively. Recently, Yadav et al. (2019) proposed the Bayesian estimation of stress-strength reliability parameter for ILD using observations under a progressive type-II censoring (PCS-II) setup. Analyzing survival data under a progressive censoring strategy has garnered significant attention over a few decades because as opposed to usual type-I and type-II censoring, it offers a convenient flexibility of removing experimental subjects during the experiment. Progressive censoring was properly discussed by Cohen (1963) to deal with more sophisticated life testing experiments. PCS-II can be described as follows; suppose n units are subjected to a certain life test and the experimenter needs to obtain only m failures. At the time of first failure $X_{1:m:n}$, R_1 of the surviving units are randomly removed from the remaining $(n-1)$ units and the experiment continues further. When the second failure $X_{2:m:n}$ is observed, R_2 of the surviving units are randomly removed from the remaining $(n-2-R_1)$ units. The experiment continues in a similar fashion until the m^{th} failure $X_{m:m:n}$ is observed, and all the remaining R_m surviving units are removed from the experiment. Such a sample of size m is known as progressive type-II censored sample with censoring scheme $(R_1, R_2, ..., R_m)$ provided $n = m + \sum_{i=1}^{m} R_i$. If all $R_i; i = 1, 2, ..., m-1$ are zero and $R_m = n-m$, then the above censoring scheme reduces to conventional type-II censoring scheme and the case of complete sample is obtained by considering all $R_i; i = 1, 2, ..., m$ equal to zero. The theoretical aspects along with the applications of the PCS-II in life testing experiments are discussed by several statisticians(see Cohen, 1963; Balakrishnan and Aggarwala, 2000; Balakrishnan, 2007, for details). Various parametric inferential procedures for different lifetime models under PCS-II are explored by several researchers, viz., Kundu (2008), Pradhan and Kundu (2009), Rastogi and Tripathi (2014), Singh et al. (2015a), Singh et al. (2015b), Dey et al. (2016), Kayal et al. (2017), Maurya et al. (2017) etc. For recent accounts of the estimation of model parameters for different probability distributions under PCS-II, the readers may go through Lodhi et al. (2021), Maurya et al. (2019), Yadav et al. (2019), and Yadav et al. (2022).

The main objective of the present article is twofold. First; to derive some additional survival and distributional properties not considered by Rahman et al. (2013), and second; to address the classical and the Bayes estimation techniques of the survival and hazard function of ILD under PCS-II. Further, the asymptotic confidence interval (ACI) and different bootstrap confidence intervals (BCIs), namely, standard bootstrap (s-boot), percentile bootstrap (p-boot), and student-t bootstrap (t-boot), and the Bayes credible intervals are constructed for the survival and hazard function for the same design of censoring parameters. A Monte Carlo simulation study has been performed to investigate the performances of the obtained estimators and corresponding intervals in terms of mean squared errors (MSEs) and width of the intervals, respectively. To the best of our knowledge, no attempt has been made to estimate the survival characteristics for ILD under PCS-II till now. Hence, the present article has been molded to fill up this research gap.

The sectional bifurcation of the study is as follows; the introduction of the proposed study and model is given in Section 1. Section 2 describes some distributional properties of ILD. The likelihood function under PCS-II and maximum likelihood estimator (MLE) are discussed in Section 3. Section 4 is devoted to the estimation of survival characteristics. Different bootstrap confidence intervals (s-boot, p-boot, t-boot) under PCS-II are obtained in Section 5. Section 6 describes the Bayes estimation procedure along with the Bayes computation under symmetric and asymmetric loss functions using informative gamma prior. Comparative study among the proposed estimators is performed by conducting Monte Carlo simulation in Section 7. The practical application of the study has been illustrated through two medical data sets in Section 8. Finally, the concluding remarks of the paper are presented in Section 9.

2. DIFFERENT CHARACTERIZATIONS OF ILD

In this section, various survival and distributional properties of ILD such as survival and hazard rate, residual life function, moments and inverse moment based characterizations etc have been discussed, and important expressions related to them are derived.

2.1. Survival and hazard functions

The survival and hazard functions are the two vital characteristics of any lifetime distribution. Hazard rate (better known as instantaneous failure rate) is the dynamic speed with which a system or component fails, expressed in failures per unit of time, whereas the survival function gives the probability that an object of interest will survive beyond any specified time. So if the phenomenon under consideration resembles the characteristics of ILD then the survival and hazard of ILD for $t > 0$ are expressed as

$$S(t;\zeta,\vartheta) = 1 - F(t;\zeta,\vartheta) = 1 - \left(1 + \frac{\vartheta}{t}\right)^{-\zeta} \tag{2.1}$$

and

$$h(t;\zeta,\vartheta) = \frac{f(t;\zeta,\vartheta)}{S(t;\zeta,\vartheta)} = \frac{\frac{\zeta\vartheta}{t^2}(1+\frac{\vartheta}{t})^{-(\zeta+1)}}{1-(1+\frac{\vartheta}{t})^{-\zeta}}, \tag{2.2}$$

respectively.

2.2. Aging intensity

Aging is an interesting phenomenon and an inherent property of a system(may be living or non-living) in life tests. It is obtained by using the hazard rate and is a measure of deterioration of a unit (system) over time. It plays a vital role in depicting the failure pattern of an unit. Jiang et al. (2003) defined aging intensity (AI) function as a quantitative measure of aging. Larger value of the AI function indicates increasing tendency of aging of the random variable under consideration. It is interesting to note that for a given failure rate there exists a unique AI function but not vice versa. Using Equation (2.1) and (2.2), the AI function for ILD is

$$L_x(t;\zeta,\vartheta) = \frac{-tf(t;\zeta,\vartheta)}{S(t;\zeta,\vartheta)\log S(t;\zeta,\vartheta)} = \frac{\frac{-(\zeta\vartheta)}{t}(1+\frac{\vartheta}{t})^{-(\zeta+1)}}{1-\left(1+\frac{\vartheta}{t}\right)^{-\zeta}\log\left(1-\left(1+\frac{\vartheta}{t}\right)^{-\zeta}\right)}. \tag{2.3}$$

Singh et al. (2016) demonstrated the shape of hazard rate of ILD to be inverted bathtub shaped. A comparative study illustrates that the monotonicity of hazard rate is not, in general, transmitted to the monotonicity of the AI function (see Nanda et al., 2007).

2.3. Characterization of (reversed) residual lifetime distribution

The residual lifetime $R(t)$ can be interpreted as the additional life beyond which an unit survives under the constraint that it has survived up to time x i.e; $R(t) = X - x|X > x, \;\; x \geq 0$. As a dual of this concept, the reverse residual lifetime $\bar{R}(t)$ is defined as the time glided to reach the failure time given that the unit has lifetime less than or equal to x, i.e, $\bar{R}(t) = x - X|X \leq x, \;\; x \geq 0$. The survival function of $R(t)$ for ILD is given by

$$S_{R(t)} = P(X - x > t|X > x) = \frac{S(x+t;\zeta,\vartheta)}{S(x;\zeta,\vartheta)} = \frac{1-\left(1+\frac{\vartheta}{t+x}\right)^{-\zeta}}{1-\left(1+\frac{\vartheta}{x}\right)^{-\zeta}} \tag{2.4}$$

and corresponding PDF of $R(t)$ is obtained as
(2.5)

$$f_{R(t)} = \frac{\zeta\vartheta\left[\frac{\left(1+\frac{\vartheta}{t+x}\right)^{-(\zeta+1)}}{(t+x)^2} - \frac{\left(1+\frac{\vartheta}{x}\right)^{-(\zeta+1)}}{(x)^2}\right] - \zeta\vartheta\left(1+\frac{\vartheta}{t+x}\right)^{-(\zeta+1)}\left(1+\frac{\vartheta}{x}\right)^{-(\zeta+1)}\left[\frac{\left(1+\frac{\vartheta}{x}\right)}{(t+x)^2} - \frac{\left(1+\frac{\vartheta}{x+t}\right)}{(x)^2}\right]}{\left[1-\left(1+\frac{\vartheta}{x}\right)^{-\zeta}\right]^2}.$$

The hazard function of $R(t)$ denoted by $h_{R(t)}$ is given as
(2.6)

$$h_{R(t)} = \frac{\zeta\vartheta\left[\frac{\left(1+\frac{\vartheta}{t+x}\right)^{-(\zeta+1)}}{(t+x)^2} - \frac{\left(1+\frac{\vartheta}{x}\right)^{-(\zeta+1)}}{(x)^2}\right] - \zeta\vartheta\left(1+\frac{\vartheta}{t+x}\right)^{-(\zeta+1)}\left(1+\frac{\vartheta}{x}\right)^{-(\zeta+1)}\left[\frac{\left(1+\frac{\vartheta}{x}\right)}{(t+x)^2} - \frac{\left(1+\frac{\vartheta}{x+t}\right)}{(x)^2}\right]}{\left[1-\left(1+\frac{\vartheta}{x}\right)^{-\zeta}\right]\left[1-\left(1+\frac{\vartheta}{x+t}\right)^{-\zeta}\right]}.$$

Similarly, the expressions of survival function, PDF and hazard function for $\bar{R}(t)$, $t \geq 0$ are given as

$$S_{\bar{R}(t)} = P(x-X>t|X\leq x) = \frac{F(x-t;\zeta,\vartheta)}{F(x;\zeta,\vartheta)} = \frac{\left(1+\frac{\vartheta}{x-t}\right)^{-\zeta}}{\left(1+\frac{\vartheta}{x}\right)^{-\zeta}}, \tag{2.7}$$

$$f_{\bar{R}(t)} = \zeta\vartheta\left(1+\frac{\vartheta}{x-t}\right)^{-(\zeta+1)}\left(1+\frac{\vartheta}{x}\right)^{-(\zeta+1)}\left[\frac{\left(1+\frac{\vartheta}{x}\right)}{(x-t)^2} - \frac{\left(1+\frac{\vartheta}{x-t}\right)}{x^2}\right], \tag{2.8}$$

and

$$h_{\bar{R}(t)} = \zeta\vartheta\left(1+\frac{\vartheta}{x-t}\right)^{-1}\left(1+\frac{\vartheta}{x}\right)^{-(2\zeta+1)}\left[\frac{\left(1+\frac{\vartheta}{x}\right)}{(x-t)^2} - \frac{\left(1+\frac{\vartheta}{x-t}\right)}{x^2}\right], \tag{2.9}$$

respectively.

2.4. Moment based characterization

Moments are crucial measures for determining the shape and nature of any distribution. The expression for r^{th} raw moment of ILD is obtained as follows:

$$\mu_r = E(x^r) = \int_0^\infty \zeta\vartheta\, x^{r-2}\left(1+\frac{\vartheta}{x}\right)^{-(1+\zeta)} dx. \tag{2.10}$$

The integral stated above involves the term $1/x^2$ which is divergent, therefore moments of ILD do not exist for $r \geq 1$. To overcome this drawback, truncation can be considered as a viable option. Sometimes truncating the distribution at both ends holds significance when finite mean (first order moment) of the distribution does not exist. Truncation plays a crucial role in understanding the population characteristics when some units are lost or deliberately removed. A truncated distributions are the conditional distributions with a restricted domain, and it is categorized as doubly truncated (restricted at both ends), right truncated (restricted at the right end), and left truncated (restricted at the left end). The PDF of both end truncated ILD in $(a,b); a<b$ is given as

$$f_{TILD}(x;\zeta,\vartheta) = \frac{\vartheta\zeta\left(1+\frac{\vartheta}{x}\right)^{-(\zeta+1)}}{x^2\left[\left(1+\frac{\vartheta}{b}\right)^{-\zeta} - \left(1+\frac{\vartheta}{a}\right)^{-\zeta}\right]} \quad ; a<x<b. \tag{2.11}$$

Hence, the expression for r^{th} moments of TILD can be obtained as follows:

$$E(X^r_{TILD}) = \frac{\vartheta\zeta}{\left[\left(1+\frac{\vartheta}{b}\right)^{-\zeta} - \left(1+\frac{\vartheta}{a}\right)^{-\zeta}\right]} \int_a^b x^{r-2}\left(1+\frac{\vartheta}{x}\right)^{-(\zeta+1)} dx.$$

However, after careful consideration it can be seen that even after truncating, the finite moments do not exist in closed form.

2.5. Characterization based on inverse moments

Several studies have asserted that for an inverse family of distributions, moments do not exist. The probable reason is that the integral under consideration for the calculation of moments might not be absolutely convergent. Therefore, the need to evaluate inverse moments and subsequently inverse moment generating function arises. The r^{th} inverse moment about origin is calculated as follows:

$$\mu'_{r-1} = E\left(\frac{1}{x^r}\right) = \int_0^\infty \frac{1}{x^r}\frac{\zeta\vartheta}{x^2}\left(1+\frac{\vartheta}{x}\right)^{-(1+\zeta)} dx.$$

after simplification of the above integral, we get

$$\mu'_{r-1} = \frac{\zeta}{\vartheta^r}\,\mathfrak{B}(r+1,\zeta-1)\,\mathfrak{B}_2(r+1,\zeta-1), \tag{2.12}$$

where $\mathfrak{B}(\cdot,\cdot)$ and $\mathfrak{B}_2(\cdot,\cdot)$ is the beta function and beta function of second kind, respectively. For a particular value of $r = 1$, inverse moment converts into the harmonic mean H.

$$H = \frac{1}{(\zeta-1)\vartheta}, \qquad \zeta > 1. \tag{2.13}$$

Inverse moment generating function is derived as follows and denoted by $G_x(t)$. The other generating function can be obtained in a similar fashion.

$$G_x(t) = \sum_0^\infty \frac{t^r}{r!}\,\mu'_{r-1}.$$

which yields

$$G_x(t) = \sum_0^\infty \frac{t^r}{r!}\frac{\zeta}{\vartheta^r}\,\mathfrak{B}(r+1,\zeta-1)\,\mathfrak{B}_2(r+1,\zeta-1). \tag{2.14}$$

2.6. Entropy

Renyi entropy is a common measure for quantifying information that provides a generalization to various notions of entropy. It is particularly useful in the field of statistical inference, econometrics, and pattern recognition in computer science. For ILD, the entropy function is given by

$$\eta = \frac{1}{1-\delta_1}\log\int_0^\infty f(x;\zeta,\vartheta)^{\delta_1}\,dx = \frac{1}{1-\delta_1}\log\int_0^\infty \left(\frac{\zeta\vartheta}{x^2}\left(1+\frac{\vartheta}{x}\right)^{-(\zeta+1)}\right)^{\delta_1} dx.$$

After simplifications, the above integral becomes

$$\eta = \frac{1}{1-\delta_1}\log\left(\frac{\zeta^{\delta_1}}{\vartheta^{\delta_1}-1}\,\mathfrak{B}[2\delta_1-1,(\zeta-1)\delta_1+1]\right); \qquad \delta_1 \neq 1. \tag{2.15}$$

2.7. Stochastic ordering

A random variable X is said to be stochastically greater than another random variable Y ($Y \leq_{st} X$) if $F_Y(x) \leq F_X(x)$ for all x. Similar assertions can be firmly stated in

- hazard rate order ($Y \leq_{hr} X$) if $h_Y(x) \leq h_X(x)$ for all x.
- mean residual life order ($Y \leq_{mlr} X$) if $e_Y(x) \leq e_X(x)$ for all x.
- likelihood ratio order ($Y \leq_{lr} X$) if $\frac{f_X(x)}{f_Y(x)}$ is an increasing function of x.

The results due to Shaked and Shanthikumar (1994) are well known for establishing the implications regarding the stochastic ordering of distributions. The inference drawn using the likelihood ratio implies that the same will be obtained if we consider either hazard rate or mean residual life.

Theorem 2.1. *Let X and Y be two independent random variables that follow ILD with shape parameters ζ_1 and ζ_2 and scale parameters ϑ_1 and ϑ_2 respectively. If $\zeta_1 > \zeta_2$ and $\vartheta_1 > \vartheta_2$ then $(Y \leq_{lr} X)$ for all x.*

Proof: For given X $\sim$ ILD(ζ_1, ϑ_1) and Y $\sim$ ILD(ζ_2,ϑ_2), we have

$$\psi = \frac{f(x;\zeta_1,\vartheta_1)}{f(x;\zeta_2,\vartheta_2)}$$

$$\psi = \frac{\zeta_1\vartheta_1}{\zeta_2\vartheta_2}\left(1+\frac{\vartheta_1}{x}\right)^{-(\zeta_1+1)}\left(1+\frac{\vartheta_2}{x}\right)^{\zeta_2+1}.$$

$$\frac{d}{dx}\psi = \left(\frac{\zeta_1\vartheta_1}{\zeta_2\vartheta_2}\right)\frac{\left(1+\frac{\vartheta_1}{x}\right)^{-(\zeta_1+2)}\left(1+\frac{\vartheta_2}{x}\right)^{\zeta_2}}{x^2}\left[\left(1+\frac{\vartheta_2}{x}\right)\vartheta_1(\zeta_1+1)-\left(1+\frac{\vartheta_1}{x}\right)\vartheta_2(\zeta_2+1)\right].$$

the above equation increases in x for all $\zeta_1 > \zeta_2$ and $\vartheta_1 > \vartheta_2$. □

Corollary 2.1. *Let X $\sim$ ILD(ζ_1, ϑ_1) and Y $\sim$ ILD(ζ_2,ϑ_2). If $\zeta_1 > \zeta_2$ and $\vartheta_1 > \vartheta_2$ then $(Y \leq_{lr} X)$. Hence, $(Y \leq_{hr} X)$ and $(Y \leq_{mrl} X)$ and $(Y \leq_{st} X)$.*

2.8. Characterization based on order statistics

2.8.1. Distribution of extreme order statistics

Let $x_{(1)} \leq x_{(2)} \leq ... \leq x_{(n)}$ be the ordered sample taken from ILD with PDF in Equation (1.1) and CDF in Equation (1.2). Then, the PDF of minimum order statistics for the ILD, denoted by $f_{X_{(1)}}(x_{(1)})$, is given as

$$f_{X_{(1)}}(x_{(1)}) = \begin{cases} \frac{n\zeta\vartheta}{x_{(1)}^2}\left(1-\left(1+\frac{\vartheta}{x_{(1)}}\right)^{-\zeta}\right)^{n-1}\left(1+\frac{\vartheta}{x_{(1)}}\right)^{-(\zeta+1)} & x_{(1)} > 0 \\ 0 \quad \text{otherwise.} \end{cases} \tag{2.16}$$

and the PDF of maximum order statistics for ILD, denoted by $f_{X_{(n)}}(x_{(n)})$, is given as

$$f_{X_{(n)}}(x_{(n)}) = \begin{cases} \frac{n\zeta\vartheta}{x_{(n)}^2}(1+\frac{\vartheta}{x_{(n)}})^{-(\zeta n+1)} & x_{(n)} > 0 \\ 0 \quad \text{otherwise.} \end{cases}, \tag{2.17}$$

respectively.

2.8.2. Distribution of range

Range is defined as the difference between maximum order statistics and minimum order statistics, given as

$$r = x_{(n)} - x_{(1)}.$$

Let, $s = x_{(1)}$. Now the joint distribution of (r, s) is given as follows:

(2.18)

$$\mathfrak{h}(r,s) = \frac{n(n-1)\zeta^2\vartheta^2}{(r+s)s}\left[\left(1+\frac{\vartheta}{r+s}\right)^{-\zeta} - \left(1+\frac{\vartheta}{s}\right)^{-\zeta}\right]^{n-2}\left[\left(1+\frac{\vartheta}{r+s}\right)\left(1+\frac{\vartheta}{s}\right)\right]^{-(\zeta+1)}.$$

To obtain the distribution of range r, integrate $h(r,s)$ with respect to s and we get following as the PDF of range r and is given as

(2.19)

$$\mathfrak{h}(r) = \zeta^2\vartheta^2 n(n-1)\frac{\sum_{i=0}^{n-2}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{n-2-i+l-j-k+\zeta(i+1-n)}\binom{-(\zeta(i+1)+1)}{j}\binom{-(\zeta(n-i-1)+1)}{k}(r+j)^j\vartheta^j r^{n-2-i-j-k+\zeta(i+1-n)}}{\sum_{l=0}^{\infty}\binom{-\zeta(i+1)}{l}(l-j-k-\zeta(n-i-1)-1)}.$$

2.8.3. Distribution of median

Let $x_{(1)} \le x_{(2)} \le ... \le x_{(n)}$ be the ordered sample taken from ILD with PDF (1.1) and CDF (1.2). Then median is defined as the value corresponding to that observation that exceeds and is exceeded by the same number of observations. There are two cases to deal with:

First case is when the number of observations (n) is odd i.e., $n = 2m+1$ where m is any positive integer. In this case median is given by $(n+1)/2$ th observation and the PDF for the median is as follows:

$$\mathfrak{h}(x_{med}) = \frac{\zeta\vartheta(2m+1)!}{(m!)^2 x_{med}^2}\left[1+\frac{\vartheta}{x_{med}}\right]^{-\zeta(m+1)+1}\left[1-\left(1+\frac{\vartheta}{x_{med}}\right)^{-\zeta}\right]^m. \tag{2.20}$$

The Second case, when the number of observations (n) is even i.e., $n = 2m$ where m is any positive integer. In this case median is given as the average value of $(n/2)^{th}$ observation and $(n/2)^{th+1}$ observation. For the PDF of median in this case we first evaluate the joint pdf of $u = \left(X_{(n)} + X_{(n+1)}\right)/2$ and $v = X_{(n)}$ which is

(2.21)

$$\mathfrak{h}(u,v) = \frac{2m!\zeta\vartheta}{(m-1)^2!}\frac{1}{u^2(2u-v)^2}\int_0^u\left[\left(1+\frac{\vartheta}{v}\right)^{-\zeta}\left(1-\left(1+\frac{\vartheta}{2u-v}\right)^{-\zeta}\right)\right]^{m-1}\left[\left(1+\frac{\vartheta}{v}\right)\left(1-(1+\frac{\vartheta}{2u-v})^{-}\zeta\right)\right]^{-(\zeta+1)}.$$

Now the PDF of median (obtained by integrating with respect to v) is as follows:

$$\mathfrak{h}'(u) = \frac{(2m)!\sum_{r=0}^{m-1}\sum_0^{\infty}\sum_0^{\infty}(-1)^{i+k+j-1}\binom{m-1}{i}\binom{-(2\zeta+1)}{j}\binom{-\zeta i}{k}\vartheta^{-(2\zeta+1+i)}(2u+\vartheta)^{-(\zeta i+k)}u^{j+k+(3+i)\zeta-1}}{\sum_0^{\infty}\binom{-\zeta(i+1)-1}{l}2^{-(\zeta(i+1)+l+1)}}. \tag{2.22}$$

3. LIKELIHOOD FUNCTION AND MLE OF THE PARAMETERS

In this section, the method of maximum likelihood estimation has been discussed to obtain the estimate of the $\mathcal{S}$ for $t > 0$, where $\mathcal{S} = [S(t), h(t)]$. Let us assume that (X, R, m, n) is the progressively censored data observed from ILD, i.e.,

$$(X, R, m, n) \Rightarrow (X_{1:m:n}, R_1), (X_{2:m:n}, R_2), ..., (X_{m:m:n}, R_m)$$

and the likelihood function of (ζ, ϑ) for observed data is written as:

$$\begin{aligned} L(\zeta, \vartheta|\underline{\mathrm{X}}) &= \zeta_1 \prod_{i=1}^{m} f(x_i; \zeta, \vartheta)[S(x_i; \zeta, \vartheta)]^{R_i} \\ &= \zeta_1 \zeta^m \vartheta^m \prod_{i=1}^{m} \frac{1}{x_i^2} \left(1 + \frac{\vartheta}{x_i}\right)^{-(\zeta+1)} \left[1 - \left(1 + \frac{\vartheta}{x_i}\right)^{-\zeta}\right]^{R_i} \end{aligned} \tag{3.1}$$

where

$$\zeta_1 = n(n-1-R_1)(n-2-R_2-R_1)...(n-m+1-R_1...-R_{m-1}).$$

The log-likelihood function is

$$\begin{aligned} \ln L(\zeta, \vartheta) = \ln \zeta + m \ln \zeta + m \ln \vartheta - (1+\zeta) \sum_{i=1}^{m} \ln \left(1 + \frac{\vartheta}{x_i}\right) - 2 \sum_{i=1}^{m} \ln x_i \\ + \sum_{i=1}^{m} R_i \ln \left[1 - \left(1 + \frac{\vartheta}{x_i}\right)^{-\zeta}\right]. \end{aligned} \tag{3.2}$$

The MLEs of the parameters ζ and ϑ are computed by differentiating Equation (3.2) with respect to the parameters and equate it to zero, i.e,

$$\frac{\partial \ln L(\zeta, \vartheta)}{\partial \zeta} = \frac{m}{\zeta} - \sum_{i=1}^{m} \ln \left(1 + \frac{\vartheta}{x_i}\right) + \sum_{i=1}^{m} \frac{R_i \left(1 + \frac{\vartheta}{x_i}\right)^{-\zeta} \ln \left(1 + \frac{\vartheta}{x_i}\right)}{\left[1 - \left(1 + \frac{\vartheta}{x_i}\right)^{-\zeta}\right]} = 0 \tag{3.3}$$

and

$$\frac{\partial \ln L}{\partial \vartheta} = \frac{m}{\vartheta} - (1+\zeta) \sum_{i=1}^{m} \frac{1}{x_i + \vartheta} + \sum_{i=1}^{m} \frac{\zeta R_i \left(1 + \frac{\vartheta}{x_i}\right)^{-(\zeta+1)}}{x_i \left[1 - \left(1 + \frac{\vartheta}{x_i}\right)^{-\zeta}\right]} = 0. \tag{3.4}$$

From the obtained likelihood equations, given in Equations (3.3) and (3.4), it is observed that the MLEs can not be computed in closed form due to the implicit form of the likelihood equation. Therefore, one may use any iterative procedure like Newton-Raphson (N-R) to obtain the MLEs of the parameter.

4. ESTIMATION OF THE SURVIVAL CHARACTERISTICS

In parametric inferential theory, the estimation of the survival characteristics for the complete sample case is somehow manageable due to the explicit form of the associated distribution, but the same is not true in case of censored sample. Therefore, the invariance property of MLE may be used to obtain the estimates of the survival characteristics. Once, the MLEs $(\hat{\zeta}, \hat{\vartheta})$ of the parameters (ζ, ϑ) are obtained, the MLEs of the survival and hazard function for any $t > 0$ can be calculated by simply plugging the MLEs of the parameters into the corresponding functions. Hence, the MLEs of $S(t)$, $h(t)$ are given as

$$\hat{S}(t) = 1 - \left(1 + \frac{\hat{\vartheta}}{t}\right)^{-\hat{\zeta}} \tag{4.1}$$

and

$$\hat{h}(t) = \frac{\hat{\zeta}\,\hat{\vartheta}\left(1 + \frac{\hat{\vartheta}}{t}\right)^{-(1+\hat{\zeta})}}{t^2\left[1 - \left(1 + \frac{\hat{\vartheta}}{t}\right)^{-\hat{\zeta}}\right]}, \tag{4.2}$$

respectively.

4.1. Asymptotic distribution of survival characteristics

From the previous section, it is observed that the exact distribution of MLE of the survival characteristics $\mathcal{S} = [S(t), h(t)]$ are not available. Hence, in this section, the asymptotic distribution of $\mathcal{S}$ with the help of asymptotic distribution of (ζ, ϑ) based on asymptotic properties and general condition of MLEs are derived. Generally, for large samples, the asymptotic distribution of the parameters approximately follow a normal distribution. i.e.

$$[\sqrt{n_1}(\hat{\zeta} - \zeta), \sqrt{n_2}(\hat{\vartheta} - \vartheta)] \to N_2\left(0, \frac{1}{I(\hat{\Theta})}\right)$$

where $\Theta = (\zeta, \vartheta)$ and

$$I(\hat{\Theta}) = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}. \tag{4.3}$$

The matrix $I(\hat{\Theta})$ is called the Fisher information matrix.

$$\phi_{11} = -E\left(\frac{\partial^2 \ln L}{\partial \zeta^2}\right), \phi_{12} = -E\left(\frac{\partial^2 \ln L}{\partial \zeta \partial \vartheta}\right), \phi_{21} = -E\left(\frac{\partial^2 \ln L}{\partial \zeta \partial \vartheta}\right), \phi_{22} = -E\left(\frac{\partial^2 \ln L}{\partial \vartheta^2}\right)$$

where,

$$\begin{aligned}
\frac{\partial^2 \ln L}{\partial \zeta^2} &= \frac{m}{\zeta^2} + \sum_{i=1}^{m} R_i \log\left(1 + \frac{\vartheta}{x_i}\right)^2 \frac{(1 + \frac{\vartheta}{x_i})^{\zeta}}{\left[\left(1 + \frac{\vartheta}{x_i}\right)^{\zeta} - 1\right]^2}, \\
\frac{\partial^2 \ln L}{\partial \vartheta^2} &= \frac{m}{\vartheta^2} + \sum_{i=0}^{m}\left(\frac{\zeta + 1}{(x_i + \vartheta)^2}\right) + \zeta \sum_{i=0}^{m} R_i \left(\frac{(\zeta+1)(1 + \frac{\vartheta}{x_i})^{\zeta} - 1}{x_i^2\left[(1 + \frac{\vartheta}{x_i})^{\zeta+1} - (1 + \frac{\vartheta}{x_i})\right]^2}\right), \\
\frac{\partial^2 \ln L}{\partial \zeta \partial \vartheta} &= \frac{\partial^2 \ln L}{\partial \vartheta \partial \zeta} \\
&= \sum_{i=1}^{m} \frac{1}{(\vartheta + x_i)} - \sum_{i=1}^{m} \frac{R_i\left[\frac{1}{x_i+\vartheta}\left[(1 + \frac{\vartheta}{x_i})^{\zeta} - 1\right] - \zeta \log(1 + \frac{\vartheta}{x_i})(1 + \frac{\vartheta}{x_i})^{\zeta-1}\frac{1}{x_i}\right]}{(1 + \frac{\vartheta}{x_i})^{\zeta} - 1}.
\end{aligned}$$

We are interested in constructing the confidence interval for $\mathcal{S}$. Since the explicit distribution of $\mathcal{S}$ is not easily obtained, thus the concept of large sample theory has been used to construct an asymptotic confidence interval. For large samples, it can be easily verified that,

$$Z = \frac{\hat{\mathcal{S}} - \mathcal{S}}{\sqrt{Var(\hat{\mathcal{S}})}} \to N(0,1) \ when \ n_i \to \infty; i = 1, 2. \tag{4.4}$$

From Equation (4.4), it is clear that an estimated variance of $\mathcal{S}$ is needed. Thus, the concept of delta method is used to obtain the variance of $\mathcal{S}$.

$$Var(\mathcal{S}) = \frac{1}{n_1\phi_{11}}\left(\frac{\partial \mathcal{S}}{\partial \zeta}\right)^2 + \frac{1}{n_2\phi_{22}}\left(\frac{\partial \mathcal{S}}{\partial \vartheta}\right)^2$$

where,

$$\left(\frac{\partial \mathcal{S}}{\partial \zeta}\right)^2 = \left[\left(\frac{\partial S(t)}{\partial \zeta}\right)^2, \quad \left(\frac{\partial h(t)}{\partial \zeta}\right)^2\right], \qquad \left(\frac{\partial \mathcal{S}}{\partial \vartheta}\right)^2 = \left[\left(\frac{\partial h(t)}{\partial \vartheta}\right)^2, \quad \left(\frac{\partial h(t)}{\partial \vartheta}\right)^2\right]$$

and

$$\frac{\partial S(t)}{\partial \zeta} = \left(1 + \frac{\vartheta}{t}\right)^{-\zeta} \log\left(1 + \tfrac{\vartheta}{t}\right), \qquad \frac{\partial S(t)}{\partial \vartheta} = \frac{\zeta}{t}\left(1 + \tfrac{\vartheta}{t}\right)^{-(\zeta+1)},$$

$$\frac{\partial h(t)}{\partial \zeta} = \frac{\frac{\zeta\vartheta}{t^2}(1 + \frac{\vartheta}{t})^{-(\zeta+1)}}{1 - (1 + \frac{\vartheta}{t})^{-\zeta}}\left[\frac{1}{\zeta} - \log\left(1 + \tfrac{\vartheta}{t}\right) - \frac{(1+\frac{\vartheta}{t})^{-\zeta}\log\left(1+\frac{\vartheta}{t}\right)}{1-(1+\frac{\vartheta}{t})^{-\zeta}}\right],$$

$$\frac{\partial h(t)}{\partial \vartheta} = \frac{\frac{\zeta\vartheta}{t^2}(1 + \frac{\vartheta}{t})^{-(\zeta+1)}}{1 - (1 + \frac{\vartheta}{t})^{-\zeta}}\left[\frac{1}{\vartheta} - \frac{1+\zeta}{(t+\vartheta)} - \frac{\zeta(1+\frac{\vartheta}{t})^{-\zeta-1}}{t\left(1-\left(1+\frac{\vartheta}{t}\right)^{-\zeta}\right)}\right],$$

respectively. Hence, using the asymptotic distribution of $\mathcal{S}$ given in Equation (4.4) the $100(1-\tau)\%$ ACI for $\mathcal{S}$ is obtained as

$$[\hat{\mathcal{S}}_L, \hat{\mathcal{S}}_U] \in [\hat{\mathcal{S}} \mp Z_{\frac{\tau}{2}}\sqrt{Var(\hat{\mathcal{S}})}].$$

5. BOOTSTRAP CONFIDENCE INTERVAL

The number of observations obtained through any life testing experiments are often not large enough, therefore the ACI may not be an appropriate choice. Thus, in this section, an alternative confidence interval construction procedure, known as the bootstrap method, suggested by Efron and Tibshirani (1993), is considered. This method of finding confidence interval is the most efficient sampling and re-sampling procedures without the need of pivotal quantity. Here, we discuss the different types of BCIs, namely $(s-\text{boot})$, $(p-\text{boot})$, and $(t-\text{boot})$. The following steps may be used to construct different 95% BCIs.

- Specify the values of censoring parameters n, m, R_i and model parameters ζ, ϑ.
- Generate $X_{1:m:n}, X_{2:m:n}, ..., X_{m:m:n}$ ordered PCS-II from ILD such that $m \leq n$.
- Compute MLE $(\hat{\zeta}, \hat{\vartheta})$ of the parameters (ζ, ϑ) and obtain $\hat{S}(t), \hat{h}(t)$ of $\mathcal{S}$ using $X_{1:m:n}, ..., X_{m:m:n}$.
- Again generate PCS-II bootstrap samples $X^*_{1:m:n}, X^*_{2:m:n}, ..., X^*_{m:m:n}$ from ILD using $\hat{\Theta} = (\hat{\zeta}, \hat{\vartheta})$ as population parameters and calculate the MLE $\hat{\Theta}^*$.
- Compute MLE $\hat{\mathcal{S}}^*_1 = [\hat{S}^*_1(t), \hat{h}^*_1(t)]$ of $\mathcal{S}$ using $\hat{\Theta}^*$.
- Repeat above steps B times to generate $\hat{\mathcal{S}}^*_i$; $i = 2, ..., B$.

5.1. s-boot

Let $\bar{\mathcal{S}}^*$ and V^* be the sample mean and sample standard deviation of $\hat{S}^*, i = 1, 2, ..., B$.

$$\bar{\mathcal{S}}^* = \frac{1}{B}\sum_{i=1}^{B} \hat{\mathcal{S}}^*_i \quad \textit{and} \quad V^* = \sqrt{\frac{1}{B}\sum_{i=1}^{B}(\hat{\mathcal{S}}^*_i - \bar{\mathcal{S}}^*)^2},$$

respectively. Thus, $100(1-\tau)\%$ s-boot confidence interval for $\mathcal{S}$ is given by

$$\left(\hat{\mathcal{S}}^s_L \qquad \hat{\mathcal{S}}^s_U\right) \in \left(\hat{\mathcal{S}}^* - Z_{\tau/2}\, V^* \qquad \hat{\mathcal{S}}^* + Z_{\tau/2}\, V^*\right).$$

5.2. p-boot

Let $\hat{\mathcal{S}}^{*(\delta)}$ be the δ-percentile of $(\hat{\mathcal{S}}^*_{(i)};\ i = 1, 2, ..., B)$ and $\hat{\mathcal{S}}^{*(\delta)}$ is such that

$$\frac{1}{B}\sum_{i=1}^{B} I(\mathcal{S}^*_{(i)} \leq \hat{\mathcal{S}}^{*(\delta)}) = \delta \quad : 0 \leq \delta \leq 1.$$

where, $I(.)$ is the indicator function. Then $100(1-\tau)\%$ p-boot confidence interval is given by

$$\left(\hat{\mathcal{S}}^p_L \qquad \hat{\mathcal{S}}^p_U\right) \in \left(\hat{\mathcal{S}}^{*[B\frac{\tau}{2}]} \qquad \hat{\mathcal{S}}^{*[B\frac{1-\tau}{2}]}\right).$$

5.3. t-boot

The student's t-bootstrap confidence interval is obtained by the following additional steps:

- Generate again bootstrap sample $X^{**}_{1:m:n}, X^{**}_{2:m:n}, ..., X^{**}_{m:m:n}$ of size $m \leq n$ using $\hat{\Theta}^*$.
- Compute MLE of Θ say $\hat{\Theta_i}^{**}$ and obtain $\hat{\mathcal{S}}^{**}_i$, MLE of $\mathcal{S}$, $\forall i = 1, ..., B$.
- Calculate $V^{**} = \sqrt{\frac{1}{B}\sum_{i=1}^{B}(\hat{\mathcal{S}}^{**}_i - \bar{\mathcal{S}}^{**})^2}$ where $\bar{\mathcal{S}}^{**} = \frac{1}{B}\sum_{i=1}^{B}\hat{\mathcal{S}}^{**}_i$.

- Compute the statistic $T = \dfrac{\hat{\mathcal{S}}^{**} - \bar{\mathcal{S}}^{**}}{V^{**}}$. The $100(1-\tau)\%$ t-boot confidence interval for $\mathcal{S}$ is given by

$$\left(\hat{\mathcal{S}}_L^p \qquad \hat{\mathcal{S}}_U^p\right) \quad \in \quad \left(\bar{\mathcal{S}}^{**} - t^{\tau/2}\, V^{**} \qquad \bar{\mathcal{S}}^{**} + t^{\tau/2}\, V^{**}\right).$$

To study the different CIs, we consider their estimated average widths ($\mathcal{W}$) and coverage probability ($\mathcal{P}$). For each of the considered methods, the average width of the BCIs are computed based on B different trials. The average width and coverage probability are given by

$$\mathcal{W} = \frac{\sum_{i=1}^{B} (U_i - L_i)}{B},$$

$$\mathcal{P} = \frac{\#\,(L \leq \;\Theta\; \leq U)}{B},$$

where L and U are the lower and upper limit of the corresponding $100(1-\tau)\%$ CI based on B replicates, respectively.

6. BAYESIAN ESTIMATION

In this section, Bayes estimators for $\mathcal{S}$ are derived using the posterior distribution of ζ and ϑ. It is well known that in the Bayesian paradigm, model parameter(s) are treated as a random variable that follows some standard probability distribution, quantified as the prior distribution. The accuracy of the Bayes estimators is measured by appropriately chosen loss functions. Here, we have chosen gamma prior and general class of asymmetric loss function (ASLF) in our study. The considered prior is more flexible in terms of accommodating variety of shapes of other standard distributions. The prior for ζ and ϑ are given as

$$g_1(\zeta;\mu,\omega) = \frac{\omega^{\mu}}{\Gamma\mu} e^{-\zeta\omega} \zeta^{\mu-1} \tag{6.1a}$$

and

$$g_2(\vartheta;\kappa,\nu) = \frac{\nu^{\kappa}}{\Gamma\kappa} e^{-\vartheta\nu} \vartheta^{\kappa-1}, \tag{6.1b}$$

respectively, where, μ, ω, κ & ν are the hyper-parameters which are assumed to be known and positive. Since, the considered priors are independent in nature, the joint posterior given data using the Equations (3.1) and (6.1a-6.1b) is obtained as follows:

$$P(\zeta,\vartheta|data) = \Lambda\, e^{-(\omega\zeta+\nu\vartheta)} \zeta^{\mu+m-1} \vartheta^{\omega+m-1} \prod_{i=1}^{m} \left(1+\frac{\vartheta}{x_i}\right)^{-(\zeta+1)} \left[1-\left(1+\frac{\vartheta}{x_i}\right)^{-\zeta}\right]^{R_i}, \tag{6.2}$$

where Λ is the normalizing constant, given as

$$\Lambda = \int_0^{\infty}\int_0^{\infty} e^{-(\omega\zeta+\nu\vartheta)} \zeta^{\mu+m-1} \vartheta^{\kappa+m-1} \prod_{i=1}^{m} \left(1+\frac{\vartheta}{x_i}\right)^{-(\zeta+1)} \left[1-\left(1+\frac{\vartheta}{x_i}\right)^{-\zeta}\right]^{R_i} d\zeta\, d\vartheta.$$

The general form of the generalized class of ASLF was proposed by Calabria and Pulcini (1996) as the modified version of the Linex loss function. The general mathematical form of generalized ASLF is given as

$$L_{as}(\delta,\hat{\delta}) \propto \left(\frac{\hat{\delta}}{\delta}\right)^{\varepsilon} - \varepsilon \ln\left(\frac{\hat{\delta}}{\delta}\right) - 1.$$

where ε is the loss parameter that reflects the departure from symmetry. The Bayes estimates of the parameter δ concerning the considered loss function is obtained by using the following expression:

$$\hat{\delta}_{as} = \left[E(\delta^{-\varepsilon})\right]^{-\frac{1}{\varepsilon}}, \tag{6.3}$$

provided the expectation exists and is finite. It is to be noted that for different values of ε as -2, -1, 1, 2 the ASLF is reduced to precautionary loss function (PLF), squared error loss function (SELF), entropy loss function (ELF) and general entropy loss function (GELF), respectively. Hence, using the Equation (6.3), the Bayes estimator of $\mathcal{S}$ under ASLF are obtained as

$$\hat{S}(t)_{as} = \left[\Lambda \int_0^\infty \int_0^\infty \left[1-\left(1+\frac{\vartheta}{t}\right)^{-\zeta}\right]^{-\varepsilon} e^{-(\omega\zeta+\nu\vartheta)}\zeta^{\mu+m-1}\vartheta^{\kappa+m-1}\prod_{i=1}^{m}\left(1+\frac{\vartheta}{x_i}\right)^{-(\zeta+1)}\left[1-\left(1+\frac{\vartheta}{x_i}\right)^{-\zeta}\right]^{R_i} d\zeta\, d\vartheta\right]^{-1/\varepsilon}, \tag{6.4}$$

$$\hat{h}(t)_{as} = \left[\Lambda \int_0^\infty \int_0^\infty \left[\frac{\zeta\vartheta}{t^2+\vartheta t}\right]^{-\varepsilon} e^{-(\omega\zeta+\nu\vartheta)}\zeta^{\mu+m-1}\vartheta^{\kappa+m-1}\prod_{i=1}^{m}\left(1+\frac{\vartheta}{x_i}\right)^{-(\zeta+1)}\left[1-\left(1+\frac{\vartheta}{x_i}\right)^{-\zeta}\right]^{R_i} d\zeta\, d\vartheta\right]^{-1/\varepsilon}. \tag{6.5}$$

Since all the Bayesian estimates obtained above involve a ratio of two integrals, they are not easy to compute analytically. Therefore, resorting to approximation techniques or sample generation techniques from high dimensional posterior setting is an immediate option. Here, we used the Markov Chain Monte Carlo method to obtain the approximated Bayes estimates of the survival characteristics using the generated posterior sample.

6.1. Markov Chain Monte Carlo method

Markov Chain Monte Carlo (MCMC) method is one of the most appropriate and efficient computational procedures to approximate the Equations (6.4-6.5) based on generated sequences of ζ and ϑ. MCMC comprises various Bayes computational techniques. Among all of the accept and reject MCMC methods, the M-H algorithm suggested by Hastings (1970) is used extensively by the researchers. This method efficiently works when the marginal posterior distribution does not assume any standard form. The detailed description of this method may be found in Smith and Roberts (1993), Gelfand and Smith (1990), Upadhyay et al. (2001), Yadav et al. (2022). The marginal posterior distribution of ζ and ϑ are given as

$$\pi_\zeta(\zeta, |data, \vartheta) \propto e^{-\omega\zeta}\, \zeta^{\mu+m-1}\prod_{i=1}^{m}\left(1+\frac{\vartheta}{x_i}\right)^{-(\zeta+1)}\left[1-\left(1+\frac{\vartheta}{x_i}\right)^{-\zeta}\right]^{R_i} \tag{6.6}$$

and

$$\pi_\vartheta(\vartheta, |data, \zeta) \propto e^{-\nu\vartheta}\, \vartheta^{\kappa+m-1}\prod_{i=1}^{m}\left(1+\frac{\vartheta}{x_i}\right)^{-(\zeta+1)}\left[1-\left(1+\frac{\vartheta}{x_i}\right)^{-\zeta}\right]^{R_i}, \tag{6.7}$$

respectively. The following steps are used to extract the posterior sample from the above marginal densities, given in the Equation (6.6-6.7).

- start with initial values (ζ^0, ϑ^0).

- generate N sequences of ζ, ϑ as $\zeta_1, \zeta_2, ..., \zeta_N$, $\vartheta_1, \vartheta_2, ..., \vartheta_N$ using normal distribution as proposal density.
- generate the sequence for $\mathcal{S}$ for any mission time $t > 0$ as

$$S_1, S_2, ..., S_N \quad \text{and} \quad h_1, h_2, ..., h_N.$$

- The Bayes estimates of $\mathcal{S}$ under ASLF are obtained as

$$\hat{S}(t)_{as} = \left(\frac{1}{N - N_b} \sum_{j=1}^{N-N_b} S_j^{-\varepsilon} \right)^{-1/\varepsilon} \qquad \hat{h}(t)_{as} = \left(\frac{1}{N - N_b} \sum_{j=1}^{N-N_b} h_j^{-\varepsilon} \right)^{-1/\varepsilon},$$

Where, N_b denotes the burn-in period. The Bayes credible/highest posterior density interval (HPDI) for survival characteristics may be constructed by employing the algorithm by Chen and Shao (1999).

7. MONTE CARLO SIMULATIONS

From the previous section, it is clear that the exact expressions of MLE and Bayes estimators are not available in explicit form, thus the theoretical comparison among the obtained estimators is not possible. Hence, in this section, Monte Carlo simulations have been performed to compare the performances of the proposed estimators. The simulation study has been conducted for different variations of the censoring parameters (n, m, R_i) [see, Table 1] along with the parameters' variation as $[(0.5, 0.75), (0.85, 1), (1.5, 0.85)$ & $(2, 2)]$. The

Table 1: Censoring schemes for simulation.

n, m	Schemes	Place of removals R_i
20, 10	$R_1 : (20, 10, 10, 0^9)$	Removals occurs at first stage
	$R_2 : (20, 10, 0^9, 10)$	Removals occurs at last stage
	$R_3 : (20, 10, 1^{10})$	Removals occurs at each stage
	$R_4 : (20, 10, 5, 0^8, 5)$	Removals occurs at first and last stages
	$R_5 : (20, 10, 0^4, 5^2, 0^4)$	Removals occurs at some intermediate stages
40, 20	$R_6 : (40, 20, 5^4, 0^{16})$	Removals occurs at some beginning stages
	$R_7 : (40, 20, 0^{16}, 5^4)$	Removals occurs at some last stage
	$R_8 : (40, 20, 1^{20})$	Removals occurs at each stage
	$R_9 : (40, 20, 2^5, 0^{10}, 2^5)$	Removals occurs at first and last stages
	$R_{10} : (40, 20, 0^8, 5^4, 0^8)$	Removals occurs at some intermediate stages

sequence of PCS-II samples is generated using the algorithm suggested by Balakrishnan and Sandhu (1995). The value of effective sample size m is chosen such that the sample information is 50% censored. The MLEs of the parameters are calculated using the N-R method and invariance property has been used to construct MLEs of survival characteristics for known mission time $t = 4$. The point estimates of the $\mathcal{S}$ are compared in terms of average mean square (MSE) and interval estimates are compared in terms of the average width of the interval. Bayes estimators are obtained with gamma informative priors under ASLF. The values of loss parameter ε are chosen as $(-2, -1, 1, 2)$. A positive value of ε indicates that over-estimation is more serious than under-estimation and vice versa. The Bayes estimators for the considered values of loss parameter correspond to the one obtained under PLF, SELF,

ELF, and GELF respectively. Since the Bayes estimator assumes the ratio of two integrals, hence MCMC technique has been used to obtain the Bayes point and interval estimates. The values of the hyper-parameters are obtained in such a way that the prior mean is informative (less prior variance). The ACIs of $\mathcal{S}$ are constructed by following the idea of the delta method. Different parametric BCIs (s-boot, p-boot, t-boot) are computed. HPDIs are also computed for the same setup and compared with ACIs and BCIs. All simulation procedures has been performed using R-software and results are reported based on 3000 replications [see Tables 2-6]. In the reported tables, $\hat{S}(t)_M$, $\hat{S}(t)_P$, $\hat{S}(t)_S$, $\hat{S}(t)_E$, $\hat{S}(t)_G$ and $\hat{h}(t)_M$, $\hat{h}(t)_P$, $\hat{h}(t)_S$, $\hat{h}(t)_E$, $\hat{h}(t)_G$ denote the MLEs and Bayes estimates of $\mathcal{S}$ and R_M, R_P, R_S, R_E, R_G denotes the corresponding MSEs of the same obtained under PLF, SELF, ELF and GELF respectively. The following conclusions are made based on Monte Carlo simulations.

- The MSEs of the MLEs and Bayes estimators are decreasing by increasing sample sizes and effective sample sizes.
- For $S(t)$, the Bayes estimators under ELF are less penalized as compared to the MLE and other Bayes estimators.
- For $h(t)$, the Bayes estimators under PLF outperform MLE and other Bayes estimators. However, for parametric combination $\zeta = 2$ and $\vartheta = 2$ for some censoring schemes Bayes estimator under SELF and PLF perform equally well.
- The width of the BCIs is lesser than the ACIs for $S(t)$ as well as for $h(t)$. Although boot-p has the smallest width among BCIs for $S(t)$, some variation is reported when $\zeta = 2$ and $\vartheta = 2$ where boot-t bags the position.
- Similarly boot-t has smallest width among BCIs for $h(t)$ but for $\zeta = 2$ and $\vartheta = 2$ for some censoring schemes boot-p hails.
- The width of HPDIs is smaller than that of ACIs and BCIs for both $S(t)$ and $h(t)$.

8. REAL DATA APPLICATIONS

This section explored the application of the proposed procedures through two medical data sets. The data sets are taken from Efron (1988) and Collett (2023). Data set-I represents the survival times of 58 head and neck cancer patients treated with radiotherapy and data set-II represents the survival times of 38 prostate cancer patients. The fitting of the ILD model to the considered data sets is appropriately explained by Yadav et al. (2019) in the stress-strength reliability estimation study. They have fitted ILD for these two data sets and also shown the model compatibility among the most popular family of inverted distributions such as inverted exponential distribution, generalized inverted exponential distribution, inverse Weibull distribution, and remarked that ILD might be an alternative choice to explain the real phenomenon with hump type hazard rate. Hence, here we have taken the same data sets to estimate the survival characteristics with different censoring schemes, given in Table 7. In Table 7, the quantity $a_1^{b_1}$ represents that the number a_1 is repeated b_1 times. For the considered data sets, MLEs and Bayes estimates of the survival characteristics are obtained using censoring schemes reported in Table 7 for arbitrarily chosen mission time $t = 50, 200$ for data set-I and $t = 50, 65$ for data set-II. The boxplot summary of the generated PCS-II samples is given in Figure 1 & 2 respectively. The Bayes estimates are obtained with non-informative prior ($\mu, \omega, \kappa, \nu \to 0.0001$) under different loss functions using the MCMC technique. To implement MCMC technique in the real data set, the stationarity of Markov

Table 2: Average estimates and MSEs of the estimator $S(t)|_{t=4}$.

Schemes	Parameter	$\hat{S}(t)_M$	$\hat{S}(t)_P$	$\hat{S}(t)_S$	$\hat{S}(t)_E$	$\hat{S}(t)_G$	R_M	R_P	R_S	R_E	R_G
R_1		0.14646	0.13854	0.13050	0.11354	0.10440	0.00565	0.00310	0.00287	0.00282	0.00305
R_2		0.13810	0.13605	0.13060	0.11942	0.11358	0.00331	0.00232	0.00220	0.00215	0.00224
R_3		0.13968	0.13657	0.13040	0.11770	0.11105	0.00395	0.00260	0.00246	0.00242	0.00253
R_4		0.13707	0.13369	0.12759	0.11490	0.10815	0.00410	0.00273	0.00262	0.00262	0.00276
R_5	0.5, 0.75	0.13774	0.13378	0.12750	0.11468	0.10806	0.00443	0.00271	0.00259	0.00260	0.00274
R_6		0.13749	0.13667	0.13233	0.12333	0.11861	0.00282	0.00214	0.00204	0.00197	0.00200
R_7		0.13382	0.13455	0.13155	0.12546	0.12232	0.00168	0.00143	0.00139	0.00137	0.00139
R_8		0.13629	0.13609	0.13265	0.12561	0.12196	0.00226	0.00175	0.00169	0.00164	0.00165
R_9		0.13716	0.13713	0.13374	0.12679	0.12318	0.00214	0.00172	0.00166	0.00160	0.00161
R_{10}		0.13608	0.13610	0.13250	0.12516	0.12139	0.00243	0.00190	0.00182	0.00175	0.00175
R_1		0.17978	0.16737	0.15896	0.14114	0.13144	0.00848	0.00411	0.00402	0.00430	0.00474
R_2		0.17339	0.16918	0.16322	0.15091	0.14443	0.00481	0.00302	0.00296	0.00307	0.00325
R_3		0.17522	0.16884	0.16219	0.14839	0.14108	0.00564	0.00329	0.00322	0.00337	0.00361
R_4		0.17370	0.16797	0.16127	0.14725	0.13972	0.00534	0.00322	0.00316	0.00335	0.00361
R_5	0.85, 1.0	0.17012	0.16398	0.15730	0.14354	0.13635	0.00599	0.00335	0.00333	0.00359	0.00388
R_6		0.16965	0.16761	0.16291	0.15315	0.14801	0.00407	0.00273	0.00270	0.00277	0.00288
R_7		0.17250	0.17186	0.16848	0.16157	0.15800	0.00267	0.00199	0.00195	0.00194	0.00197
R_8		0.16804	0.16711	0.16337	0.15567	0.15168	0.00306	0.00223	0.00221	0.00227	0.00235
R_9		0.17348	0.17199	0.16818	0.16033	0.15625	0.00299	0.00215	0.00210	0.00210	0.00215
R_{10}		0.16869	0.16797	0.16407	0.15609	0.15196	0.00332	0.00237	0.00234	0.00238	0.00246
R_1		0.25891	0.24748	0.23779	0.21736	0.20637	0.01033	0.00582	0.00577	0.00624	0.00682
R_2		0.25491	0.25008	0.24346	0.22973	0.22251	0.00598	0.00408	0.00402	0.00418	0.00440
R_3		0.25781	0.25156	0.24416	0.22878	0.22068	0.00752	0.00486	0.00476	0.00490	0.00515
R_4		0.26042	0.25358	0.24591	0.22986	0.22132	0.00762	0.00504	0.00491	0.00501	0.00526
R_5	1.5, 0.85	0.25184	0.24650	0.23915	0.22398	0.21605	0.00685	0.00437	0.00435	0.00464	0.00497
R_6		0.25056	0.25027	0.24523	0.23486	0.22948	0.00512	0.00373	0.00367	0.00371	0.00381
R_7		0.25196	0.25217	0.24860	0.24132	0.23758	0.00315	0.00262	0.00258	0.00258	0.00262
R_8		0.25109	0.25125	0.24730	0.23923	0.23507	0.00401	0.00316	0.00311	0.00311	0.00316
R_9		0.25204	0.25177	0.24772	0.23943	0.23514	0.00381	0.00299	0.00295	0.00295	0.00300
R_{10}		0.24844	0.24912	0.24513	0.23701	0.23285	0.00397	0.00308	0.00305	0.00308	0.00315
R_1		0.55021	0.52093	0.51302	0.49598	0.48668	0.01354	0.00874	0.00948	0.01148	0.01282
R_2		0.55507	0.53545	0.52979	0.51779	0.51136	0.00962	0.00645	0.00679	0.00771	0.00832
R_3		0.55617	0.53402	0.52787	0.51479	0.50777	0.01049	0.00652	0.00691	0.00799	0.00871
R_4		0.55671	0.53304	0.52657	0.51275	0.50529	0.01146	0.00730	0.00773	0.00892	0.00972
R_5	2,2	0.55302	0.53271	0.52666	0.51380	0.50692	0.01011	0.00661	0.00700	0.00808	0.00878
R_6		0.55058	0.53913	0.53490	0.52610	0.52150	0.00750	0.00459	0.00479	0.00534	0.00568
R_7		0.55396	0.54659	0.54351	0.53713	0.53382	0.00516	0.00363	0.00373	0.00398	0.00415
R_8		0.54650	0.54061	0.53719	0.53011	0.52643	0.00598	0.00418	0.00433	0.00471	0.00495
R_9		0.55276	0.54358	0.54009	0.53287	0.52911	0.00603	0.00405	0.00418	0.00453	0.00475
R_{10}		0.55091	0.54381	0.54043	0.53345	0.52983	0.00580	0.00386	0.00398	0.00432	0.00453

Chain has been investigated via tuning of the variance, trace plot, auto-correlation function plot, etc. About 500 generated posterior samples from full conditional densities in the very beginning of the process (burn-in period, N_b) are excluded from the total generated sequence of samples and it is observed that the generated sequence of posterior samples is well mixed and follows the Markov property. The MCMC convergence and estimated density plots of the survival and hazard function based on generated MCMC samples for $t = 50$ are given in Figure 3 & 4, respectively. Different interval estimates (ACIs, BCIs, and HPDIs) for both survival characteristics have been obtained for the same set up of design and reported in Table 6. Table 6 indicates that the width of the BCIs is less as compared to the ACIs for all set up of designs. However, the width of HPD intervals are better throughout the analysis than these two in terms of the width of the interval. Further, it is noticed that for $S(t)$, t-boot has lesser width among all BCIs for data set-I and p-boot for data set-II. Similarly for $h(t)$, p-boot is preferable for the first data set whereas t-boot is for the second set.

Table 3: Average estimates and MSEs of the estimators $h(t)|_{t=4}$.

Schemes	Parameter	$\hat{h}(t)_M$	$\hat{h}(t)_P$	$\hat{h}(t)_S$	$\hat{h}(t)_E$	$\hat{h}(t)_G$	R_M	R_P	R_S	R_E	R_G
R_1		0.20999	0.21500	0.21454	0.21358	0.21307	0.00063	0.00027	0.00028	0.00031	0.00032
R_2		0.21330	0.21545	0.21515	0.21451	0.21419	0.00040	0.00023	0.00024	0.00025	0.00026
R_3		0.21282	0.21557	0.21523	0.21450	0.21413	0.00044	0.00024	0.00025	0.00027	0.00028
R_4		0.21309	0.21589	0.21556	0.21486	0.21449	0.00047	0.00026	0.00027	0.00029	0.00030
R_5	0.5, 0.75	0.21309	0.21628	0.21592	0.21517	0.21477	0.00053	0.00026	0.00026	0.00028	0.00029
R_6		0.21308	0.21466	0.21442	0.21391	0.21365	0.00032	0.00022	0.00023	0.00024	0.00024
R_7		0.21456	0.21513	0.21497	0.21463	0.21446	0.00021	0.00016	0.00017	0.00017	0.00018
R_8		0.21323	0.21431	0.21411	0.21370	0.21350	0.00028	0.00020	0.00020	0.00021	0.00022
R_9		0.21332	0.21428	0.21409	0.21371	0.21351	0.00026	0.00019	0.00020	0.00020	0.00021
R_{10}		0.21355	0.21462	0.21441	0.21397	0.21375	0.00029	0.00021	0.00021	0.00022	0.00023
R_1		0.20135	0.20780	0.20725	0.20609	0.20548	0.00088	0.00034	0.00035	0.00038	0.00039
R_2		0.20428	0.20724	0.20684	0.20602	0.20559	0.00054	0.00028	0.00029	0.00030	0.00031
R_3		0.20338	0.20732	0.20688	0.20595	0.20547	0.00062	0.00030	0.00031	0.00032	0.00033
R_4		0.20401	0.20768	0.20724	0.20634	0.20588	0.00060	0.00030	0.00030	0.00032	0.00033
R_5	0.85, 1.0	0.20515	0.20909	0.20865	0.20772	0.20724	0.00070	0.00033	0.00033	0.00035	0.00036
R_6		0.20484	0.20693	0.20662	0.20596	0.20563	0.00046	0.00028	0.00029	0.00030	0.00031
R_7		0.20434	0.20550	0.20526	0.20478	0.20453	0.00033	0.00023	0.00024	0.00024	0.00025
R_8		0.20508	0.20650	0.20624	0.20572	0.20545	0.00037	0.00025	0.00026	0.00026	0.00027
R_9		0.20352	0.20512	0.20486	0.20432	0.20404	0.00036	0.00024	0.00024	0.00025	0.00026
R_{10}		0.20542	0.20683	0.20656	0.20600	0.20571	0.00039	0.00026	0.00026	0.00027	0.00028
R_1		0.19262	0.19784	0.19712	0.19559	0.19477	0.00086	0.00035	0.00037	0.00040	0.00043
R_2		0.19478	0.19704	0.19654	0.19551	0.19497	0.00054	0.00028	0.00029	0.00031	0.00033
R_3		0.19380	0.19678	0.19621	0.19502	0.19440	0.00065	0.00032	0.00033	0.00036	0.00038
R_4		0.19371	0.19672	0.19615	0.19494	0.19431	0.00063	0.00031	0.00032	0.00035	0.00037
R_5	1.5, 0.85	0.19519	0.19798	0.19742	0.19626	0.19565	0.00061	0.00029	0.00030	0.00033	0.00034
R_6		0.19659	0.19753	0.19715	0.19636	0.19596	0.00044	0.00028	0.00028	0.00030	0.00031
R_7		0.19616	0.19653	0.19625	0.19570	0.19542	0.00028	0.00020	0.00021	0.00021	0.00022
R_8		0.19660	0.19708	0.19678	0.19616	0.19584	0.00037	0.00025	0.00025	0.00026	0.00027
R_9		0.19603	0.19670	0.19639	0.19576	0.19544	0.00035	0.00024	0.00024	0.00025	0.00026
R_{10}		0.19713	0.19752	0.19721	0.19658	0.19626	0.00037	0.00024	0.00025	0.00026	0.00026
R_1		0.13215	0.14127	0.13976	0.13652	0.13474	0.00104	0.00038	0.00037	0.00037	0.00038
R_2		0.13336	0.13834	0.13720	0.13480	0.13354	0.00077	0.00031	0.00031	0.00032	0.00033
R_3		0.13236	0.13862	0.13739	0.13478	0.13339	0.00087	0.00032	0.00031	0.00032	0.00033
R_4		0.13198	0.13878	0.13749	0.13475	0.13329	0.00090	0.00035	0.00034	0.00036	0.00037
R_5	2,2	0.13333	0.13898	0.13777	0.13520	0.13383	0.00083	0.00031	0.00031	0.00031	0.00032
R_6		0.13454	0.13765	0.13674	0.13484	0.13384	0.00072	0.00027	0.00027	0.00027	0.00028
R_7		0.13501	0.13630	0.13561	0.13421	0.13349	0.00051	0.00024	0.00024	0.00024	0.00024
R_8		0.13720	0.13800	0.13726	0.13573	0.13495	0.00057	0.00026	0.00026	0.00026	0.00026
R_9		0.13442	0.13661	0.13584	0.13426	0.13345	0.00058	0.00025	0.00025	0.00026	0.00026
R_{10}		0.13539	0.13698	0.13623	0.13468	0.13388	0.00056	0.00024	0.00024	0.00024	0.00025

9. CONCLUDING REMARKS

In this paper, some distributional properties including some survival characteristics of ILD have been studied. ILD can be used as a suitable lifetime model in several cancer or tumor-related studies where hazard initially increases and decreases after a certain point. Analyzing such data under PCS-II gives an edge to practitioners where subjects often leave the study during the experiment for several reasons. Further, MLE and Bayes estimators for $\mathcal{S}$ are discussed for specified t when the observed sample information is obtained under PCS-II. The MLEs for the same are computed by using the invariance property. The Bayes estimators are derived with two independent gamma prior under ASLF, and computed via MCMC method also. Different confidence interval estimation procedures namely ACIs, BCIs (boot-s, boot-p, and boot-t) & HPDIs are computed numerically for $\mathcal{S}$. ACIs for the same are computed by using the concept of delta method. A Monte Carlo simulation study has been performed to investigate the performances of the proposed estimators for the different censoring parameters. From this extensive simulation study, it is noted that Bayes estimators

Table 4: Average width ($\mathcal{W}$) and coverage probability ($\mathcal{P}$) of ACIs and HPDIs for $\mathcal{S}$.

Schemes	Parameter	ACI				HPDI			
		S(t)		h(t)		S(t)		h(t)	
		$\mathcal{W}$	$\mathcal{P}$	$\mathcal{W}$	$\mathcal{P}$	$\mathcal{W}$	$\mathcal{P}$	$\mathcal{W}$	$\mathcal{P}$
R_1	0.5, 0.75	0.59439	0.982	0.13671	0.961	0.17478	0.838	0.04704	0.879
R_2		0.49384	0.988	0.11239	0.967	0.14513	0.825	0.03917	0.842
R_3		0.53742	0.992	0.12221	0.972	0.15341	0.919	0.04138	0.845
R_4		0.51180	0.993	0.11558	0.970	0.15156	0.889	0.04077	0.871
R_5		0.55872	0.991	0.12837	0.963	0.15228	0.895	0.04149	0.826
R_6		0.38232	0.990	0.09041	0.965	0.13115	0.896	0.03574	0.913
R_7		0.33347	0.978	0.07757	0.970	0.10915	0.879	0.02996	0.922
R_8		0.34861	0.983	0.08254	0.960	0.11726	0.861	0.03243	0.902
R_9		0.34067	0.985	0.07997	0.967	0.11713	0.897	0.03195	0.723
R_{10}		0.36988	0.993	0.08759	0.959	0.11939	0.908	0.03315	0.814
R_1	0.85, 1.0	0.68697	0.980	0.15588	0.949	0.19845	0.893	0.05225	0.849
R_2		0.61617	0.987	0.13673	0.973	0.17033	0.886	0.04513	0.856
R_3		0.62798	0.991	0.14202	0.969	0.17880	0.911	0.04740	0.851
R_4		0.61815	0.994	0.13753	0.970	0.17961	0.914	0.04707	0.841
R_5		0.68605	0.993	0.15267	0.963	0.17562	0.898	0.04663	0.905
R_6		0.45690	0.987	0.10584	0.952	0.15196	0.916	0.04064	0.892
R_7		0.41171	0.988	0.09450	0.973	0.13162	0.931	0.03560	0.897
R_8		0.41675	0.994	0.09675	0.960	0.13625	0.902	0.03699	0.903
R_9		0.42230	0.979	0.09774	0.968	0.13959	0.889	0.03773	0.958
R_{10}		0.44285	0.992	0.10264	0.965	0.13870	0.879	0.03771	0.918
R_1	1.5, 0.85	1.23244	0.973	0.23652	0.978	0.26140	0.907	0.05967	0.941
R_2		1.09173	0.940	0.20572	0.977	0.22025	0.881	0.05049	0.893
R_3		1.18641	0.969	0.22391	0.977	0.23249	0.871	0.05340	0.913
R_4		1.16277	0.943	0.21935	0.979	0.23800	0.881	0.05404	0.913
R_5		1.17021	0.933	0.22264	0.981	0.22884	0.865	0.05280	0.882
R_6		0.85517	0.923	0.16247	0.983	0.19315	0.857	0.04438	0.873
R_7		0.72049	0.972	0.13767	0.985	0.16438	0.868	0.03819	0.907
R_8		0.76727	0.952	0.14556	0.987	0.17222	0.853	0.03978	0.926
R_9		0.74240	0.949	0.14193	0.989	0.17471	0.860	0.04030	0.888
R_{10}		0.79104	0.982	0.15036	0.991	0.17188	0.871	0.04006	0.948
R_1	2,2	2.20791	0.972	0.47009	0.993	0.34879	0.909	0.07879	0.934
R_2		1.98079	0.961	0.42167	0.996	0.30033	0.914	0.06798	0.924
R_3		2.05598	0.951	0.43625	0.988	0.31230	0.924	0.07082	0.933
R_4		2.03495	0.941	0.43308	0.990	0.32008	0.915	0.07240	0.919
R_5		2.10546	0.931	0.44696	0.989	0.30987	0.928	0.07038	0.930
R_6		1.57591	0.921	0.33118	0.968	0.26156	0.909	0.06095	0.915
R_7		1.38112	0.940	0.29048	0.943	0.22560	0.934	0.05259	0.892
R_8		1.47554	0.980	0.30985	0.973	0.23614	0.928	0.05498	0.904
R_9		1.38589	0.949	0.29213	0.949	0.23907	0.927	0.05571	0.917
R_{10}		1.44112	0.980	0.30372	0.979	0.23528	0.927	0.05521	0.908

for the designed set up of simulation provides more accurate and precise results in terms of MSE as compared to the classical (MLE) estimators. Subsequently, the HPDIs show a shorter length of the interval in comparison to ACIs and BCIs. However, the interval obtained through the bootstrap method is better in terms of the length of the interval than

Table 5: Average width and corresponding coverage probability of different BCIs for the $\mathcal{S}$.

Schemes	Parameter	s-boot				p-boot				t-boot			
		S(t)		h(t)		S(t)		h(t)		S(t)		h(t)	
		$\mathcal{W}$	$\mathcal{P}$	$\mathcal{W}$	$\mathcal{P}$	$\mathcal{W}$	$\mathcal{P}$	$\mathcal{W}$	$\mathcal{P}$	$\mathcal{W}$	$\mathcal{P}$	$\mathcal{W}$	$\mathcal{P}$
R_1	0.5, 0.75	0.29544	0.873	0.07523	0.891	0.28378	0.886	0.07324	0.835	0.34628	0.895	0.05505	0.895
R_2		0.21038	0.859	0.07314	0.842	0.20573	0.936	0.07104	0.896	0.24435	0.942	0.05714	0.888
R_3		0.24127	0.889	0.07045	0.908	0.23423	0.889	0.07528	0.897	0.28671	0.879	0.05471	0.897
R_4		0.23853	0.869	0.07411	0.893	0.23301	0.966	0.07069	0.914	0.28454	0.946	0.05193	0.897
R_5		0.23643	0.918	0.08954	0.901	0.22909	0.942	0.08528	0.913	0.28475	0.952	0.05879	0.895
R_6		0.20426	0.910	0.07011	0.895	0.20010	0.928	0.06828	0.926	0.24157	0.936	0.05224	0.886
R_7		0.15742	0.892	0.05649	0.869	0.15501	0.920	0.05525	0.908	0.17670	0.923	0.04606	0.871
R_8		0.17541	0.902	0.06093	0.877	0.17263	0.934	0.05945	0.907	0.20083	0.936	0.04800	0.823
R_9		0.17308	0.932	0.05935	0.877	0.17041	0.922	0.05797	0.917	0.19856	0.925	0.04699	0.853
R_{10}		0.18643	0.894	0.06420	0.843	0.18310	0.922	0.06233	0.886	0.21522	0.942	0.04946	0.859
R_1	0.85, 1.0	0.31824	0.843	0.08922	0.879	0.30892	0.848	0.08057	0.867	0.38068	0.898	0.06723	0.867
R_2		0.23656	0.861	0.08339	0.868	0.23125	0.870	0.07946	0.880	0.26713	0.888	0.06639	0.887
R_3		0.27553	0.870	0.08568	0.875	0.26815	0.956	0.08173	0.878	0.31452	0.943	0.06277	0.902
R_4		0.25835	0.900	0.09012	0.887	0.25231	0.940	0.08617	0.900	0.30549	0.950	0.06978	0.885
R_5		0.30012	0.850	0.09317	0.872	0.29171	0.930	0.08824	0.898	0.33972	0.920	0.06900	0.867
R_6		0.23756	0.890	0.07857	0.868	0.23329	0.917	0.07655	0.903	0.27315	0.923	0.06178	0.881
R_7		0.18867	0.896	0.06437	0.854	0.18568	0.929	0.06299	0.898	0.20540	0.923	0.05499	0.890
R_8		0.20558	0.897	0.06899	0.862	0.20233	0.921	0.06737	0.894	0.22904	0.917	0.05674	0.898
R_9		0.20092	0.896	0.06785	0.871	0.19771	0.932	0.06637	0.909	0.22418	0.931	0.05625	0.884
R_{10}		0.21548	0.891	0.07447	0.863	0.21147	0.915	0.07256	0.897	0.23939	0.911	0.06030	0.898
R_1	1.5, 0.85	0.36231	0.882	0.10451	0.860	0.35456	0.930	0.09965	0.927	0.43190	0.934	0.07256	0.886
R_2		0.27880	0.891	0.08134	0.847	0.27348	0.926	0.07812	0.909	0.31423	0.932	0.06353	0.878
R_3		0.30380	0.875	0.08848	0.837	0.29800	0.924	0.08490	0.904	0.34757	0.930	0.06529	0.921
R_4		0.30945	0.899	0.08723	0.848	0.30335	0.932	0.08379	0.911	0.35705	0.944	0.06391	0.900
R_5		0.30762	0.886	0.08911	0.889	0.30105	0.919	0.08494	0.879	0.34854	0.923	0.06491	0.898
R_6		0.26792	0.884	0.07936	0.850	0.26395	0.923	0.07704	0.901	0.29845	0.924	0.06365	0.897
R_7		0.21408	0.912	0.06396	0.838	0.21132	0.928	0.06247	0.884	0.22868	0.932	0.05570	0.898
R_8		0.22800	0.895	0.06930	0.844	0.22495	0.916	0.06750	0.887	0.24731	0.921	0.05867	0.910
R_9		0.23282	0.898	0.07046	0.873	0.22967	0.928	0.06882	0.919	0.25259	0.930	0.06003	0.908
R_{10}		0.23319	0.874	0.07193	0.842	0.22967	0.914	0.06981	0.891	0.25031	0.907	0.06072	0.886
R_1	2,2	0.44109	0.898	0.13105	0.873	0.43231	0.931	0.12650	0.948	0.44720	0.897	0.11339	0.858
R_2		0.36824	0.910	0.10699	0.863	0.36314	0.932	0.10361	0.930	0.36708	0.927	0.09854	0.874
R_3		0.38664	0.899	0.11435	0.861	0.38031	0.926	0.11052	0.929	0.38435	0.916	0.10481	0.902
R_4		0.39495	0.898	0.11599	0.873	0.38847	0.917	0.11222	0.937	0.39559	0.895	0.10499	0.868
R_5		0.38736	0.891	0.11455	0.851	0.38084	0.918	0.11065	0.921	0.38019	0.902	0.10416	0.826
R_6		0.33297	0.896	0.10286	0.842	0.32817	0.922	0.09988	0.903	0.31694	0.901	0.09924	0.887
R_7		0.27225	0.911	0.08629	0.879	0.26956	0.927	0.08418	0.911	0.26040	0.909	0.08732	0.834
R_8		0.29191	0.935	0.09239	0.872	0.28866	0.945	0.09001	0.913	0.27892	0.934	0.09280	0.882
R_9		0.29427	0.928	0.09188	0.869	0.29071	0.950	0.08955	0.912	0.28156	0.930	0.09108	0.892
R_{10}		0.29404	0.918	0.09288	0.853	0.29057	0.934	0.09030	0.916	0.27453	0.914	0.09292	0.900

that of ACIs. Consequently, the coverage probability goes down for the shorter length of the confidence interval. Hence, the proposed estimation procedures under PCS-II might be helpful for the lifetime practitioner/researchers to develop the guidelines for choosing the best estimation techniques.

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Table 6: Estimates of the survival characteristics $\mathcal{S}$ for different t in case of real data.

Scheme	time	$\hat{S}(t)_M$	$\hat{S}(t)_P$	$\hat{S}(t)_S$	$\hat{S}(t)_E$	$\hat{S}(t)_G$	$\hat{h}(t)_M$	$\hat{h}(t)_P$	$\hat{h}(t)_S$	$\hat{h}(t)_E$	$\hat{h}(t)_G$
S_c^1		0.78344	0.77581	0.77476	0.77261	0.77151	0.00602	0.00621	0.00615	0.00603	0.00596
S_1^1		0.93249	0.92325	0.92275	0.92172	0.92118	0.00247	0.00280	0.00268	0.00244	0.00231
S_2^1	t=50	0.81057	0.80386	0.80295	0.80109	0.80014	0.00477	0.00496	0.00490	0.00477	0.00470
S_3^1		0.92374	0.91639	0.91596	0.91508	0.91463	0.00307	0.00334	0.00324	0.00304	0.00293
S_4^1		0.84957	0.84302	0.84235	0.84098	0.84029	0.00307	0.00324	0.00318	0.00307	0.00302
S_c^1		0.39219	0.38911	0.38679	0.38207	0.37967	0.00344	0.00346	0.00345	0.00344	0.00344
S_1^1		0.61751	0.60872	0.60527	0.59804	0.59421	0.00251	0.00256	0.00255	0.00251	0.00249
S_2^1	t=200	0.45579	0.45241	0.44974	0.44428	0.44147	0.00301	0.00303	0.00302	0.00301	0.00300
S_3^1		0.56303	0.55908	0.55649	0.55115	0.54839	0.00288	0.00290	0.00289	0.00287	0.00286
S_4^1		0.58007	0.57479	0.57229	0.56709	0.56439	0.00212	0.00217	0.00215	0.00212	0.00210
S_c^2		0.50168	0.49672	0.49353	0.48698	0.48360	0.01252	0.01262	0.01258	0.01250	0.01246
S_1^2		0.59031	0.58177	0.57829	0.57100	0.56714	0.01093	0.01113	0.01107	0.01094	0.01088
S_2^2	t=50	0.54012	0.53310	0.52976	0.52282	0.51921	0.01156	0.01172	0.01167	0.01157	0.01152
S_3^2		0.58363	0.57580	0.57260	0.56595	0.56248	0.01110	0.01127	0.01121	0.01110	0.01104
S_4^2		0.54913	0.54376	0.54058	0.53399	0.53056	0.01135	0.01148	0.01143	0.01133	0.01128
S_c^2		0.42186	0.41721	0.41407	0.40761	0.40427	0.01067	0.01075	0.01073	0.01069	0.01067
S_1^2		0.50626	0.49833	0.49448	0.48652	0.48237	0.00959	0.00972	0.00969	0.00962	0.00958
S_2^2	t=65	0.45971	0.45490	0.45153	0.44460	0.44102	0.01000	0.01009	0.01006	0.01001	0.00998
S_3^2		0.49947	0.49607	0.49243	0.48486	0.48089	0.00971	0.00977	0.00974	0.00967	0.00964
S_4^2		0.46863	0.46541	0.46195	0.45480	0.45109	0.00985	0.00991	0.00988	0.00982	0.00979

Scheme	time	ACIs		BCIs						HPDIs	
		$S(t)$	$h(t)$	$S(t)$ $[\mathcal{W}]$			$h(t)$ $[\mathcal{W}]$			$S(t)$	$h(t)$
		$\mathcal{W}$	$\mathcal{W}$	s-boot	p-boot	t-boot	s-boot	p-boot	t-boot	$\mathcal{W}$	$\mathcal{W}$
S_c^1		0.63162	0.01284	0.17508	0.17762	0.17712	0.00489	0.00481	0.00565	0.15711	0.00339
S_1^1		0.39312	0.01103	0.12814	0.12863	0.10409	0.00370	0.00385	0.00468	0.11203	0.00311
S_2^1	t=50	0.57287	0.01112	0.16770	0.16900	0.16619	0.00458	0.00440	0.00561	0.14882	0.00315
S_3^1		0.46221	0.01390	0.11355	0.11089	0.10422	0.00361	0.00345	0.00387	0.10520	0.00308
S_4^1		0.41529	0.00689	0.16008	0.15436	0.15314	0.00331	0.00336	0.00445	0.12863	0.00231
S_c^1		0.65159	0.00253	0.20771	0.20474	0.20857	0.00127	0.00124	0.00125	0.16599	0.00074
S_1^1		0.90544	0.00402	0.29154	0.29052	0.26534	0.00194	0.00184	0.00197	0.25247	0.00121
S_2^1	t=200	0.70909	0.00294	0.23085	0.22830	0.22120	0.00160	0.00156	0.00162	0.18984	0.00093
S_3^1		0.94641	0.00400	0.24873	0.24948	0.25422	0.00163	0.00161	0.00165	0.21125	0.00095
S_4^1		0.65089	0.00281	0.26672	0.26874	0.21355	0.00181	0.00178	0.00215	0.21436	0.00107
S_c^2		0.97569	0.01566	0.27779	0.27633	0.28298	0.00645	0.00636	0.00586	0.22187	0.00385
S_1^2		0.98259	0.01701	0.29669	0.28999	0.28198	0.00766	0.00732	0.00732	0.25065	0.00453
S_2^2	t=50	0.95709	0.01595	0.27914	0.27271	0.27563	0.00696	0.00669	0.00701	0.23077	0.00417
S_3^2		0.99988	0.01722	0.27835	0.27135	0.27176	0.00722	0.00702	0.00710	0.23845	0.00432
S_4^2		0.94550	0.01588	0.29541	0.30014	0.28853	0.00700	0.00688	0.00686	0.22929	0.00426
S_c^2		0.90097	0.01046	0.26271	0.26208	0.28382	0.00074	0.00071	0.00055	0.19974	0.00252
S_1^2		0.94943	0.01174	0.30240	0.29761	0.30065	0.00104	0.00095	0.00072	0.24200	0.00316
S_2^2	t=65	0.90499	0.01094	0.26274	0.26014	0.26846	0.00094	0.00092	0.00069	0.21692	0.00287
S_3^2		0.96209	0.01182	0.28030	0.27269	0.27761	0.00091	0.00088	0.00068	0.23597	0.00309
S_4^2		0.89885	0.01096	0.27326	0.26975	0.27870	0.00101	0.00098	0.00073	0.22157	0.00295

Table 7: Censoring schemes for the considered real data sets.

n, m	Schemes	Place of removals R_i
58, 30	$S_c^1 : (58, 58, 0^{58})$	Complete sample: no removals
	$S_1^1 : (58, 30, 28, 0^{29})$	Removals occurs at first stage
	$S_2^1 : (58, 30, 0^{29}, 28)$	Removals occurs at last stage
	$S_3^1 : (58, 30, 14, 0^{28}, 14)$	Removals occurs at first and last stage
	$S_4^1 : (58, 30, 0^{14}, 14^2, 0^{14})$	Removals occurs at some intermediate stages
38, 30	$S_c^2 : (38, 38, 0^{38})$	Complete sample: no removals
	$S_1^2 : (38, 30, 8, 0^{29})$	Removals occurs at first stage
	$S_2^2 : (38, 30, 0^{29}, 8)$	Removals occurs at last stage
	$S_3^2 : (38, 30, 4, 0^{28}, 4)$	Removals occurs at first and last stage
	$S_4^2 : (38, 30, 0^{14}, 4^2, 0^{14})$	Removals occurs at some intermediate stages

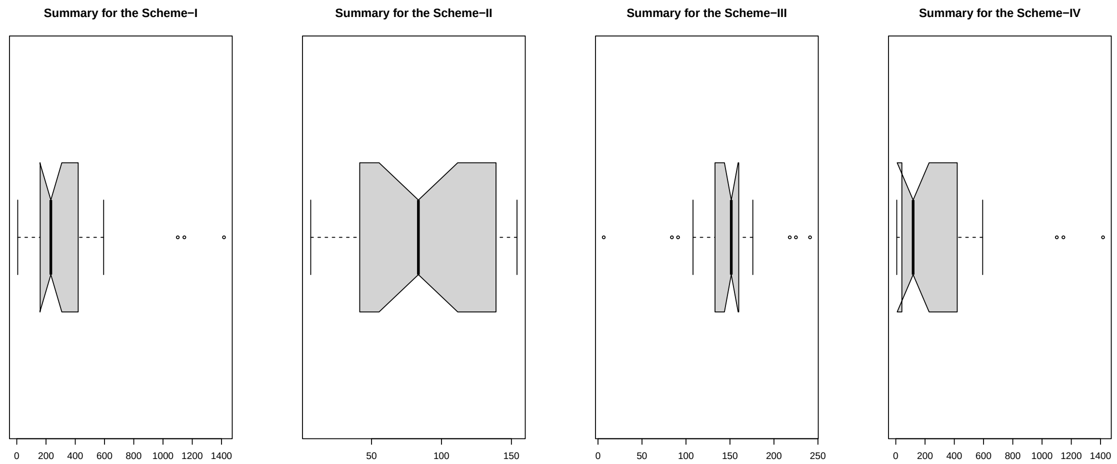

Figure 1: Box plot summary for the data set-I based on generated PCS-I samples.

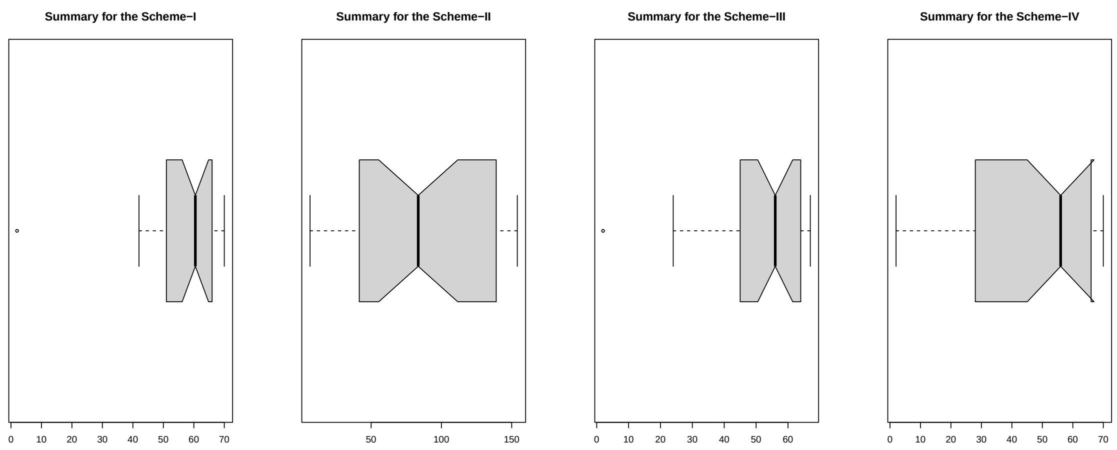

Figure 2: Box plot summary for the data set-II based on generated PCS-II samples.

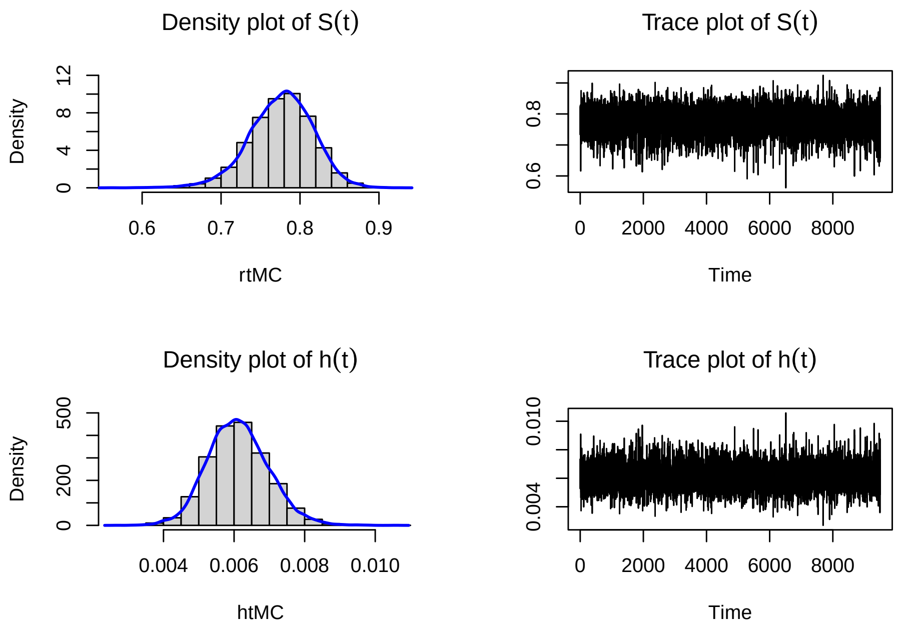

Figure 3: Posterior density and trace plot for the survival function for data set-I when $t = 50$ based on 10000 MCMC samples.

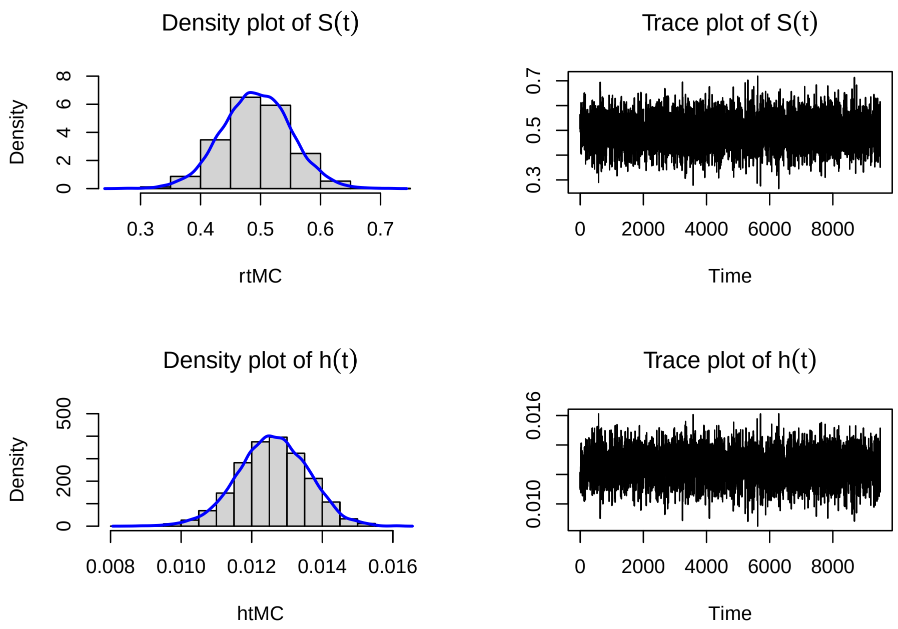

Figure 4: Posterior density and trace plot for the survival function for data set-II when $t = 50$ based on 10000 MCMC samples.

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