REVSTAT – Statistical Journal
Volume 0, Number 0, Month 0000, 000–000
https://doi.org/00.00000/revstat.v00i0.000

Testing structural hypotheses for the copula: A proofreading based on functional decomposition

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Received: Month 0000 Revised: Month 0000 Accepted: Month 0000

Abstract:

- Tests of multivariate independence may rely on asymptotically independent Cramér-von Mises statistics derived from a Möbius decomposition of the empirical copula process. We generalize this approach to other copula-based assumptions, using a functional decomposition based on commuting idempotent maps. As long as the null hypothesis reflects the stability of the copula under the action of the composition of such operators, the methodology applies. The empirical testing process, which can also be decomposed, allows the derivation of an entire family of new test statistics across various studied contexts. The asymptotic distributions are obtained. The utility of this study is illustrated through simulations on simple examples, where optimal performance is not necessarily achieved by the global statistic, but by specific statistics derived from the decomposition.

Keywords:

- *Copula models; Functional decomposition; Idempotent maps; Rank-based inference.*

AMS Subject Classification:

- 62H15, 62E10, 41A63.

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1. INTRODUCTION

The nature and strength of cross-sectional dependence is crucial for understanding economic or environmental systems. One possible measure relies on copulas, which have become popular over the last decades. In this paper, we shed new light on the formulation of structural assumptions and introduce new statistics for certain testing problems.

Consider $\mathbf{X}_1, \ldots, \mathbf{X}_n$ a sample of d-variate observations where $\mathbf{X}_j$ stands for the vector $(X_{j1}, \ldots, X_{jd})^T$. At first, one may think that this n-sample consists of independent copies of a d-dimensional random vector $\mathbf{X} = (X_1, \ldots, X_d)^T$. However, most of the results hold true for some strictly stationary time series. We assume that the cumulative distribution function (c.d.f.) F of the representative vector $\mathbf{X}$ has continuous univariate margins denoted by $F_1, \ldots, F_d$. There exists then a unique copula $C : [0,1]^d \to [0,1]$, that is a d-dimensional c.d.f. with standard uniform margins such that $F(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d))$ for all $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$. This representation, due to Sklar (1959), illustrates that the copula C characterizes the dependence between the components of $\mathbf{X}$.

The present work is concerned with testing structural hypotheses for the copula. There exists indeed a large number of copula families, and testing procedures help guide the choice of the most appropriate one. Tests based on empirical copula processes have been successfully proposed in the literature. For instance, Deheuvels (1981), Beran et al. (2007), Kojadinovic and Holmes (2009), Kojadinovic et al. (2011), Genest et al. (2019) or Bücher and Pakzad (2022) have handled the independence, serial independence, independence by blocks, or broader classes such as extreme value copulas. Whereas these references focus on one hypothesis at a time, the aim of this paper is to demonstrate that several structural hypotheses for dependence share a common pattern. The main intention is thus to illuminate the connections between different statistical questions that seemed isolated, providing a key to transition from one example to another.

Our procedure can be roughly illustrated by the pioneering idea of Deheuvels (1981), which reveals independence through the Möbius decomposition of the empirical process. The null hypothesis is thus equivalent to the intersection of a finite set of hypotheses since all secondary terms of the decomposition vanish. We generalize this method by applying another functional decomposition, chosen in accordance with the structural assumption being tested. Again, a collection of sub-hypotheses holds true under the null hypothesis. In consequence, new test statistics are defined by extracting and combining all the associated information.

The remainder of this paper is organized as follows. Section 2 recalls the functional decomposition based on operators and makes it explicit in the context of dependence structures. Section 3 is devoted to the theoretical definition of the testing process and the study of its asymptotic behavior. New test statistics are introduced, and their practical implementation is discussed. Several experiments based on simulations are provided in Section 4. Concluding remarks are given in Section 5.

2. DEPENDENCE STRUCTURES AND ASSOCIATED OPERATORS

Our aim in this section is to explain how a general functional decomposition, based on commuting idempotent maps, allows us to address various null hypotheses of dependence using a common mechanism. Such a null hypothesis is defined by the stability of the copula under a composition of certain operators. The second part of this section provides a list of examples. Finally, a collection of sub-hypotheses naturally emerges, leading to the definition of new test statistics.

2.1. A general functional decomposition and the null hypothesis

Let $\mathcal{F}$ be the linear space of real-valued functions acting on $[0,1]^d$. Let $\mathbf{I} : \mathcal{F} \to \mathcal{F}$ denote the identity map. For $i \in \{1,\dots,d\}$, let $\mathbf{P}_i : \mathcal{F} \to \mathcal{F}$ be an operator. We assume that the collection of functionals $\mathbf{P}_1,\dots,\mathbf{P}_d$ commutes. The composition of the maps $\mathbf{P}_i$ for $i \in \mathcal{A}$ will be denoted as $\prod_{i\in\mathcal{A}} \mathbf{P}_i$ and equals $\mathbf{I}$ in the case where $\mathcal{A} = \emptyset$.

Every $f \in \mathcal{F}$ can be decomposed as

$$f = \prod_{i=1}^{d}(\mathbf{I} - \mathbf{P}_i + \mathbf{P}_i)(f) = \sum_{\mathcal{A}\in\mathcal{P}_d} \mathbf{M}_{\mathcal{A}}(f) \tag{2.1}$$

where $\mathcal{P}_d$ stands for the superset of $\{1,\dots,d\}$ and where $\mathbf{M}_{\mathcal{A}}$ is defined as

$$\mathbf{M}_{\mathcal{A}} = \prod_{i\in\mathcal{A}}(\mathbf{I} - \mathbf{P}_i) \prod_{i\notin\mathcal{A}} \mathbf{P}_i \ . \tag{2.2}$$

Another way of writing the equation (2.1) is

$$f - \mathbf{M}_{\emptyset}(f) = \sum_{\mathcal{A}\in\mathcal{P}_d^\star} \mathbf{M}_{\mathcal{A}}(f) \tag{2.3}$$

where $\mathcal{P}_d^\star = \mathcal{P}_d \setminus \emptyset$ and where $\mathbf{M}_{\emptyset} = \prod_{i=1}^{d} \mathbf{P}_i$ from (2.2).

Consider now copula functions C associated with continuous random vectors $\mathbf{X}$. The main objective of this section is to identify, for some copula-based structural dependence, their associated set of operators $\{\mathbf{P}_1,\dots,\mathbf{P}_d\}$ that allows to write the dependence null hypothesis as

$$(\mathcal{H})\, C = \mathbf{M}_{\emptyset}(C) = \prod_{i=1}^{d} \mathbf{P}_i(C) \tag{2.4}$$

the composition of all functionals.

2.2. A first list of examples

Testing independence and independence by blocks. The complete independence among all components of $\mathbf{X}$, written in terms of copulas as $(\mathcal{H}_{0,1})\, C(\mathbf{x}) = x_1 \times \cdots \times x_d$, corresponds to the map

$$\mathbf{P}_i(C)(\mathbf{x}) = x_i \times C(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_d). \tag{2.5}$$

Let us structure $\mathbf{X}$ as $(\mathbf{X}_{\{1\}}, \ldots, \mathbf{X}_{\{p\}})$ the concatenation of p subvectors of dimension $d_1, \ldots, d_p$. Therefore $d = d_1 + \cdots + d_p$. The assertion $(\mathcal{H}_{0,2})\, \mathbf{X}_{\{1\}}, \ldots, \mathbf{X}_{\{p\}}$ are independent, or equivalently

$$(\mathcal{H}_{0,2})\, C(\mathbf{x}) = C(\mathbf{x}_{\{1\}}, \mathbf{1}_{-\{1\}}) \times \cdots \times C(\mathbf{x}_{\{i\}}, \mathbf{1}_{-\{i\}}) \times \cdots \times C(\mathbf{x}_{\{p\}}, \mathbf{1}_{-\{p\}})$$

is associated with

$$\mathbf{P}_i(C)(\mathbf{x}) = C(\mathbf{x}_{\{i\}}, \mathbf{1}_{-\{i\}}) \times C(\mathbf{1}_{\{i\}}, \mathbf{x}_{-\{i\}}). \tag{2.6}$$

Testing specific Archimedean copula. Now, the independence assumptions are set aside to focus on Archimedean copulas. Fix φ the generator of interest that is a non-negative, continuous, strictly decreasing and convex function defined on $[0, 1]$ satisfying $\varphi(1) = 0$. Its pseudo-inverse, denoted $\varphi^{[-1]}$, is defined as the usual inverse on $[0, \varphi(0)]$ and equals 0 elsewhere. Writing the specific Archimedean copula generated by φ

$$(\mathcal{H}_{0,3;\varphi})\, C(\mathbf{x}) = \varphi^{[-1]}\left[\varphi(x_1) + \cdots + \varphi(x_d)\right]$$

corresponds to the choice

$$\mathbf{P}_i(C)(\mathbf{x}) = \varphi^{[-1]}\left[\varphi\left(C(x_i, \mathbf{1}_{-i})\right) + \varphi\left(C(1_i, \mathbf{x}_{-i})\right)\right]. \tag{2.7}$$

The symmetric logistic extreme value copula is a particular case. Let $\ell : [0, \infty]^d \to [0, \infty]$ be a stable tail dependence function. Recall that an extreme value copula can be written as $C(\mathbf{x}) = \exp\left[-\ell\left\{-\ln(x_1), \ldots, -\ln(x_d)\right\}\right]$. See for instance Chapter 7 of de Haan and Ferreira (2006) for more details on the ℓ function. It is called the symmetric logistic extreme value copula model when there exists a real $\theta \in [1, \infty[$ such that $\ell(x_1, \ldots, x_d) = (x_1^\theta + \cdots + x_d^\theta)^{1/\theta}$. Testing the symmetric logistic extreme value model

$$C(\mathbf{x}) = \exp\left[-\left\{(-\ln(x_1))^\theta + \cdots + (-\ln(x_d))^\theta\right\}^{1/\theta}\right]$$

corresponds to the combination of the last expression (2.7) of the map with $\varphi_\theta(t) = (-\ln(t))^\theta$.

Testing specific Archimax copula. As before, consider φ a generator associated with an Archimedean structure. And consider $\ell : [0, \infty]^d \to [0, \infty]$ a stable tail dependence function. Recall from Charpentier et al. (2014) and Chatelain et al. (2020) that

$$C(\mathbf{x}) = \varphi^{-1}\left[\ell\left(\varphi(x_1), \ldots, \varphi(x_d)\right)\right]$$

is called an Archimax copula. We restrict here the form of ℓ as following

$$\ell(x_1, \ldots, x_d) = g^{-1}\left[g\{\ell(\mathbf{x}_{\{1\}}, \mathbf{0}_{-\{1\}})\} + \cdots + g\{\ell(\mathbf{0}_{-\{p\}}, \mathbf{x}_{\{p\}})\}\right]$$

where g is a continuous bijection from $\mathbb{R}_+$ to $\mathbb{R}_+$ satisfying $g(1) = 1$. From Theorem 6 in Ressel (2022), one knows that $g(x) = x^\theta$ for some $\theta \geq 1$. For the sake of simplicity, set $\varphi\mathbf{x}_{\{i\}} = \sum_{j \in \{i\}} \varphi(x_j)\mathbf{e}_j$. For $i \in \{1, \ldots, p\}$, let define $\mathbf{P}_i$ by

$$\mathbf{P}_i(C)(\mathbf{x}) = \varphi^{-1}\left[\left\{\left(\ell\left(\varphi\mathbf{x}_{\{i\}}, \mathbf{0}_{-\{i\}}\right)\right)^\theta + \left(\varphi \circ C\left(\mathbf{1}_{\{i\}}, \mathbf{x}_{-\{i\}}\right)\right)^\theta\right\}^{1/\theta}\right]$$

completed by $\mathbf{P}_{p+1} = \ldots = \mathbf{P}_d = \mathbf{I}$, to describe the null hypothesis

$$(\mathcal{H}_{0,4;\varphi,\ell,\theta})\, C(\mathbf{x}) = \varphi^{-1}\left[\left\{\left(\ell(\varphi\mathbf{x}_{\{1\}}, \mathbf{0}_{-\{1\}})\right)^{\theta} + \cdots + \left(\ell(\mathbf{0}_{-\{p\}}, \varphi\mathbf{x}_{\{p\}})\right)^{\theta}\right\}^{1/\theta}\right].$$

This list concludes the enumeration of examples that completely characterize the copula. This point is important because it will be possible to make a decision for the test by using parametric bootstrap, primarily, and somewhat non-parametric. Most of these examples will be used numerically in Section 4 to illustrate the interest of the functional decomposition and the new statistics that will be presented in the next section.

2.3. A list of hypotheses that do not completely characterize the copula

For now, we will continue the list of examples linking structure hypotheses and choice of operator. However, these will not be used in the numerical section. This is purely a theoretical presentation. The reason is simple. The following hypotheses do not completely characterize the copula. Consequently, the use of more advanced numerical techniques (for example, the multiplier bootstrap) is necessary to understand the law under the null hypothesis. This point is not within the scope of this paper, which aims to lay the groundwork for the use of functional decompositions in the definition of new test statistics.

Testing max-stability. For a given positive integer r, let us consider the null hypothesis

$$(\mathcal{H}_{0,5;r})\, C(\mathbf{x}) = C^r(\mathbf{x}^{1/r}) \quad \forall\, \mathbf{x} \in [0,1]^d .$$

The max-stability assumption, which is the intersection $(\mathcal{H}_{0,5}) = \cap_{r\in\mathbb{N}^\star}(\mathcal{H}_{0,5;r})$, corresponds to the choice $\mathbf{P}_i(C)(\mathbf{x}) = C^{r_i}(\mathbf{x}^{1/r_i})$ as functional. This null hypothesis has been handled in Kojadinovic et al. (2011).

Testing symmetry. Similarly, let $\mathfrak{S}_d$ be the set of all permutations of $\{1,\ldots,d\}$ and for any $\sigma \in \mathfrak{S}_d$ set $\mathbf{x}_\sigma = (x_{\sigma(1)},\ldots,x_{\sigma(d)})$. Testing symmetry of the copula

$$(\mathcal{H}_{0,6})\, C(\mathbf{x}) = C(\mathbf{x}_\sigma) \quad \forall \mathbf{x} \in [0,1]^d \text{ and } \forall \sigma \in \mathfrak{S}_d$$

can be handled in a very similar way to the previous one. Let $T_{1,d}$ denote the set consisting of the $d-1$ transpositions $\tau_i = (1i)$ for $i = 2,\ldots,d$. Noting that $T_{1,d}$ generates $\mathfrak{S}_d$, it is also possible to write here that $(\mathcal{H}_{0,6}) = \bigcap_{i=2}^{d}(\mathcal{H}_{0,6;\tau_i})$. It is thus sufficient to consider $\mathbf{P}_i(C)(\mathbf{x}) = C(\mathbf{x}_{\tau_i})$.

Testing Archimedean structure by blocks. Recall that the random vector $\mathbf{X}$ might be seen as the concatenation of p subvectors $\mathbf{X}_{\{1\}},\ldots,\mathbf{X}_{\{p\}}$. The independence by blocks could be replaced by an Archimedean structure by blocks associated with φ. Then the null hypothesis

$$(\mathcal{H}_{0,7;\varphi})\, C(\mathbf{x}) = \varphi^{-1}\left[\varphi(C(\mathbf{x}_{\{1\}}, \mathbf{1}_{-\{1\}})) + \cdots + \varphi(C(\mathbf{1}_{-\{p\}}, \mathbf{x}_{\{p\}}))\right]$$

could be obtained using $\mathbf{P}_i(C)(\mathbf{x}) = \varphi^{-1}\left[\varphi(C(\mathbf{x}_{\{i\}}, \mathbf{1}_{-\{i\}})) + \varphi(C(\mathbf{x}_{-\{i\}}, \mathbf{1}_{\{i\}}))\right]$. Copulas which satisfy the corresponding null hypothesis have an easy interpretation. Only p-uplets of variables, each belonging to one of the p blocks, are completely specified: Their dependence structure follows the Archimedean copula generated by φ. The dependence within any groups of variables belonging partially to the same block is not fixed. This differs from the notion of nested or hierarchical copulas.

2.4. An associated collection of sub-hypotheses

The null hypothesis (2.4) can be written as following $(\mathcal{H})\, C - \mathbf{M}_\emptyset(C) = 0$. From (2.3),

$$C - \mathbf{M}_\emptyset(C) = \sum_{\mathcal{A}\in\mathcal{P}_d^\star} \mathbf{M}_\mathcal{A}(C)$$

so that the summation $\sum_{\mathcal{A}\in\mathcal{P}_d^\star} \mathbf{M}_\mathcal{A}(C)$ vanishes when $(\mathcal{H})$ holds true. It is thus interesting to consider for any $\mathcal{A} \in \mathcal{P}_d^\star$ the sub-hypothesis

$$(\mathcal{H}^\mathcal{A})\, \mathbf{M}_\mathcal{A}(C) = 0. \tag{2.8}$$

An immediate property is $\cap_\mathcal{A}(\mathcal{H}^\mathcal{A}) \subseteq (\mathcal{H})$. So that a relevant question is to analyze whether any $(\mathcal{H}^\mathcal{A})$ holds true under the null hypothesis $(\mathcal{H})$. What is its link exactly with the intersection? In the next proposition, we answer part of the question.

Proposition 2.1. *Let $\mathbf{P}_1, \ldots, \mathbf{P}_d$ be a commuting collection of idemptotent operators on $\mathcal{F}$. Then, the null hypothesis (2.4) is linked to the sub-hypotheses (2.8) through the equality*

$$(\mathcal{H}) = \bigcap_{\mathcal{A}\in\mathcal{P}_d^\star} (\mathcal{H}^\mathcal{A}) .$$

Proof of Proposition 2.1: Suppose $\mathbf{M}_\mathcal{A}(C) \equiv 0$ for any non-empty subset $\mathcal{A}$ of $\{1, \ldots, d\}$. By application of (2.1), one obtains $C - \mathbf{M}_\emptyset(C) \equiv 0$ which is $(\mathcal{H})$. Reciprocally, if $(\mathcal{H})$ holds true, then $C = \mathbf{M}_\emptyset(C) = (\prod_{j=1}^d \mathbf{P}_j)(C)$. Combined with (2.2), it yields by commutativity,

$$\begin{aligned}
\mathbf{M}_\mathcal{A}(C) &= \left(\prod_{i\in\mathcal{A}}(\mathbf{I}-\mathbf{P}_i)\prod_{i\notin\mathcal{A}}\mathbf{P}_i\right)(C) = \left(\prod_{i\in\mathcal{A}}(\mathbf{I}-\mathbf{P}_i)\prod_{i\notin\mathcal{A}}\mathbf{P}_i\right)\left(\prod_{j=1}^d \mathbf{P}_j(C)\right) \\
&= \left(\prod_{i\in\mathcal{A}}(\mathbf{P}_i-\mathbf{P}_i^2)\prod_{i\notin\mathcal{A}}\mathbf{P}_i\right)\left(\prod_{j\notin\mathcal{A}} \mathbf{P}_j(C)\right)
\end{aligned}$$

which vanishes for $\mathcal{A} \neq \emptyset$, since $\mathbf{P}_i = \mathbf{P}_i^2$ by the idempotence assumption. □

Naturally, one may wonder whether the list of examples satisfies these conditions or not. More precisely, the question is as follows: under $(\mathcal{H})$, are the associated operators $\mathbf{P}_1, \ldots, \mathbf{P}_d$ commutative and idempotent? The answer is affirmative in each case presented in Sections 2.2 and 2.3. Their commutativity property is evident. As for idempotence, we regularly need to use the following: $C(\mathbf{1}) = 1$ and $\varphi(1) = 0$.

3. THE TESTING PROCEDURE

The purpose of this section is to introduce the empirical testing processes. Consider a structural dependence hypothesis for copulas given by (2.4) that is

$$(\mathcal{H})\, C = \mathbf{M}_\emptyset(C) = \prod_{i=1}^{d} \mathbf{P}_i(C).$$

Starting from a copula estimator C_n, it is natural to construct the testing process as

$$(\sqrt{n}(C_n - \mathbf{M}_\emptyset(C_n))(\mathbf{x}), \mathbf{x} \in [0,1]^d)$$

when considering $(\mathcal{H})$. This is precisely what is done in the literature. Recall now that $\{\mathbf{M}_\mathcal{A}\}_{\mathcal{A}\in\mathcal{P}_d}$ is the set depending, through Formula (2.2), on $\{\mathbf{P}_1, \ldots, \mathbf{P}_d\}$ a collection of operators defined on $\mathcal{F}$. It is assumed that $\{\mathbf{P}_1, \ldots, \mathbf{P}_d\}$ are commuting and idempotent maps, at least when $(\mathcal{H})$ holds true. From Proposition 2.1, $(\mathcal{H})$ implies any sub-hypothesis $(\mathcal{H}^\mathcal{A})\, \mathbf{M}_\mathcal{A}(C) = 0$. As a consequence, another choice of empirical testing processes is possible. In this section, these empirical processes will be studied and used to define new test statistics.

3.1. Weak convergence of the concatenated empirical process

Consider $\mathbf{X}_1, \ldots, \mathbf{X}_n$ a sample of d-variate observations of $\mathbf{X}$ where $\mathbf{X}_j$ stands for $(X_{j1}, \ldots, X_{jd})^T$. Set $\mathbf{U}_j = (F_1(X_{j1}), \ldots, F_d(X_{jd}))$ for $j \in \{1, \ldots, n\}$. The empirical cumulative distribution function based on $\mathbf{U}_1, \ldots, \mathbf{U}_n$ is denoted by G_n and we set $\mathbb{G}_n = \sqrt{n}(G_n - C)$. Under regular conditions, the empirical process $\mathbb{G}_n$ converges weakly in $\ell^\infty([0,1]^d)$ to a tight centered Gaussian process $\mathbb{G}_C$ concentrated on

$$\begin{aligned} \mathcal{C}_0 = \Big\{ h : [0,1]^d \to \mathbb{R} \ \text{continuous such that}\, h(\mathbf{1}) = 0 \ \text{and} \\ h(\mathbf{x}) = 0 \ \text{if some components of}\ \mathbf{x}\ \text{are equal to}\, 0 \Big\}\,. \end{aligned} \tag{3.1}$$

Throughout the paper, we assume the existence and the paths continuity of

$$\mathbb{W}_C(\mathbf{x}) = \mathbb{G}_C(\mathbf{x}) - \sum_{i=1}^{d} \partial C_i(\mathbf{x}) \mathbb{G}_C(x_i, \mathbf{1}_{-i}), \qquad \mathbf{x} \in [0,1]^d\,. \tag{3.2}$$

We introduce and study in this section the concatenated empirical testing process

$$\Big(\sqrt{n}(C_n - \mathbf{M}_\emptyset(C_n)), \{\sqrt{n}\mathbf{M}_\mathcal{A}(C_n)\}_{\mathcal{A}\in\mathcal{P}_d^\star} \Big)\,.$$

Theorem 3.1. *Assume, at least when (2.4) holds true, that*
- The operators $\{\mathbf{P}_1, \ldots, \mathbf{P}_d\}$ are commuting and idempotent maps.
- The maps $\{\mathbf{M}_\mathcal{A}\}_{\mathcal{A}\in\mathcal{P}_d}$ derived from (2.2) are Hadamard-differentiable at C tangentially to $\mathcal{C}_0$.
Consider an empirical copula C_n such that, as n tends to infinity, the empirical copula process $\sqrt{n}(C_n - C)$ converges weakly in $\ell^\infty([0,1]^d)$ to $\mathbb{W}_C$ given in (3.2).

Then, under $(\mathcal{H})$ and as n tends to infinity, the joint empirical processes converge weakly in $\{\ell^\infty([0,1]^d)\}^{2^d}$ as following

$$\begin{aligned}&\left(\sqrt{n}(C_n - \mathbf{M}_\emptyset(C_n)), \{\sqrt{n}\mathbf{M}_\mathcal{A}(C_n)\}_{\mathcal{A}\in\mathcal{P}_d^\star}\right) \xrightarrow[n\to\infty]{w} \\ &\qquad \left(\mathbb{W}_C - \mathbf{M}'_\emptyset(C;\mathbb{W}_C), \{\mathbf{M}'_\mathcal{A}(C;\mathbb{W}_C)\}_{\mathcal{A}\in\mathcal{P}_d^\star}\right). \end{aligned} \tag{3.3}$$

Proof of Theorem 3.1: By assumption, $\sqrt{n}(C_n - C) \xrightarrow[n\to\infty]{w} \mathbb{W}_C$ in $\ell^\infty([0,1]^d)$ and any $\mathbf{M}_\mathcal{A}$ is Hadamard-differentiable at C. The functional version of the Delta method (see Section 3.9 of van der Vaart and Wellner (1996)) applied to $f \mapsto (f, \mathbf{M}_\emptyset(f), \{\mathbf{M}_\mathcal{A}(f)\}_{\mathcal{A}\in\mathcal{P}_d^\star})$ yields

$$\begin{aligned}&\left(\sqrt{n}(C_n - C), \sqrt{n}(\mathbf{M}_\emptyset(C_n) - \mathbf{M}_\emptyset(C)), \{\sqrt{n}(\mathbf{M}_\mathcal{A}(C_n) - \mathbf{M}_\mathcal{A}(C))\}_{\mathcal{A}\in\mathcal{P}_d^\star}\right) \xrightarrow[n\to\infty]{w} \\ &\qquad \left(\mathbb{W}_C, \mathbf{M}'_\emptyset(C;\mathbb{W}_C), \{\mathbf{M}'_\mathcal{A}(C;\mathbb{W}_C)\}_{\mathcal{A}\in\mathcal{P}_d^\star}\right)\end{aligned}$$

in $\left\{\ell^\infty([0,1]^d)\right\}^{2^d+1}$. From the continuous mapping theorem applied to the functional $T(f, g, \{h_\mathcal{A}\}_\mathcal{A}) = (f - g, \{h_\mathcal{A}\}_\mathcal{A})$, we obtain the weak convergence of

$$\left(\sqrt{n}(C_n - \mathbf{M}_\emptyset(C_n)) - \sqrt{n}(C - \mathbf{M}_\emptyset(C)), \{\sqrt{n}(\mathbf{M}_\mathcal{A}(C_n) - \mathbf{M}_\mathcal{A}(C))\}_{\mathcal{A}\in\mathcal{P}_d^\star}\right)$$

in $\left(\ell^\infty([0,1]^d)\right)^{2^d}$ to $\left(\mathbb{W}_C - \mathbf{M}'_\emptyset(C;\mathbb{W}_C), \{\mathbf{M}'_\mathcal{A}(C;\mathbb{W}_C)\}_{\mathcal{A}\in\mathcal{P}_d^\star}\right)$. Now, recall the definition of the sub-hypotheses (2.8). When (2.4) holds true, the collection of maps $\{\mathbf{P}_i\}_{i=1,\dots,d}$ is assumed to form an idempotent and commuting family. Consequentlty, Proposition 2.1 applies, $\cap_{\mathcal{A}\in\mathcal{P}_d^\star}(\mathcal{H}^\mathcal{A})$ holds true so that $C - \mathbf{M}_\emptyset(C) = 0$ as well as any sub-hypothesis, that is $\mathbf{M}_\mathcal{A}(C) = 0$ for any $A \in \mathcal{P}_d^\star$. Then, the left hand side of the last convergence reduces to the process under study. □

Several copula estimates C_n satisfy the required convergence. The last lines of Section 2 in Kojadinovic and Stemikovskaya (2019) list carefully the conditions under which $\sqrt{n}(C_n - C) \xrightarrow[n\to\infty]{w} \mathbb{W}_C$ in $\ell^\infty([0,1]^d)$ for the following list of well-known empirical copulas

- the non-parametric estimators $\tilde{C}_n(\mathbf{x}) = \frac{1}{n}\sum_{j=1}^n \prod_{i=1}^d \mathbf{1}_{\{F_{nj}(X_{ji})\le x_i\}}$ and $\hat{C}_n(\mathbf{x}) = \frac{1}{n}\sum_{j=1}^n \prod_{i=1}^d \mathbf{1}_{\{R_{ji,n}/(n+1)\le x_i\}}$ where $R_{ji,n}$ = rank of X_{ji} among $X_{1i}, \dots, X_{ni}$,
- the checkerboard version $C_n^\#(\mathbf{x}) = \frac{1}{n}\sum_{j=1}^n \prod_{i=1}^d \min\{\max\{nx_i - R_{ji,n}, 0\}, 1\}$,
- and the empirical beta copula, $C_n^\beta(\mathbf{x}) = \frac{1}{n}\sum_{j=1}^n \prod_{i=1}^d \mathrm{F}_{n,R_{ji,n}}(x_i)$ where $\mathrm{F}_{n,r}$ stands for the probability distribution function of the Beta distribution $\mathcal{B}(r, n+1-r)$.

3.2. New collection of test statistics

Natural measures of departure from the null hypothesis are Cramér-von Mises statistics. Consider the null hypothesis $(\mathcal{H})$ described in (2.4) and depending through (2.2) on the operators $\mathbf{P}_1, \ldots, \mathbf{P}_d$. It is assumed, at least when $(\mathcal{H})$ holds true, that they are commuting and idempotent maps.

For the sake of clearness, let us introduce some names for the empirical processes studied in Theorem 3.1. When $\mathcal{A} = \emptyset$, the definition of $D_{\mathcal{A},n}$ is specific since it is

$$D_{\emptyset,n} = C_n - \mathbf{M}_{\emptyset}(C_n) = C_n - (\prod_{i=1}^{d} \mathbf{P}_i)(C_n).$$

Whereas, if $\mathcal{A} \neq \emptyset$, the process is

$$D_{\mathcal{A},n} = \mathbf{M}_{\mathcal{A}}(C_n) = (\prod_{i\in\mathcal{A}} (\mathbf{I} - \mathbf{P}_i) \prod_{i\notin\mathcal{A}} \mathbf{P}_i)(C_n).$$

Note that the empirical version of the functional decomposition is

$$D_{\emptyset,n} = \sum_{\mathcal{A}\in\mathcal{P}_d^\star} D_{\mathcal{A},n}. \tag{3.4}$$

A new collection of 2^d Cramér-von Mises (CvM) test statistics, indexed by the subsets $\mathcal{A} \in \mathcal{P}_d$, is now introduced as

$$S_{\mathcal{A},n} = \int_{[0,1]^d} \{D_{\mathcal{A},n}(\mathbf{x})\}^2 \, dC_n(\mathbf{x}). \tag{3.5}$$

Except under (block) independence, this collection is new. Their asymptotic limit, under assumptions of Theorem 3.1, can be characterized as

$$\{nS_{\mathcal{A},n}\}_{\mathcal{A}} \xrightarrow[n\to\infty]{d} \left\{ \mathbb{S}_{\mathcal{A}} = \int_{[0,1]^d} \{\mathbb{D}_{\mathcal{A}}(\mathbf{x})\}^2 \, dC(\mathbf{x}) \right\}_{\mathcal{A}}$$

where $\mathbb{D}_\emptyset = \mathbb{W}_C - \mathbf{M}'_\emptyset(C; \mathbb{W}_C)$ and $\mathbb{D}_{\mathcal{A}} = \mathbf{M}'_{\mathcal{A}}(C; \mathbb{W}_C)$.

When C_n is taken as $\hat{C}_n$, the notation $S_{\mathcal{A},n}$ becomes $\hat{S}_{\mathcal{A},n}$. Let $R_{ji,n}$ denote the rank of X_{ji} among $X_{1i}, \ldots, X_{ni}$ and set $\hat{\mathbf{U}}_{j\cdot,n} = (R_{j1,n}/n, \ldots, R_{jd,n}/n)$. Then

$$\hat{S}_{\mathcal{A},n} = \frac{1}{n} \sum_{j=1}^{n} \left\{ D_{\mathcal{A},n}(\hat{\mathbf{U}}_{j\cdot,n}) \right\}^2. \tag{3.6}$$

In some particular cases, it is possible to provide the expression of $\hat{S}_{\mathcal{A},n}$ in terms of the pseudo-observations only.

3.3. Linear combinations of the collection of test statistics

Under $(\mathcal{H}_{0,1})$ or $(\mathcal{H}_{0,2})$ the test statistics are asymptotically mutually independent (with respect to $\mathcal{A}$). As a consequence, individual critical values can be chosen to achieve an asymptotic global significance level. Furthermore, it is possible to combine individual p-values and get a global p-value thanks to the method *à la Fisher* (the resulting statistics is denoted by W_n) as well as *à la Tippett* (denoted T_n). For more details, we refer to the discussion in Genest and Rémillard (2004) or the paragraph "Combining p-values" in Section 3 of Kojadinovic and Holmes (2009).

In general, under other types of null hypotheses, the asymptotic mutual independence is no more true. However, we propose to study some linear combinations of the previous collection. Let $w = (w^{\mathcal{A}})_{\mathcal{A}\in\mathcal{P}_d}$ be a vector of positive weights. The latter reflects the importance we put in the test $(\mathcal{H})\, C = \mathbf{M}_{\emptyset}(C)$ through $w^{\emptyset}$, or in the test $(\mathcal{H}^{\mathcal{A}})\ \mathbf{M}_{\mathcal{A}}(C) = 0$ through $w^{\mathcal{A}}$. We introduce the w-combined test statistics by

$$S_{w,n} = \sum_{\mathcal{A}\in\mathcal{P}_d} w^{\mathcal{A}} S_{\mathcal{A},n} \tag{3.7}$$

as well as its tractable version $\hat{S}_{w,n} = \sum_{\mathcal{A}\in\mathcal{P}_d} w^{\mathcal{A}} \hat{S}_{\mathcal{A},n}$.

The focus on five weights. Only five distinct weights will be under study in the numerical experiments. The first weight only measures the distance between C_n and C as it is specified by $(\mathcal{H})$. To summarise, one can write

$$w_1 = (\underbrace{1}_{|\mathcal{A}|=0}, \underbrace{0, \dots, 0}_{|\mathcal{A}|>0}).$$

Proposition 2.1 invites us to consider the weight

$$w_2 = (\underbrace{0}_{|\mathcal{A}|=0}, \underbrace{1, \dots, 1}_{|\mathcal{A}|>0})$$

that assigns a similar weight to each sub-hypothesis. A natural question is whether the mixture of both previous weights improves performance or not. Consequently, the weight with ones everywhere

$$w_3 = (1, 1, \dots, 1)$$

combines information from the original statistics with that derived from each term of the decomposition.

On several examples, the first terms of the decomposition (those associated with singletons $\mathcal{A}$) are always zero. This is why the method becomes interesting when $d > 2$. For example, under $(\mathcal{H}_{0,1})$, it simply reflects the fact that $C(x_i, \mathbf{1}_{-i}) = x_i$ allowing for simplification. Consequently, since singletons are uninformative, the focus here shifts to subsets $\mathcal{A}$ describing pairs. Then

$$w_4 = (\underbrace{0}_{|\mathcal{A}|=0}, \underbrace{0, \dots, 0}_{|\mathcal{A}|=1}, \underbrace{1, \dots, 1}_{|\mathcal{A}|=2}, \underbrace{0, \dots, 0}_{|\mathcal{A}|>2})$$

where the first $d+1$ components vanish, along with the terms from position $d(d-1)/2+d+2$. Finally, the last weight under study in the next section measures the information of highest order in the decomposition, that is

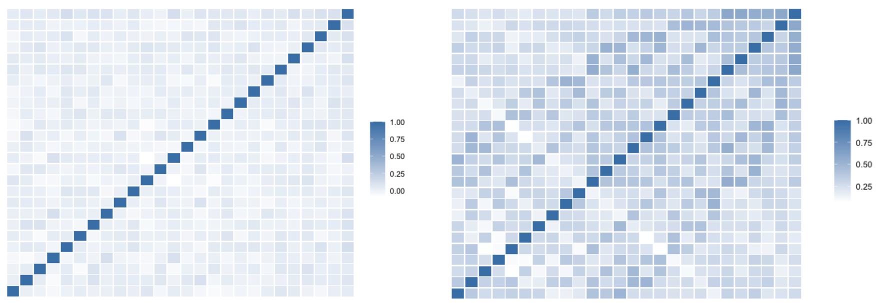
$$w_5 = (\underbrace{0, \dots, 0}_{|\mathcal{A}|<d}, \underbrace{1}_{|\mathcal{A}|=d}).$$

3.4. Additional empirical insights on the new statistics

The act of combining different statistics using a weight w can raise questions. It is important to note that we are not asserting independence among individual contributions.

In this manner, we construct a statistic, denoted as $\hat{S}_{w,n}$, which is a linear combination of individual contributions, where only the variability may be affected by this combination.

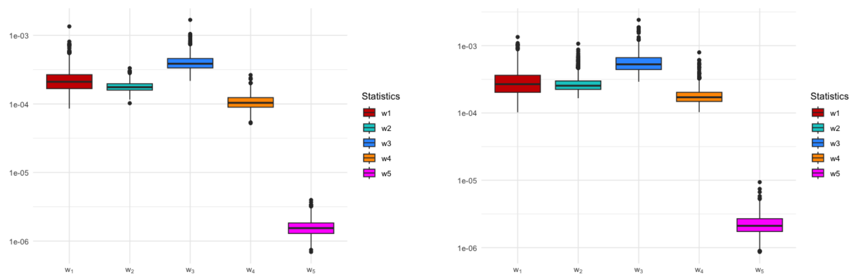
To illustrate this dependency practically, 1 000 decomposition statistics $\{S_{\mathcal{A},n=100}\}_{\mathcal{A}}$ have been evaluated under $(\mathcal{H}_{0,1})$ and under $(\mathcal{H}_{0,3;\varphi_0})$ where φ_0 is the generator associated with the Clayton copula having a high bivariate Kendall's tau of 0.8. Statistics associated with $|\mathcal{A}| = 1$ are excluded because they are always zero. Heatmaps of the correlation matrices are presented in Figure 1. As expected, the contributions obtained from the decomposition outside the independence assumption reveal stronger dependence.



$(\mathcal{H}_{0,1})$ $(\mathcal{H}_{0,3;\varphi_0})$

Figure 1: Heatmap of the correlation matrix of $\{\hat{S}_{\mathcal{A},n=100}\}_{|\mathcal{A}|\neq 1}$ under $(\mathcal{H}_{0,1})$ and $(\mathcal{H}_{0,3;\varphi_0})$ in a 5-dimensional setting based on 1 000 experiments.

The natural next question is whether this has a significant consequence on the variability of the w-linear combinations. This point is illustrated in Figure 2. On the contrary, a slight contraction in the range of the boxplots is observed. We are not comparing the values on the left graph with those on the right graph here. Instead, we are comparing the size of the boxplots for the combined statistics $\hat{S}_{w_2,n}, \ldots, \hat{S}_{w_5,n}$ with that associated with w_1, which is the classical global statistic.



$(\mathcal{H}_{0,1})$ $(\mathcal{H}_{0,3;\varphi_0})$

Figure 2: Boxplots of $\{\hat{S}_{w_1,n=100}, \ldots, \hat{S}_{w_5,n=100}\}$ under $(\mathcal{H}_{0,1})$ and $(\mathcal{H}_{0,3;\varphi_0})$ in a 5-dimensional setting based on 1 000 experiments. The ordinate axis has been scaled logarithmically.

4. NUMERICAL EXPERIMENTS

In this section, we shall consider the use of the functional decomposition in four experiments and analyze the results. The studies encompass tests for independence, blockwise independence, a hypothesis defined by a specific Archimedean copula, and a test for a given class of copulas. The results demonstrate variability, although it is noted that incorporating information from the right-hand member of the decomposition enhances statistical power. When possible, the results are compared with those obtained using tools from the R package copula.

The code for this section is available at the following address
`http://math.univ-lyon1.fr/~mercadier/functionaldecomposition.html`
Please contact the author if you encounter any access issues.

4.1. Complete independence

To begin, consider a simple example. Let $\mathbf{X} \sim \mathcal{N}_5(0, \Sigma)$ with $\Sigma = I + \rho R$, where the off-diagonal elements of R equal 1 and the other ones are 0. The value $\rho = 0$ leads to the null hypothesis $(\mathcal{H}_{0,1})$ of total independence. The significance level is arbitrarily set to 5% and evaluated at $\rho = 0$. This empirical nominal level and the powers of the test statistics $\hat{S}_{w_1,n}, \ldots, \hat{S}_{w_5,n}$ are assessed for $\rho \in [0, 0.3]$. Their performances are also compared to those of tests available in the R copula package: I_n the global CvM statistics as well as W_n and T_n, the Fisher and the Tippett combination. These procedures have been previously mentioned in the beginning of Section 3.3.

First, we observe in Figure 3 that the weight leading to the most powerful test statistic is w_3, which evenly distributes weights on both sides of the equation (3.4) to construct $\hat{S}_{w_3,n}$. Very close behind is the original statistic with weight w_1, that measures the distance from the left hand side of (3.4). Surprisingly, the information from second-order terms (associated with weight w_4) captures here all the information from the right-hand side, overlapping with the values obtained with weight w_2. The weight w_5 leads to a weak performance under such an experiment.
The graph also includes the global CvM statistic I_n, which, similar to $\hat{S}_{w_1,n}$, is derived from the left-hand side of (3.4). The decision rule for I_n is computed using another procedure[1], explaining why it does not overlap with $\hat{S}_{w_1,n}$. We observe that T_n remains the least powerful test for any $\rho \leq 0.2$, performing even worse than $\hat{S}_{w_5,n}$ whereas W_n's performance remains in between throughout the curve.

These conclusions do not provide a universal hierarchical order among these procedures. These results are intrinsically linked to the framework considered, namely total independence versus a specific alternative under the Gaussian assumption. However, this study illustrates that several CvM statistics can be derived from the decomposition. Finally, the comparison allows us to observe a slight benefit from combining both sides of the equation (3.4)

[1] R documentation of `copula::indepTest` "simulation step, which consists of simulating the distribution of the test statistics under independence for the sample size under consideration".

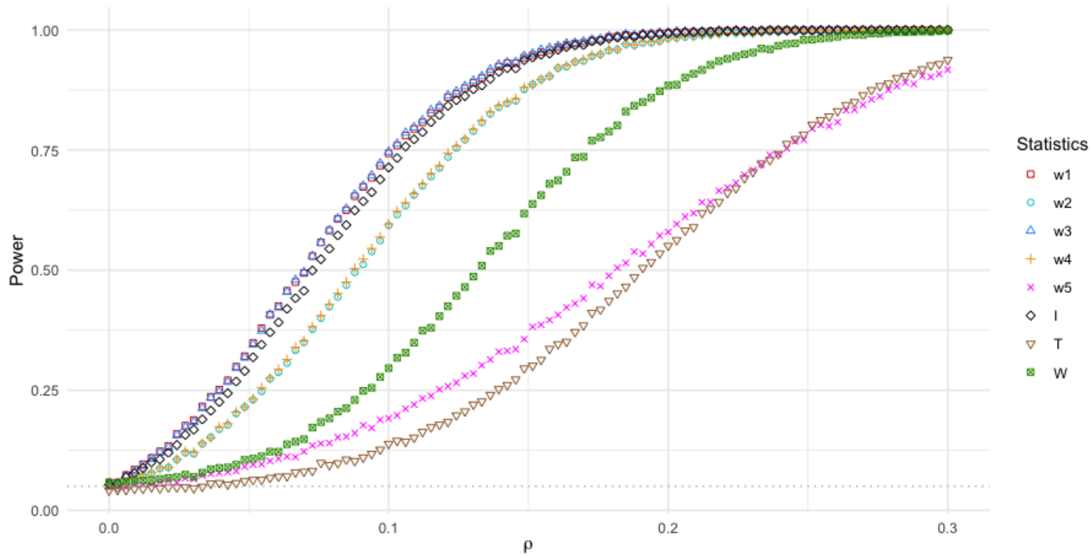


Figure 3: Percentage of rejection of the null hypothesis ($\mathcal{H}_{0,1}$) with respect to the value of ρ involved in the variance matrix $\Sigma = I + \rho R$ of the simulated centered gaussian vector. The graph is obtained for $n = 100$, $d = 5$ and from 10 000 repetitions of the non parametric bootstrap with $N = 1\,000$.

4.2. Independence between sub random vectors

We adapt here Section 4 from Kojadinovic and Holmes (2009) that implements testing procedures for block independence. Let $\mathbf{X} = (X_1, \ldots, X_{12})$ and consider the 3 groups $\{X_1, \ldots, X_4\}$, $\{X_5, \ldots, X_8\}$ and $\{X_9, \ldots, X_{12}\}$ so that $p = 3$ and $d = 12$ in ($\mathcal{H}_{0,2}$). The dependence is described by the normal copula. The $d \times d$ correlation matrices Σ are structured by block as follows

	X_1	$\ldots$	X_4	X_5	$\ldots$	X_8	X_9	$\ldots$	X_{12}
X_1	1		ρ_{intra}						
$\vdots$					ρ_{inter}			ρ_{inter}	
X_4	ρ_{intra}		1						
X_5				1		ρ_{intra}			
$\vdots$		ρ_{inter}						ρ_{inter}	
X_8				ρ_{intra}		1			
X_9							1		ρ_{intra}
$\vdots$		ρ_{inter}			ρ_{inter}				
X_{12}							ρ_{intra}		1

The quantity ρ_{inter} (resp. ρ_{intra}) controls the amount of dependence among (resp. within) the three random vectors. Under the normal copula, the values $\rho_{\text{inter}} \in \{0.000, 0.025, \ldots, 0.300\}$ for $\rho_{\text{intra}} = 0.5$ are considered. We generate 1 000 samples composed of $n = 100$ independent realizations of $\mathbf{X}$. Note that, in all the simulations, the number of randomized samples is set to $N = 1\,000$.

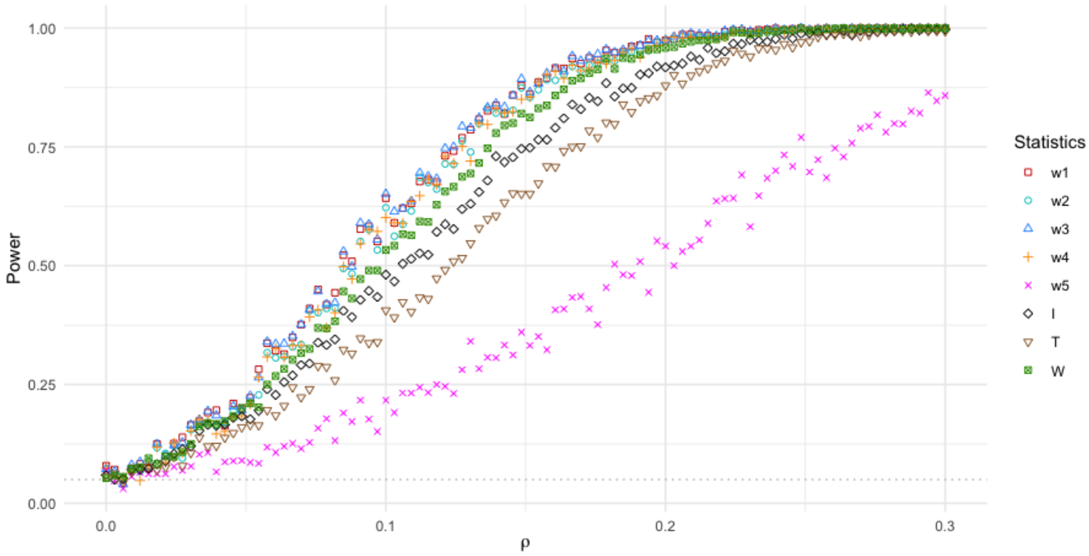


Figure 4: Percentage of rejection of the null hypothesis $(\mathcal{H}_{0,2})$ with respect to the value of ρ_{inter} involved in the variance matrix defined by blocks. The graph is obtained for $n = 100$, $d = 12$, $p = 3$ and from 1 000 repetitions of the non parametric bootstrap with $N = 1\,000$.

Figure 4 shows the proportion of times that the different tests reject $(\mathcal{H}_{0,2})$ with respect to the value of ρ_{inter}. The significance level is arbitrarily set to 5% and measured at $\rho_{\text{inter}} = 0$. The global CvM statistic I_n, as well as W_n and T_n are those studied in Figure 3 of Kojadinovic and Holmes (2009). The order of performance between $\hat{S}$-type statistics is the same as for the previous study. On the other hand, this is not the case for the statistics group I_n, T_n and W_n. On the one hand T_n behaves better here, and on the other hand, the hierarchy between I_n and W_n has been reversed. $\hat{S}_{w_3,n}$ is again a slightly better choice.

4.3. Simple null Archimedean hypothesis

In this section, the aim is to compare the power of the statistics in discriminating a null hypothesis $(\mathcal{H}_{0,3;\varphi_0})$ $C = C_{\varphi_0}$ against the alternative of the form $(\mathcal{H}_1)$ $C = (1-p)C_{\varphi_0} + pC_{\varphi_1}$. The graphs summarize the resulting performances as a function of p. We set $d = 4$ and $n = 100$. To avoid trivial powers, the copulas C_{φ_0} and C_{φ_1} share the same value of the bivariate Kendall's tau, arbitrarily set to 0.2.

For each graph among Figure 5, 6 and 7, there is 1 000 repetitions of each experiment and the null distribution is learnt using $N = 1\,000$ parametric bootstrap replications. The most efficient is the one based on the statistic $\hat{S}_{w_5,n}$, which measures the highest order term of the decomposition (the fourth order here). And this is true in all three configurations studied. Even though the order of the ordinates varies according to the graphs, and even though the performance of the other estimators is more or less fluctuating, that of $\hat{S}_{w_5,n}$ dominates. The fact that some of the statistics fail to discriminate the hypotheses might be explained by the small size of n and by the choice of the respective values of the copula parameters, so that strength of dependence are identical. However, it is more difficult to understand why the highest order is so informative under Archimedean assumptions whereas it was not the case in previous subsections while studying dependent alternatives.

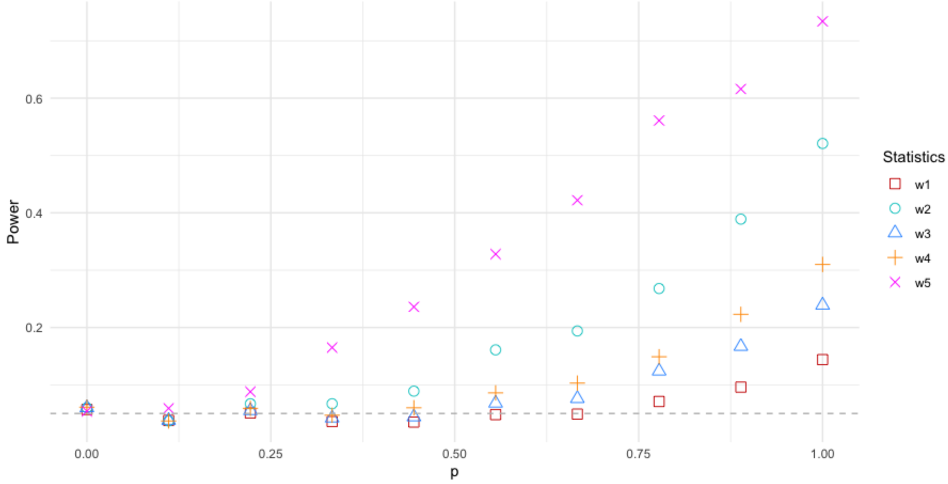


Figure 5: Percentage of rejection of the null hypothesis ($\mathcal{H}_{0,3;\varphi_0}$) with respect to the value of the probability mixture p. Here C_{φ_0} is the Clayton copula with parameter 0.5 and C_{φ_1} is the Gumbel copula with parameter 1.25.

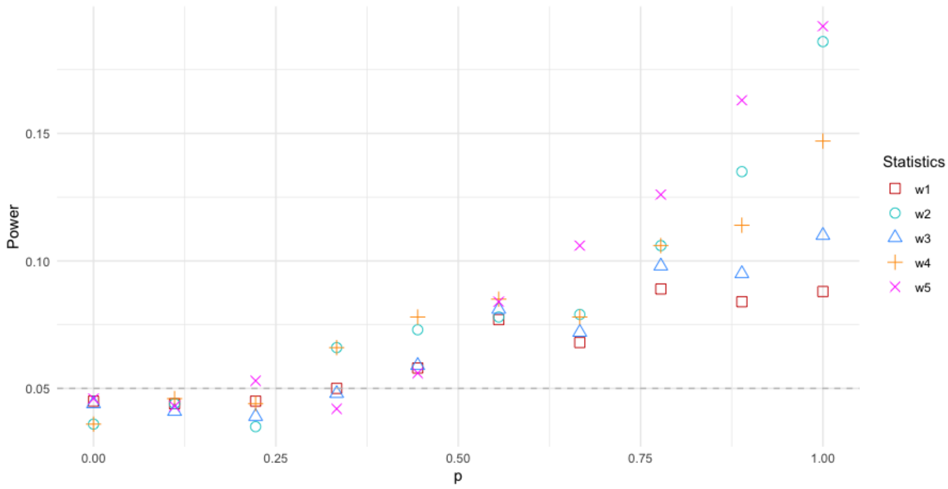


Figure 6: Percentage of rejection of the null hypothesis ($\mathcal{H}_{0,3;\varphi_0}$) with respect to the value of the probability mixture p. Here C_{φ_0} is the Gumbel copula with parameter 1.25 and C_{φ_1} is the Frank copula with parameter 1.860884.

4.4. Goodness-of-fit tests for Archimedean copulas

Turning to the Archimedean goodness-of-fit tests, we consider the Clayton or the Gumbel family in a 3-dimensional setting. These classes will both be used as the generator of datasets or as the family being tested. The difference from the previous section is significant. Here, the null hypothesis is not $(H_{0,3;\varphi})$. Indeed, φ is not fully specified. We only know the form of the generator φ, but the knowledge of its parameter is missing. Only a class of

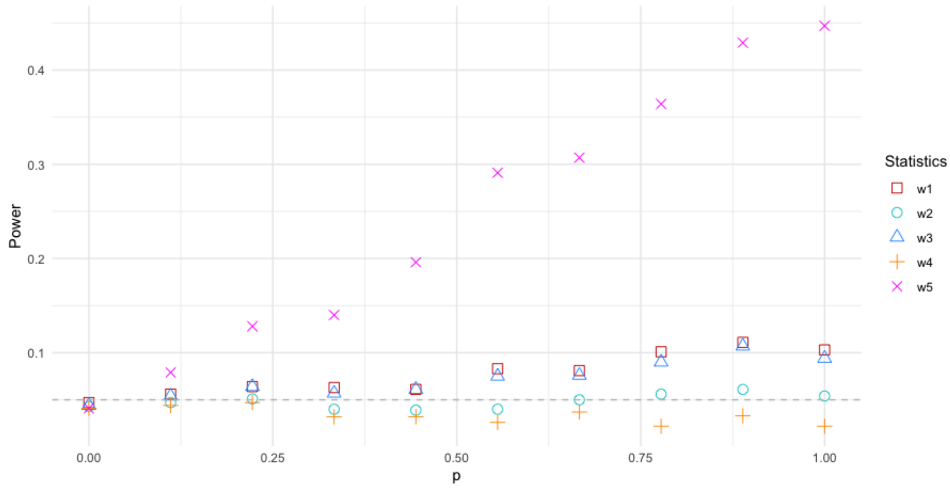


Figure 7: Percentage of rejection of the null hypothesis $(\mathcal{H}_{0,3;\varphi_0})$ with respect to the value of the probability mixture p. Here C_{φ_0} is the Frank copula with parameter 1.860884 and C_{φ_1} is the Clayton copula with parameter 0.5.

Archimedean copulas is given. The parameter associated with the generator φ is estimated at each step as the empirical mean of the Kendall's τ estimates. The p-value is computed using $N = 500$ parametric bootstrap evaluations from the estimated copula.

To generate the original samples, three values of the bivariate Kendall's τ are chosen: $\tau = .1, .15$ and $.2$. The sample size is fixed to $n = 200$. The rejection rates are estimated through 1 000 repetitions of each experiment.

The results are provided in Table 1. The nominal level is arbitrarily fixed at 0.05. The first lines are dedicated to testing whether the dependence structure is given by the Clayton copula. Similarly, Gumbel copula is tested in the last lines of the table. First, note that on the right upper corner of the table, I_n[2] always dominates in discriminating true Gumbel from supposed Clayton. Whereas $\hat{S}_{w_1,n}$ and $\hat{S}_{w_3,n}$ are more powerful in the left bottom corner.

5. CONCLUDING REMARKS

Identifying and modeling dependencies with copulas remain an important topic, which has become very popular over the last decades since it has been applied in almost every discipline. The aim in this paper is to unify various papers, as Deheuvels (1981), Genest and Rémillard (2004), Genest et al. (2007), Beran et al. (2007), Kojadinovic and Holmes (2009), Kojadinovic et al. (2011) among others, that derive copula-based tests of the structure of dependence. The solution here is to dip them in a functional decomposition context in order to reveal a common pattern.

The numerical section presents four simple studies. In the first one, which tests $(\mathcal{H}_{0,1})$, our statistics provide a very slight improvement and also allow us to position their perfor-

[2] I_n is obtained from the R command `copula::gofCopula` with `itau` method.

		Simulated copulas					
		Clayton			Gumbel		
		$\tau = .1$	$\tau = .15$	$\tau = .2$	$\tau = .1$	$\tau = .15$	$\tau = .2$
Tested copulas: Clayton	I_n	0.032	0.047	0.057	0.455	0.764	0.925
	$\hat{S}_{w_1,n}$	0.031	0.050	0.052	0.322	0.645	0.866
	$\hat{S}_{w_2,n}$	0.048	0.051	0.048	0.255	0.593	0.847
	$\hat{S}_{w_3,n}$	0.035	0.051	0.058	0.329	0.649	0.885
	$\hat{S}_{w_4,n}$	0.048	0.050	0.051	0.204	0.517	0.782
	$\hat{S}_{w_5,n}$	0.044	0.050	0.044	0.293	0.587	0.758
Tested copulas: Gumbel	I_n	0.492	0.841	0.973	0.043	0.035	0.069
	$\hat{S}_{w_1,n}$	0.610	0.898	0.985	0.047	0.040	0.064
	$\hat{S}_{w_2,n}$	0.483	0.816	0.972	0.046	0.043	0.065
	$\hat{S}_{w_3,n}$	0.585	0.890	0.984	0.044	0.033	0.067
	$\hat{S}_{w_4,n}$	0.397	0.731	0.944	0.051	0.052	0.068
	$\hat{S}_{w_5,n}$	0.367	0.660	0.872	0.042	0.050	0.051

Table 1: Rejection rates of the null hypothesis of being in a given Archimedean copula class. In the first lines, Clayton copula is being tested, whereas Gumbel copula is tested in the last lines of the table. The datasets are simulated for three different bivariate Kendall's τ. The sample size is set to $n = 200$, the parametric bootstrap size is $N = 500$ and the number of repetitions of the experiment is 1 000.

mance relative to the statistics already available in the literature. Similar conclusions are drawn from the study of the block independence hypothesis ($\mathcal{H}_{0,2}$). In the third one, associated with a simple Archimedean hypothesis ($\mathcal{H}_{0,3;\varphi_0}$), we observe the high power of the statistic that measures the higher-order term in the decomposition. In this context, we gain a significant benefit from the fact that the copula is completely specified and thus the reference law under ($\mathcal{H}_{0,3;\varphi_0}$) is well learned numerically. This explains why this hierarchy is not found in the last study, which again concerns Archimedean copulas but in the sense of an entire class. Despite everything, in part of the results, we noted good performances of our statistics. Providing insights into why certain tests performed well in specific situations, while others did not, is a task that still needs further investigation.

In order to implement this methodological generalization to other test hypotheses, those listed at the end of Section 2.2 but not tested in Section 4, it will be necessary to apply a multiplier bootstrap procedure. It will then be interesting to see if the good qualities of the statistic associated with the highest order term in the decomposition prove to be as effective in an experiment similar to the one conducted in Section 4.4.

The dimensions d or p are small in our experiments. Nevertheless, the current paper offers an interesting perspective on high dimensional problems. The practical implementation of the tests relies indeed on a trade-off between exhaustivity (all subsets of $\mathcal{P}_d$) and dimensionality (exponential growth in d). When d becomes larger, it could be interesting to use only part of the decomposition. With the help of the weight w introduced in the definition of the combined test statistics, we can focus only on a given size of subsets or on all sizes that do not exceed a given size. In this way, we can control the underlying complexity of the method.

The question then becomes: to what extent does this selection affect the corresponding power of the test procedure?

ACKNOWLEDGMENTS

The author expresses her sincere thanks to the editor-in-chief and two anonymous reviewers for their constructive comments which improved the quality of the paper.

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