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Testing the Validity of Lindley Model Based on Nonparametric Probability Density Functions of Entropy Estimators

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Abstract:

- The Lindley distribution is one of the fundamental models applied for reliability models and in the present article, we propose some test statistics for testing the validity of Lindley model based on correcting moments of nonparametric probability density functions of entropy estimators. Critical points and type I error of the tests are obtained and power values of the tests are computed by Monte Carlo simulation. We show that the proposed tests are more powerful than competitor tests. Finally, the proposed tests are illustrated by a real data example.

Keywords:

- *Lindley distribution, Entropy, Kullback-Leibler information, Critical points, Test power, Monte Carlo simulation.*

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1. INTRODUCTION

Entropy, denoted as H, is a measure of uncertainty and information content associated with a continuous random variable. It is a fundamental concept in information theory and provides a measure of the average amount of information required to describe or predict the outcomes of a random variable. Introduced by Shannon in 1948 [24], entropy has been a subject of interest for many researchers in the field of statistics. In statistics, entropy estimation is a method used to quantify the uncertainty or information content associated with a random variable or a probability distribution. Entropy estimation is particularly useful when the underlying probability distribution is unknown or when it deviates significantly from known parametric distributions. Non-parametric entropy estimation methods offer a robust alternative for quantifying uncertainty in data without imposing restrictive assumptions. Various estimators for entropy have been proposed, including those by Vasicek (1976) [26], van Es (1992) [25], Ebrahimi et al. (1994) [11], Correa (1995) [9], Wieczorkowski-Grzegorewski (1999) [28], and Alizadeh (2010) [3].

Entropy estimation has various applications in statistics and related fields. It is widely used in information theory, machine learning, data compression, pattern recognition, and data mining. In statistical analysis, entropy estimation can help identify the complexity or variability of a dataset, assess the amount of information present in a random variable, and compare the uncertainty between different datasets or probability distributions.

In summary, entropy estimation in statistics is a non-parametric method used to quantify the uncertainty or information content associated with a random variable or a probability distribution. It provides a measure of the average amount of information required to describe or predict outcomes.

The entropy $H(f)$, of a continuous random variable X with a density function $f(x)$ is defined as

$$H(f) = -\int_{-\infty}^{\infty} f(x)\, \log f(x)\, dx\,.$$

Consider a random sample $X_1, ..., X_n$ of size n, where the order statistics are denoted as $X_{(1)} \leq X_{(2)} \leq ... \leq X_{(n)}$. Vasicek (1976) [26] introduced an entropy estimator defined as:

$$HV_{mn} = \frac{1}{n}\sum_{i=1}^{n} \log\left\{\frac{n}{2m}(X_{(i+m)} - X_{(i-m)})\right\},$$

where the window size m is a positive integer smaller than $n/2$, $X_{(i)} = X_{(1)}$ if $i < 1$, $X_{(i)} = X_{(n)}$ if $i > n$. He proved the consistency of HV_{mn} as an estimator for the population entropy $H(f)$.

In statistics, a probability density function (pdf) is a function that describes the probability distribution of a continuous random variable. It assigns probabilities to different values or ranges of values that the random variable can take. Traditional parametric pdfs, such as the normal (Gaussian) distribution or the exponential distribution, are defined by a fixed set of parameters that determine their shape, location, and scale. On the other hand, non-parametric pdfs are flexible alternatives that do not rely on specific parametric assumptions about the underlying data distribution. Non-parametric pdf estimation methods aim to estimate the shape and characteristics of the pdf directly from the observed data, without imposing a predefined functional form.

Non-parametric pdf estimation provides several advantages. It allows for more flexibility in modeling complex and diverse data distributions, as it does not restrict the shape of the pdf to a specific parametric form. It also provides a more data-driven approach, relying solely on the observed data to estimate the pdf. This makes non-parametric pdf estimation suitable for a wide range of applications, including exploratory data analysis, hypothesis testing, and modeling in fields such as finance, ecology, and engineering.

Park and Park (2003) [21] derived the nonparametric pdf of Vasicek's estimator as

$$g_v(x) = \begin{cases} 0 & x < \xi_1 \ or \ x > \xi_{n+1} \\ \frac{1}{n}\frac{2m}{x_{(i+m)} - x_{(i-m)}} & \xi_i < x \leq \xi_{i+1} \quad i = 1, ..., n, \end{cases}$$

where $\xi_i = \left(x_{(i-m)} + ... + x_{(i+m-1)}\right)/2m$, and $x_{(i)} = x_{(1)}$ if $i < 1$, $x_{(i)} = x_{(n)}$ if $i > n$.

Ebrahimi et al. (1994) [11] modified the Vasicek's estimator as

$$HE_{mn} = \frac{1}{n}\sum_{i=1}^{n} \log\left\{\frac{n}{c_i m}(X_{(i+m)} - X_{(i-m)})\right\},$$

where

$$c_i = \begin{cases} 1 + \frac{i-1}{m}, & 1 \leq i \leq m, \\ 2, & m+1 \leq i \leq n-m, \\ 1 + \frac{n-i}{m}, & n-m+1 \leq i \leq n. \end{cases}$$

They proved that $HE_{mn} \to H(f)$ as $n \to \infty, m \to \infty$, $m/n \to 0$.

The nonparametric pdf of Ebrahimi et al.'s estimator was derived by Park and Park (2003) [21] as:

$$g_e(x) = \begin{cases} 0 & x < \eta_1 \ or \ x > \eta_{n+1} \\ \frac{1}{n}\frac{1}{\eta_{i+1} - \eta_i} & \eta_i < x \leq \eta_{i+1} \quad i = 1, ..., n, \end{cases}$$

where

$$\eta_i = \begin{cases} \xi_{m+1} - \sum\limits_{k=i}^{m} \frac{1}{m+k-1}\left(x_{(m+k)} - x_{(1)}\right) & if \ \ 1 \leq i \leq m, \\ \frac{\left(x_{(i-m)} + ... + x_{(i+m-1)}\right)}{2m} & if \ \ m+1 \leq i \leq n-m+1, \\ \xi_{n-m+1} + \sum\limits_{k=n-m+2}^{i} \frac{1}{n+m-k+1}\left(x_{(n)} - x_{(k-m-1)}\right) & if \ \ n-m+2 \leq i \leq n+1, \end{cases}$$

Alizadeh and Arghami (2010) [4] proposed another entropy estimator as

$$HA_{mn} = \frac{1}{n}\sum_{i=1}^{n} \log\left\{\frac{n}{a_i m}(X_{(i+m)} - X_{(i-m)})\right\},$$

where

$$a_i = \begin{cases} 1 & 1 \le i \le m, \\ 2 & m+1 \le i \le n-m, \\ 1 & n-m+1 \le i \le n. \end{cases}$$

and $X_{(i-m)} = X_{(1)}$ for $i \le m$ and $X_{(i+m)} = X_{(n)}$ for $i \ge n-m$. They proved the consistency of HA_{mn} as an estimator for $H(f)$. The nonparametric pdf of their estimator was derived as:

$$g_a(x) = \begin{cases} 0 & x < \gamma_1 \; or \; x > \gamma_{n+1} \\ \frac{1}{n}\frac{1}{\gamma_{i+1}-\gamma_i} & \gamma_i < x \le \gamma_{i+1} \quad i = 1, ..., n, \end{cases}$$

where

$$\gamma_i = \begin{cases} \xi_{m+1} - \frac{1}{m}\sum\limits_{k=i}^{m}\left(x_{(m+k)} - x_{(1)}\right) & if \;\; 1 \le i \le m, \\ \frac{\left(x_{(i-m)} + ... + x_{(i+m-1)}\right)}{2m} & if \;\; m+1 \le i \le n-m+1, \\ \xi_{n-m+1} + \frac{1}{m}\sum\limits_{k=n-m+2}^{i}\left(x_{(n)} - x_{(k-m-1)}\right) & if \;\; n-m+2 \le i \le n+1, \end{cases}$$

and $\xi_i = \left(x_{(i-m)} + ... + x_{(i+m-1)}\right)/2m$, $x_{(i)} = x_{(1)}$ if $i < 1$, and $x_{(i)} = x_{(n)}$ if $i > n$.

Park and Park (2003) [21] applied the moments of the nonparametric pdfs of entropy estimators proposed by Vasicek (1976) [26] and Ebrahimi et al. (1994) [11] to construct tests for normality and exponentiality. Additionally, Alizadeh and Arghami (2013) [5] introduced other tests for normal and exponential distributions based on the moments of the nonparametric pdfs of their estimators. In this study, we utilize the nonparametric pdfs of entropy estimators and propose goodness-of-fit tests for the Lindley distribution.

The Lindley distribution is an important statistical model for analyzing reliability data with positive support. This distribution was proposed by Lindley (1958) in the context of Bayesian statistics, as a counter example of fiducial statistics. Its density is

$$f(x;\theta) = \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}, \qquad x > 0, \;\; \theta > 0.$$

The mean and variance of this distribution are

$$\mu = E(X) = \frac{\theta+2}{\theta(\theta+1)},$$

and

$$\sigma^2 = Var(X) = \frac{\theta^2 + 4\theta + 2}{\theta^2(\theta+1)^2},$$

respectively.

Suppose $X_1, ..., X_n$ is a random sample from the Lindley distribution, both the maximum likelihood estimate (MLE) and method of moments estimate of the parameter θ coincides and is

$$\hat{\theta} = \frac{-\left(\bar{X}-1\right) + \sqrt{\left(\bar{X}-1\right)^2 + 8\bar{X}}}{2\bar{X}}, \qquad \bar{X} > 0.$$

Ghitany et al. (2008) [13] conducted a detailed study about various properties of Lindley distribution including skewness, kurtosis, hazard rate function, mean residual life function, stochastic ordering, stress-strength reliability, among other things; estimation of its parameter and application to model waiting time data in a bank. They showed that the estimator $\hat{\theta}$ of θ is positively biased: $E(\hat{\theta}) - \theta > 0$, and it is consistent and asymptotically normal $\sqrt{n}\left(\hat{\theta} - \theta\right) \to N(0, 1/\sigma^2)$.

In the literature of survival analysis and reliability theory, the exponential distribution is widely used as a model of lifetime data. However, the exponential distribution only provides a reasonable fit for modeling phenomenon with constant failure rates. Distributions like gamma, Weibull and lognormal have become suitable alternatives to the exponential distribution in many practical situations. Ghitany et al. (2008) [13] found that the Lindley distribution can be a better model than one based on the exponential distribution.

Shanker et al. (2015) [23] discussed a comparative study of Lindley and exponential distributions for modelling various lifetime data sets from biomedical science and engineering, and concluded that even though there are lifetime data where exponential distribution gives better fit than Lindley distribution and in majority of data sets Lindley distribution gives better fit than exponential distribution.

In complete sample case, Ghitany et al. (2008) [13] developed different distributional properties, reliability characteristics and some inferential procedures for the Lindley distribution. Krishna and Kumar (2011) [16] discussed reliability estimation in Lindley distribution with progressively type II right censored sample. Gupta and Singh (2013) [14] gave parameter estimation of Lindley distribution with hybrid censored data. Also, Al-Mutairi et al. (2013) [6] studied inferences on stress-strength reliability for Lindley distribution with complete sample information. Kumar et al. (2015) [18] discussed estimation of stress-strength reliability using progressively first failure censoring. These studies suggest that in many real-life situations Lindley distribution serves as a better lifetime model than the so far popular distributions like exponential, gamma, Rayleigh, Weibull etc. For more applications of the Lindley distribution one can see Dey et al. (2019) [10], Khan et al. (2020) [15], Nadarajah and Chan (2020) [19], Oliveira, et al. (2021) [20], Al-Babtain, et al. (2021) [2], Ghitany and Wang (2022) [12], Alrasheedi, et al. (2022) [7], Ahsan-ul-Haq et al. (2022) [1], Wang and Weib (2023) [27].

Constructing powerful goodness-of-fit tests for the Lindley distribution holds significant importance. In line with this objective, we propose a set of distribution-free goodness-of-fit tests based on the nonparametric pdfs of entropy estimators. In Section 2, we introduce novel goodness-of-fit tests specifically designed for assessing the composite Lindley hypothesis. We present the details of these tests, including their formulation and statistical properties. Section 3 focuses on the critical values and power analysis of the newly developed tests. We provide insights into the determination of critical values and evaluate the statistical power of the tests under different scenarios. To demonstrate the practical efficacy of the proposed tests, we present two real data examples in Section 4. Through these examples, we evaluate the performance and effectiveness of the tests in practice. Finally, in Section 5, we summarize our findings and conclusions drawn from the study. We discuss the implications of the proposed tests and highlight their potential applications in assessing the goodness of fit for the Lindley distribution.

2. The proposed tests

The Kullback-Leibler (KL) divergence is a measure that quantifies how close a probability density $f(x)$ is to a model density $f_0(x)$. For density functions f and f_0, the KL divergence of f from f_0 is defined (Kullback and Leibler (1951) [17]) by

$$D(f, f_0) = \int f(x) \log \frac{f(x)}{f_0(x)} \, dx \, .$$

Note that $D(f, f_0) = 0$ if and only if $f(x) = f_0(x)$ almost everywhere.
Let f represents the true density and $F_0 = \{f_0(., \theta) : \theta \in \Omega\}$ represents a parametric model for f, where Ω is a subset of R^p. If $f \in F_0$, the minimum value, $\min\limits_{\theta\in\Omega} D(f,\ f_0(.,\theta))$, of the KL divergence is zero and if $f \notin F_0$ the minimum KL divergence is strictly positive.
Let $X_1, ..., X_n$ be a random sample from an unknown continuous cumulative density function $F(x)$ with a density $f(x)$. We want to test the hypothesis

$$H_0 : \ f(x) = f_0(x;\theta) = \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}, \qquad x > 0, \ \ for\ some\ \theta \in \Theta \, ,$$

where θ is unspecified and $\Theta = R^+$. The alternative to H_0 is

$$H_1 : \ f(x) \neq f_0(x;\theta) \qquad for\ any\ \theta \in \Theta \, .$$

Here, we construct tests for the Lindley based on the KL information and the moments of nonparametric pdfs of the entropy estimators as follows.
The asymmetric KL distance of f from f_0 is

$$\begin{aligned} D(f, f_0) &= \textstyle\int f(x) \log \frac{f(x)}{f_0(x;\theta)} \, dx \\ &= -H(f) - 2\log(\theta) + \log(\theta+1) - E_f\left(\log(1+X)\right) + \theta E_f\left(X\right) \, , \end{aligned}$$

$D(f, f_0)$ is minimum (zero) if and only if $f = f_0$, where

$$\begin{aligned} D(f, f_0) &= -H(f) - 2\log(\theta) + \log(\theta+1) - E_f\left(\log(1+X)\right) + \tfrac{\theta+2}{\theta+1} \\ &= \log(\theta+1) - 2\log(\theta) - E_f\left(\log(1+X)\right) + \tfrac{\theta+2}{\theta+1} - H(f), \end{aligned}$$

which can be estimated by

$$TV_{mn} = \log(\hat{\theta}_v + 1) - 2\log(\hat{\theta}_v) - E_{g_v}\left(\log(1+X)\right) + \frac{\hat{\theta}_v + 2}{\hat{\theta}_v + 1} - HV_{mn},$$

where

$$\hat{\theta}_v = \frac{-\left(E_{g_v}(X) - 1\right) + \sqrt{\left(E_{g_v}(X) - 1\right)^2 + 8E_{g_v}(X)}}{2E_{g_v}(X)},$$

and HV_{mn} is Vasicek entropy estimator. We reject H_0 for large values of TV_{mn}. Based on the other entropy estimators, different test statistics can be obtained. We have the following test statistics.

$$TE_{mn} = \log(\hat{\theta}_e + 1) - 2\log(\hat{\theta}_e) - E_{g_e}\left(\log(1+X)\right) + \frac{\hat{\theta}_e + 2}{\hat{\theta}_e + 1} - HE_{mn},$$

$$TA_{mn} = \log(\hat{\theta}_a + 1) - 2\log(\hat{\theta}_a) - E_{g_a}\left(\log(1+X)\right) + \frac{\hat{\theta}_a + 2}{\hat{\theta}_a + 1} - HA_{mn},$$

where

$$\hat{\theta}_e = \frac{-\left(E_{g_e}(X) - 1\right) + \sqrt{\left(E_{g_e}(X) - 1\right)^2 + 8E_{g_e}(X)}}{2E_{g_e}(X)},$$

$$\hat{\theta}_a = \frac{-\left(E_{g_a}(X) - 1\right) + \sqrt{\left(E_{g_a}(X) - 1\right)^2 + 8E_{g_a}(X)}}{2E_{g_a}(X)},$$

and HE_{mn} and HA_{mn} are Ebrahimi et al. (1994) [11]'s and Alizadeh and Arghami (2010) [4]'s entropy estimator, respectively. We reject H_0 for large values of test statistics.
Here, we obtain $E(X)$ and $E\left(\log(1+X)\right)$ under the distributions of g_v, g_e and g_a.

Lemma 2.1. *For the distribution of g_e, we have*

$$E_{g_e}(X) = \frac{1}{n}\left(\frac{\eta_1}{2} + \sum_{i=2}^{n} \eta_i + \frac{\eta_{n+1}}{2}\right),$$

and for g_v and g_a, η_i's replace with ξ_i's and γ_i's, respectively.

Proof: Consider the distribution g_e.

$$\begin{aligned} E_{g_e}(X) &= \int_{\eta_1}^{\eta_{n+1}} x g_e(x) dx \\ &= \sum_{i=1}^{n} \int_{\eta_i}^{\eta_{i+1}} \frac{1}{n} \frac{x}{\eta_{i+1}-\eta_i} dx = \sum_{i=1}^{n} \frac{1}{n} \frac{(\eta_{i+1}^2 - \eta_i^2)}{2(\eta_{i+1}-\eta_i)} \\ &= \frac{1}{2n} \sum_{i=1}^{n} (\eta_{i+1} + \eta_i) = \frac{1}{n}\left(\frac{\eta_1}{2} + \sum_{i=2}^{n} \eta_i + \frac{\eta_{n+1}}{2}\right). \end{aligned}$$

Similarly under other distributions $E(X)$ can be computed. □

Lemma 2.2. *For the distribution of g_e, we have*

$$E_{g_e}\left(\log(1+X)\right) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{(\eta_{i+1}+1)\log(\eta_{i+1}+1) - (\eta_i+1)\log(\eta_i+1)}{\eta_{i+1} - \eta_i}\right) - 1,$$

and for g_v and g_a, η_i's replace with ξ_i's and γ_i's, respectively.

Proof: Consider the distribution g_e.

$$
\begin{aligned}
E_{g_e}(\log(1+X)) &= \int_{\eta_1}^{\eta_{n+1}} \log(1+x) g_e(x) dx \\
&= \sum_{i=1}^{n} \int_{\eta_i}^{\eta_{i+1}} \frac{1}{n} \frac{\log(1+x)}{\eta_{i+1}-\eta_i} dx = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{\eta_{i+1}-\eta_i} \int_{\eta_i}^{\eta_{i+1}} \log(1+x) dx \\
&= \frac{1}{n} \sum_{i=1}^{n} \frac{1}{\eta_{i+1}-\eta_i} \left((\eta_{i+1}+1)\log(\eta_{i+1}+1) - (\eta_{i+1}+1) - (\eta_i+1)\log(\eta_i+1) + (\eta_i+1)\right) \\
&= \frac{1}{n} \sum_{i=1}^{n} \left\{ \frac{(\eta_{i+1}+1)\log(\eta_{i+1}+1) - (\eta_i+1)\log(\eta_i+1)}{\eta_{i+1}-\eta_i} - 1 \right\} \\
&= \frac{1}{n} \sum_{i=1}^{n} \left\{ \frac{(\eta_{i+1}+1)\log(\eta_{i+1}+1) - (\eta_i+1)\log(\eta_i+1)}{\eta_{i+1}-\eta_i} \right\} - 1.
\end{aligned}
$$

Similar to the above argument, the others can be computed. □

Theorem 2.1. *Let F be a distribution with a positive support and F_0 be the Lindley distribution, with the unspecified parameter. Then, under H_0,*

$$TV_{mn} \to 0\,, \qquad \text{in probability as } n \to \infty,\ m \to \infty \text{ and } m/n \to 0\,,$$

and, under H_1, TV_{mn} is a consistent test.

Proof: We note that under H_0,

$$
\begin{aligned}
&\log(\hat{\theta}_v + 1) - 2\log(\hat{\theta}_v) - E_{g_v}(\log(1+X)) + \frac{\hat{\theta}_v+2}{\hat{\theta}_v+1} \\
&\qquad \to \log(\theta+1) - 2\log(\theta) - E_{f_0}(\log(1+X)) + \frac{\theta+2}{\theta+1} = H(f_0),
\end{aligned}
$$

and from Vasicek (1976) [26] when $n \to \infty$, $m \to \infty$, and $m/n \to 0$, we have

$$HV_{mn} \to H(f_0)$$

So

$$TV_{mn} \to 0.$$

Also, under H_1,

$$HV_{mn} \to H(f),$$

Thus

$$TV_{mn} \to H(f_0) - H(f) > 0.$$

(all limits are in probability). □

The above theorem is indeed satisfied for the other tests.
Finally, we can write the proposed test statistics as follow.

$$TV_{mn} = 2 + \log\left(\frac{\hat{\theta}_v+1}{\hat{\theta}_v^2}\right) + \frac{1}{\hat{\theta}_v+1} - \frac{1}{n} \sum_{i=1}^{n} \left\{ \frac{(\xi_{i+1}+1)\log(\xi_{i+1}+1) - (\xi_i+1)\log(\xi_i+1)}{\xi_{i+1}-\xi_i} \right\} - HV_{mn},$$

$$TE_{mn} = 2 + \log\left(\frac{\hat{\theta}_e+1}{\hat{\theta}_e^2}\right) + \frac{1}{\hat{\theta}_e+1} - \frac{1}{n} \sum_{i=1}^{n} \left\{ \frac{(\eta_{i+1}+1)\log(\eta_{i+1}+1) - (\eta_i+1)\log(\eta_i+1)}{\eta_{i+1}-\eta_i} \right\} - HE_{mn},$$

$$TA_{mn} = 2 + \log\left(\frac{\hat{\theta}_a+1}{\hat{\theta}_a^2}\right) + \frac{1}{\hat{\theta}_a+1} - \frac{1}{n} \sum_{i=1}^{n} \left\{ \frac{(\gamma_{i+1}+1)\log(\gamma_{i+1}+1) - (\gamma_i+1)\log(\gamma_i+1)}{\gamma_{i+1}-\gamma_i} \right\} - HA_{mn}.$$

3. Critical values and power study

The null hypothesis H_0, at the significance level α is rejected if

$$TV_{mn}\,(TE_{mn}, TA_{mn}) \geq C(\alpha),$$

where the critical point $C(\alpha)$ is determined by the $\alpha-$quantile of the distribution of the test statistics under the null hypothesis.
In order to obtain the critical points of the test statistics, 100,000 samples of size n were generated from the Lindley distribution with the parameter one. For each sample ($n = 10, 20, 30, 40, 50, 75, 100$) the test statistics was computed and by using these values the critical points $C(\alpha)$, were determined. The following steps outline the methodology for determining the critical values:

1. Generate a random sample, denoted as $X_1, ..., X_n$, of size n, drawn from the Lindley(1) distribution.
2. Calculate the test statistic based on the sample $X_1, ..., X_n$.
3. Repeat steps 1 and 2 a significant number of times, creating a large number of simulated datasets and corresponding test statistic.
4. Determine the $(1 - \alpha)$th quantile of the test statistic obtained from the simulations. This quantile represents the critical value for the test at the desired significance level α.

The critical points of the statistics TV_{mn}, TE_{mn} and TA_{mn} are presented in Table 1.

Table 1: Critical values of the proposed test statistics for $\alpha = 0.05$

n	m	TV_{mn}	TE_{mn}	TA_{mn}
10	2	0.7007	0.5277	0.4824
20	3	0.4067	0.2999	0.2710
30	4	0.3012	0.2168	0.2009
40	5	0.2464	0.1729	0.1665
50	5	0.2109	0.1531	0.1497
75	6	0.1608	0.1176	0.1209
100	7	0.1330	0.0975	0.1053

Figures 1 and 2 show the empirical probability density functions of the test statistics with Monte Carlo samples. From figures, TA_{mn} have closer values to 0 than the other statistics. Then the bias of TA_{mn} is smallest.

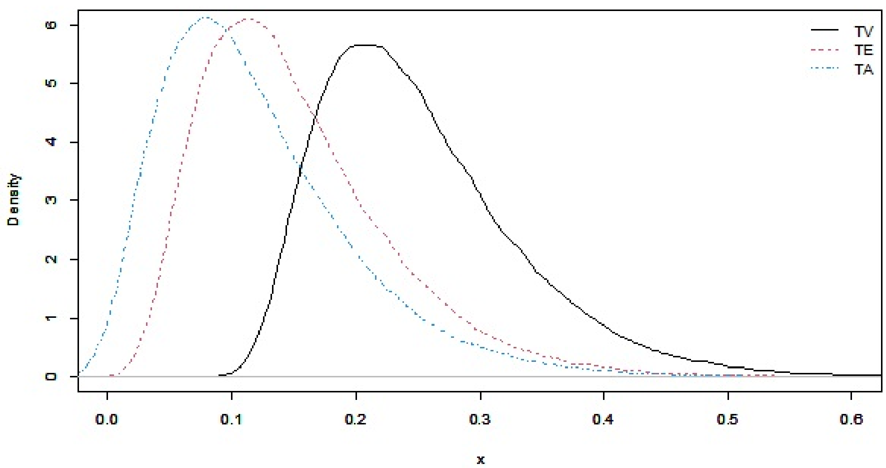

Figure 1: Empirical densities of the test statistics based on 100,000 simulations under the Lindley hypothesis and $n = 20$.

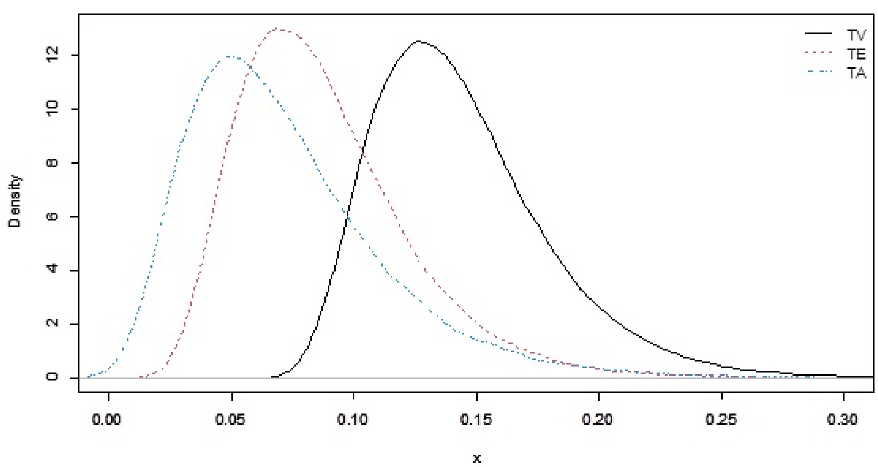

Figure 2: Empirical densities of the test statistics based on 100,000 simulations under the Lindley hypothesis and $n = 50$.

We also evaluate in Table 2 the estimated type I error control using the 0.05 percentiles of the proposed test ($\alpha = 0.05$). We generated random samples from a spectrum of Lindley populations and then obtained the actual sizes of the tests. The results are presented in Table 2.
It is evident, from Table 2, that the actual sizes of the tests are approximately equal to the nominal size 0.05. Therefore, we can conclude that the empirical percentiles presented in Table 1 provides an excellent type I error control.
In practice, a general guide for the choice of m for a fixed n is valuable to the users. Our simulations show that the optimal m (in terms of power) depends on the sample size and the alternative hypothesis. In other hand, there is no a value m that is optimal against all alternatives. Therefore, if one wants to guard against all alternatives a compromise should be made. Based our power study, we recommend for all the test statistics $m = 2$ for $n = 10$, $m = 3$ for $n = 20$, $m = 4$ for $n = 30$, and $m = 5$ for $n = 50$ as the optimal values which the tests attain good (not best) power values against all alternatives. Generally, with increasing n, an optimal choice of m also increases, while the ratio m/n tends to zero.

Table 2: Type I error control of the tests for the nominal significance level $\alpha = 0.05$.

Alternative	n	TV_{mn}	TE_{mn}	TA_{mn}
Lindley(0.5)	10	0.0515	0.0529	0.0517
	20	0.0495	0.0517	0.0524
	30	0.0496	0.0524	0.0517
	50	0.0482	0.0506	0.0491
Lindley(2)	10	0.0514	0.0508	0.0504
	20	0.0485	0.0482	0.0489
	30	0.0473	0.0466	0.0460
	50	0.0491	0.0488	0.0483
Lindley(4)	10	0.0498	0.0493	0.0485
	20	0.0508	0.0491	0.0478
	30	0.0502	0.0490	0.0465
	50	0.0512	0.0506	0.0461
Lindley(4)	10	0.0515	0.0509	0.0491
	20	0.0526	0.0511	0.0463
	30	0.0515	0.0491	0.0428
	50	0.0531	0.0508	0.0416

For power comparison, we consider the well-known tests based on the empirical distribution function (EDF) that used widely in practice. These tests are Cramer von Mises W^2, Kolmogorov-Smirnov D, Anderson-Darling A^2, Kuiper V, and Watson U^2. The test statistics of the EDF-tests are briefly described as follows.

For more details about these tests, see D'Agostino and Stephens (1986). Let $X_{(1)} \leq X_{(2)} \leq ... \leq X_{(n)}$ are the order statistics based on the random sample $X_1, ..., X_n$.
1. The Cramer-von Mises statistic (1931):

$$W^2 = \frac{1}{12n} + \sum_{i=1}^{n} \left(\frac{2i-1}{2n} - F_0(X_{(i)}; \hat{\theta}) \right)^2 .$$

2. The Watson statistic (1961):

$$U^2 = W^2 - n\big(\bar{P} - 0.5\big)^2,$$

where $\bar{P}$ is the mean of $F_0(X_{(i)}; \hat{\theta})$, $i = 1, ..., n$. 3. The Kolmogorov-Smirnov statistic (1933):

$$D = \max(D^+, D^-),$$

where

$$D^+ = \max_{1 \leq i \leq n} \left\{ \frac{i}{n} - F_0(X_{(i)}; \hat{\theta}) \right\}; \quad D^- = \max_{1 \leq i \leq n} \left\{ F_0(X_{(i)}; \hat{\theta}) - \frac{i-1}{n} \right\}.$$

4. The Kuiper statistic (1960):

$$V = D^+ + D^-.$$

5. The Anderson-Darling statistic (1952):

$$A^2 = -n - \frac{1}{n} \sum_{i=1}^{n} (2i-1) \left\{ \log F_0(X_{(i)}; \hat{\theta}) + \log \left[1 - F_0(X_{(n-i+1)}; \hat{\theta}) \right] \right\}.$$

In the above test statistics, $F_0(x)$ is the cumulative density distribution function of the Lindley distribution and $\hat{\theta}$ is the maximum likelihood estimate of the parameter θ.
We compute the power of the considered tests and the proposed tests against various distributions. In power comparison, we considered the following alternatives.

- the Weibull distribution with density $\theta x^{\theta-1} \exp\left(-x^{\theta}\right)$, denoted by $W(\theta)$;
- the gamma distribution with density $\Gamma(\theta)^{-1} x^{\theta-1} \exp(-x)$, denoted by $\Gamma(\theta)$;
- the lognormal distribution $LN(\theta)$ with density $(\theta x)^{-1} (2\pi)^{-1/2} \exp\left(-(\log x)^2 \big/ (2\theta^2)\right)$;
- the half-normal HN distribution with density $\Gamma(2/\pi)^{1/2} \exp\left(-x^2/2\right)$;
- the uniform distribution U with density 1, $0 \leq x \leq 1$;
- the modified extreme value $EV(\theta)$, with distribution function $1 - \exp\left(\theta^{-1}(1 - e^{x})\right)$;
- the linear increasing failure rate law $LF(\theta)$ with density $(1+\theta x) \exp\left(-x - \theta x^2/2\right)$;

- Dhillon's (1981) distribution with distribution function $1-\exp\left(-(\log(x+1))^{\theta+1}\right)$;
- Chen's (2000) distribution $CH(\theta)$, with distribution function $1-\exp\left(2\left(1-e^{x^{\theta}}\right)\right)$.

These alternatives include densities f with decreasing failure rates (DFR), increasing failure rates (IFR) as well as models with unimodal failure rate (UFR) functions and bathtub failure rate (BFR) functions.
To assess the power values of the tests, we generate 100,000 random samples from the alternative hypothesis for different choices of sample sizes and then the test statistics are calculated. Then power of the corresponding test is computed by the frequency of the event "the statistic is in the critical region".
Here's our algorithm for calculating the power of tests.

1. Set the desired significance level (α) for your hypothesis test.
2. Choose the alternative hypothesis you want to test against.
3. Specify the sample size (n) for the test.
4. Set the number of iterations (e.g., 100,000) for generating random samples.
5. Initialize a counter for the number of times the test statistic falls into the critical region.
6. For each iteration:
   - Simulate a random sample from the alternative hypothesis distribution, with the chosen sample size (n).
   - Calculate the test statistic for the generated sample.
   - Determine the critical value corresponding to the chosen significance level (α) and the null hypothesis distribution.
   - If the test statistic falls into the critical region, increment the counter.
7. Calculate the power as the ratio of the counter to the total number of iterations.
8. Return the estimated power value.

By simulating random samples from the alternative hypothesis distribution and comparing the test statistic to the critical value, the algorithm determines the frequency at which the test statistic falls into the critical region, allowing for the estimation of the power of the test.
Tables 3 and 4 display and compares the power values of the tests for sample sizes $n = 10, 20, 30, 50$ at the significance level $\alpha = 0.05$.

Table 3: Empirical powers of the tests against IFR alternatives at significance level 5%.

Alternative	n	W^2	D	V	U^2	A^2	TV_{mn}	TE_{mn}	TA_{mn}
$W(1.4)$	10	0.1303	0.1174	0.1104	0.1170	0.0894	0.1445	0.1345	0.1183
	20	0.2258	0.1966	0.1761	0.1884	0.1917	0.2171	0.1985	0.1456
	30	0.3237	0.2691	0.2330	0.2635	0.2967	0.2920	0.2599	0.1486
	50	0.5098	0.4231	0.3736	0.4167	0.5036	0.4088	0.3629	0.1786
$\Gamma(2)$	10	0.1175	0.1028	0.1101	0.1188	0.0810	0.1486	0.1452	0.1315
	20	0.2011	0.1754	0.1772	0.1935	0.1800	0.2323	0.2226	0.1790
	30	0.2879	0.2412	0.2369	0.2687	0.2827	0.3233	0.3072	0.2078
	50	0.4745	0.4014	0.3875	0.4408	0.5104	0.4654	0.4460	0.2727
HN	10	0.0952	0.0887	0.0844	0.0875	0.0678	0.0970	0.0896	0.0773
	20	0.1364	0.1234	0.1084	0.1149	0.1076	0.1286	0.1125	0.0735
	30	0.1835	0.1552	0.1340	0.1446	0.1492	0.1579	0.1298	0.0583
	50	0.2839	0.2321	0.1960	0.2139	0.2445	0.2124	0.1667	0.0526
U	10	0.3386	0.2647	0.3088	0.2957	0.2615	0.4008	0.3658	0.3110
	20	0.6318	0.4888	0.6071	0.5477	0.5793	0.7798	0.7317	0.5952
	30	0.8309	0.6764	0.8143	0.7416	0.8056	0.9494	0.9236	0.7780
	50	0.9756	0.9000	0.9777	0.9417	0.9756	0.9987	0.9971	0.9649
$CH(1)$	10	0.0937	0.0868	0.0772	0.0789	0.0673	0.0967	0.0883	0.0750
	20	0.1364	0.1220	0.0998	0.1061	0.1074	0.1264	0.1095	0.0689
	30	0.1826	0.1557	0.1230	0.1332	0.1477	0.1596	0.1290	0.0563
	50	0.2796	0.2301	0.1810	0.1933	0.2379	0.2145	0.1654	0.0472
$CH(1.5)$	10	0.4268	0.3505	0.3359	0.3553	0.3348	0.3993	0.3714	0.3251
	20	0.7600	0.6343	0.6239	0.6480	0.7160	0.7109	0.6684	0.5439
	30	0.9200	0.8205	0.8176	0.8370	0.9071	0.8898	0.8544	0.6785
	50	0.9943	0.9684	0.9736	0.9763	0.9943	0.9880	0.9796	0.8865
$LF(2)$	10	0.1386	0.1235	0.1113	0.1187	0.0972	0.1366	0.1253	0.1069
	20	0.2282	0.1943	0.1706	0.1802	0.1851	0.1937	0.1697	0.1141
	30	0.3292	0.2723	0.2327	0.2527	0.2828	0.0.2573	0.2161	0.1053
	50	0.5133	0.4204	0.3663	0.3955	0.4662	0.3645	0.3026	0.1185
$LF(4)$	10	0.2056	0.1790	0.1594	0.1700	0.1469	0.1879	0.1732	0.1485
	20	0.3777	0.3160	0.2752	0.2980	0.3192	0.3016	0.2675	0.1884
	30	0.5308	0.4386	0.3864	0.4204	0.4758	0.4157	0.3630	0.2032
	50	0.7680	0.6595	0.6067	0.6401	0.7313	0.5868	0.5207	0.2643
$EV(0.5)$	10	0.0923	0.0861	0.0749	0.0782	0.0670	0.0951	0.0872	0.0744
	20	0.1384	0.1221	0.1020	0.1074	0.1068	0.1293	0.1115	0.0713
	30	0.1833	0.1557	0.1242	0.1345	0.1467	0.1591	0.1285	0.0537
	50	0.2779	0.2262	0.1803	0.1933	0.2378	0.2153	0.1658	0.0454
$EV(1.5)$	10	0.1681	0.1456	0.1420	0.1547	0.1170	0.1745	0.1594	0.1344
	20	0.3359	0.2706	0.2529	0.2634	0.2658	0.3060	0.2696	0.1771
	30	0.4612	0.3645	0.3618	0.3805	0.4152	0.4370	0.3734	0.1929
	50	0.7218	0.5906	0.5811	0.5943	0.6880	0.6424	0.5659	0.2656

Table 4: Empirical powers of the tests against UFR, DFR and BFR alternatives at significance level 5%.

Alternative	n	W^2	D	V	U^2	A^2	TV_{mn}	TE_{mn}	TA_{mn}
$LN(0.8)$	10	0.1413	0.1302	0.1279	0.1403	0.1068	0.1592	0.1710	0.1823
	20	0.2221	0.1968	0.2204	0.2448	0.2110	0.3197	0.3549	0.3806
	30	0.3180	0.2720	0.3268	0.3652	0.3440	0.4968	0.5451	0.5404
	50	0.5147	0.4436	0.5541	0.6054	0.6131	0.7713	0.8079	0.7577
$LN(1.5)$	10	0.5140	0.4823	0.3849	0.4001	0.5544	0.2743	0.3490	0.4208
	20	0.8027	0.7664	0.6690	0.6869	0.8197	0.6437	0.7177	0.7922
	30	0.9257	0.9020	0.8342	0.8489	0.9306	0.8459	0.8892	0.9265
	50	0.9900	0.9842	0.9642	0.9697	0.9905	0.9775	0.9861	0.9904
$DL(1)$	10	0.0877	0.0813	0.0809	0.0862	0.0629	0.1033	0.1096	0.1132
	20	0.1185	0.1064	0.1139	0.1236	0.1041	0.1503	0.1694	0.1878
	30	0.1486	0.1274	0.1445	0.1619	0.1445	0.2112	0.2433	0.2564
	50	0.2123	0.1771	0.2245	0.2533	0.2394	0.3284	0.3734	0.3649
$DL(1.5)$	10	0.1999	0.1735	0.1751	0.1937	0.1462	0.2305	0.2242	0.2063
	20	0.3844	0.3271	0.3228	0.3634	0.3601	0.3950	0.3849	0.3254
	30	0.5568	0.4783	0.4598	0.5241	0.5677	0.5569	0.5479	0.4206
	50	0.8123	0.7363	0.7129	0.7832	0.8509	0.7780	0.7685	0.6043
$W(0.8)$	10	0.1960	0.1750	0.1288	0.1366	0.2748	0.0471	0.0703	0.1019
	20	0.3570	0.3095	0.2295	0.2438	0.4417	0.1259	0.1818	0.2693
	30	0.4933	0.4319	0.3201	0.3476	0.5752	0.2312	0.3076	0.4151
	50	0.7062	0.6330	0.5093	0.5395	0.7720	0.4505	0.5335	0.6151
$\Gamma(0.4)$	10	0.5137	0.4712	0.3701	0.3914	0.7163	0.1941	0.2425	0.2961
	20	0.8109	0.7663	0.6579	0.6850	0.9222	0.5510	0.6146	0.6838
	30	0.9354	0.9074	0.8310	0.8551	0.9810	0.7874	0.8304	0.8673
	50	0.9943	0.9894	0.9697	0.9762	0.9990	0.9693	0.9766	0.9800
$CH(0.5)$	10	0.3912	0.3546	0.2711	0.2860	0.5728	0.1104	0.1449	0.1812
	20	0.6670	0.6127	0.4979	0.5281	0.8141	0.3468	0.4061	0.4780
	30	0.8331	0.7839	0.6733	0.7102	0.9251	0.5728	0.6320	0.6848
	50	0.9669	0.9464	0.8924	0.9137	0.9903	0.8670	0.8927	0.8995

Based on the power values in Table 3, it is seen that the tests based on W^2 and TV_{mn} statistics have the most power against IFR alternatives. Although for this type of alternatives the tests W^2 and TV_{mn} have the most power but the power differences of these tests with each other are small and we can select one of the tests based on W^2 or TV_{mn} statistic as a powerful test.
From Table 4, it is evident that the tests based on A^2 or TA_{mn} statistics have the most power against UFR alternatives and power differences between these

tests and the other tests are substantial.
Tables 4 reveals a superiority of the test based on A^2 statistic to all other tests as we can say that this test outperforms all other tests against DFR and BFR alternatives.
Although there is no uniformly most powerful test against all alternatives, the tests based on W^2, A^2, TV_{mn} and TA_{mn} statistics can be recommended in practice. Depending on type of alternative, we can say that among EDF-based tests, the tests W^2 and A^2 have the most power and among entropy-based tests, the tests TV_{mn}, TE_{mn} and TA_{mn} have the most power. In general, we can conclude that the tests W^2, A^2, TV_{mn} and TA_{mn} have a good performance and therefore can be used in practice.

4. Applications to real data

In this section, to show the behavior of the proposed tests in real cases, two real data sets are analyzed. In Figures 3 and 4, we depict the histogram of these data sets.

Example 4.1. The following data set gives data of the failure times of 25 ball bearings in endurance test presented by Caroni (2002) [8].
17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.48, 51.84, 51.96, 54.12, 55.56, 67.80, 67.80, 67.80, 68.64, 86.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.40.
Our procedures can be used to investigate whether the data come from a Lindley distribution. The values of the proposed test statistics are

$$TV_{mn} = 0.2636,\ TE_{mn} = 0.1822,\ TA_{mn} = 0.1409,$$

and the critical values at the 5% are 0.3492, 0.2651, 0.2449, respectively.
Since the values of the test statistics are smaller than the corresponding critical values, the Lindley assumption is not rejected at the significance level of 0.05. Therefore, the null hypothesis that the failure times of ball bearings follow a Lindley distribution is not rejected.

Example 4.2. The following data set is derived from days between air-conditioning failure in Boeing 720 aircraft. These data are presented by Pearson and Hartley (1972) [22].
0.013, 0.052, 0.143, 0.208, 0.234, 0.234, 0.234, 0.312, 0.404, 0.508, 0.599, 0.664, 0.703, 0.820, 0.885, 1.002, 1.041, 1.067, 1.263, 1.380, 1.445, 1.836, 1.849, 2.122, 2.486, 2.682, 2.812.
The proposed tests can be used to investigate whether these data come from a Lindley distribution. The values of the test statistics are computed as

$$TV_{mn} = 0.1842,\ TE_{mn} = 0.0822,\ TA_{mn} = 0.0311,$$

and the critical values at the significance level 5% are obtained as 0.3285, 0.2176, 0.1993, respectively.
It is evident that the values of the test statistics are smaller than the corresponding critical values and consequently the Lindley assumption is not rejected at the significance level of 0.05. Therefore, the null hypothesis that the days between air-conditioning failure in Boeing 720 aircraft follow a Lindley distribution is not rejected.

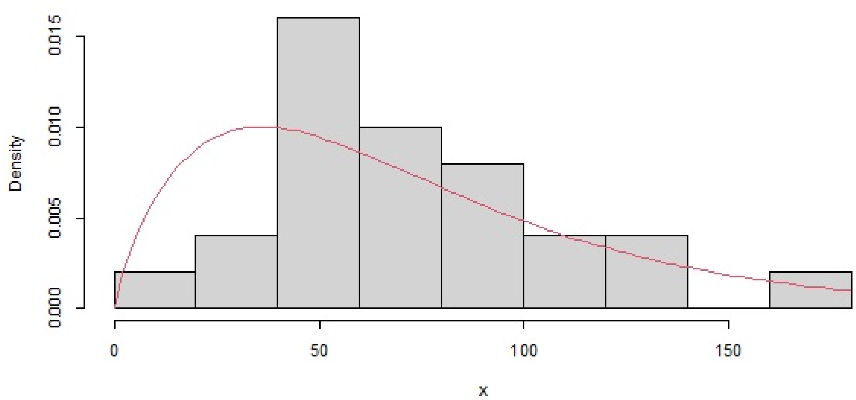

Figure 3: Histogram of failure times of ball bearings and a fitted Lindley density function.

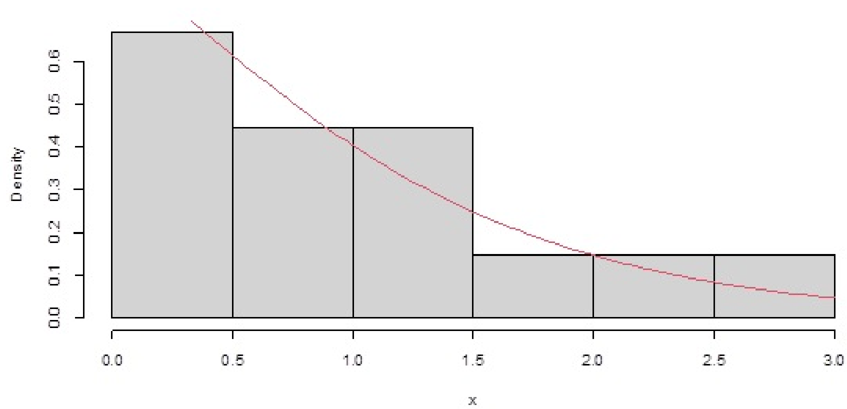

Figure 4: Histogram of days between air-conditioning failure in Boeing 720 aircraft and a fitted Lindley density function.

5. Conclusions

In this paper, we obtained the minimum discriminant information (MDI) loss estimator for the unknown parameter of the Lindley distribution. Then, based on the nonparametric distribution functions of the entropy estimators, we proposed goodness of fit test statistics for the Lindley distribution. We obtained the power values of the proposed test statistics with Monte Carlo simulation and compared them with the competing test statistics against various alternatives. Generally, we concluded that among the proposed and competing test statistics, the proposed tests have a good performance against different alternatives. Therefore, these tests can be confidently recommended in practice. Finally, we illustrated the performance of the new test statistics in real cases.

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REFERENCES

[1] Ahsan-ul-Haq, M., Choudhary, S.M., AL-Marshadi, A.H. and Aslam, M. (2022). A new generalization of Lindley distribution for modeling of wind speed data, *Energy Reports*, **8**, 1–11.

[2] Al-Babtain, A.A., Kumar, D., Gemeay, A.M., Dey, S. and Afify, A.Z. (2021). Modeling engineering data using extended power-Lindley distribution: Properties and estimation methods, *Journal of King Saud University - Science*, **33**, 101582.

[3] Alizadeh, H.N. (2010). A new estimator of entropy and its application in testing normality, *Journal of Statistical Computation and Simulation*, **80**, 1151–1162.

[4] Alizadeh, H.N. and Arghami N.R. (2010). A new estimator of entropy, *Journal of the Iranian Statistical Society*, **9**, 53–64.

[5] Alizadeh, H.N. and Arghami N.R. (2013). Goodness-of-fit tests based on correcting moments of entropy estimators, *Communications in Statistics – Simulation and Computation*, **42**, 499–513.

[6] Al-Mutairi, D.K., Ghitany, M.E. and Kundu, D. (2013). Inferences on the stress-strength reliability from Lindley distributions, *Communications in Statistics–Theory and Methods*, **42**, 1443–1463.

[7] Alrasheedi, A., Abouammoh, A. and Kayid, M. (2022). A new flexible extension of the Lindley distribution with applications, *Journal of King Saud University - Science*, **34**, 101714.

[8] Caroni, C. (2002). The correct ball bearing data, *Lifetime Data Analysis*, **8**, 395–399.

[9] Correa, J.C. (1995). A new estimator of entropy, *Communications in Statistics–Theory and Methods*, **24**, 2439–2449.

[10] Dey, S., Nassar, M. and Kumar, D. (2019). Alpha power transformed inverse Lindley distribution: A distribution with an upside-down bathtub-shaped hazard function, *Journal of Computational and Applied Mathematics*, **348**, 130–145.

[11] Ebrahimi, N., Pflughoeft, K. and Soofi, E. (1994). Two measures of sample entropy, *Statistics and Probability Letters*, **20**, 225–234.

[12] Ghitany, M.E. and Wang, S. (2022). A note on parameter asymptotics for weighted Lindley distribution, *Communications in Statistics - Simulation and Computation*, **51**, 1513–1524.

[13] Ghitany, M.E., Atieh, B. and Nadarajah, S. (2008). Lindley distribution and its Application, *Mathematics Computing and Simulation*, **78**, 493–506.

[14] Gupta, P.K. and Singh, B. (2013). Parameter estimation of Lindley distribution with hybrid censored data, *International Journal of System Assurance Engineering and Management*, **4**, 378–385.

[15] Khan, M.J.S. Sharma, A. and Iqrar, S. (2020). On Moments of Lindley Distribution Based on Generalized Order Statistics, *American Journal of Mathematical and Management Sciences*, **39**, 214–233.

[16] Krishna, H. and Kumar, K. (2011). Reliability estimation in Lindley distribution with progressively type II right censored sample, *Mathematics and Computers in Simulation*, **82**, 281–294.

[17] Kullback, S. and Leibler, R.A. (1951). On information and sufficiency, *The Annals of Mathematical Statistics*, **22**, 79–86.

[18] Kumar, K., Krishna, H. and Garg, R. (2015). Estimation of P(Y¡X) in Lindley distribution using progressively first failure censoring, *International Journal of System Assurance Engineering and Management*, **22**, 330–341.

[19] Nadarajah, S. and Chan, S. (2020). On moments of the unit Lindley distribution, *Journal of Applied Statistics*, **47**, 947–949.

[20] Oliveira, R.P., Achcar, J.A., Mazucheli, J. and Bertoli, W. (2021). A new class of bivariate Lindley distributions based on stress and shock models and some of their reliability properties, *Reliability Engineering & System Safety*, **211**, 107528.

[21] Park, S. and Park, D. (2003). Correcting moments for goodness of fit tests based on two entropy estimates, *Journal of Statistical Computation and Simulation*, **73**, 685–694.

[22] Pearson, E.S. and Hartley, H.O. (1972). *Biometrika Tables for Statisticians*, Volume H. Cambridge University Press, Cambridge.

[23] Shanker, R., Hagos, F. and Sujatha, S. (2015). On modeling of Lifetimes data using exponential and Lindley distributions, *Biometrics & Biostatistics International Journal*, **2**, 1–9.

[24] Shannon, C.E. (1948). A mathematical theory of communications, *Bell System Technical Journal*, **27**, 379–423; 623–656.

[25] van Es, B. (1992). Estimating functionals related to a density by class of statistics based on spacings, *Scandinavian Journal of Statistics*, **19**, 61–72.

[26] Vasicek, O. (1976). A test for normality based on sample entropy, *Journal of the Royal Statistical Society, Series B*, **38**, 54–59.

[27] Wang, S. and Weib, C.H. (2023). New characterizations of the (discrete) Lindley distribution and their applications, *Mathematics and Computers in Simulation*, **212**, 310–322.

[28] Wieczorkowski, R. and Grzegorzewsky, P. (1999). Entropy estimators improvements and comparisons, *Communications in Statistics–Simulation and Computation*, **28**, 541–567.