Supplementary result for

"A refined extreme quantile estimator for Weibull tail-distributions"

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A.3. Proofs of auxiliary results

Proof of Lemma 1.1. Let $\theta = 1/v$ and $x > 0$. A second-order Taylor expansion yields, as $t \to \infty$,

$$\begin{aligned} U(tx) - U(t) &= \lambda(\log t)^\theta \left(\left(1 + \frac{\log x}{\log t}\right)^\theta - 1 \right) \\ &= \lambda\theta(\log x)(\log t)^{\theta-1} \left(1 + \frac{\theta-1}{2}\frac{\log x}{\log t}(1+o(1))\right). \end{aligned}$$

Remarking that $K_0(x) = \log x$ and letting $a(t) = \lambda\theta(\log t)^{\theta-1}$, it follows

$$\frac{U(tx)-U(t)}{a(t)} - \log x = \frac{\theta-1}{2}\frac{(\log x)^2}{\log t}(1+o(1)) = A(t)\int_1^x \frac{\log s}{s} ds\ (1+o(1)),$$

where $A(t) := (\theta - 1)/\log t$, the result is thus proved. □

Proof of Lemma 1.2. The result is straightforwardly true for $\beta = 0$. Let us then focus on the case where $\beta \in (0, 1/2]$ and consider $u \in (0, 1]$. Differentiating three times, one gets

$$\begin{aligned} g'_\beta(u) &= \frac{1}{u}(1 - 2u + u^\beta(1 + \beta\log(u))) =: \frac{\tilde{g}_\beta(u)}{u}, \\ \tilde{g}'_\beta(u) &= -2 + \beta u^{\beta-1}(2 + \beta\log(u)), \\ \tilde{g}''_\beta(u) &= \beta u^{\beta-2}(3\beta - 2 + (\beta^2 - \beta)\log(u)). \end{aligned}$$

Let $u_0(\beta) = \exp\left(\frac{2-3\beta}{\beta(\beta-1)}\right)$ be the unique point in $u \in (0, 1]$ such that $\tilde{g}''_\beta(u) = 0$ when $\beta \in (0, 1/2]$. It is easily checked that $\tilde{g}''_\beta(u) \geq 0$ when $u \in (0, u_0(\beta)]$ while $\tilde{g}''_\beta(u) \leq 0$ when $u \in [u_0(\beta), 1]$. As a consequence, $\tilde{g}'_\beta(\cdot)$ has a global maxima $m(\beta)$ on $(0, 1]$ at $u_0(\beta)$ given by

$$m_0(\beta) := \tilde{g}'_\beta(u_0(\beta)) = -2 - \frac{\beta^2}{\beta-1}\exp\left(\frac{2-3\beta}{\beta}\right).$$

The sign of $m_0(\beta)$ depends on $\beta \in (0, 1/2]$. Observe that $m_0(\beta) \to +\infty$ as $\beta \to 0$, $m_0(1/2) = -2 + e/2 < 0$ and

$$m'_0(\beta) = -\frac{\beta^2 - 4\beta + 2}{(\beta-1)^2}\exp\left(\frac{2-3\beta}{\beta}\right) < 0,$$

for all $\beta \in (0, 1/2]$. As a consequence, there exists a unique $\beta_0 \in (0, 1/2]$ such that $m_0(\beta) = 0$ and $m_0(\beta) \geq 0$ for all $\beta \in (0, \beta_0]$ while $m_0(\beta) \leq 0$ when $\beta \in [\beta_0, 1/2]$.

Two cases appear:

- If $\beta \in [\beta_0, 1/2]$, then $m_0(\beta) \leq 0$ and consequently $\tilde{g}'_\beta(u) \leq 0$ for all $u \in [0,1]$. The function $\tilde{g}_\beta(\cdot)$ is thus decreasing on $(0,1]$, and taking account of $\tilde{g}_\beta(1) = 0$ yields $\tilde{g}_\beta(u) > 0$ for all $u \in (0,1)$.

- If $\beta \in (0, \beta_0]$, then $m_0(\beta) \geq 0$ and there exist two unique points $u_1(\beta) \in (0, u_0(\beta)]$ and $u_2(\beta) \in [u_0(\beta), 1]$ such that $\tilde{g}'_\beta(u) = 0$. The function $\tilde{g}_\beta(\cdot)$ is thus decreasing on $(0, u_1(\beta)]$, increasing on $[u_1(\beta), u_2(\beta)]$ and decreasing on $[u_2(\beta), 1]$ see Figure 5 for an illustration. It has two local minima given by
$$\tilde{g}_\beta(u_1(\beta)) = 1 - u_1^\beta(\beta) + 2u_1(\beta)\left(\frac{1-\beta}{\beta}\right) > 0,$$
and $\tilde{g}_\beta(1) = 0$. This proves that $\tilde{g}_\beta(u) > 0$ for all $u \in (0,1)$.

As a conclusion, in both cases, $\tilde{g}_\beta(u) > 0$ and $g'_\beta(u) > 0$ for all $u \in (0,1)$ which implies that $g_\beta(\cdot)$ is an increasing function on $(0,1)$. Since $g_\beta(1) = 0$ then $g_\beta(u) < 0$ for all $u \in (0,1)$ and $\beta \in (0, 1/2]$. □

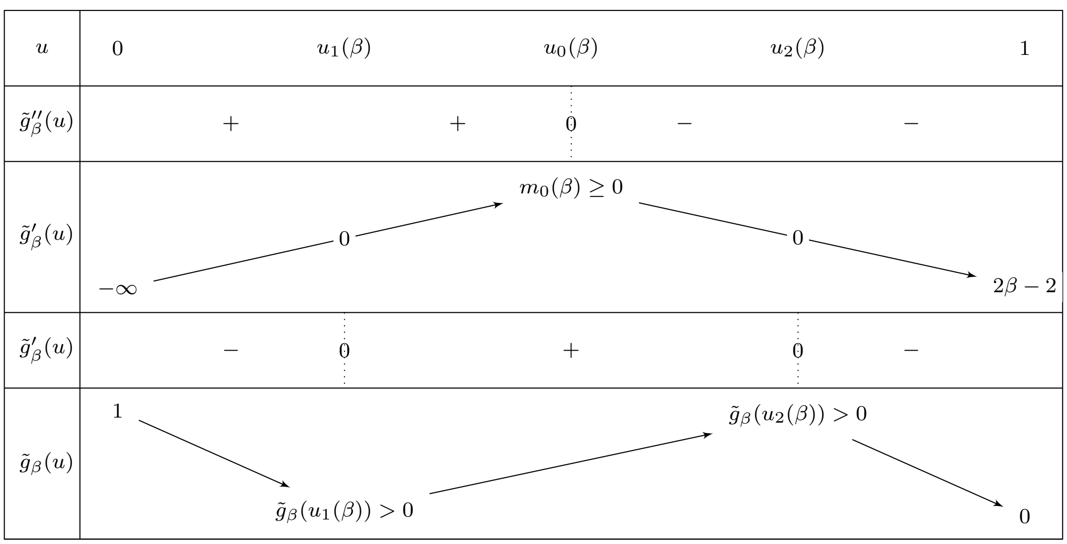

Figure 5: Variations of the function $\tilde{g}_\beta(\cdot)$, $\beta \in (0, 1/2]$ studied in the proof of Lemma 1.2.

Proof of Lemma 1.3. **(i)** A first-order Taylor expansion shows that $h_y(\cdot)$ can be extended by continuity letting $h_y(1) = 1$ for all $y \leq 0$.
(ii) Differentiating twice, one gets for all $y < 0$ and $\tau > 1$:

$$h'_y(\tau) = \frac{y}{(\tau^y - 1)^2}\left(\frac{\tau^y - 1}{\tau} - y\tau^{y-1}\log(\tau)\right) = \frac{y}{\tau(\tau^y-1)^2}\left(\tau^y - 1 - y\tau^y \log(\tau)\right)$$
$$=: \frac{y}{\tau(\tau^y-1)^2}\tilde{h}_y(\tau).$$

with $\tilde{h}'_y(\tau) = -y^2\tau^{y-1}\log(\tau) < 0$, for all $y < 0$ and $\tau > 1$. As a consequence $\tilde{h}_y(\cdot)$ is a decreasing function for all $y < 0$. Besides, since $\tilde{h}_y(1) = 0$, it follows that $\tilde{h}_y(\tau) < 0$ and $a'_y(\tau) > 0$ for all $y < 0$ and $\tau > 1$. This proves that $h_y(\cdot)$ is an increasing function for all $y < 0$.
(iii) $h_y(\tau) > 1$ for all $\tau > 1$ and $y < 0$ is a direct consequence of Lemma 1.3(i,ii). To prove the second inequality, let $\Delta_y(\tau) := h_y(\tau)\tau^{y/2} - 1$ for all $\tau > 1$ and $y < 0$. Differentiating three times yields

$$\Delta'_y(\tau) = \frac{y\tau^{y/2-1}}{(\tau^y-1)^2}\left(\tau^y - 1 - \frac{1}{2}y\log\tau(\tau^y+1)\right) =: \frac{y\tau^{y/2-1}}{(\tau^y-1)^2}\tilde{\Delta}_y(\tau),$$
$$\tilde{\Delta}'_y(\tau) = \frac{y}{2\tau}\left(\tau^y(1 - y\log\tau) - 1\right) =: \frac{y}{2\tau}\tilde{\tilde{\Delta}}_y(\tau),$$
$$\tilde{\tilde{\Delta}}'_y(\tau) = -y^2(\log\tau)\tau^{y-1} < 0.$$

It thus appears that $\tilde{\tilde{\Delta}}_y(\cdot)$ is a decreasing function on $(1,\infty)$ with $\tilde{\tilde{\Delta}}_y(1) = 0$ so that $\tilde{\tilde{\Delta}}_y(\tau) < 0$ for all $\tau > 1$. As a consequence $\tilde{\Delta}'_y(\tau) > 0$ and $\tilde{\Delta}_y(\cdot)$ is an increasing function on $(1,\infty)$ with $\tilde{\Delta}_y(1) = 0$ so that $\tilde{\Delta}_y(\tau) > 0$ for all $\tau > 1$. Finally, $\Delta'_y(\tau) < 0$ and $\Delta_y(\cdot)$ is thus a decreasing function on $(1,\infty)$ with $\Delta_y(1) = 0$ so that $\Delta_y(\tau) < 0$ for all $\tau > 1$. The result is proved. □

Proof of Lemma 1.4. **(i)** A second-order Taylor expansion shows that $f_y(\cdot)$ can be extended by continuity letting $f_y(1) = 1/2$ for all $y \leq 0$.
(ii) Let $y < 0$ and $\tau > 1$. Differentiating twice, one has:

$$f'_y(\tau) = -\frac{1}{\tau y\log(\tau)^2}\left(1 - \log h_y(\tau) - \tau^y h_y(\tau)\right) =: -\frac{\tilde{f}_y(\tau)}{\tau y \log(\tau)^2},$$
$$\tilde{f}'_y(\tau) = \frac{h_y^2(\tau)\tau^y - 1}{\tau\log\tau} < 0,$$

in view of Lemma 1.3(iii). This implies that $\tilde{f}_y(\cdot)$ is a decreasing function on $(1,\infty)$. Besides $\tilde{f}_y(1) = 0$ since $h_y(1) = 1$ from Lemma 1.3(i), and therefore $\tilde{f}_y(\tau) < 0$ and $f'_y(\tau) < 0$ for all $\tau > 1$ which implies that $f_y(\cdot)$ is a decreasing function on $(1,\infty)$.
(iii) $f_y(\tau) < 1/2$ for all $y < 0$ and $\tau > 1$ is a direct consequence of Lemma 1.4(i,ii) while $f_y(\tau) > 0$ follows from Lemma 1.3(iii). □

Proof of Proposition 1.1. First, note that for all $\tau > 1$, $\rho \leq 0$ and $y \leq 0$, one has

$$\begin{aligned} B(1,\tau,\rho) - B(\beta^\star(\tau,y),\tau,\rho) &= \log(\tau) - K_\rho(\tau) - \log(\tau)\left(\frac{K_y(\tau)}{\log(\tau)}\right)^{\rho/y} + K_\rho(\tau) \\ &= \log(\tau)(1 - h_y(\tau)^{-\rho/y}). \end{aligned}$$

Clearly, $B(1,\tau,0) - B(\beta^\star(\tau,y),\tau,0) = 0$ and $B(1,\tau,0) = 0$ in view of (2.6). Let us thus focus on the case where $\tau > 1$, $\rho < 0$ and $y \leq 0$. Combining $\log(\tau) > 0$, $\rho/y > 0$ and Lemma 1.3(iii) yields

$$B(1,\tau,\rho) - B(\beta^\star(\tau,y),\tau,\rho) > 0. \tag{1.6}$$

Second,

$$\rho\{B(1,\tau,\rho) + B(\beta^\star(\tau,y),\tau,\rho)\} = \rho\left(\log(\tau) + \log(\tau)\left(\frac{K_y(\tau)}{\log(\tau)}\right)^{\rho/y} - 2K_\rho(\tau)\right),$$

for all $\tau > 1$, $\rho < 0$ and $y \leq 0$. The change of variable $\rho \mapsto u = \tau^\rho$ yields

$$\begin{aligned} \rho\{B(1,\tau,\rho) + B(\beta^\star(\tau,y),\tau,\rho)\} &= \log(u) + \log(u)\left(\frac{K_y(\tau)}{\log(\tau)}\right)^{\log(u)/(y\log(\tau))} - 2(u-1) \\ &= \log(u)\left(1 + u^{f_y(\tau)}\right) - 2(u-1), \\ &= g_{f_y(\tau)}(u), \end{aligned}$$

with

$$\begin{aligned} g_\beta(u) &= \log(u)(1+u^\beta) - 2(u-1), && \text{(see Lemma 1.2)}, \\ h_y(\tau) &= \frac{K_0(\tau)}{K_y(\tau)}, && \text{(see Lemma 1.3)}, \\ f_y(\tau) &= \frac{1}{y\log(\tau)}\log\left(\frac{K_y(\tau)}{\log(\tau)}\right) = -\frac{\log(h(\tau))}{y\log(\tau)}, && \text{(see Lemma 1.4)}. \end{aligned}$$

Lemma 1.4(iii) shows that $0 < f_y(\tau) \leq 1/2$ for all $y \leq 0$, $\tau > 1$ and thus one can apply Lemma 1.2 to obtain $g_{f_y(\tau)}(u) < 0$ and consequently

$$B(1,\tau,\rho) + B(\beta^\star(\tau,y),\tau,\rho) = \frac{g_{f_y(\tau)}(\tau^\rho)}{\rho} > 0. \tag{1.7}$$

Collecting (1.6) and (1.7) concludes the proof. □