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Stochastic Orders on the Univariate Unified Skew Normal Family of Distributions

Authors: Souhila Merabet ✉
– Department of Probability and Statistics, Faculty of Mathematics, USTHB, Algiers, Algeria
souhila_merabet@yahoo.fr

Rabah Messaci
– Department of Probability and Statistics, Faculty of Mathematics, USTHB, Algiers, Algeria
rrabmes@gmail.com

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Abstract:

- The family of Unified Skew Normal distributions is associated with probability distributions encountered in various problems, notably those of the selection of individuals in a normal population. In this work, we focus on the ordering of the real subfamily of this class of distributions with respect to its parameters and for some stochastic orders (usual order, increasing convex order, increasing concave order and likelihood ratio order). Our results are applied to a reliability problem as well as a selection problem.

Keywords:

- *Unified skew-normal distribution; extended skew-normal distribution; stochastic orders; selection distribution; logconcavity.*

AMS Subject Classification:

- 62E15, 60E15, 62H10.

✉ Corresponding author

1. INTRODUCTION

The Unified Skew Normal family $\mathrm{SUN}_{(p,q)}$, which was introduced by Arellano-Valle and Azzalini [1], is a family of asymmetric distributions generalizing the normal distribution and incorporating parameters controlling this asymmetry. The $\mathrm{SUN}_{(p,q)}$ familly also generalizes the basic asymmetric family of Skew Normal (SN) distributions, which was introduced by Azzalini [7] and Azzalini and Dalla Valle [11]. In practice, the $\mathrm{SUN}_{(p,q)}$ familly is used as an alternative to the gaussian distribution for modelling non-normal features as skewness. Among the numerous methods used to generate the $\mathrm{SUN}_{(p,q)}$ distributions, we find the method known as stochastic representation by conditioning which is achieved as folows: if $(\boldsymbol{U}^t, \boldsymbol{V}^t)^t$ is a Gaussian vector of (p, q) order, then the distribution of $\boldsymbol{U}|\boldsymbol{V} > \mathbf{0}$ belongs to the family of $\mathrm{SUN}_{(p,q)}$ distributions. As a result, the obtained distribution is a special case of the so-called selection distributions (see [3]). In this work, we consider the case of $\mathrm{SUN}_{(1,q)}$ distributions where U is real and $\boldsymbol{V}$ is a vector and we intend to study the influence of the underlying $(\boldsymbol{U}^t, \boldsymbol{V}^t)^t$ vector parameters on the induced selection distribution. More specifically, we are interested in studying the stochastic ordering of the latter, relative to their parameters for some stochastic orders: usual stochastic order (first dominance stochastic order), increasing concave order (second dominance stochastic order), increasing convex order and the likelihood ratio order.

Azzalini [8] has established the usual stochastic order for SN distributions with respect to the skewness parameter α, whereas Loperfido *et al* [20] have shown the existence of the likelihood ratio order for the location parameter μ. Blasi and Scarlatti [15] have also addressed the stochastic ordering of the SN distribution with respect to μ, α and the dispersion parameter σ.

The present work is structured as follows: in Section 1, we recall the general definition of the SUN distributions with some of their properties. Section 2 introduces some of the stochastic orders that we shall consider in this work. In Section 3, we provide the main results on the ordering of the $\mathrm{SUN}_{(1,q)}$ distributions. Section 4 deals with the application of the results obtained to both a reliability and a selection problems.

2. THE $\mathrm{SUN}_{(p,q)}$ DISTRIBUTION

Let $\boldsymbol{U}$ and $\boldsymbol{V}$ be two Gaussian random vectors such that:

$$\begin{pmatrix} \boldsymbol{U} \\ \boldsymbol{V} \end{pmatrix} \sim \boldsymbol{N}_{p+q}(\boldsymbol{\xi}, \boldsymbol{\Omega}^*), \qquad \boldsymbol{\xi} = \begin{pmatrix} \boldsymbol{\mu} \\ \gamma \end{pmatrix}, \tag{2.1}$$

where $\boldsymbol{\mu} \in \mathbb{R}^p$, $\boldsymbol{\gamma} \in \mathbb{R}^q$ and $\boldsymbol{\Omega^*}$ is a non-singular variance covariance matrix. The correlation matrix associated with $\boldsymbol{\Omega^*}$ is:

$$\bar{\boldsymbol{\Omega}}^* = \begin{pmatrix} \bar{\boldsymbol{\Omega}} & \boldsymbol{\Delta} \\ \boldsymbol{\Delta^t} & \boldsymbol{\Gamma} \end{pmatrix}$$

where $\bar{\boldsymbol{\Omega}} \in \mathbb{R}^p \times \mathbb{R}^p$, $\boldsymbol{\Gamma} \in \mathbb{R}^q \times \mathbb{R}^q$, $\boldsymbol{\Delta} \in \mathbb{R}^p \times \mathbb{R}^q$.
Let $\boldsymbol{\Omega} = \boldsymbol{\sigma}\bar{\boldsymbol{\Omega}}\boldsymbol{\sigma}$, where $\boldsymbol{\sigma}$ is a diagonal matrix of order p with elements $\sigma_1, \sigma_2, \ldots, \sigma_p$ representing the standard deviations of the $\mathbf{U}$ components.

Definition 2.1. A random vector $\mathbf{X}$ with values in $\mathbb{R}^p$ is said to have a $\mathrm{SUN}_{(p,q)}$ distribution if

$$\boldsymbol{X} \stackrel{d}{=} \boldsymbol{U}|(\boldsymbol{V} > \mathbf{0}), \tag{2.2}$$

where $\boldsymbol{V} > \mathbf{0}$ means that each component of $\boldsymbol{V}$ is greater than 0.

The probability density function of $\boldsymbol{X}$ is then given by:

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \phi_p(\boldsymbol{x} - \boldsymbol{\mu}; \boldsymbol{\Omega}) \frac{\Phi_q(\gamma + \boldsymbol{\Delta^t}\bar{\boldsymbol{\Omega}}^{-1}\boldsymbol{\sigma}^{-1}(\boldsymbol{x} - \boldsymbol{\mu}); \boldsymbol{\Gamma} - \boldsymbol{\Delta^t}\bar{\boldsymbol{\Omega}}^{-1}\boldsymbol{\Delta})}{\Phi_q(\boldsymbol{\gamma}; \boldsymbol{\Gamma})}, \tag{2.3}$$

where $\mathbf{x} \in \mathbb{R}^p$ and $\phi_n(., \Sigma)$, $\Phi_n(., \Sigma)$ are the probability density function (pdf) and the cumulative distribution function (cdf) of the multivariate normal distribution $N_n(\mathbf{0}, \boldsymbol{\Sigma})$, respectively.
The distribution of $\mathbf{X}$ is denoted $\mathrm{SUN}_{(p,q)}(\boldsymbol{\mu}, \boldsymbol{\Omega}, \boldsymbol{\gamma}, \boldsymbol{\Gamma}, \boldsymbol{\Delta})$ where $\boldsymbol{\mu}$ is the location parameter, $\boldsymbol{\Omega}$ the dispersion parameter, $\boldsymbol{\gamma}$ the truncation parameter and $\boldsymbol{\Delta}$ the shape parameter. Its expectation is given by (see [9]):

$$E(X) = \boldsymbol{\mu} + \boldsymbol{\sigma}\boldsymbol{\Delta}\frac{\nabla\Phi_q(\boldsymbol{\gamma}, \boldsymbol{\Gamma})}{\Phi_q(\boldsymbol{\gamma}, \boldsymbol{\Gamma})}, \tag{2.4}$$

where $\nabla\Phi_q(\boldsymbol{\gamma}, \boldsymbol{\Gamma})$ is the gradient vector at point $\boldsymbol{\gamma}$ of $\Phi_q(., \boldsymbol{\Gamma})$.
If $p = 1$ and $q = 1$, we get $\Omega = \sigma^2$. In addition, for $\Gamma = 1$ and by letting $\Delta = \rho$, this distribution is the so-called Extended Skew Normal distribution (ESN) and it admits the following pdf [5]:

$$f(x) = \frac{\phi(\frac{x-\mu}{\sigma})\Phi(\frac{\gamma+\rho(\frac{x-\mu}{\sigma})}{\sqrt{1-\rho^2}})}{\sigma\Phi(\gamma)}, \qquad x \in \mathbb{R}, \tag{2.5}$$

where $\phi(.)$ and $\Phi(.)$ are the standard normal pdf and cdf respectively.
The expectation of X simplifies to:

$$E(X) = \mu + \sigma\rho\lambda(\gamma), \tag{2.6}$$

where $\lambda(\gamma) = \frac{\phi(\gamma)}{\Phi(\gamma)}$ is the inverse Mills' ratio.

Remark 2.1. Arnold and Beaver [5] wrote the density in (2.5) using a different parametrization:

$$f(x) = \frac{\phi(\frac{x-\mu}{\sigma})\Phi(\alpha_0 + \alpha_1(\frac{x-\mu}{\sigma}))}{\sigma\Phi(\frac{\alpha_0}{\sqrt{1+\alpha_1^2}})}, \qquad \text{where} \quad \alpha_0 \in \mathbb{R},\ \alpha_1 \in \mathbb{R}, \tag{2.7}$$

with, $$\alpha_0 = \frac{\gamma}{\sqrt{1-\rho^2}} \qquad \text{and} \qquad \alpha_1 = \frac{\rho}{\sqrt{1-\rho^2}}.$$

Alternatively, we may as well use:

$$\rho = \frac{\alpha_1}{\sqrt{1+\alpha_1^2}} \qquad \text{and} \qquad \gamma = \frac{\alpha_0}{\sqrt{1+\alpha_1^2}}.$$

When $\gamma = 0$, the ESN distribution reduces to the $\mathrm{SN}(\mu, \sigma^2, \alpha_1)$ distribution introduced by Azzalini [7], while for $\gamma = 0$ and $\rho = 0$ the normal distribution $N(\mu, \sigma^2)$ is obtained. The distribution of X is denoted $\mathrm{ESN}(\mu, \sigma, \gamma, \rho)$ or $\mathrm{ESN}(\mu, \sigma, \alpha_0, \alpha_1)$.

3. STOCHASTIC ORDERS

In this section, we recall the definitions and some properties of the stochastic orders that we will use in the sequel. These are mainly based on the following references: Shaked and Shanthikumar [26] and Muller and Stoyan [25].

Definition 3.1. Let X_1 and X_2 be two real random variables with cdf F_1 and F_2 and pdf f_1 and f_2, respectively.

- X_1 is said to be smaller than X_2 in the sense of the usual stochastic order (or smaller in distribution) and we denote $X_1 \leq_{st} X_2$, if:

  $$\bar{F_1}(x) \leq \bar{F_2}(x), \quad \forall x \in \mathbb{R}, \tag{3.1}$$

  where $\bar{F_1}, \bar{F_2}$ represent the survival functions of X_1 and X_2, respectively, i.e,
  $\bar{F_1}(x) = 1 - F_1(x)$ and $\bar{F_2}(x) = 1 - F_2(x)$, or equivalently, if:

  $$F_1(x) \geq F_2(x), \quad \forall x \in \mathbb{R}. \tag{3.2}$$

  This stochastic or distributional order is known in economic theory as first-order stochastic dominance.

- X_1 is said to be smaller than X_2 in the sense of the likelihood ratio order and we denote $X_1 \leq_{lr} X_2$ if:

  $$\frac{f_2(x)}{f_1(x)} \quad \text{is increasing in x}, \ \forall x \in I; \tag{3.3}$$

  where I is the union of the supports of X_1 and X_2.

- X_1 is said to be smaller than X_2 in the sense of increasing convex order and we denote $X_1 \leq_{icx} X_2$ if:

$$\int_y^{\infty} \bar{F}_1(x)dx \leq \int_y^{\infty} \bar{F}_2(x)dx, \quad \forall y \in \mathbb{R}. \tag{3.4}$$

- X_1 is said to be smaller than X_2 in the sense of increasing concave order and we denote $X_1 \leq_{icv} X_2$ if:

$$\int_{-\infty}^{y} F_1(x)dx \geq \int_{-\infty}^{y} F_2(x)dx, \quad \forall y \in \mathbb{R}. \tag{3.5}$$

  The increasing concave stochastic order is also known in the literature as second-order stochastic dominance.

- X_1 is smaller than X_2 in the sense of the less dangerous order, denoted by $X_1 \leq_D X_2$, if:
  - $\exists t_0 \in \mathbb{R}$ such that $F_1(t) \leq F_2(t)$, $\forall t < t_0$ and $F_1(t) \geq F_2(t)$, $\forall t \geq t_0$;
  - $E(X_1) \leq E(X_2)$.

The following proposition connects these stochastic orders.

Proposition 3.1. *The above partial orders verify the following implications:*

- $X_1 \leq_{lr} X_2 \Rightarrow X_1 \leq_{st} X_2 \Rightarrow X_1 \leq_{icv} X_2$;
- $X_1 \leq_{st} X_2 \Rightarrow X_1 \leq_{icx} X_2$;
- $X_1 \leq_D X_2 \Rightarrow X_1 \leq_{icx} X_2$.

Several properties stem from these definitions. We give some of them in the following:

Proposition 3.2.

1. $X_1 \leq_{st} X_2$ *if and only if there is a positive* Y *random variable such that* $X_2 \stackrel{d}{=} X_1 + Y$;
2. *If* $X_1 \leq_{st} X_2$ *then* $E(X_1) \leq E(X_2)$;
3. *If* $X_1 \leq_{st} X_2$ *and* $E(X_1) = E(X_2)$, *then* X_1 *et* X_2 *have the same distribution, i.e.* $F_1 = F_2$;

4. ORDERING OF SUN$_{(1,q)}$ DISTRIBUTIONS

In this section, we study the stochastic ordering of the $SUN_{(1,q)}$ distributions for the orders defined previously and relatively to each of its parameters, assuming that the others are held constant.

Choosing $p = 1$ in (2.1), we obtain the SUN $_{(1,q)}$ density:

$$f_X(x) = \phi(x-\mu;\sigma^2)\frac{\Phi_q(\boldsymbol{\gamma} + \boldsymbol{\Delta}^{\boldsymbol{t}}(\frac{x-\mu}{\sigma}); \boldsymbol{\Gamma} - \boldsymbol{\Delta}^t\boldsymbol{\Delta})}{\Phi_q(\boldsymbol{\gamma};\boldsymbol{\Gamma})}, \qquad x \in \mathbb{R}. \tag{4.1}$$

where $\mu \in \mathbb{R}$, $\Omega = \sigma^2 \in \mathbb{R}_+^*$, $\boldsymbol{\gamma} \in \mathbb{R}^q$ and $\boldsymbol{\Delta} \in \mathbb{R}^q$.

Now, we state the main result of our work.

Theorem 4.1. *Let X_1 and X_2 be two random variables with pdf f_1 and f_2 and cdf F_1 and F_2, respectively. We have:*

1. *If $X_1 \sim SUN_{(1,q)}(\mu_1, \sigma, \boldsymbol{\gamma}, \boldsymbol{\Gamma}, \boldsymbol{\Delta})$, $X_2 \sim SUN_{(1,q)}(\mu_2, \sigma, \boldsymbol{\gamma}, \boldsymbol{\Gamma}, \boldsymbol{\Delta})$ and if $\mu_1 \leq \mu_2$ then:*
$$X_1 \leq_{lr} X_2.$$

2. *If $X_1 \sim SUN_{(1,q)}(\mu, \sigma, \boldsymbol{\gamma}, \boldsymbol{\Gamma}, \boldsymbol{\Delta})$ and $X_2 \sim SUN_{(1,q)}(\mu, \sigma, \boldsymbol{\gamma}', \boldsymbol{\Gamma}, \boldsymbol{\Delta})$ with $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_{i-1}, \gamma_i, \gamma_{i+1}, \ldots, \gamma_q)$ and $\boldsymbol{\gamma}' = (\gamma_1, \ldots, \gamma_{i-1}, \gamma_i', \gamma_{i+1}, \ldots, \gamma_q)$ and if $\gamma_i \leq \gamma_i'$ then:*
$$X_1 \geq_{lr} X_2, \ \text{for} \quad \boldsymbol{\Delta} \geq \mathbf{0};$$
$$X_1 \leq_{lr} X_2, \ \text{for} \quad \boldsymbol{\Delta} \leq \mathbf{0}.$$

3. *If $X_1 \sim SUN_{(1,q)}(\mu, \sigma, \boldsymbol{\gamma}, \boldsymbol{\Gamma}, \boldsymbol{\Delta})$ and $X_2 \sim SUN_{(1,q)}(\mu, \sigma, \boldsymbol{\gamma}, \boldsymbol{\Gamma}, \boldsymbol{\Delta}')$ with $\boldsymbol{\Delta} = (\delta_1, \ldots, \delta_{i-1}, \delta_i, \delta_{i+1}, \ldots, \delta_q)$ and $\boldsymbol{\Delta}' = (\delta_1, \ldots, \delta_{i-1}, \delta_i', \delta_{i+1}, \ldots, \delta_q)$, and if $\delta_i \leq \delta_i'$ then:*
$$X_1 \leq_{st} X_2.$$

4. *If $X_1 \sim SUN_{(1,q)}(\mu, \sigma_1, \boldsymbol{\gamma}, \boldsymbol{\Gamma}, \boldsymbol{\Delta})$, $X_2 \sim SUN_{(1,q)}(\mu, \sigma_2, \boldsymbol{\gamma}, \boldsymbol{\Gamma}, \boldsymbol{\Delta})$ and if $\sigma_1 \leq \sigma_2$ then:*
$$X_1 \leq_{icx} X_2, \ \text{when } \boldsymbol{\Delta} \geq \mathbf{0};$$
$$X_1 \geq_{icv} X_2, \ \text{when } \boldsymbol{\Delta} \leq \mathbf{0}.$$

Proof:

1. Arellano-Valle and Azzalini [2] have established the log-concavity of the density SUN$_{(p,q)}$. Moreover, we know that if g is a log-concave density in

$\mathbb{R}$, then the family $g(x-\theta)$ has a monotone likelihood ratio with respect to θ (see [17]), i.e.

$$\text{If } \ \theta_1 < \theta_2, \ \frac{g(x-\theta_2)}{g(x-\theta_1)} \quad \text{is an increasing monotone function of x.}$$

As a result, for all $\mu_1 < \mu_2$ the ratio

$$\frac{f_2(x)}{f_1(x)} = \frac{f(x-\mu_2, 0, \sigma^2, \boldsymbol{\gamma}, \boldsymbol{\Gamma}, \boldsymbol{\Delta})}{f(x-\mu_1, 0, \sigma^2, \boldsymbol{\gamma}, \boldsymbol{\Gamma}, \boldsymbol{\Delta})}$$

is an increasing monotone function of x. ■

2. We have

$$\frac{f_2(x)}{f_1(x)} = \frac{\phi(\frac{x-\mu}{\sigma})\Phi_q(\boldsymbol{\gamma}' + \boldsymbol{\Delta}^t(\frac{x-\mu}{\sigma}), \boldsymbol{\Gamma} - \boldsymbol{\Delta}^t\boldsymbol{\Delta})}{\phi(\frac{x-\mu}{\sigma})\Phi_q(\boldsymbol{\gamma} + \boldsymbol{\Delta}^t(\frac{x-\mu}{\sigma}), \boldsymbol{\Gamma} - \boldsymbol{\Delta}^t\boldsymbol{\Delta})} \frac{\Phi_q(\boldsymbol{\gamma}, \boldsymbol{\Gamma})}{\Phi_q(\boldsymbol{\gamma}', \boldsymbol{\Gamma})};$$

Note that, $\frac{\Phi_q(\boldsymbol{\gamma},\boldsymbol{\Gamma})}{\Phi_q(\boldsymbol{\gamma}',\boldsymbol{\Gamma})}$ is a positive constant independent of x.
Consider the function g given by:

$$g(z) = \frac{\Phi_q(\boldsymbol{\gamma}' + \boldsymbol{\Delta}z, \boldsymbol{\Gamma} - \boldsymbol{\Delta}\boldsymbol{\Delta}^t)}{\Phi_q(\boldsymbol{\gamma} + \boldsymbol{\Delta}z), \boldsymbol{\Gamma} - \boldsymbol{\Delta}\boldsymbol{\Delta}^t)}, \quad \text{where } z = \frac{x-\mu}{\sigma}.$$

We have

$$\frac{dg(z)}{dz} = \frac{\frac{d\Phi_q(\boldsymbol{\gamma}' + \boldsymbol{\Delta}z, \boldsymbol{\Gamma} - \boldsymbol{\Delta}\boldsymbol{\Delta}^t)}{dz}\Phi_q(\boldsymbol{\gamma} + \boldsymbol{\Delta}z, \boldsymbol{\Gamma} - \boldsymbol{\Delta}\boldsymbol{\Delta}^t) - \frac{d\Phi_q(\boldsymbol{\gamma} + \boldsymbol{\Delta}z, \boldsymbol{\Gamma} - \boldsymbol{\Delta}\boldsymbol{\Delta}^t)}{dz}\Phi_q(\boldsymbol{\gamma}' + \boldsymbol{\Delta}z, \boldsymbol{\Gamma} - \boldsymbol{\Delta}\boldsymbol{\Delta}^t)}{\Phi_q^2(\boldsymbol{\gamma} + \boldsymbol{\Delta}z, \boldsymbol{\Gamma} - \boldsymbol{\Delta}\boldsymbol{\Delta}^t)}.$$

Let $\boldsymbol{u} = \boldsymbol{\gamma} + \boldsymbol{\Delta}z$ and $\boldsymbol{u}' = \boldsymbol{\gamma}' + \boldsymbol{\Delta}z$. The derivative may then be written as follows:

$$\frac{dg(z)}{dz} = \frac{\sum_{j=1}^q \frac{du_j'}{dz}\frac{\partial\Phi_q(\boldsymbol{u}', \boldsymbol{\Gamma} - \boldsymbol{\Delta}\boldsymbol{\Delta}^t)}{\partial u_j}\Phi_q(\boldsymbol{u}, \boldsymbol{\Gamma} - \boldsymbol{\Delta}\boldsymbol{\Delta}^t) - \sum_{j=1}^q \frac{du_j}{dz}\frac{\partial\Phi_q(\boldsymbol{u}, \boldsymbol{\Gamma} - \boldsymbol{\Delta}\boldsymbol{\Delta}^t)}{\partial u_j}\Phi_q(\boldsymbol{u}', \boldsymbol{\Gamma} - \boldsymbol{\Delta}\boldsymbol{\Delta}^t)}{\Phi_q^2(\mathbf{u}, \boldsymbol{\Gamma} - \boldsymbol{\Delta}\boldsymbol{\Delta}^t)}.$$

Moreover, Φ_q has a decreasing reversed hazard rate (DRHR) since it is log-concave (see [21]), i.e.

$$\frac{\partial \ln \Phi_q(u_1, u_2, \ldots, u_q)}{\partial u_j} \quad \text{is decreasing in } u_j, \forall j = \overline{1, q}.$$

For any $\boldsymbol{\Delta} \geq \mathbf{0}$, it holds then

$$\sum_{j=1}^q \delta_j \frac{\partial\Big(\ln \Phi_q(\boldsymbol{u}', \boldsymbol{\Gamma} - \boldsymbol{\Delta}\boldsymbol{\Delta}^t)\Big)}{\partial u_j} \leq \sum_{j=1}^q \delta_j \frac{\partial\Big(\ln \Phi_q(\boldsymbol{u}, \boldsymbol{\Gamma} - \boldsymbol{\Delta}\boldsymbol{\Delta}^t)\Big)}{\partial u_j},$$

$$\Leftrightarrow \sum_{j=1}^q \delta_j \frac{\frac{\partial\Phi_q(\boldsymbol{u}', \boldsymbol{\Gamma} - \boldsymbol{\Delta}\boldsymbol{\Delta}^t)}{\partial u_j}}{\Phi_q(\boldsymbol{u}', \boldsymbol{\Gamma} - \boldsymbol{\Delta}\boldsymbol{\Delta}^t)} \leq \sum_{j=1}^q \delta_j \frac{\frac{\partial\Phi_q(\mathbf{u}, \boldsymbol{\Gamma} - \boldsymbol{\Delta}\boldsymbol{\Delta}^t)}{\partial u_j}}{\Phi_q(\boldsymbol{u}, \boldsymbol{\Gamma} - \boldsymbol{\Delta}\boldsymbol{\Delta}^t)},$$

$$\Leftrightarrow \frac{dg(z)}{dz} \leq 0, \ \forall z \in \mathbb{R}.$$

Similarly, for any $\boldsymbol{\Delta} \leq \mathbf{0}$, it holds then

$$\sum_{j=1}^{q} \delta_j \frac{\partial\Big(\ln \Phi_q(\boldsymbol{u}', \boldsymbol{\Gamma} - \boldsymbol{\Delta}\boldsymbol{\Delta}^t)\Big)}{\partial u_j} \geq \sum_{j=1}^{q} \delta_j \frac{\partial\Big(\ln \Phi_q(\boldsymbol{u}, \boldsymbol{\Gamma} - \boldsymbol{\Delta}\boldsymbol{\Delta}^t)\Big)}{\partial u_j},$$

which is equivalent to:

$$\frac{dg(z)}{dz} \geq 0, \forall z \in \mathbb{R}.$$

In conclusion, if $\gamma_i \leq \gamma_i^{'}, \quad \forall i = \overline{1, q}$, we have:

$$X_1 \geq_{lr} X_2, \quad \text{for } \boldsymbol{\Delta} \geq \mathbf{0}$$

$$\text{and } X_1 \leq_{lr} X_2, \quad \text{for } \boldsymbol{\Delta} \leq \mathbf{0}. \qquad \blacksquare$$

3. Without loss of generality, we take $i = 1$ in the proof.

Referring to Azzalini and Bacchieri [9], the cdf of X_1 and X_2 can be written as follows:

$$F_1(x) = \frac{\Phi_{1+q}(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{\Omega}}_{\mathbf{1}})}{\Phi_q(\boldsymbol{\gamma}, \boldsymbol{\Gamma})}; \qquad F_2(x) = \frac{\Phi_{1+q}(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{\Omega}}_{\mathbf{2}})}{\Phi_q(\boldsymbol{\gamma}, \boldsymbol{\Gamma})},$$

where $\tilde{\boldsymbol{x}} = \begin{pmatrix} \frac{x-\mu}{\sigma} \\ \boldsymbol{\gamma} \end{pmatrix}$, $\tilde{\boldsymbol{\Omega}}_{\mathbf{1}} = \begin{pmatrix} 1 & -\boldsymbol{\Delta} \\ -\boldsymbol{\Delta}^{\boldsymbol{t}} & \boldsymbol{\Gamma} \end{pmatrix}$ and $\tilde{\boldsymbol{\Omega}}_{\mathbf{2}} = \begin{pmatrix} 1 & -\boldsymbol{\Delta}' \\ -\boldsymbol{\Delta}'^{\boldsymbol{t}} & \boldsymbol{\Gamma} \end{pmatrix}$;

On the other hand, we know, from the Slepian's inequality [27], that:
if $\boldsymbol{X} \sim N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma} = (\sigma_{ij}))$ and $\boldsymbol{Y} \sim N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}' = (\sigma_{ij}^{'}))$ with $\sigma_{ii} = \sigma_{ii}^{'}$ $\forall i = \overline{1, n}$ and $\sigma_{ij} \geq \sigma_{ij}^{'}$ $\forall i \neq j$ then:

$$\Phi_n(\boldsymbol{a}, \boldsymbol{\Sigma}) \geq \Phi_n((\boldsymbol{a}, \boldsymbol{\Sigma}') \quad \forall \boldsymbol{a} \in \mathbb{R}^n.$$

In our case, if $\delta_1 \leq \delta_1^{'}$, we deduce that:

$$\Phi_{1+q}(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{\Omega}}_{\mathbf{1}}) \geq \Phi_{1+q}(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{\Omega}}_{\mathbf{2}}),$$

and then,

$$F_1(x) \geq F_2(x), \ \forall x \in \mathbb{R}. \qquad \blacksquare$$

4. Using the following notation:

$$K(\sigma) = \int_{-\infty}^{x} F(t)dt,$$

where F is the $\text{SUN}_{(1,q)}$ cdf, we have:

$$\begin{aligned}
\frac{dK(\sigma)}{d\sigma} &= \frac{d}{d\sigma}\Big(\int_{-\infty}^{x} F(t)dt\Big), \\
&= \frac{d}{d\sigma}\Big[\int_{-\infty}^{x}\Big(\int_{-\infty}^{\frac{t-\mu}{\sigma}} \frac{\phi(z)\Phi_q(\boldsymbol{\gamma} + \boldsymbol{\Delta}^t z, \boldsymbol{\Gamma} - \boldsymbol{\Delta}^t\boldsymbol{\Delta})}{\Phi_q(\boldsymbol{\gamma}, \boldsymbol{\Gamma})} dz\Big)dt\Big], \\
&= \frac{1}{\Phi_q(\boldsymbol{\gamma}, \boldsymbol{\Gamma})} \int_{-\infty}^{x} \frac{d}{d\sigma}\Big(\int_{-\infty}^{\frac{t-\mu}{\sigma}} \phi(z)\Phi_q(\boldsymbol{\gamma} + \boldsymbol{\Delta}^t z, \boldsymbol{\Gamma} - \boldsymbol{\Delta}^t\boldsymbol{\Delta})dz\Big)dt, \\
&= \frac{1}{\Phi_q(\boldsymbol{\gamma}, \boldsymbol{\Gamma})} \int_{-\infty}^{x} -\frac{(t-\mu)}{\sigma^2}\phi(\frac{t-\mu}{\sigma})\Phi_q(\boldsymbol{\gamma} + \boldsymbol{\Delta}^t(\frac{t-\mu}{\sigma}), \boldsymbol{\Gamma} - \boldsymbol{\Delta}^t\boldsymbol{\Delta})dt.
\end{aligned}$$

Integrating by parts and using the change of variable $u = \frac{t-\mu}{\sigma}$, we get:

$$\frac{dK(\sigma)}{d\sigma} = \frac{1}{\Phi_q(\boldsymbol{\gamma}, \boldsymbol{\Gamma})} \int_{-\infty}^{\frac{(x-\mu)}{\sigma}} -u\phi(u)\Phi_q(\boldsymbol{\gamma} + \boldsymbol{\Delta}^t u, \boldsymbol{\Psi})du,$$

where $\boldsymbol{\Psi} = \boldsymbol{\Gamma} - \boldsymbol{\Delta}^t\boldsymbol{\Delta}$.
Noting that $\phi(u)' = -u\phi(u)$, we get:

$$K'(\sigma) = \frac{1}{\Phi_q(\boldsymbol{\gamma}, \boldsymbol{\Gamma})}\Big[\phi(\frac{x-\mu}{\sigma})\Phi_q\big(\boldsymbol{\gamma} + \boldsymbol{\Delta}^t(\frac{x-\mu}{\sigma}), \boldsymbol{\Psi}\big)\Big] - \int_{-\infty}^{\frac{x-\mu}{\sigma}} \phi(u) \sum_{j=1}^{q} \delta_j (\nabla\Phi_q)_j du;$$

where $(\nabla\Phi_q)_j$ is the j^{th} element of the gradient vector of $\Phi_q(., \boldsymbol{\Psi})$ at the point $\boldsymbol{\gamma} + \boldsymbol{\Delta}^t u$. Recall that each gradient component of $\Phi_q(., .)$ is positive. We deduce that $K'(\sigma) \geq 0$ if $\boldsymbol{\Delta} \leq \mathbf{0}$. Thus, $X_1 \geq_{icv} X_2$.
If $\boldsymbol{\Delta} \geq \mathbf{0}$, we can show that $X_1 \leq_D X_2$, which according to Proposition 3.1, will lead to $X_1 \leq_{icx} X_2$.

1) If $\sigma_1 < \sigma_2$ and $t < \mu$, then we have:

$$\int_{-\infty}^{\frac{t-\mu}{\sigma_1}} \phi(z)\frac{\Phi_q(\boldsymbol{\gamma} + \boldsymbol{\Delta}^t z, \boldsymbol{\Gamma} - \boldsymbol{\Delta}^t\boldsymbol{\Delta})}{\Phi_q(\boldsymbol{\gamma}, \boldsymbol{\Gamma})}dz \leq \int_{-\infty}^{\frac{t-\mu}{\sigma_2}} \phi(z)\frac{\Phi_q(\boldsymbol{\gamma} + \boldsymbol{\Delta}^t z, \boldsymbol{\Gamma} - \boldsymbol{\Delta}^t\boldsymbol{\Delta})}{\Phi_q(\boldsymbol{\gamma}, \boldsymbol{\Gamma})}dz,$$

This is true because it is an integral of a product of positive functions which is positive and increasing with respect to the bound. Thus we get:

$$F_1(t) \leq F_2(t) \quad \forall t < \mu \quad \text{and} \quad F_1(t) \geq F_2(t) \quad \forall t \geq \mu. \tag{4.2}$$

2) Referring to Equation (2.4), each component of the gradient vector $\nabla\Phi_q$ being positive, we conclude that:
if $\sigma_1 < \sigma_2$ and $\boldsymbol{\Delta} \geq \mathbf{0}$, we have: $E(X_1) \leq E(X_2)$.

Thus, both conditions for the "D" order are verified. We deduce that $X_1 \leq_{icx} X_2$.

□

Remark 4.1.

- From Theorem 4.1, we deduce, that: if $\boldsymbol{\Delta} = (\delta_1, \delta_2, \dots, \delta_q)$ and $\boldsymbol{\Delta}' = (\delta_1', \delta_2', \dots, \delta_q')$ with $\boldsymbol{\Delta'} \leq \boldsymbol{\Delta}$, then $X_1 \leq_{st} X_2$. Similarly, if $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \dots, \gamma_q)$ and $\boldsymbol{\gamma}' = (\gamma_1', \gamma_2', \dots, \gamma_q')$ with $\boldsymbol{\gamma} \leq \boldsymbol{\gamma}'$ and if $\boldsymbol{\Delta} \geq \mathbf{0}$ ($\boldsymbol{\Delta} \leq \mathbf{0}$), then $X_1 \geq_{lr} X_2$ ($X_1 \leq_{lr} X_2$ respectively).

The following results on the stochastic ordering of ESN distributions stem immediately from those established for $\text{SUN}_{(1,q)}$ distributions.

Corollary 4.1. *Let X_1 and X_2 be two random variables of ESN distribution. We have:*

1. *If $X_1 \sim ESN(\mu_1, \sigma, \gamma, \rho)$ and $X_2 \sim ESN(\mu_2, \sigma, \gamma, \rho)$ with $\mu_1 \leq \mu_2$, then:*

$$X_1 \leq_{lr} X_2.$$

2. *If $X_1 \sim ESN(\mu, \sigma, \gamma_1, \rho)$ and $X_2 \sim ESN(\mu, \sigma, \gamma_2, \rho)$ with $\gamma_1 \leq \gamma_2$, then:*

$$X_1 \geq_{lr} X_2, \quad \text{for} \quad \rho \geq 0;$$

$$X_1 \leq_{lr} X_2, \quad \text{for} \quad \rho \leq 0.$$

3. *If $X_1 \sim ESN(\mu, \sigma, \gamma, \rho_1)$ and $X_2 \sim ESN(\mu, \sigma, \gamma, \rho_2)$ with $\rho_1 \leq \rho_2$, then:*

$$X_1 \leq_{st} X_2.$$

4. *If $X_1 \sim ESN(\mu, \sigma_1, \gamma, \rho)$ and $X_2 \sim ESN(\mu, \sigma_2, \gamma, \rho)$ with $\sigma_1 \leq \sigma_2$, then:*

$$X_1 \leq_{icx} X_2, \quad \textit{for} \quad \rho \geq 0;$$

$$X_1 \geq_{icv} X_2, \quad \textit{for} \quad \rho \leq 0.$$

Remark 4.2. By adopting the parametrization $\text{ESN}(\mu, \sigma, \alpha_0, \alpha_1)$, each of the following results is a direct consequence of Corollary 4.1:

1. If $X_1 \sim \text{ESN}(\mu, \sigma, \alpha_0, \alpha_1)$, $X_2 \sim \text{ESN}(\mu, \sigma, \alpha_0^{'}, \alpha_1)$ and if $\alpha_0 \leq \alpha_0^{'}$, then we have:

$$X_1 \geq_{lr} X_2, \quad \text{for } \alpha_1 \geq 0;$$

$$X_1 \leq_{lr} X_2, \quad \text{for } \alpha_1 \leq 0.$$

2. If $X_1 \sim \text{ESN}(\mu, \sigma, \alpha_0, \alpha_1)$, $X_2 \sim \text{ESN}(\mu, \sigma, \alpha_0, \alpha_1^{'})$ and if $\alpha_1 \leq \alpha_1^{'}$, then we have:

$$X_1 \leq_{st} X_2.$$

5. APPLICATIONS

In this section, we apply the previous results to two practical problems: the lifetime of a system in reliability and a selection problem encountered in education.

5.1. Lifetime of a system

Let $X_1, X_2, \ldots, X_n$ be the lifetimes of the n components of a system. Let us denote by Z the lifetime of the system distributed in parallel and U the lifetime of the system distributed in series. We have:

$$Z = \max(X_1, X_2, \ldots, X_n)$$

$$U = \min(X_1, X_2, \ldots, X_n)$$

Assume that $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)^t$ follows an exchangeable normal distribution $N_n(\boldsymbol{\mu^*}, \boldsymbol{\Omega})$ with $\boldsymbol{\mu^*} = \mu\,\mathbf{1}_{(n)}$, where $\mathbf{1}_{(n)} = (1, 1, \ldots, 1)^t \in \mathbb{R}^n$,
$\boldsymbol{\Omega} = \left(\sigma^2\rho_{ij}\right)_{1\leq i\leq j\leq n}$ and ρ_{ij} is given as follow:

$$\rho_{ij} = \begin{cases} \rho, & \text{if } \; i \neq j \;\forall i, j \in \{1, \ldots n\} \\ 1, & \text{if } \; i = j. \end{cases}$$

The cdf of Z is given by:

$$\begin{aligned} F_Z(z) &= P(\max(X_1, X_2, \ldots, X_n) \leq z) \\ &= \sum_{i=1}^{n} P\Big(X_i \leq z | \bigcap_{\substack{j=1\\ j\neq i}}^{n} \Big\{X_j \leq X_i\Big\}\Big) P\Big(\bigcap_{\substack{j=1\\ j\neq i}}^{n} \Big\{X_j \leq X_i\Big\}\Big). \end{aligned}$$

The assumption of exchangeability implies that:

$$X_i | \bigcap_{\substack{j=1\\ j\neq i}}^{n} \Big\{X_i - X_j \geq 0\Big\} \sim SUN_{(1,n-1)}(\mu, \mathbf{0}_{\mathbb{R}^{n-1}}, \boldsymbol{\Omega^*}), \quad \forall i = \overline{1, n};$$

with $\boldsymbol{\Omega^*} = \sigma^2(1-\rho)[\boldsymbol{A}_{(n)} + \mathbf{1}_{(n)}\mathbf{1}^t_{(n)}]$ and $\boldsymbol{A}_{(n)}$ given by:

$$\boldsymbol{A}_{(n)} = \begin{pmatrix} \frac{\rho}{1-\rho} & 0 & \ldots & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix} \in \mathbb{R}^n \times \mathbb{R}^n.$$

We deduce that: $Z \sim SUN_{(1,n-1)}(\mu, \mathbf{0}_{\mathbb{R}^{n-1}}, \boldsymbol{\Omega^*})$ because

$$\sum_{i=1}^{n} P\Big(\bigcap_{\substack{j=1\\ j\neq i}}^{n} \Big\{X_j \leq X_i\Big\}\Big) = 1.$$

The associated correlation matrix $\bar{\boldsymbol{\Omega}}^*$ is then:

$$\bar{\boldsymbol{\Omega}}^* = \begin{pmatrix} 1 & \boldsymbol{\Delta} \\ \boldsymbol{\Delta^t} & \boldsymbol{\Gamma} \end{pmatrix}, \tag{5.1}$$

where $\boldsymbol{\Delta} = \sqrt{\frac{1-\rho}{2}}\;\mathbf{1}^{\boldsymbol{t}}_{\boldsymbol{(n-1)}}$, $\boldsymbol{\Gamma} = \frac{1}{2}\boldsymbol{I}_{\boldsymbol{(n-1)}} + \frac{1}{2}\mathbf{1}_{\boldsymbol{(n-1)}}\mathbf{1}^{\boldsymbol{t}}_{\boldsymbol{(n-1)}}$ and $\boldsymbol{I}_{(n-1)}$ is the identity matrix of order $n-1$.

On the other hand, we have $U = -\max(-X_1, -X_2, \ldots, -X_n)$. We deduce that $U \sim SUN_{(1,n-1)}(\mu, \mathbf{0}_{\mathbb{R}^{n-1}}, \mathbf{\Omega^{**}})$ where $\mathbf{\Omega}^{**} = \sigma^2(1-\rho)[\boldsymbol{B}_{(n)} + \mathbf{1}_{(n)}\mathbf{1}^t_{(n)}]$ and $\boldsymbol{B}_{(n)}$ is the following matrix

$$\boldsymbol{B}_{(n)} = \begin{pmatrix} \frac{\rho}{1-\rho} & -2 & \ldots & \ldots & -2 \\ -2 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -2 & 0 & \ldots & 0 & 1 \end{pmatrix} \in \mathbb{R}^n \times \mathbb{R}^n.$$

From (5.1), we obtain the associated correlation matrix below:

$$\bar{\mathbf{\Omega}}^{**} = \begin{pmatrix} 1 & -\mathbf{\Delta} \\ -\mathbf{\Delta^t} & \mathbf{\Gamma} \end{pmatrix}; \tag{5.2}$$

The results of Theorem 4.1 allow us to conclude that:

- When all other parameters are held constant, the survival function is an increasing function of μ for both variables Z and U.
- When all other parameters are held constant, the survival function decreases with ρ for the variable Z and increases with ρ for the variable U.
- From Equation (4.2), we deduce that the survival function of the variable Z increases with σ for $z \geq \mu$ while it decreases with σ for $z < \mu$. The same result holds for the variable U.

Corollary 5.1. *Let Z_1 and Z_2 be the lifetimes of two parallel systems which are characterized by the following parameters: μ_1, ρ_1 and σ^2 for Z_1 and μ_2, ρ_2 and σ^2 for Z_2. The two systems have the same lifetime if and only if:*

$$\mu_1 - \mu_2 = \sigma\Big(\sqrt{\frac{1-\rho_2}{2}} - \sqrt{\frac{1-\rho_1}{2}}\Big)\mathbf{1}^t_{(n-1)}\frac{\nabla\Phi_{n-1}(\mathbf{0}_{\mathbb{R}^{n-1}}, \mathbf{\Gamma})}{\Phi_{n-1}((\mathbf{0}_{\mathbb{R}^{n-1}}, \mathbf{\Gamma})}. \tag{5.3}$$

Proof:
It is obvious that if $F_{Z_1}(z) = F_{Z_2}(z)$ then Equation (5.3) holds and therefore $(\mu_1 - \mu_2)(\rho_1 - \rho_2) < 0$.
Now, for the sufficient condition, let $Z_1^{'}$ be the lifetime of a parallel system which is characterized by the parameters μ_2, ρ_1 and σ^2.
From the results of Theorem 4.1, we have:

$$\mu_1 \leq \mu_2 \Rightarrow Z_1 \leq_{st} Z_1^{'}, \tag{5.4}$$

$$\rho_1 \geq \rho_2 \Rightarrow Z_1^{'} \leq_{st} Z_2. \tag{5.5}$$

Thus, from (5.4) and (5.5), we find that:

$$\rho_1 \geq \rho_2 \text{ and } \mu_1 \leq \mu_2 \Rightarrow Z_1 \leq_{st} Z_2.$$

According to the last property of Proposition 3.2, to have $F_{Z_1}(z) = F_{Z_2}(z)$, it is sufficient that $E(Z_1) = E(Z_2)$.
This is equivalent to:

$$\mu_1 - \mu_2 = \sigma\big(\sqrt{\frac{1-\rho_2}{2}} - \sqrt{\frac{1-\rho_1}{2}}\big)\mathbf{1}^t_{(n-1)}\frac{\nabla\Phi_{n-1}(\mathbf{0}_{\mathbb{R}^{n-1}},\mathbf{\Gamma})}{\Phi_{n-1}((\mathbf{0}_{\mathbb{R}^{n-1}},\mathbf{\Gamma})}.$$

□

5.2. Selection problems

We now consider selection problems which have been studied in particular by Birnbaum [13] and Birnbaum and Chapman [14]. These problems invoke the so-called selection distributions [3] which reduce in some cases to SUN distributions.

Equation (2.2) can be interpreted as the selection of individuals in a population using the variable of interest $\mathbf{U}$ under the constraint $\boldsymbol{V} > \mathbf{0}$, where $\mathbf{V}$ is the truncation variable. The distribution of $\mathbf{Z} \overset{d}{=} \mathbf{U}|\mathbf{V} > \boldsymbol{\tau}$, $\boldsymbol{\tau} \in \mathbb{R}^{\boldsymbol{q}}$ is called a selection distribution and when the vector $\boldsymbol{(U^t, V^t)^t}$ follows a normal distribution, we recover the SUN distribution.

For example, U can stand for a student's baccalaureate grade and V for his Math grade. In this case, since $p = 1$, $q = 1$ and under the assumption of normality made on the vector $\boldsymbol{(U^t, V^t)^t}$, Z follows the ESN distribution. According to (2.5), the pdf of Z, is:

$$f_Z(z) = \frac{1}{\sigma}\phi(\frac{z-\mu}{\sigma})\Phi\left(\frac{\rho\,(\dfrac{z-\mu}{\sigma}) + \tilde{\gamma}}{\sqrt{1-\rho^2}}\right)\Big/\Phi(\tilde{\gamma}),$$

where $\tilde{\gamma} = -(\frac{\tau-\gamma}{\sigma'})$ and σ' is the scale parameter of V. Then $Z \sim ESN(\mu, \sigma^2, \rho, \tilde{\gamma})$.

We select the individuals according to the grade obtained at the baccalaureate exam under the condition that the Math grade exceeds τ. The proportion of students retained (admission rate) is equal to:

$$\begin{aligned} P(Z \geq z_\alpha) &= P(U \geq u_\alpha | V > \tau), \\ &= \alpha. \end{aligned} \tag{5.6}$$

In this example, V can be a vector, if for instance we consider the vector $\boldsymbol{V} = (V_1, V_2)$ of Math and Physics grades. In this case, we have:

$$Z \overset{d}{=} U|V_1 > \tau_1, V_2 > \tau_2,$$

Under the assumption of normality made on the vector $(\boldsymbol{U}, \boldsymbol{V^t})^t$,
$Z \sim SUN_{(1,2)}\Big(\mu, \sigma^2, \mathbf{\Delta}, \tilde{\boldsymbol{\gamma}}, \mathbf{\Gamma}\Big)$ with $\tilde{\boldsymbol{\gamma}} = -\boldsymbol{\sigma}'^{-1}(\boldsymbol{\tau} - \boldsymbol{\gamma})$. Here, $\boldsymbol{\sigma}'$ is the diagonal

matrix of order 2 of elements $\sigma_1^{'}, \sigma_2^{'}$ which correspond to the standard deviations of V_1 and V_2 respectively.

In the first case ($p = 1$, $q = 1$), we are interested in studying the variation of the proportion of students retained as a function of some parameters of Z which are μ, σ^2, ρ and $\tilde{\gamma}$, the other parameters being held constant. Note that γ and $\sigma^{'}$ are only involved through the truncation parameter $\tilde{\gamma}$. The results stated in Corollary 4.1 imply that:

1. The admission rate is an increasing function of μ when the other parameters are held constant, i.e:

$$\mu_1 \leq \mu_2 \;\Rightarrow\; \bar{F}_1(z) \leq \bar{F}_2(z), \quad \forall z \in \mathbb{R}$$

2. The admission rate decreases with τ if $\rho \leq 0$ and increases with τ if $\rho \geq 0$ when the other parameters are held constant, i.e:

$$\tau_1 \geq \tau_2 \Rightarrow \begin{cases} \rho \geq 0, & \bar{F}_1(z) \geq \bar{F}_2(z), \quad \forall z \in \mathbb{R}; \\ \rho \leq 0, & \bar{F}_1(z) \leq \bar{F}_2(z), \quad \forall z \in \mathbb{R}. \end{cases}$$

3. The admission rate increases with ρ when the other parameters are held constant, i.e:

$$\rho_1 \leq \rho_2 \Rightarrow \bar{F}_1(z) \leq \bar{F}_2(z), \quad \forall z \in \mathbb{R}.$$

Remark 5.1. The problem of determining the selection threshold τ for a given α when the other parameters are held constant can be solved numerically by expressing (5.6) using the bivariate normal distribution function (see [9], or [8]). This gives:

$$\frac{\Phi_2(\tilde{\mathbf{z}}_\alpha, -\rho)}{\Phi(\tilde{\gamma})} = 1 - \alpha, \qquad \text{whre} \quad \tilde{\mathbf{z}}_\alpha = (\frac{z_\alpha - \mu}{\sigma}, \tilde{\gamma})^t.$$

5.2.1. Equality of two selection distributions

Consider two Gaussian vectors such that :

$$\begin{pmatrix} U_1 \\ V_1 \end{pmatrix} \sim N_2\left(\begin{pmatrix} \mu_1 \\ \xi_1 \end{pmatrix}, \Omega_1^*\right) \quad \text{and} \quad \begin{pmatrix} U_2 \\ V_2 \end{pmatrix} \sim N_2\left(\begin{pmatrix} \mu_2 \\ \xi_2 \end{pmatrix}, \Omega_2^*\right).$$

We define $Z_1 \overset{d}{=} U_1 | V_1 > \tau_1$, $Z_2 \overset{d}{=} U_2 | V_2 > \tau_2$ and F_1, F_2 their respective cdf.

We know that $Z_1 \sim ESN(\mu_1, \sigma_1^2, \rho_1, \tilde{\gamma}_1)$ and $Z_2 \sim ESN(\mu_2, \sigma_2^2, \rho_2, \tilde{\gamma}_2)$.

We are interested to know under which conditions it holds that $\bar{F}_1(z) = \bar{F}_2(z)$ $\forall z \in \mathbb{R}$. For instance, if we consider the same variables as those defined in the previous

example with U_1 and V_1 corresponding to high school A, U_2 and V_2 corresponding to high school B, we may ask the question: "under which conditions will the admission rate in both high schools be the same?"
We propose to solve the problem when two parameters are held constant.

Corollary 5.2. *Let $Z_1 \sim ESN(\mu_1, \sigma_1^2, \rho_1, \tilde{\gamma}_1)$ and $Z_2 \sim ESN(\mu_2, \sigma_2^2, \rho_2, \tilde{\gamma}_2)$. Then $\bar{F}_1(z) = \bar{F}_2(z)$ if and only if one of the following holds:*

i) *When $\tilde{\gamma}_1 = \tilde{\gamma}_2 = \tilde{\gamma}$ and $\sigma_1 = \sigma_2 = \sigma$,*

$$\mu_1 - \mu_2 = \sigma\lambda(\tilde{\gamma})(\rho_2 - \rho_1). \tag{5.7}$$

ii) *When $\mu_1 = \mu_2 = \mu$ and $\sigma_1 = \sigma_2 = \sigma$,*

$$\frac{\lambda(\tilde{\gamma}_1)}{\lambda(\tilde{\gamma}_2)} = \frac{\rho_2}{\rho_1}. \tag{5.8}$$

iii) *When $\sigma_1 = \sigma_2 = \sigma$ and $\rho_1 = \rho_2 = \rho$,*

$$\mu_1 - \mu_2 = \sigma\rho\big(\lambda(\tilde{\gamma}_2) - \lambda(\tilde{\gamma}_1)\big). \tag{5.9}$$

Proof: The proof is analogous to the one given for Corollary 5.1 □

Remark 5.2. Corollary 5.2 allows one to draw the following conclusions for the considered example:

- To prevent a change in the admission rate if the correlation ρ between the baccalaureate grade and the Math grade inceases when τ and σ are held constant, it is necessary to lower the average baccalaureate grade.
- To get a similar admission rate in the two high schools when μ and σ are held constant, it is necessary to vary the selection threshold τ and ρ in opposite directions. This problem has been discussed by Birnbaum [13] who established that the selection threshold τ is a decreasing function of $|\rho|$.
- To get a similar admission rate in both high schools when ρ and σ are held constant, it is necessary to vary the selection threshold τ and the average baccalaureate grade μ in opposite directions when ρ is positive and vary them in the same direction when ρ is negative .

6. CONCLUSION AND FUTURE WORK

In the present paper, we compare the univariate Unified Skew Normal distributions according to some classical criteria (usual stochastic order, increasing concave order, increasing convex order and the likelihood ratio order) and give two applications to both a reliability and a selection problems. A natural sequel of this work concerns the extension to the multivariate Unified Skew Normal family $SUN_{(p,q)}$ with $p > 1$ and to the more general class of the multivariate Unified Skew Elliptical distributions $SUE_{p,q}$ [1]. This requires the use of multivariate stochastic orders as defined, for example, in Shaked and Shanthikumar [26]. In this connection, Yin [29] recently considered the special case of the multivariate Skew Elliptical distributions [10] for some criteria: Hessian order, increasing Hessian order as well as many of their special cases.

As mentionned in the introduction, both $SUN_{(p,q)}$ and $SUE_{(p,q)}$ families introduce skweness, in addition $SUE_{(p,q)}$ introduce kurtosis. So, it would be of great interest to compare the above distributions relatively to these features. In the litterature, several skewness and kurtosis orderings and measures have been defined, among them the well-known convex transform order of Van Zwet [28]. For the univariate case, we can refer to Arnold and Groeneveld [6] and MacGillivray [22]. Much less work has been devoted to the multivariate case. Belzunce *et al* [12] extended the convex transform order to the multivariate setting. For the particular case of skew-normal vectors, Arevalillo and Navarro [4] have introduced a new multivariate skewness order based on the canonical transformation of these vectors. He also established that the univariate Skew Normal family is ordered for the skewness parameter α according to the convex transform order.

On the other hand, Loperfido [18] revisited some usual measures of the multivariate skewness: Mardia's skewness [24], partial skewness [16], directional skewness [23] and established relationships between them. Later, Loperfido [19] defined a new kurtosis matrix as alternative for the existing measures of multivariate kurtosis.

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