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Information matrices for composite type distributions

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Abstract:

- Composite type distributions are being increasingly used to model insurance data. Yet no expressions seem to be available for observed information matrices. In this paper, we give expressions for the matrices for two-piece, three-piece and m-piece composite distributions in their most general forms. Expressions for a number of particular cases and a simulation study showing practical use are also given.

Keywords:

- *m-piece model; Three-piece model; Two-piece model*

AMS Subject Classification:

- Primary 62E15.

1. Introduction

Composite type distributions have attracted much attention for modeling insurance data lately. Nearly all of the proposed models are made up of two pieces. The most general two-piece composite distribution has its probability density and cumulative distribution functions specified by

$$f(x) = \begin{cases} \dfrac{1}{1+\Phi} \dfrac{f_1(x)}{F_1(\theta)}, & x \leq \theta, \\[2ex] \dfrac{\Phi}{1+\Phi} \dfrac{f_2(x)}{1-F_2(\theta)}, & \theta < x, \end{cases} \tag{1.1}$$

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and

$$F(x)=\begin{cases}\dfrac{1}{1+\Phi}\dfrac{f_1(x)}{F_1(\theta)}, & x<\theta,\\[2ex] \dfrac{\Phi}{1+\Phi}\left[1+\Phi\dfrac{F_2(x)-F_2(\theta)}{1-F_2(\theta)}\right], & \theta<x,\end{cases} \tag{1.2}$$

respectively, where f_1, f_2 are smooth and valid probability density functions, F_1, F_2 are the corresponding cumulative distribution functions and $\Phi>0$.

Various particular cases of (1.1) and (1.2) have been studied in the literature: Cooray and Ananda [9], Scollnik [33], Teodorescu [35], Lee [18], Pigeon and Denuit [31], Eliazar and Cohen [13], Pak [29], Bee [3], Cooray and Cheng [10], Luckstead and Devadoss [22], Luckstead *et al.* [23] and Mutali and Vernic [26] considered lognormal and Pareto distributions for f_1 and f_2, respectively; Teodorescu and Vernic [37], Aminzadeh and Deng [1] and Gencturk *et al.* [14] considered exponential and Pareto distributions for f_1 and f_2, respectively; Cooray [8], Teodorescu and Panaitescu [36], Scollnik and Sun [34], Calderin-Ojeda [6] and Deng and Aminzadeh [12] considered Weibull and Pareto distributions for f_1 and f_2, respectively; Cooray *et al.* [11] considered Weibull and inverse Weibull distributions for f_1 and f_2, respectively; Teodorescu and Vernic [38] considered the Pareto distribution for f_2; Nadarajah and Bakar [27, 28] considered the lognormal distribution for f_1; Maghsoudi *et al.* [25] considered Weibull and inverse transformed gamma distributions for f_1 and f_2, respectively; Calderin-Ojeda [5] considered Weibull and Burr distributions for f_1 and f_2, respectively; Calderin-Ojeda and Kwok [7] considered the Stoppa distribution for f_1; Kim *et al.* [16] and Park and Baek [30] considered lognormal and generalized Pareto distributions for f_1 and f_2, respectively; Maghsoudi and Abu Bakar [24] considered transformed gamma and inverse transformed gamma distributions for f_1 and f_2, respectively; Aminzadeh and Deng [2] considered inverse gamma and Pareto distributions for f_1 and f_2, respectively; Kim *et al.* [17] considered gamma and generalized Pareto distributions for f_1 and f_2, respectively; Benatmane *et al.* [4] considered Rayleigh and Pareto distributions for f_1 and f_2, respectively; and so on. Liu and Ananda [19, 20] studied an exponentiated version of (1.1) and (1.2). Grün and Miljkovic [15] provided a comprehensive analysis on two-piece composite distributions. Liu and Ananda [21] proposed an exponentiated version of inverse gamma-Pareto composite distribution.

Yet we are aware of no work giving the observed information matrix for the general form of composite distributions. Even the particular cases mentioned in the cited papers do not appear to have derived the observed information matrices. Observed information matrices are important for tests of hypothesis and interval estimation.

Observed information matrices are commonly computed numerically. This can be prone to errors especially for composite type distributions because they involve continuity and differentiability conditions at θ. The aim of this paper is to derive explicit expressions for the observed information matrices for the most general two-piece, three-piece and m-piece composite distributions. These expressions are given in Sections 2, 3 and 4. Some particular cases of these expressions when f_1 is taken to correspond to exponential and Weibull distributions are given in the supplementary file. A simulation study showing practical use of the expressions in Section 4 is described in Section 5.

m-piece composite distributions when $m>2$ can be useful models for portfolio losses when the portfolio consists of more than two assets behaving differently. Examples could include portfolios consisting of losses relating to food, energy and metals.

All of the differentiation was performed manually. The correctness of differentiation was verified numerically. Mathematica code implementing the expressions in Sections 2 to 4 can be obtained from the corresponding author.

2. General two-piece composite distribution

Suppose that the body and tail are described probability density functions f_1 and f_2, respectively. Then the probability density and cumulative distribution functions of the most general two-piece composite distribution are (1.1) and (1.2), respectively. Suppose f_1 and f_2 are parameterized by $\mathbf{\Omega}_1$ and $\mathbf{\Omega}_2$, respectively. Conditions for continuity and differentiability at θ ensure that $\theta=\theta\left(\mathbf{\Omega}_1,\mathbf{\Omega}_2\right)$ and $\Phi=\Phi\left(\mathbf{\Omega}_1,\mathbf{\Omega}_2\right)=\frac{f_1(\theta)\left[1-F_2(\theta)\right]}{F_1(\theta)f_2(\theta)}$. If $x_1,x_2,\ldots,x_n$ is a random sample from (1.1)-(1.2) then the log-likelihood function of $(\mathbf{\Omega}_1, \mathbf{\Omega}_2)$ is

$$\log L\left(\mathbf{\Omega}_1, \mathbf{\Omega}_2\right) = \sum_{x_i \leq \theta} \log f_1\left(x_i\right) + \sum_{\theta < x_i} \log f_2\left(x_i\right) - a\left[\log(1+\Phi) + \log F_1(\theta)\right]$$
$$+(n-a)\left\{\log \Phi - \log(1+\Phi) - \log\left[1 - F_2(\theta)\right]\right\},$$

where

$$a = \sum_{i=1}^{n} I\left\{x_i \leq \theta\right\}.$$

Given the facts that

$$\frac{\partial}{\partial \theta} I\left\{x \geq \theta\right\} = \Delta(\theta)$$

and

$$\frac{\partial}{\partial \theta} I\left\{x < \theta\right\} = -\Delta(\theta)$$

(see `https://en.wikipedia.org/wiki/Indicator_function`), where $\Delta(\cdot)$ denotes the Dirac-delta function, we obtain the first order partial derivatives as

$$\frac{\partial \log L}{\partial \mathbf{\Omega}_1} = \sum_{x_i \leq \theta} \frac{\frac{\partial f_1(x_i)}{\partial \mathbf{\Omega}_1}}{f_1\left(x_i\right)} + \frac{\partial \theta}{\partial \mathbf{\Omega}_1} \sum_{i=1}^{n} \Delta\left(x_i - \theta\right)\left[\log f_1\left(x_i\right) - \log f_2\left(x_i\right)\right]$$
$$-\frac{\partial a}{\partial \mathbf{\Omega}_1}\left[\log(1+\Phi) + \log F_1(\theta)\right]$$
$$-\frac{\partial a}{\partial \mathbf{\Omega}_1}\left\{\log \Phi - \log(1+\Phi) - \log\left[1 - F_2(\theta)\right]\right\}$$
$$-a\left[\frac{\frac{\partial \Phi}{\partial \mathbf{\Omega}_1}}{1+\Phi} + \frac{f_1(\theta)}{F_1(\theta)} \frac{\partial \theta}{\partial \mathbf{\Omega}_1}\right] + (n-a)\left[\left(\frac{1}{\Phi} - \frac{1}{1+\Phi}\right) \frac{\partial \Phi}{\partial \mathbf{\Omega}_1} + \frac{f_2(\theta)}{1 - F_2(\theta)} \frac{\partial \theta}{\partial \mathbf{\Omega}_1}\right]$$

and

$$\frac{\partial \log L}{\partial \mathbf{\Omega}_2} = \sum_{\theta < x_i} \frac{\frac{\partial f_2(x_i)}{\partial \mathbf{\Omega}_2}}{f_2\left(x_i\right)} + \frac{\partial \theta}{\partial \mathbf{\Omega}_2} \sum_{i=1}^{n} \Delta\left(x_i - \theta\right)\left[\log f_1\left(x_i\right) - \log f_2\left(x_i\right)\right]$$
$$-\frac{\partial a}{\partial \mathbf{\Omega}_2}\left[\log(1+\Phi) + \log F_1(\theta)\right]$$
$$-\frac{\partial a}{\partial \mathbf{\Omega}_2}\left\{\log \Phi - \log(1+\Phi) - \log\left[1 - F_2(\theta)\right]\right\}$$
$$-a\left[\frac{\frac{\partial \Phi}{\partial \mathbf{\Omega}_2}}{1+\Phi} + \frac{f_1(\theta)}{F_1(\theta)} \frac{\partial \theta}{\partial \mathbf{\Omega}_2}\right] + (n-a)\left[\left(\frac{1}{\Phi} - \frac{1}{1+\Phi}\right) \frac{\partial \Phi}{\partial \mathbf{\Omega}_2} + \frac{f_2(\theta)}{1 - F_2(\theta)} \frac{\partial \theta}{\partial \mathbf{\Omega}_2}\right].$$

Hence, the observed information matrix is

$$\mathbf{J} = -\begin{pmatrix} \frac{\partial^2 \log L}{\partial \mathbf{\Omega}_1^2} & \frac{\partial^2 \log L}{\partial \mathbf{\Omega}_1 \partial \mathbf{\Omega}_2} \\ \frac{\partial^2 \log L}{\partial \mathbf{\Omega}_1 \partial \mathbf{\Omega}_2} & \frac{\partial^2 \log L}{\partial \mathbf{\Omega}_2^2} \end{pmatrix}, \tag{2.1}$$

where

$$
\begin{aligned}
\frac{\partial^2 \log L}{\partial \mathbf{\Omega}_1^2} &= \sum_{x_i \leq \theta} \frac{\frac{\partial^2 f_1(x_i)}{\partial \mathbf{\Omega}_1^2} f_1\left(x_i\right) - \left[\frac{\partial f_1(x_i)}{\partial \mathbf{\Omega}_1}\right]^2}{\left[f_1\left(x_i\right)\right]^2} + 2 \frac{\partial \theta}{\partial \mathbf{\Omega}_1} \sum_{i=1}^n \Delta\left(x_i - \theta\right) \frac{\frac{\partial f_1(x_i)}{\partial \mathbf{\Omega}_1}}{f_1\left(x_i\right)} \\
&+ \frac{\partial^2 \theta}{\partial \mathbf{\Omega}_1^2} \sum_{i=1}^n \Delta\left(x_i - \theta\right) \left[\log f_1\left(x_i\right) - \log f_2\left(x_i\right)\right] \\
&- \frac{\partial^2 a}{\partial \mathbf{\Omega}_1^2} \left[\log(1+\Phi) + \log F_1(\theta)\right] - \frac{\partial^2 a}{\partial \mathbf{\Omega}_1^2} \left\{\log \Phi - \log(1+\Phi) - \log\left[1 - F_2(\theta)\right]\right\} \\
&- 2 \frac{\partial a}{\partial \mathbf{\Omega}_1} \left[\frac{\frac{\partial \Phi}{\partial \mathbf{\Omega}_1}}{1+\Phi} + \frac{f_1(\theta)}{F_1(\theta)} \frac{\partial \theta}{\partial \mathbf{\Omega}_1}\right] - 2 \frac{\partial a}{\partial \mathbf{\Omega}_1} \left[\left(\frac{1}{\Phi} - \frac{1}{1+\Phi}\right) \frac{\partial \Phi}{\partial \mathbf{\Omega}_1} + \frac{f_2(\theta)}{1 - F_2(\theta)} \frac{\partial \theta}{\partial \mathbf{\Omega}_1}\right] \\
&- a \frac{\frac{\partial^2 \Phi}{\partial \mathbf{\Omega}_1^2}(1+\Phi) - \left(\frac{\partial \Phi}{\partial \mathbf{\Omega}_1}\right)^2}{(1+\Phi)^2} - a \frac{\frac{\partial f_1(\theta)}{\partial \mathbf{\Omega}_1} F_1(\theta) - \frac{\partial \theta}{\partial \mathbf{\Omega}_1}\left[f_1(\theta)\right]^2}{\left[F_1(\theta)\right]^2} \frac{\partial \theta}{\partial \mathbf{\Omega}_1} - a \frac{f_1(\theta)}{F_1(\theta)} \frac{\partial^2 \theta}{\partial \mathbf{\Omega}_1^2} \\
&+ (n-a) \left[\frac{1}{(1+\Phi)^2} - \frac{1}{\Phi^2}\right] \left(\frac{\partial \Phi}{\partial \mathbf{\Omega}_1}\right)^2 + (n-a) \left(\frac{1}{\Phi} - \frac{1}{1+\Phi}\right) \frac{\partial^2 \Phi}{\partial \mathbf{\Omega}_1^2} \\
&+ (n-a) \frac{\frac{\partial f_2(\theta)}{\partial \mathbf{\Omega}_1}\left[1 - F_2(\theta)\right] + \left[f_2(\theta)\right]^2 \frac{\partial \theta}{\partial \mathbf{\Omega}_1}}{\left[1 - F_2(\theta)\right]^2} \frac{\partial \theta}{\partial \mathbf{\Omega}_1} + (n-a) \frac{f_2(\theta)}{1 - F_2(\theta)} \frac{\partial^2 \theta}{\partial \mathbf{\Omega}_1^2},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 \log L}{\partial \mathbf{\Omega}_1 \partial \mathbf{\Omega}_2} &= \frac{\partial \theta}{\partial \mathbf{\Omega}_2} \sum_{i=1}^n \Delta\left(x_i - \theta\right) \frac{\frac{\partial f_1(x_i)}{\partial \mathbf{\Omega}_1}}{f_1\left(x_i\right)} + \frac{\partial^2 \theta}{\partial \mathbf{\Omega}_1 \partial \mathbf{\Omega}_2} \sum_{i=1}^n \Delta\left(x_i - \theta\right) \left[\log f_1\left(x_i\right) - \log f_2\left(x_i\right)\right] \\
&- \frac{\partial \theta}{\partial \mathbf{\Omega}_1} \sum_{i=1}^n \Delta\left(x_i - \theta\right) \frac{\frac{\partial f_2(x_i)}{\partial \mathbf{\Omega}_2}}{f_2\left(x_i\right)} - \frac{\partial^2 a}{\partial \mathbf{\Omega}_1 \partial \mathbf{\Omega}_2} \left[\log(1+\Phi) + \log F_1(\theta)\right] \\
&- \frac{\partial^2 a}{\partial \mathbf{\Omega}_1 \partial \mathbf{\Omega}_2} \left\{\log \Phi - \log(1+\Phi) - \log\left[1 - F_2(\theta)\right]\right\} - \frac{\partial a}{\partial \mathbf{\Omega}_2} \left[\frac{\frac{\partial \Phi}{\partial \mathbf{\Omega}_1}}{1+\Phi} + \frac{f_1(\theta)}{F_1(\theta)} \frac{\partial \theta}{\partial \mathbf{\Omega}_1}\right] \\
&- \frac{\partial a}{\partial \mathbf{\Omega}_1} \left[\frac{\frac{\partial \Phi}{\partial \mathbf{\Omega}_2}}{1+\Phi} + \frac{f_1(\theta)}{F_1(\theta)} \frac{\partial \theta}{\partial \mathbf{\Omega}_2}\right] - \frac{\partial a}{\partial \mathbf{\Omega}_1} \left[\left(\frac{1}{\Phi} - \frac{1}{1+\Phi}\right) \frac{\partial \Phi}{\partial \mathbf{\Omega}_2} + \frac{f_2(\theta)}{1 - F_2(\theta)} \frac{\partial \theta}{\partial \mathbf{\Omega}_2}\right] \\
&- \frac{\partial a}{\partial \mathbf{\Omega}_2} \left[\left(\frac{1}{\Phi} - \frac{1}{1+\Phi}\right) \frac{\partial \Phi}{\partial \mathbf{\Omega}_1} + \frac{f_2(\theta)}{1 - F_2(\theta)} \frac{\partial \theta}{\partial \mathbf{\Omega}_1}\right] \\
&- a \frac{\frac{\partial^2 \Phi}{\partial \mathbf{\Omega}_1 \partial \mathbf{\Omega}_2}(1+\Phi) - \frac{\partial \Phi}{\partial \mathbf{\Omega}_1} \frac{\partial \Phi}{\partial \mathbf{\Omega}_2}}{(1+\Phi)^2} - a \frac{\frac{\partial f_1(\theta)}{\partial \mathbf{\Omega}_2} F_1(\theta) - \left[f_1(\theta)\right]^2 \frac{\partial \theta}{\partial \mathbf{\Omega}_2}}{\left[F_1(\theta)\right]^2} \frac{\partial \theta}{\partial \mathbf{\Omega}_1} \\
&- a \frac{f_1(\theta)}{F_1(\theta)} \frac{\partial^2 \theta}{\partial \mathbf{\Omega}_1 \partial \mathbf{\Omega}_2} \\
&+ (n-a) \left[\frac{1}{(1+\Phi)^2} - \frac{1}{\Phi^2}\right] \frac{\partial \Phi}{\partial \mathbf{\Omega}_1} \frac{\partial \Phi}{\partial \mathbf{\Omega}_2} + (n-a) \left(\frac{1}{\Phi} - \frac{1}{1+\Phi}\right) \frac{\partial^2 \Phi}{\partial \mathbf{\Omega}_1 \partial \mathbf{\Omega}_2} \\
&+ (n-a) \frac{\partial \theta}{\partial \mathbf{\Omega}_1} \frac{\frac{\partial f_2(\theta)}{\partial \mathbf{\Omega}_2}\left[1 - F_2(\theta)\right] + \left[f_2(\theta)\right]^2 \frac{\partial \theta}{\partial \mathbf{\Omega}_2}}{\left[1 - F_2(\theta)\right]^2} + \frac{f_2(\theta)}{1 - F_2(\theta)} \frac{\partial^2 \theta}{\partial \mathbf{\Omega}_1 \partial \mathbf{\Omega}_2}
\end{aligned}
$$

and

$$\frac{\partial^2 \log L}{\partial \mathbf{\Omega}_2^2} = \sum_{x_i > \theta} \frac{\frac{\partial^2 f_2(x_i)}{\partial \mathbf{\Omega}_2^2} f_2(x_i) - \left[\frac{\partial f_2(x_i)}{\partial \mathbf{\Omega}_2}\right]^2}{[f_2(x_i)]^2} + 2\frac{\partial \theta}{\partial \mathbf{\Omega}_2} \sum_{i=1}^{n} \Delta(x_i - \theta) \frac{\frac{\partial f_2(x_i)}{\partial \mathbf{\Omega}_2}}{f_2(x_i)}$$

$$+\frac{\partial^2 \theta}{\partial \mathbf{\Omega}_2^2} \sum_{i=1}^{n} \Delta(x_i - \theta) [\log f_1(x_i) - \log f_2(x_i)]$$

$$-\frac{\partial^2 a}{\partial \mathbf{\Omega}_2^2} [\log(1+\Phi) + \log F_1(\theta)] - \frac{\partial^2 a}{\partial \mathbf{\Omega}_2^2} \{\log \Phi - \log(1+\Phi) - \log[1 - F_2(\theta)]\}$$

$$-2\frac{\partial a}{\partial \mathbf{\Omega}_2} \left[\frac{\frac{\partial \Phi}{\partial \mathbf{\Omega}_2}}{1+\Phi} + \frac{f_1(\theta)}{F_1(\theta)} \frac{\partial \theta}{\partial \mathbf{\Omega}_2}\right] - 2\frac{\partial a}{\partial \mathbf{\Omega}_2} \left[\left(\frac{1}{\Phi} - \frac{1}{1+\Phi}\right) \frac{\partial \Phi}{\partial \mathbf{\Omega}_2} + \frac{f_2(\theta)}{1 - F_2(\theta)} \frac{\partial \theta}{\partial \mathbf{\Omega}_2}\right]$$

$$-a \frac{\frac{\partial^2 \Phi}{\partial \mathbf{\Omega}_2^2}(1+\Phi) - \left(\frac{\partial \Phi}{\partial \mathbf{\Omega}_2}\right)^2}{(1+\Phi)^2} - a \frac{\frac{\partial f_1(\theta)}{\partial \mathbf{\Omega}_2} F_1(\theta) - \frac{\partial \theta}{\partial \mathbf{\Omega}_2} [f_1(\theta)]^2}{[F_1(\theta)]^2} \frac{\partial \theta}{\partial \mathbf{\Omega}_2} - a \frac{f_1(\theta)}{F_1(\theta)} \frac{\partial^2 \theta}{\partial \mathbf{\Omega}_2^2}$$

$$+(n-a)\left[\frac{1}{(1+\Phi)^2} - \frac{1}{\Phi^2}\right] \left(\frac{\partial \Phi}{\partial \mathbf{\Omega}_2}\right)^2 + (n-a)\left(\frac{1}{\Phi} - \frac{1}{1+\Phi}\right) \frac{\partial^2 \Phi}{\partial \mathbf{\Omega}_2^2}$$

$$+(n-a) \frac{\frac{\partial f_2(\theta)}{\partial \mathbf{\Omega}_2} [1 - F_2(\theta)] + [f_2(\theta)]^2 \frac{\partial \theta}{\partial \mathbf{\Omega}_2}}{[1 - F_2(\theta)]^2} \frac{\partial \theta}{\partial \mathbf{\Omega}_2} + (n-a) \frac{f_2(\theta)}{1 - F_2(\theta)} \frac{\partial^2 \theta}{\partial \mathbf{\Omega}_2^2}.$$

3. General three-piece composite distribution

Suppose that the lower tail, middle part and the upper tail are described probability density functions f_1, f_2 and f_3, respectively. Then the probability density and cumulative distribution functions of the most general three-piece composite distribution are

$$f(x) = \begin{cases} \frac{1}{1+\Phi_2+\Phi_3} \frac{f_1(x)}{F_1(\theta_1)}, & x \le \theta_1, \\ \frac{\Phi_2}{1+\Phi_2+\Phi_3} \frac{f_2(x)}{F_2(\theta_2) - F_2(\theta_1)}, & \theta_1 < x \le \theta_2, \\ \frac{\Phi_3}{1+\Phi_2+\Phi_3} \frac{f_3(x)}{1 - F_3(\theta_2)}, & \theta_2 < x \end{cases} \tag{3.1}$$

and

$$F(x) = \begin{cases} \frac{1}{1+\Phi_2+\Phi_3} \frac{F_1(x)}{F_1(\theta_1)}, & x \le \theta_1, \\ \frac{1}{1+\Phi_2+\Phi_3} \left[1 + \Phi_2 \frac{F_2(x) - F_2(\theta_1)}{F_2(\theta_2) - F_2(\theta_1)}\right], & \theta_1 < x \le \theta_2, \\ \frac{1}{1+\Phi_2+\Phi_3} \left[1 + \Phi_2 + \Phi_3 \frac{F_3(x) - F_3(\theta_2)}{1 - F_3(\theta_2)}\right], & \theta_2 < x, \end{cases} \tag{3.2}$$

respectively. Suppose f_1, f_2 and f_3 are parameterized by $\mathbf{\Omega}_1$, $\mathbf{\Omega}_2$ and $\mathbf{\Omega}_3$, respectively. Conditions for continuity and differentiability at θ_1, θ_2 ensure that $\theta_1 = \theta_1(\mathbf{\Omega}_1, \mathbf{\Omega}_2, \mathbf{\Omega}_3)$, $\theta_2 = \theta_2(\mathbf{\Omega}_1, \mathbf{\Omega}_2, \mathbf{\Omega}_3)$, $\Phi_2 = \Phi_2(\mathbf{\Omega}_1, \mathbf{\Omega}_2, \mathbf{\Omega}_3) = \frac{f_1(\theta_1)[F_2(\theta_2) - F_2(\theta_1)]}{F_1(\theta_1) f_2(\theta_1)}$ and $\Phi_3 = \Phi_3(\mathbf{\Omega}_1, \mathbf{\Omega}_2, \mathbf{\Omega}_3) = \frac{f_1(\theta_1) f_2(\theta_2)[1 - F_3(\theta_2)]}{F_1(\theta_1) f_2(\theta_1) f_3(\theta_2)}$. If $x_1, x_2, \ldots, x_n$ is a random sample from (3.1)-(3.2) then the log-likelihood function of $(\mathbf{\Omega}_1, \mathbf{\Omega}_2, \mathbf{\Omega}_3)$ is

$$\log L(\mathbf{\Omega}_1, \mathbf{\Omega}_2, \mathbf{\Omega}_3) = \sum_{x_i \le \theta_1} \log f_1(x_i) - a \log F_1(\theta_1) - n \log(1 + \Phi_2 + \Phi_3)$$

$$+b\{\log \Phi_2 - \log[F_2(\theta_2) - F_2(\theta_1)]\} + \sum_{\theta_1 \le x_i \le \theta_2} \log f_2(x_i) + c\{\log \Phi_3 - \log[1 - F_3(\theta_2)]\}$$

$$+\sum_{\theta_2 < x_i} \log f_3(x_i),$$

where

$$a = \sum_{i=1}^{n} I\left\{x_i \leq \theta_1\right\},$$

$$b = \sum_{i=1}^{n} I\left\{\theta_1 < x_i \leq \theta_2\right\}$$

and

$$c = \sum_{i=1}^{n} I\left\{\theta_2 < x_i\right\}$$

with $a + b + c = n$. The first order partial derivatives are

$$\begin{aligned}
\frac{\partial \log L}{\partial \mathbf{\Omega}_1} = & \frac{\partial \theta_1}{\partial \mathbf{\Omega}_1} \sum_{i=1}^{n} \Delta\left(x_i - \theta_1\right) \log f_1\left(x_i\right) + \sum_{x_i \leq \theta_1} \frac{\frac{\partial f_1(x_i)}{\partial \mathbf{\Omega}_1}}{f_1\left(x_i\right)} - \frac{\partial a}{\partial \mathbf{\Omega}_1} \log F_1\left(\theta_1\right) - \frac{a f_1(\theta_1) \frac{\partial \theta_1}{\partial \mathbf{\Omega}_1}}{F_1\left(\theta_1\right)} \\
& - \frac{n\left(\frac{\partial \Phi_2}{\partial \mathbf{\Omega}_1} + \frac{\partial \Phi_3}{\partial \mathbf{\Omega}_1}\right)}{1 + \Phi_2 + \Phi_3} + \frac{\partial b}{\partial \mathbf{\Omega}_1}\left\{\log \Phi_2 - \log\left[F_2\left(\theta_2\right) - F_2\left(\theta_1\right)\right]\right\} \\
& + b\left[\frac{\frac{\partial \Phi_2}{\partial \mathbf{\Omega}_1}}{\Phi_2} - \frac{\frac{f_2(\theta_2)\partial \theta_2}{\partial \mathbf{\Omega}_1} - \frac{f_2(\theta_1)\partial \theta_1}{\partial \mathbf{\Omega}_1}}{F_2\left(\theta_2\right) - F_2\left(\theta_1\right)}\right] - \frac{\partial \theta_1}{\partial \mathbf{\Omega}_1} \sum_{x_i \leq \theta_2} \Delta\left(x_i - \theta_1\right) \log f_2\left(x_i\right) \\
& + \frac{\partial \theta_2}{\partial \mathbf{\Omega}_1} \sum_{\theta_1 \leq x} \Delta\left(x_i - \theta_2\right) \log f_2\left(x_i\right) \\
& + \frac{\partial c}{\partial \mathbf{\Omega}_1}\left\{\log \Phi_3 - \log\left[1 - F_3\left(\theta_2\right)\right]\right\} + c\left[\frac{\frac{\partial \Phi_3}{\partial \mathbf{\Omega}_1}}{\Phi_3} + \frac{\frac{f_3(\theta_2)\partial \theta_2}{\partial \mathbf{\Omega}_1}}{1 - F_3\left(\theta_2\right)}\right] - \frac{\partial \theta_2}{\partial \mathbf{\Omega}_1} \sum_{i=1}^{n} \Delta\left(x_i - \theta_2\right) \log f_3\left(x_i\right),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \log L}{\partial \mathbf{\Omega}_2} = & \frac{\partial \theta_1}{\partial \mathbf{\Omega}_2} \sum_{i=1}^{n} \Delta\left(x_i - \theta_1\right) \log f_1\left(x_i\right) - \frac{\partial a}{\partial \mathbf{\Omega}_2} \log F_1\left(\theta_1\right) - \frac{a f_1\left(\theta_1\right) \frac{\partial \theta_1}{\partial \mathbf{\Omega}_2}}{F_1\left(\theta_1\right)} \\
& - \frac{n\left(\frac{\partial \Phi_2}{\partial \mathbf{\Omega}_2} + \frac{\partial \Phi_3}{\partial \mathbf{\Omega}_2}\right)}{1 + \Phi_2 + \Phi_3} + \frac{\partial b}{\partial \mathbf{\Omega}_2}\left\{\log \Phi_2 - \log\left[F_2\left(\theta_2\right) - F_2\left(\theta_1\right)\right]\right\} \\
& + b\left[\frac{\frac{\partial \Phi_2}{\partial \mathbf{\Omega}_2}}{\Phi_2} - \frac{\frac{f_2(\theta_2)\partial \theta_2}{\partial \mathbf{\Omega}_2} - \frac{f_2(\theta_1)\partial \theta_1}{\partial \mathbf{\Omega}_2}}{F_2\left(\theta_2\right) - F_2\left(\theta_1\right)}\right] - \frac{\partial \theta_1}{\partial \mathbf{\Omega}_2} \sum_{x_i \leq \theta_2} \Delta\left(x_i - \theta_1\right) \log f_2\left(x_i\right) \\
& + \frac{\partial \theta_2}{\partial \mathbf{\Omega}_2} \sum_{\theta_1 \leq x_i} \Delta\left(x_i - \theta_2\right) \log f_2\left(x_i\right) \\
& + \sum_{\theta_1 \leq x_i \leq \theta_2} \frac{\frac{\partial f_2(x_i)}{\partial \mathbf{\Omega}_2}}{f_2\left(x_i\right)} + \frac{\partial c}{\partial \mathbf{\Omega}_2}\left\{\log \Phi_3 - \log\left[1 - F_3\left(\theta_2\right)\right]\right\} + c\left[\frac{\frac{\partial \Phi_3}{\partial \mathbf{\Omega}_2}}{\Phi_3} + \frac{\frac{f_3(\theta_2)\partial \theta_2}{\partial \mathbf{\Omega}_2}}{1 - F_3\left(\theta_2\right)}\right] \\
& - \frac{\partial \theta_2}{\partial \mathbf{\Omega}_2} \sum_{i=1}^{n} \Delta\left(x_i - \theta_2\right) \log f_3\left(x_i\right)
\end{aligned}$$

and

$$
\begin{aligned}
\frac{\partial \log L}{\partial \mathbf{\Omega}_3} &= \frac{\partial \theta_1}{\partial \mathbf{\Omega}_3} \sum_{i=1}^{n} \Delta\left(x_i - \theta_1\right) \log f_1\left(x_i\right) - \frac{\partial a}{\partial \mathbf{\Omega}_3} \log F_1\left(\theta_1\right) - \frac{a f_1\left(\theta_1\right) \frac{\partial \theta_1}{\partial \mathbf{\Omega}_3}}{F_1\left(\theta_1\right)} \\
&- \frac{n\left(\frac{\partial \Phi_2}{\partial \mathbf{\Omega}_3} + \frac{\partial \Phi_3}{\partial \mathbf{\Omega}_3}\right)}{1 + \Phi_2 + \Phi_3} + \frac{\partial b}{\partial \mathbf{\Omega}_3} \left\{\log \Phi_2 - \log\left[F_2\left(\theta_2\right) - F_2\left(\theta_1\right)\right]\right\} \\
&+ b\left[\frac{\frac{\partial \Phi_2}{\partial \mathbf{\Omega}_3}}{\Phi_2} - \frac{\frac{f_2(\theta_2)\partial \theta_2}{\partial \mathbf{\Omega}_3} - \frac{f_2(\theta_1)\partial \theta_1}{\partial \mathbf{\Omega}_3}}{F_2\left(\theta_2\right) - F_2\left(\theta_1\right)}\right] - \frac{\partial \theta_1}{\partial \mathbf{\Omega}_3} \sum_{x_i \leq \theta_2} \Delta\left(x_i - \theta_1\right) \log f_2\left(x_i\right) \\
&+ \frac{\partial \theta_2}{\partial \mathbf{\Omega}_3} \sum_{\theta_1 \leq x_i} \Delta\left(x_i - \theta_2\right) \log f_2\left(x_i\right) \\
&+ \frac{\partial c}{\partial \mathbf{\Omega}_3} \left\{\log \Phi_3 - \log\left[1 - F_3\left(\theta_2\right)\right]\right\} + c\left[\frac{\frac{\partial \Phi_3}{\partial \mathbf{\Omega}_3}}{\Phi_3} + \frac{\frac{f_3(\theta_2)\partial \theta_2}{\partial \mathbf{\Omega}_3}}{1 - F_3\left(\theta_2\right)}\right] - \frac{\partial \theta_2}{\partial \mathbf{\Omega}_3} \sum_{i=1}^{n} \Delta\left(x_i - \theta_2\right) \log f_3\left(x_i\right) \\
&+ \sum_{\theta_2 < x_i} \frac{\frac{\partial f_3(x_i)}{\partial \mathbf{\Omega}_3}}{f_3\left(x_i\right)}.
\end{aligned}
$$

Hence, the observed information matrix is

$$
\mathbf{J} = -\begin{pmatrix}
\frac{\partial^2 \log L}{\partial \mathbf{\Omega}_1^2} & \frac{\partial^2 \log L}{\partial \mathbf{\Omega}_1 \partial \mathbf{\Omega}_2} & \frac{\partial^2 \log L}{\partial \mathbf{\Omega}_1 \partial \mathbf{\Omega}_3} \\
\frac{\partial^2 \log L}{\partial \mathbf{\Omega}_1 \partial \mathbf{\Omega}_2} & \frac{\partial^2 \log L}{\partial \mathbf{\Omega}_2^2} & \frac{\partial^2 \log L}{\partial \mathbf{\Omega}_2 \partial \mathbf{\Omega}_3} \\
\frac{\partial^2 \log L}{\partial \mathbf{\Omega}_1 \partial \mathbf{\Omega}_3} & \frac{\partial^2 \log L}{\partial \mathbf{\Omega}_2 \partial \mathbf{\Omega}_3} & \frac{\partial^2 \log L}{\partial \mathbf{\Omega}_3^2}
\end{pmatrix},
$$

where

$$
\begin{aligned}
\frac{\partial^2 \log L}{\partial \boldsymbol{\Omega}_1^2} &= \frac{\partial^2 \theta_1}{\partial \boldsymbol{\Omega}_1^2} \sum_{i=1}^{n} \Delta\left(x_i - \theta_1\right) \log f_1\left(x_i\right) + \frac{2\partial \theta_1}{\partial \boldsymbol{\Omega}_1} \sum_{i=1}^{n} \Delta\left(x_i - \theta_1\right) \frac{\frac{\partial f_1(x_i)}{\partial \boldsymbol{\Omega}_1}}{f_1\left(x_i\right)} \\
&+ \sum_{x_i \leq \theta_1} \frac{\frac{\partial^2 f_1(x_i)}{\partial \boldsymbol{\Omega}_1^2} f_1\left(x_i\right) - \left[\frac{\partial f_1(x_i)}{\partial \boldsymbol{\Omega}_1}\right]^2}{\left[f_1\left(x_i\right)\right]^2} \\
&- \frac{\partial^2 a}{\partial \boldsymbol{\Omega}_1^2} \log F_1\left(\theta_1\right) - \frac{\left[\frac{\partial a}{\partial \boldsymbol{\Omega}_1} f_1\left(\theta_1\right) \frac{\partial \theta_1}{\partial \boldsymbol{\Omega}_1} + a \frac{\partial f_1(\theta_1)}{\partial \boldsymbol{\Omega}_1} \frac{\partial \theta_1}{\partial \boldsymbol{\Omega}_1} + a f_1(\theta_1) \frac{\partial^2 \theta_1}{\partial \boldsymbol{\Omega}_1^2}\right] F_1\left(\theta_1\right) - a\left[f_1\left(\theta_1\right)\right]^2 \left(\frac{\partial \theta_1}{\partial \boldsymbol{\Omega}_1}\right)^2}{\left[F_1\left(\theta_1\right)\right]^2} \\
&- \frac{\partial a}{\partial \boldsymbol{\Omega}_1} \frac{f_1\left(\theta_1\right) \frac{\partial \theta_1}{\partial \boldsymbol{\Omega}_1}}{F_1\left(\theta_1\right)} - \frac{n\left[\left(\frac{\partial^2 \Phi_2}{\partial \boldsymbol{\Omega}_1^2} + \frac{\partial^2 \Phi_3}{\partial \boldsymbol{\Omega}_1^2}\right)\left(1 + \Phi_2 + \Phi_3\right) - \left(\frac{\partial \Phi_2}{\partial \boldsymbol{\Omega}_1} + \frac{\partial \Phi_3}{\partial \boldsymbol{\Omega}_1}\right)^2\right]}{\left(1 + \Phi_2 + \Phi_3\right)^2} \\
&+ \frac{\partial^2 b}{\partial \boldsymbol{\Omega}_1^2} \left\{\log \Phi_2 - \log\left[F_2\left(\theta_2\right) - F_2\left(\theta_1\right)\right]\right\} + 2 \frac{\partial b}{\partial \boldsymbol{\Omega}_1} \left[\frac{\frac{\partial \Phi_2}{\partial \boldsymbol{\Omega}_1}}{\Phi_2} - \frac{f_2\left(\theta_2\right) \frac{\partial \theta_2}{\partial \boldsymbol{\Omega}_1} - f_2\left(\theta_1\right) \frac{\partial \theta_1}{\partial \boldsymbol{\Omega}_1}}{F_2\left(\theta_2\right) - F_2\left(\theta_1\right)}\right] \\
&+ \frac{b \frac{\partial^2 \Phi_2}{\partial \boldsymbol{\Omega}_1^2} \Phi_2 - b\left(\frac{\partial \Phi_2}{\partial \boldsymbol{\Omega}_1}\right)^2}{\Phi_2^2} \\
&- \frac{b\left[\frac{\partial f_2(\theta_2)}{\partial \boldsymbol{\Omega}_1} \frac{\partial \theta_2}{\partial \boldsymbol{\Omega}_1} + f_2\left(\theta_2\right) \frac{\partial^2 \theta_2}{\partial \boldsymbol{\Omega}_1^2} - \frac{\partial f_2(\theta_1)}{\partial \boldsymbol{\Omega}_1} \frac{\partial \theta_1}{\partial \boldsymbol{\Omega}_1} - f_2\left(\theta_1\right) \frac{\partial^2 \theta_1}{\partial \boldsymbol{\Omega}_1^2}\right]}{F_2\left(\theta_2\right) - F_2\left(\theta_1\right)} \\
&+ \frac{b\left[f_2\left(\theta_2\right) \frac{\partial \theta_2}{\partial \boldsymbol{\Omega}_1} - f_2\left(\theta_1\right) \frac{\partial \theta_1}{\partial \boldsymbol{\Omega}_1}\right]^2}{\left[F_2\left(\theta_2\right) - F_2\left(\theta_1\right)\right]^2} \\
&- 2 \frac{\partial \theta_1}{\partial \boldsymbol{\Omega}_1} \frac{\partial \theta_2}{\partial \boldsymbol{\Omega}_1} \sum_{i=1}^{n} \Delta\left(x_i - \theta_1\right) \Delta\left(x_i - \theta_2\right) \log f_2\left(x_i\right) \\
&- \frac{\partial^2 \theta_1}{\partial \boldsymbol{\Omega}_1^2} \sum_{x_i \leq \theta_2} \Delta\left(x_i - \theta_1\right) \log f_2\left(x_i\right) + \frac{\partial^2 \theta_2}{\partial \boldsymbol{\Omega}_1^2} \sum_{\theta_1 \leq x_i} \Delta\left(x_i - \theta_2\right) \log f_2\left(x_i\right) \\
&+ \frac{\partial^2 c}{\partial \boldsymbol{\Omega}_1^2} \left\{\log \Phi_3 - \log\left[1 - F_3\left(\theta_2\right)\right]\right\} \\
&+ 2 \frac{\partial c}{\partial \boldsymbol{\Omega}_1} \left[\frac{\frac{\partial \Phi_3}{\partial \boldsymbol{\Omega}_1}}{\Phi_3} + \frac{\frac{f_3(\theta_2) \partial \theta_2}{\partial \boldsymbol{\Omega}_1}}{1 - F_3\left(\theta_2\right)}\right] + c \frac{\Phi_3 \frac{\partial^2 \Phi_3}{\partial \boldsymbol{\Omega}_1^2} - \left(\frac{\partial \Phi_3}{\partial \boldsymbol{\Omega}_1}\right)^2}{\Phi_3^2} - \frac{\partial^2 \theta_2}{\partial \boldsymbol{\Omega}_1^2} \sum_{i=1}^{n} \Delta\left(x_i - \theta_2\right) \log f_3\left(x_i\right) \\
&+ \frac{c\left[\frac{\partial f_3(\theta_2)}{\partial \boldsymbol{\Omega}_1} \frac{\partial \theta_2}{\partial \boldsymbol{\Omega}_1} + f_3\left(\theta_2\right) \frac{\partial^2 \theta_2}{\partial \boldsymbol{\Omega}_1^2}\right]\left[1 - F_3\left(\theta_2\right)\right] + c\left[f_3\left(\theta_2\right) \frac{\partial \theta_2}{\partial \boldsymbol{\Omega}_1}\right]^2}{\left[1 - F_3\left(\theta_2\right)\right]^2},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 \log L}{\partial \boldsymbol{\Omega}_1 \partial \boldsymbol{\Omega}_2} &= \frac{\partial^2 \theta_1}{\partial \boldsymbol{\Omega}_1 \partial \boldsymbol{\Omega}_2} \sum_{i=1}^{n} \Delta\left(x_i - \theta_1\right) \log f_1\left(x_i\right) + \frac{\partial \theta_1}{\partial \boldsymbol{\Omega}_2} \sum_{i=1}^{n} \Delta\left(x_i - \theta_1\right) \frac{\frac{\partial f_1(x_i)}{\partial \boldsymbol{\Omega}_1}}{f_1\left(x_i\right)} - \frac{\partial^2 a}{\partial \boldsymbol{\Omega}_1 \partial \boldsymbol{\Omega}_2} \log F_1\left(\theta_1\right) \\
&- \frac{\partial a}{\partial \boldsymbol{\Omega}_1} \frac{f_1\left(\theta_1\right) \frac{\partial \theta_1}{\partial \boldsymbol{\Omega}_2}}{F_1\left(\theta_1\right)} - \frac{n\left(\frac{\partial^2 \Phi_2}{\partial \boldsymbol{\Omega}_1 \partial \boldsymbol{\Omega}_2} + \frac{\partial^2 \Phi_3}{\partial \boldsymbol{\Omega}_1 \partial \boldsymbol{\Omega}_2}\right)\left(1 + \Phi_2 + \Phi_3\right) - n\left(\frac{\partial \Phi_2}{\partial \boldsymbol{\Omega}_1} + \frac{\partial \Phi_3}{\partial \boldsymbol{\Omega}_1}\right)\left(\frac{\partial \Phi_2}{\partial \boldsymbol{\Omega}_2} + \frac{\partial \Phi_3}{\partial \boldsymbol{\Omega}_2}\right)}{\left(1 + \Phi_2 + \Phi_3\right)^2} \\
&- \frac{\frac{\partial a}{\partial \boldsymbol{\Omega}_2} f_1\left(\theta_1\right) \frac{\partial \theta_1}{\partial \boldsymbol{\Omega}_1} + a \frac{\partial f_1(\theta_1)}{\partial \boldsymbol{\Omega}_2} \frac{\partial \theta_1}{\partial \boldsymbol{\Omega}_1} + a f_1\left(\theta_1\right) \frac{\partial^2 \theta_1}{\partial \boldsymbol{\Omega}_1 \partial \boldsymbol{\Omega}_2}}{F_1\left(\theta_1\right)} + \frac{a\left[f_1\left(\theta_1\right)\right]^2 \frac{\partial \theta_1}{\partial \boldsymbol{\Omega}_1} \frac{\partial \theta_1}{\partial \boldsymbol{\Omega}_2}}{\left[F_1\left(\theta_1\right)\right]^2} \\
&+ \frac{\partial^2 b}{\partial \boldsymbol{\Omega}_1 \partial \boldsymbol{\Omega}_2}\left\{\log \Phi_2 - \log\left[F_2\left(\theta_2\right) - F_2\left(\theta_1\right)\right]\right\} + \frac{\partial b}{\partial \boldsymbol{\Omega}_1}\left[\frac{\frac{\partial \Phi_2}{\partial \boldsymbol{\Omega}_2}}{\Phi_2} - \frac{f_2\left(\theta_2\right) \frac{\partial \theta_2}{\partial \boldsymbol{\Omega}_2} - f_2\left(\theta_1\right) \frac{\partial \theta_1}{\partial \boldsymbol{\Omega}_2}}{F_2\left(\theta_2\right) - F_2\left(\theta_1\right)}\right] \\
&+ \frac{\partial b}{\partial \boldsymbol{\Omega}_2}\left[\frac{\frac{\partial \Phi_2}{\partial \boldsymbol{\Omega}_1}}{\Phi_2} - \frac{f_2\left(\theta_2\right) \frac{\partial \theta_2}{\partial \boldsymbol{\Omega}_1} - f_2\left(\theta_1\right) \frac{\partial \theta_1}{\partial \boldsymbol{\Omega}_1}}{F_2\left(\theta_2\right) - F_2\left(\theta_1\right)}\right] + \frac{b \frac{\partial^2 \Phi_2}{\partial \boldsymbol{\Omega}_1 \partial \boldsymbol{\Omega}_2} \Phi_2 - b \frac{\partial \Phi_2}{\partial \boldsymbol{\Omega}_1} \frac{\partial \Phi_2}{\partial \boldsymbol{\Omega}_2}}{\Phi_2^2} \\
&- \frac{b\left[\frac{\partial f_2(\theta_2)}{\partial \boldsymbol{\Omega}_2} \frac{\partial \theta_2}{\partial \boldsymbol{\Omega}_1} + f_2\left(\theta_2\right) \frac{\partial^2 \theta_2}{\partial \boldsymbol{\Omega}_1 \partial \boldsymbol{\Omega}_2} - \frac{\partial f_2(\theta_2)}{\partial \boldsymbol{\Omega}_2} \frac{\partial \theta_1}{\partial \boldsymbol{\Omega}_1} - f_2\left(\theta_1\right) \frac{\partial^2 \theta_1}{\partial \boldsymbol{\Omega}_1 \partial \boldsymbol{\Omega}_2}\right]}{F_2\left(\theta_2\right) - F_2\left(\theta_1\right)} \\
&+ \frac{b\left[f_2\left(\theta_2\right) \frac{\partial \theta_2}{\partial \boldsymbol{\Omega}_1} - f_2\left(\theta_1\right) \frac{\partial \theta_1}{\partial \boldsymbol{\Omega}_1}\right]\left[f_2\left(\theta_2\right) \frac{\partial \theta_2}{\partial \boldsymbol{\Omega}_2} - f_2\left(\theta_1\right) \frac{\partial \theta_1)}{\partial \boldsymbol{\Omega}_2}\right]}{\left[F_2\left(\theta_2\right) - F_2\left(\theta_1\right)\right]^2} - \frac{\partial^2 \theta_1}{\partial \boldsymbol{\Omega}_1 \partial \boldsymbol{\Omega}_2} \sum_{x_i \leq \theta_2} \Delta\left(x_i - \theta_1\right) \log f_2\left(x_i\right) \\
&- \frac{\partial \theta_1}{\partial \boldsymbol{\Omega}_1} \frac{\partial \theta_2}{\partial \boldsymbol{\Omega}_2} \sum_{i=1}^{n} \Delta\left(x_i - \theta_1\right) \Delta\left(x_i - \theta_2\right) \log f_2\left(x_i\right) - \frac{\partial \theta_1}{\partial \boldsymbol{\Omega}_1} \sum_{x_i \leq \theta_2} \Delta\left(x_i - \theta_1\right) \frac{\frac{\partial f_2(x_i)}{\partial \boldsymbol{\Omega}_2}}{f_2\left(x_i\right)} \\
&+ \frac{\partial^2 \theta_2}{\partial \boldsymbol{\Omega}_1 \partial \boldsymbol{\Omega}_2} \sum_{\theta_1 \leq x_i} \Delta\left(x_i - \theta_2\right) \log f_2\left(x_i\right) - \frac{\partial \theta_1}{\partial \boldsymbol{\Omega}_2} \frac{\partial \theta_2}{\partial \boldsymbol{\Omega}_1} \sum_{i=1}^{n} \Delta\left(x_i - \theta_1\right) \Delta\left(x_i - \theta_2\right) \log f_2\left(x_i\right) \\
&+ \frac{\partial \theta_2}{\partial \boldsymbol{\Omega}_1} \sum_{\theta_1 \leq x_i} \Delta\left(x_i - \theta_2\right) \frac{\frac{\partial f_2(x_i)}{\partial \boldsymbol{\Omega}_2}}{f_2\left(x_i\right)} + \frac{\partial^2 c}{\partial \boldsymbol{\Omega}_1 \partial \boldsymbol{\Omega}_2}\left\{\log \Phi_3 - \log\left[1 - F_3\left(\theta_2\right)\right]\right\} \\
&+ \left[\frac{\frac{\partial \Phi_3}{\partial \boldsymbol{\Omega}_2}}{\Phi_3} + \frac{f_3\left(\theta_2\right) \frac{\partial \theta_2}{\partial \boldsymbol{\Omega}_2}}{1 - F_3\left(\theta_2\right)}\right] \frac{\partial c}{\partial \boldsymbol{\Omega}_1} + \frac{\partial c}{\partial \boldsymbol{\Omega}_2}\left[\frac{\frac{\partial \Phi_3}{\partial \boldsymbol{\Omega}_1}}{\Phi_3} + \frac{f_3\left(\theta_2\right) \frac{\partial \theta_2}{\partial \boldsymbol{\Omega}_1}}{1 - F_3\left(\theta_2\right)}\right] + \frac{c \frac{\partial^2 \Phi_3}{\partial \boldsymbol{\Omega}_1 \partial \boldsymbol{\Omega}_2} \Phi_3 - c \frac{\partial \Phi_3}{\partial \boldsymbol{\Omega}_1} \frac{\partial \Phi_3}{\partial \boldsymbol{\Omega}_2}}{\Phi_3^2} \\
&+ \frac{c\left[\frac{\partial f_3(\theta_2)}{\partial \boldsymbol{\Omega}_2} \frac{\partial \theta_2}{\partial \boldsymbol{\Omega}_1} + f_3\left(\theta_2\right) \frac{\partial^2 \theta_2}{\partial \boldsymbol{\Omega}_1 \partial \boldsymbol{\Omega}_2}\right]\left[1 - F_3\left(\theta_2\right)\right] + c\left[f_3\left(\theta_2\right)\right]^2 \frac{\partial \theta_2}{\partial \boldsymbol{\Omega}_1} \frac{\partial \theta_2}{\partial \boldsymbol{\Omega}_2}}{\left[1 - F_3\left(\theta_2\right)\right]^2} \\
&- \frac{\partial^2 \theta_2}{\partial \boldsymbol{\Omega}_1 \boldsymbol{\Omega}_2} \sum_{i=1}^{n} \Delta\left(x_i - \theta_2\right) \log f_3\left(x_i\right),
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 \log L}{\partial \boldsymbol{\Omega}_1 \partial \boldsymbol{\Omega}_3} ={}& \frac{\partial^2 \theta_1}{\partial \boldsymbol{\Omega}_1 \partial \boldsymbol{\Omega}_3} \sum_{i=1}^{n} \Delta\left(x_i-\theta_1\right) \log f_1\left(x_i\right) + \frac{\partial \theta_1}{\partial \boldsymbol{\Omega}_3} \sum_{i=1}^{n} \Delta\left(x_i-\theta_1\right) \frac{\frac{\partial f_1\left(x_i\right)}{\partial \boldsymbol{\Omega}_1}}{f_1\left(x_i\right)} - \frac{\partial^2 a}{\partial \boldsymbol{\Omega}_1 \partial \boldsymbol{\Omega}_3} \log F_1\left(\theta_1\right) \\
& - \frac{\frac{\partial a}{\partial \boldsymbol{\Omega}_3} f_1\left(\theta_1\right) \frac{\partial \theta_1}{\partial \boldsymbol{\Omega}_1} + a \frac{\partial f_1\left(\theta_1\right)}{\partial \boldsymbol{\Omega}_3} \frac{\partial \theta_1}{\partial \boldsymbol{\Omega}_1} + a f_1\left(\theta_1\right) \frac{\partial^2 \theta_1}{\partial \boldsymbol{\Omega}_1 \partial \boldsymbol{\Omega}_3}}{F_1\left(\theta_1\right)} + \frac{a\left[f_1\left(\theta_1\right)\right]^2 \frac{\partial \theta_1}{\partial \boldsymbol{\Omega}_1} \frac{\partial \theta_1}{\partial \boldsymbol{\Omega}_3}}{\left[F_1\left(\theta_1\right)\right]^2} \\
& - \frac{\partial a}{\partial \boldsymbol{\Omega}_1} \frac{f_1\left(\theta_1\right) \frac{\partial \theta_1}{\partial \boldsymbol{\Omega}_3}}{F_1\left(\theta_1\right)} - \frac{n\left(\frac{\partial^2 \Phi_2}{\partial \boldsymbol{\Omega}_1 \partial \boldsymbol{\Omega}_3} + \frac{\partial^2 \Phi_3}{\partial \boldsymbol{\Omega}_1 \partial \boldsymbol{\Omega}_3}\right)\left(1+\Phi_2+\Phi_3\right) - n\left(\frac{\partial \Phi_2}{\partial \boldsymbol{\Omega}_1} + \frac{\partial \Phi_3}{\partial \boldsymbol{\Omega}_1}\right)\left(\frac{\partial \Phi_2}{\partial \boldsymbol{\Omega}_3} + \frac{\partial \Phi_3}{\partial \boldsymbol{\Omega}_3}\right)}{\left(1+\Phi_2+\Phi_3\right)^2} \\
& + \frac{\partial^2 b}{\partial \boldsymbol{\Omega}_1 \partial \boldsymbol{\Omega}_3}\left\{\log \Phi_2 - \log\left[F_2\left(\theta_2\right)-F_2\left(\theta_1\right)\right]\right\} + \frac{\partial b}{\partial \boldsymbol{\Omega}_1}\left[\frac{\frac{\partial \Phi_2}{\partial \boldsymbol{\Omega}_3}}{\Phi_2} - \frac{f_2\left(\theta_2\right) \frac{\partial \theta_2}{\partial \boldsymbol{\Omega}_3} - f_2\left(\theta_1\right) \frac{\partial \theta_1}{\partial \boldsymbol{\Omega}_3}}{F_2\left(\theta_2\right)-F_2\left(\theta_1\right)}\right] \\
& + \frac{\partial b}{\partial \boldsymbol{\Omega}_3}\left[\frac{\frac{\partial \Phi_2}{\partial \boldsymbol{\Omega}_1}}{\Phi_2} - \frac{f_2\left(\theta_2\right) \frac{\partial \theta_2}{\partial \boldsymbol{\Omega}_1} - f_2\left(\theta_1\right) \frac{\partial \theta_1}{\partial \boldsymbol{\Omega}_1}}{F_2\left(\theta_2\right)-F_2\left(\theta_1\right)}\right] + \frac{b \frac{\partial^2 \Phi_2}{\partial \boldsymbol{\Omega}_1 \partial \boldsymbol{\Omega}_3} \Phi_2 - b \frac{\partial \Phi_2}{\partial \boldsymbol{\Omega}_1} \frac{\partial \Phi_2}{\partial \boldsymbol{\Omega}_3}}{\Phi_2^2} \\
& - \frac{b\left[\frac{\partial f_2\left(\theta_2\right)}{\partial \boldsymbol{\Omega}_3} \frac{\partial \theta_2}{\partial \boldsymbol{\Omega}_1} + f_2\left(\theta_2\right) \frac{\partial^2 \theta_2}{\partial \boldsymbol{\Omega}_1 \partial \boldsymbol{\Omega}_3} - \frac{\partial f_2\left(\theta_2\right)}{\partial \boldsymbol{\Omega}_3} \frac{\partial \theta_1}{\partial \boldsymbol{\Omega}_1} - f_2\left(\theta_2\right) \frac{\partial^2 \theta_1}{\partial \boldsymbol{\Omega}_1 \partial \boldsymbol{\Omega}_3}\right]}{F_2\left(\theta_2\right)-F_2\left(\theta_1\right)} \\
& + \frac{b\left[f_2\left(\theta_2\right) \frac{\partial \theta_2}{\partial \boldsymbol{\Omega}_1} - f_2\left(\theta_1\right) \frac{\partial \theta_1)}{\partial \boldsymbol{\Omega}_1}\right]\left[f_2\left(\theta_2\right) \frac{\partial \theta_2}{\partial \boldsymbol{\Omega}_3} - f_2\left(\theta_1\right) \frac{\partial \theta_1}{\partial \boldsymbol{\Omega}_3}\right]}{\left[F_2\left(\theta_2\right)-F_2\left(\theta_1\right)\right]^2} - \frac{\partial^2 \theta_1}{\partial \boldsymbol{\Omega}_1 \partial \boldsymbol{\Omega}_3} \sum_{x_i \leq \theta_2} \Delta\left(x_i-\theta_1\right) \log f_2\left(x_i\right) \\
& - \frac{\partial \theta_1}{\partial \boldsymbol{\Omega}_1} \frac{\partial \theta_2}{\partial \boldsymbol{\Omega}_3} \sum_{i=1}^{n} \Delta\left(x_i-\theta_1\right) \Delta\left(x_i-\theta_2\right) \log f_2\left(x_i\right) \\
& + \frac{\partial^2 \theta_2}{\partial \boldsymbol{\Omega}_1 \partial \boldsymbol{\Omega}_3} \sum_{\theta_1 \leq x_i} \Delta\left(x_i-\theta_2\right) \log f_2\left(x_i\right) - \frac{\partial \theta_1}{\partial \boldsymbol{\Omega}_3} \frac{\partial \theta_2}{\partial \boldsymbol{\Omega}_1} \sum_{i=1}^{n} \Delta\left(x_i-\theta_1\right) \Delta\left(x_i-\theta_2\right) \log f_2\left(x_i\right) \\
& + \frac{\partial^2 c}{\partial \boldsymbol{\Omega}_1 \partial \boldsymbol{\Omega}_3}\left\{\log \Phi_3 - \log\left[1-F_3\left(\theta_2\right)\right]\right\} \\
& + \left[\frac{\frac{\partial \Phi_3}{\partial \boldsymbol{\Omega}_3}}{\Phi_3} + \frac{f_3\left(\theta_2\right) \frac{\partial \theta_2}{\partial \boldsymbol{\Omega}_3}}{1-F_3\left(\theta_2\right)}\right] \frac{\partial c}{\partial \boldsymbol{\Omega}_1} + \frac{\partial c}{\partial \boldsymbol{\Omega}_3}\left[\frac{\frac{\partial \Phi_3}{\partial \boldsymbol{\Omega}_1}}{\Phi_3} + \frac{f_3\left(\theta_2\right) \frac{\partial \theta_2}{\partial \boldsymbol{\Omega}_1}}{1-F_3\left(\theta_2\right)}\right] + \frac{c \frac{\partial^2 \Phi_3}{\partial \boldsymbol{\Omega}_1 \partial \boldsymbol{\Omega}_3} \Phi_3 - c \frac{\partial \Phi_3}{\partial \boldsymbol{\Omega}_1} \frac{\partial \Phi_3}{\partial \boldsymbol{\Omega}_3}}{\Phi_3^2} \\
& + \frac{c\left[\frac{\partial f_3\left(\theta_2\right)}{\partial \boldsymbol{\Omega}_3} \frac{\partial \theta_2}{\partial \boldsymbol{\Omega}_1} + f_3\left(\theta_2\right) \frac{\partial^2 \theta_2}{\partial \boldsymbol{\Omega}_1 \partial \boldsymbol{\Omega}_3}\right]\left[1-F_3\left(\theta_2\right)\right] + c\left[f_3\left(\theta_2\right)\right]^2 \frac{\partial \theta_2}{\partial \boldsymbol{\Omega}_1} \frac{\partial \theta_2}{\partial \boldsymbol{\Omega}_3}}{\left[1-F_3\left(\theta_2\right)\right]^2} \\
& - \frac{\partial^2 \theta_2}{\partial \boldsymbol{\Omega}_1 \boldsymbol{\Omega}_3} \sum_{i=1}^{n} \Delta\left(x_i-\theta_2\right) \log f_3\left(x_i\right) - \frac{\partial \theta_2}{\partial \boldsymbol{\Omega}_1} \sum_{i=1}^{n} \Delta\left(x_i-\theta_2\right) \frac{\frac{\partial f_3\left(x_i\right)}{\partial \boldsymbol{\Omega}_3}}{f_3\left(x_i\right)},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 \log L}{\partial \mathbf{\Omega}_2^2} &= \frac{\partial^2 \theta_1}{\partial \mathbf{\Omega}_2^2} \sum_{i=1}^{n} \Delta\left(x_i - \theta_1\right) \log f_1\left(x_i\right) - \frac{\partial^2 a}{\partial \mathbf{\Omega}_2^2} \log F_1\left(\theta_1\right) - \frac{\partial a}{\partial \mathbf{\Omega}_2} \frac{f_1\left(\theta_1\right) \frac{\partial \theta_1}{\partial \mathbf{\Omega}_2}}{F_1\left(\theta_1\right)} \\
&- \frac{\left[\frac{\partial a}{\partial \mathbf{\Omega}_1} f_1\left(\theta_1\right) \frac{\partial \theta_1}{\partial \mathbf{\Omega}_2} + a \frac{\partial f_1(\theta_1)}{\partial \mathbf{\Omega}_2} \frac{\partial \theta_1}{\partial \mathbf{\Omega}_2} + a f_1\left(\theta_1\right) \frac{\partial^2 \theta_1}{\partial \mathbf{\Omega}_2^2}\right] F_1\left(\theta_1\right) - a\left[f_1\left(\theta_1\right)\right]^2 \left(\frac{\partial \theta_1}{\partial \mathbf{\Omega}_2}\right)^2}{\left[F_1\left(\theta_1\right)\right]^2} \\
&- \frac{n\left(\frac{\partial^2 \Phi_2}{\partial \mathbf{\Omega}_2^2} + \frac{\partial^2 \Phi_3}{\partial \mathbf{\Omega}_2^2}\right)\left(1 + \Phi_2 + \Phi_3\right) - n\left(\frac{\partial \Phi_2}{\partial \mathbf{\Omega}_2} + \frac{\partial \Phi_3}{\partial \mathbf{\Omega}_2}\right)^2}{\left(1 + \Phi_2 + \Phi_3\right)^2} + \frac{\partial^2 b}{\partial \mathbf{\Omega}_2^2}\left\{\log \Phi_2 - \log\left[F_2\left(\theta_2\right) - F_2\left(\theta_1\right)\right]\right\} \\
&+ \frac{2\partial b}{\partial \mathbf{\Omega}_2}\left[\frac{\frac{\partial \Phi_2}{\partial \mathbf{\Omega}_2}}{\Phi_2} - \frac{f_2\left(\theta_2\right) \frac{\partial \theta_2}{\partial \mathbf{\Omega}_2} - f_2\left(\theta_1\right) \frac{\partial \theta_1}{\partial \mathbf{\Omega}_2}}{F_2\left(\theta_2\right) - F_2\left(\theta_1\right)}\right] + \frac{b \frac{\partial^2 \Phi_2}{\partial \mathbf{\Omega}_2^2} \Phi_2 - b\left(\frac{\partial \Phi_2}{\partial \mathbf{\Omega}_2}\right)^2}{\Phi_2^2} \\
&- \frac{b\left[\frac{\partial f_2(\theta_2)}{\partial \mathbf{\Omega}_2} \frac{\partial \theta_2}{\partial \mathbf{\Omega}_2} + f_2\left(\theta_2\right) \frac{\partial^2 \theta_2}{\partial \mathbf{\Omega}_2^2} - \frac{\partial f_2(\theta_1)}{\partial \mathbf{\Omega}_2} \frac{\partial \theta_1}{\partial \mathbf{\Omega}_2} - f_2\left(\theta_1\right) \frac{\partial^2 \theta_1}{\partial \mathbf{\Omega}_2^2}\right]}{F_2\left(\theta_2\right) - F_2\left(\theta_1\right)} \\
&+ \frac{b\left[f_2\left(\theta_2\right) \frac{\partial \theta_2}{\partial \mathbf{\Omega}_2} - f_2\left(\theta_1\right) \frac{\partial \theta_1}{\partial \mathbf{\Omega}_2}\right]^2}{\left[F_2\left(\theta_2\right) - F_2\left(\theta_1\right)\right]^2} - \frac{\partial^2 \theta_1}{\partial \mathbf{\Omega}_2^2} \sum_{x_i \le \theta_2} \Delta\left(x_i - \theta_1\right) \log f_2\left(x_i\right) - \frac{\partial \theta_1}{\partial \mathbf{\Omega}_2} \sum_{x_i \le \theta_2} \Delta\left(x_i - \theta_1\right) \frac{\frac{\partial f_2(x_i)}{\partial \mathbf{\Omega}_2}}{f_2\left(x_i\right)} \\
&- 2 \frac{\partial \theta_1}{\partial \mathbf{\Omega}_2} \frac{\partial \theta_2}{\partial \mathbf{\Omega}_2} \sum_{i=1}^{n} \Delta\left(x_i - \theta_1\right) \Delta\left(x_i - \theta_2\right) \log f_2\left(x_i\right) + \frac{\partial \theta_2}{\partial \mathbf{\Omega}_2} \sum_{\theta_1 \le x_i} \Delta\left(x_i - \theta_2\right) \frac{\frac{\partial f_2(x_i)}{\partial \mathbf{\Omega}_2}}{f_2\left(x_i\right)} \\
&+ \frac{\partial^2 \theta_2}{\partial \mathbf{\Omega}_2^2} \sum_{\theta_1 \le x_i} \Delta\left(x_i - \theta_2\right) \log f_2\left(x_i\right) + \frac{\partial^2 c}{\partial \mathbf{\Omega}_2^2}\left\{\log \Phi_3 - \log\left[1 - F_3\left(\theta_2\right)\right]\right\} \\
&+ 2 \frac{\partial c}{\partial \mathbf{\Omega}_2}\left[\frac{\frac{\partial \Phi_3}{\partial \mathbf{\Omega}_2}}{\Phi_3} + \frac{f_3\left(\theta_2\right) \frac{\partial \theta_2}{\partial \mathbf{\Omega}_2}}{1 - F_3\left(\theta_2\right)}\right] + c \frac{\Phi_3 \frac{\partial^2 \Phi_3}{\partial \mathbf{\Omega}_2^2} - \left(\frac{\partial \Phi_3}{\partial \mathbf{\Omega}_2}\right)^2}{\Phi_3^2} - \frac{\partial^2 \theta_2}{\partial \mathbf{\Omega}_2^2} \sum_{i=1}^{n} \Delta\left(x_i - \theta_2\right) \log f_3\left(x_i\right) \\
&+ \frac{c\left[\frac{\partial f_3(\theta_2)}{\partial \mathbf{\Omega}_2} \frac{\partial \theta_2}{\partial \mathbf{\Omega}_2} + f_3\left(\theta_2\right) \frac{\partial^2 \theta_2}{\partial \mathbf{\Omega}_2^2}\right]\left[1 - F_3\left(\theta_2\right)\right] + c\left[f_3\left(\theta_2\right) \frac{\partial \theta_2}{\partial \mathbf{\Omega}_2}\right]^2}{\left[1 - F_3\left(\theta_2\right)\right]^2} \\
&- \frac{\partial \theta_1}{\partial \mathbf{\Omega}_2} \sum_{x_i \le \theta_2} \Delta\left(x_i - \theta_1\right) \frac{\frac{\partial f_2(x_i)}{\partial \mathbf{\Omega}_2}}{f_2\left(x_i\right)} + \frac{\partial \theta_2}{\partial \mathbf{\Omega}_2} \sum_{\theta_1 \le x_i} \Delta\left(x_i - \theta_2\right) \frac{\frac{\partial f_2(x_i)}{\partial \mathbf{\Omega}_2}}{f_2\left(x_i\right)} \\
&+ \sum_{\theta_1 \le x_i \le \theta_2} \frac{\frac{\partial^2 f_2(x_i)}{\partial \mathbf{\Omega}_2^2} f_2\left(x_i\right) - \left[\frac{\partial f_2(x_i)}{\partial \mathbf{\Omega}_2}\right]^2}{\left[f_2\left(x_i\right)\right]^2},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 \log L}{\partial \mathbf{\Omega}_2 \partial \mathbf{\Omega}_3} &= \frac{\partial^2 \theta_1}{\partial \mathbf{\Omega}_2 \partial \mathbf{\Omega}_3} \sum_{i=1}^{n} \Delta\left(x_i-\theta_1\right) \log f_1\left(x_i\right) - \frac{\partial^2 a}{\partial \mathbf{\Omega}_2 \partial \mathbf{\Omega}_3} \log F_1\left(\theta_1\right) \\
&- \frac{\frac{\partial a}{\partial \mathbf{\Omega}_3} f_1\left(\theta_1\right) \frac{\partial \theta_1}{\partial \mathbf{\Omega}_2} + a \frac{\partial f_1\left(\theta_1\right)}{\partial \mathbf{\Omega}_3} \frac{\partial \theta_1}{\partial \mathbf{\Omega}_2} + a f_1\left(\theta_1\right) \frac{\partial^2 \theta_1}{\partial \mathbf{\Omega}_2 \partial \mathbf{\Omega}_3}}{F_1\left(\theta_1\right)} + \frac{a\left[f_1\left(\theta_1\right)\right]^2 \frac{\partial \theta_1}{\partial \mathbf{\Omega}_2} \frac{\partial \theta_1}{\partial \mathbf{\Omega}_3}}{\left[F_1\left(\theta_1\right)\right]^2} \\
&- \frac{\partial a}{\partial \mathbf{\Omega}_2} \frac{f_1\left(\theta_1\right) \frac{\partial \theta_1}{\partial \mathbf{\Omega}_3}}{F_1\left(\theta_1\right)} - \frac{n\left(\frac{\partial^2 \Phi_2}{\partial \mathbf{\Omega}_2 \partial \mathbf{\Omega}_3} + \frac{\partial^2 \Phi_3}{\partial \mathbf{\Omega}_2 \partial \mathbf{\Omega}_3}\right)\left(1+\Phi_2+\Phi_3\right) - n\left(\frac{\partial \Phi_2}{\partial \mathbf{\Omega}_2} + \frac{\partial \Phi_3}{\partial \mathbf{\Omega}_2}\right)\left(\frac{\partial \Phi_2}{\partial \mathbf{\Omega}_3} + \frac{\partial \Phi_3}{\partial \mathbf{\Omega}_3}\right)}{\left(1+\Phi_2+\Phi_3\right)^2} \\
&+ \frac{\partial^2 b}{\partial \mathbf{\Omega}_2 \partial \mathbf{\Omega}_3}\left\{\log \Phi_2 - \log\left[F_2\left(\theta_2\right) - F_2\left(\theta_1\right)\right]\right\} + \frac{\partial b}{\partial \mathbf{\Omega}_2}\left[\frac{\frac{\partial \Phi_2}{\partial \mathbf{\Omega}_3}}{\Phi_2} - \frac{f_2\left(\theta_2\right) \frac{\partial \theta_2}{\partial \mathbf{\Omega}_3} - f_2\left(\theta_1\right) \frac{\partial \theta_1}{\partial \mathbf{\Omega}_3}}{F_2\left(\theta_2\right) - F_2\left(\theta_1\right)}\right] \\
&+ \frac{\partial b}{\partial \mathbf{\Omega}_3}\left[\frac{\frac{\partial \Phi_2}{\partial \mathbf{\Omega}_2}}{\Phi_2} - f_2\left(\theta_2\right) \frac{\frac{\partial \theta_2}{\partial \mathbf{\Omega}_2} - f_2\left(\theta_1\right) \frac{\partial \theta_1}{\partial \mathbf{\Omega}_2}}{F_2\left(\theta_2\right) - F_2\left(\theta_1\right)}\right] + \frac{b \frac{\partial^2 \Phi_2}{\partial \mathbf{\Omega}_2 \partial \mathbf{\Omega}_3} \Phi_2 - b \frac{\partial \Phi_2}{\partial \mathbf{\Omega}_2} \frac{\partial \Phi_2}{\partial \mathbf{\Omega}_3}}{\Phi_2^2} \\
&- \frac{b\left[\frac{\partial f_2\left(\theta_2\right)}{\partial \mathbf{\Omega}_3} \frac{\partial \theta_2}{\partial \mathbf{\Omega}_2} + f_2\left(\theta_2\right) \frac{\partial^2 \theta_2}{\partial \mathbf{\Omega}_2 \partial \mathbf{\Omega}_3} - \frac{\partial f_2\left(\theta_2\right)}{\partial \mathbf{\Omega}_3} \frac{\partial \theta_1}{\partial \mathbf{\Omega}_2} - f_2\left(\theta_1\right) \frac{\partial^2 \theta_1}{\partial \mathbf{\Omega}_2 \partial \mathbf{\Omega}_3}\right]}{F_2\left(\theta_2\right) - F_2\left(\theta_1\right)} \\
&+ \frac{b\left[f_2\left(\theta_2\right) \frac{\partial \theta_2}{\partial \mathbf{\Omega}_2} - f_2\left(\theta_1\right) \frac{\partial \theta_1}{\partial \mathbf{\Omega}_2}\right]\left[f_2\left(\theta_2\right) \frac{\partial \theta_2}{\partial \mathbf{\Omega}_3} - f_2\left(\theta_1\right) \frac{\partial \theta_1}{\partial \mathbf{\Omega}_3}\right]}{\left[F_2\left(\theta_2\right) - F_2\left(\theta_1\right)\right]^2} - \frac{\partial^2 \theta_1}{\partial \mathbf{\Omega}_2 \partial \mathbf{\Omega}_3} \sum_{x_i \le \theta_2} \Delta\left(x_i-\theta_1\right) \log f_2\left(x_i\right) \\
&- \frac{\partial \theta_1}{\partial \mathbf{\Omega}_2} \frac{\partial \theta_2}{\partial \mathbf{\Omega}_3} \sum_{i=1}^{n} \Delta\left(x_i-\theta_1\right) \Delta\left(x_i-\theta_2\right) \log f_2\left(x_i\right) \\
&- \frac{\partial \theta_1}{\partial \mathbf{\Omega}_3} \sum_{x_i \le \theta_2} \Delta\left(x_i-\theta_1\right) \frac{\frac{\partial f_2\left(x_i\right)}{\partial \mathbf{\Omega}_2}}{f_2\left(x_i\right)} + \frac{\partial \theta_2}{\partial \mathbf{\Omega}_3} \sum_{\theta_1 \le x_i} \Delta\left(x_i-\theta_2\right) \frac{\frac{\partial f_2\left(x_i\right)}{\partial \mathbf{\Omega}_2}}{f_2\left(x_i\right)} \\
&+ \frac{\partial^2 \theta_2}{\partial \mathbf{\Omega}_2 \partial \mathbf{\Omega}_3} \sum_{\theta_1 \le x_i} \Delta\left(x_i-\theta_2\right) \log f_2\left(x_i\right) - \frac{\partial \theta_1}{\partial \mathbf{\Omega}_3} \frac{\partial \theta_2}{\partial \mathbf{\Omega}_2} \sum_{i=1}^{n} \Delta\left(x_i-\theta_1\right) \Delta\left(x_i-\theta_2\right) \log f_2\left(x_i\right) \\
&+ \frac{\partial^2 c}{\partial \mathbf{\Omega}_2 \partial \mathbf{\Omega}_3}\left\{\log \Phi_3 - \log\left[1-F_3\left(\theta_2\right)\right]\right\} \\
&+ \left[\frac{\frac{\partial \Phi_3}{\partial \mathbf{\Omega}_3}}{\Phi_3} + \frac{f_3\left(\theta_2\right) \frac{\partial \theta_2}{\partial \mathbf{\Omega}_3}}{1-F_3\left(\theta_2\right)}\right] \frac{\partial c}{\partial \mathbf{\Omega}_2} + \frac{\partial c}{\partial \mathbf{\Omega}_3}\left[\frac{\frac{\partial \Phi_3}{\partial \mathbf{\Omega}_2}}{\Phi_3} + \frac{f_3\left(\theta_2\right) \frac{\partial \theta_2}{\partial \mathbf{\Omega}_2}}{1-F_3\left(\theta_2\right)}\right] + \frac{c \frac{\partial^2 \Phi_3}{\partial \mathbf{\Omega}_2 \partial \mathbf{\Omega}_3} \Phi_3 - c \frac{\partial \Phi_3}{\partial \mathbf{\Omega}_2} \frac{\partial \Phi_3}{\partial \mathbf{\Omega}_3}}{\Phi_3^2} \\
&+ \frac{c\left[\frac{\partial f_3\left(\theta_2\right)}{\partial \mathbf{\Omega}_3} \frac{\partial \theta_2}{\partial \mathbf{\Omega}_2} + f_3\left(\theta_2\right) \frac{\partial^2 \theta_2}{\partial \mathbf{\Omega}_2 \partial \mathbf{\Omega}_3}\right]\left[1-F_3\left(\theta_2\right)\right] + c\left[f_3\left(\theta_2\right)\right]^2 \frac{\partial \theta_2}{\partial \mathbf{\Omega}_2} \frac{\partial \theta_2}{\partial \mathbf{\Omega}_3}}{\left[1-F_3\left(\theta_2\right)\right]^2} \\
&- \frac{\partial^2 \theta_2}{\partial \mathbf{\Omega}_2 \mathbf{\Omega}_3} \sum_{i=1}^{n} \Delta\left(x_i-\theta_2\right) \log f_3\left(x_i\right) - \frac{\partial \theta_2}{\partial \mathbf{\Omega}_2} \sum_{i=1}^{n} \Delta\left(x_i-\theta_2\right) \frac{\frac{\partial f_3\left(x_i\right)}{\partial \mathbf{\Omega}_3}}{f_3\left(x_i\right)}
\end{aligned}
$$

and

$$\frac{\partial^2 \log L}{\partial \mathbf{\Omega}_3^2} = \frac{\partial^2 \theta_1}{\partial \mathbf{\Omega}_3^2} \sum_{i=1}^{n} \Delta\left(x_i - \theta_1\right) \log f_1\left(x_i\right) - \frac{\partial^2 a}{\partial \mathbf{\Omega}_3^2} \log F_1\left(\theta_1\right) - \frac{\partial a}{\partial \mathbf{\Omega}_3} \frac{f_1\left(\theta_1\right) \frac{\partial \theta_1}{\partial \mathbf{\Omega}_3}}{F_1\left(\theta_1\right)}$$
$$- \frac{\left[\frac{\partial a}{\partial \mathbf{\Omega}_3} f_1\left(\theta_1\right) \frac{\partial \theta_1}{\partial \mathbf{\Omega}_3} + a \frac{\partial f_1(\theta_1)}{\partial \mathbf{\Omega}_3} \frac{\partial \theta_1}{\partial \mathbf{\Omega}_3} + a f_1\left(\theta_1\right) \frac{\partial^2 \theta_1}{\partial \mathbf{\Omega}_3^2}\right] F_1\left(\theta_1\right) - a\left[f_1\left(\theta_1\right)\right]^2 \left(\frac{\partial \theta_1}{\partial \mathbf{\Omega}_3}\right)^2}{\left[F_1\left(\theta_1\right)\right]^2}$$
$$- \frac{n\left[\left(\frac{\partial^2 \Phi_2}{\partial \mathbf{\Omega}_3^2} + \frac{\partial^2 \Phi_3}{\partial \mathbf{\Omega}_3^2}\right)\left(1 + \Phi_2 + \Phi_3\right) - \left(\frac{\partial \Phi_2}{\partial \mathbf{\Omega}_3} + \frac{\partial \Phi_3}{\partial \mathbf{\Omega}_3}\right)^2\right]}{\left(1 + \Phi_2 + \Phi_3\right)^2}$$
$$+ \frac{\partial^2 b}{\partial \mathbf{\Omega}_3^2} \left\{\log \Phi_2 - \log\left[F_2\left(\theta_2\right) - F_2\left(\theta_1\right)\right]\right\} + 2 \frac{\partial b}{\partial \mathbf{\Omega}_3} \left[\frac{\frac{\partial \Phi_2}{\partial \mathbf{\Omega}_3}}{\Phi_2} - f_2\left(\theta_2\right) \frac{\frac{\partial \theta_2}{\partial \mathbf{\Omega}_3} - f_2\left(\theta_1\right) \frac{\partial \theta_1}{\partial \mathbf{\Omega}_3}}{F_2\left(\theta_2\right) - F_2\left(\theta_1\right)}\right]$$
$$+ \frac{b \frac{\partial^2 \Phi_2}{\partial \mathbf{\Omega}_3^2} \Phi_2 - b\left(\frac{\partial \Phi_2}{\partial \mathbf{\Omega}_3}\right)^2}{\Phi_2^2}$$
$$- \frac{b\left[\frac{\partial f_2(\theta_2)}{\partial \mathbf{\Omega}_3} \frac{\partial \theta_2}{\partial \mathbf{\Omega}_3} + f_2\left(\theta_2\right) \frac{\partial^2 \theta_2}{\partial \mathbf{\Omega}_3^2} - \frac{\partial f_2(\theta_1)}{\partial \mathbf{\Omega}_3} \frac{\partial \theta_1}{\partial \mathbf{\Omega}_3} - f_2\left(\theta_1\right) \frac{\partial^2 \theta_1}{\partial \mathbf{\Omega}_3^2}\right]}{F_2\left(\theta_2\right) - F_2\left(\theta_1\right)}$$
$$+ \frac{b\left[f_2\left(\theta_2\right) \frac{\partial \theta_2}{\partial \mathbf{\Omega}_3} - f_2\left(\theta_1\right) \frac{\partial \theta_1}{\partial \mathbf{\Omega}_3}\right]^2}{\left[F_2\left(\theta_2\right) - F_2\left(\theta_1\right)\right]^2}$$
$$-2 \frac{\partial \theta_1}{\partial \mathbf{\Omega}_3} \frac{\partial \theta_2}{\partial \mathbf{\Omega}_3} \sum_{i=1}^{n} \Delta\left(x_i - \theta_1\right) \Delta\left(x_i - \theta_2\right) \log f_2\left(x_i\right)$$
$$- \frac{\partial^2 \theta_1}{\partial \mathbf{\Omega}_3^2} \sum_{x_i \leq \theta_2} \Delta\left(x_i - \theta_1\right) \log f_2\left(x_i\right) + \frac{\partial^2 \theta_2}{\partial \mathbf{\Omega}_3^2} \sum_{\theta_1 \leq x_i} \Delta\left(x_i - \theta_2\right) \log f_2\left(x_i\right)$$
$$+ \frac{\partial^2 c}{\partial \mathbf{\Omega}_3^2} \left\{\log \Phi_3 - \log\left[1 - F_3\left(\theta_2\right)\right]\right\}$$
$$+2 \frac{\partial c}{\partial \mathbf{\Omega}_3} \left[\frac{\frac{\partial \Phi_3}{\partial \mathbf{\Omega}_3}}{\Phi_3} + \frac{f_3\left(\theta_2\right) \frac{\partial \theta_2}{\partial \mathbf{\Omega}_3}}{1 - F_3\left(\theta_2\right)}\right] + c \frac{\Phi_3 \frac{\partial^2 \Phi_3}{\partial \mathbf{\Omega}_3^2} - \left(\frac{\partial \Phi_3}{\partial \mathbf{\Omega}_3}\right)^2}{\Phi_3^2} - \frac{\partial^2 \theta_2}{\partial \mathbf{\Omega}_3^2} \sum_{i=1}^{n} \Delta\left(x_i - \theta_2\right) \log f_3\left(x_i\right)$$
$$+ \frac{c\left[\frac{\partial f_3(\theta_2)}{\partial \mathbf{\Omega}_3} \frac{\partial \theta_2}{\partial \mathbf{\Omega}_3} + f_3\left(\theta_2\right) \frac{\partial^2 \theta_2}{\partial \mathbf{\Omega}_3^2}\right]\left[1 - F_3\left(\theta_2\right)\right] + c\left[f_3\left(\theta_2\right) \frac{\partial \theta_2}{\partial \mathbf{\Omega}_3}\right]^2}{\left[1 - F_3\left(\theta_2\right)\right]^2}$$
$$-2 \frac{\partial \theta_2}{\partial \mathbf{\Omega}_3} \sum_{i=1}^{n} \Delta\left(x_i - \theta_2\right) \frac{\frac{\partial f_3(x_i)}{\partial \mathbf{\Omega}_3}}{f_3\left(x_i\right)} + \sum_{\theta_2 < x_i} \frac{\frac{\partial^2 f_3(x_i)}{\partial \mathbf{\Omega}_3^2} f_3\left(x_i\right) - \left[\frac{\partial f_3(x_i)}{\partial \mathbf{\Omega}_3}\right]^2}{\left[f_3\left(x_i\right)\right]^2}.$$

4. General m-piece composite distribution

Suppose every piece of the distribution is described by f_i, F_i for $i = 1, 2, \ldots, m$. Then the probability density and cumulative distribution functions of the most general m-piece composite distribution are

$$f(x) = \begin{cases} \zeta \frac{f_1(x)}{F_1\left(\theta_1\right)}, & x \leq \theta_1, \\ \zeta \phi_i \frac{f_i(x)}{F_i\left(\theta_i\right) - F_i\left(\theta_{i-1}\right)}, & \theta_{i-1} < x < \theta_i, \\ \zeta \phi_m \frac{f_m(x)}{1 - F_m\left(\theta_{m-1}\right)}, & \theta_{m-1} < x \end{cases} \tag{4.1}$$

and

$$F(x) = \begin{cases} \zeta \dfrac{F_1(x)}{F_1\left(\theta_1\right)}, & x \leq \theta_1, \\ \zeta \left[\displaystyle\sum_{j=1}^{i} \phi_j + \phi_i \dfrac{F_i(x) - F_i\left(\theta_{i-1}\right)}{F_i\left(\theta_i\right) - F_i\left(\theta_{i-1}\right)} \right], & \theta_{i-1} < x < \theta_i, \\ \zeta \phi_m \left[\displaystyle\sum_{j=1}^{m-1} \phi_j + \phi_m \dfrac{F_m(x) - F_m\left(\theta_{m-1}\right)}{1 - F_m\left(\theta_{m-1}\right)} \right], & \theta_{m-1} < x, \end{cases} \tag{4.2}$$

respectively, where θ_i, $i = 2, \ldots, m-1$ mark the section boundaries and $\zeta = 1/\left(1 + \phi_2 + \cdots + \phi_m\right)$. We take $\phi_1 = 1$. Suppose f_i are parameterized by $\mathbf{\Omega}_i$. Conditions for continuity and differentiability at θ_i ensure that $\theta_i = \theta_i\left(\mathbf{\Omega}_1, \ldots, \mathbf{\Omega}_m\right)$ and

$$\phi_i = \phi_i\left(\mathbf{\Omega}_1, \ldots, \mathbf{\Omega}_m\right) = \frac{\displaystyle\prod_{j=1}^{i-1} f_j\left(\theta_j\right) \left[F_i\left(\theta_i\right) - F_i\left(\theta_{i-1}\right)\right]}{F_1\left(\theta_1\right) \displaystyle\prod_{j=2}^{i} f_j\left(\theta_{j-1}\right)}.$$

Let

$$a_1 = \sum_{i=1}^{n} I\left\{x_i \leq \theta_1\right\},$$

$$a_m = \sum_{i=1}^{n} I\left\{\theta_{m-1} < x\right\}$$

and

$$a_j = \sum_{i=1}^{n} I\left\{\theta_{j-1} \leq x_i \leq \theta_j\right\}$$

for $j = 2, \ldots, m-1$. Note that $a_1 + a_2 + \cdots + a_m = n$. If $x_1, x_2, \ldots, x_n$ is a random sample from (4.1)-(4.2) then the log-likelihood function of $\left(\mathbf{\Omega}_1, \ldots, \mathbf{\Omega}_m\right)$ is

$$\begin{aligned} \log L\left(\mathbf{\Omega}_1, \ldots, \mathbf{\Omega}_m\right) = {} & n \log \zeta - a_1 \log\left[F_1\left(\theta_1\right)\right] - \sum_{i=2}^{m-1} a_i \log\left[F_i\left(\theta_i\right) - F_i\left(\theta_{i-1}\right)\right] \\ & - a_m \log\left[1 - F_m\left(\theta_{m-1}\right)\right] + \sum_{x_i \leq \theta_1} \log f_1\left(x_i\right) + \sum_{i=2}^{m-1} \left[a_i \log \phi_i + \sum_{\theta_{i-1} \leq x_i \leq \theta_i} \log f_i\left(x_i\right) \right] \\ & + a_m \log \phi_m + \sum_{\theta_{m-1} < x_i} \log f_m\left(x_i\right). \end{aligned}$$

The first order partial derivatives are

$$
\begin{aligned}
\frac{\partial \log L}{\partial \mathbf{\Omega}_1} &= \frac{n}{\zeta}\frac{\partial \zeta}{\partial \mathbf{\Omega}_1} - \frac{\partial a_1}{\partial \mathbf{\Omega}_1}\log\left[F_1\left(\theta_1\right)\right] - \frac{\frac{\partial \theta_1}{\partial \mathbf{\Omega}_1} a_1 f_1\left(\theta_1\right)}{F_1\left(\theta_1\right)} - \sum_{i=2}^{m-1}\frac{\partial a_i}{\partial \mathbf{\Omega}_1}\log\left[F_i\left(\theta_i\right) - F_i\left(\theta_{i-1}\right)\right] \\
&- \sum_{i=2}^{m-1} a_i \frac{\frac{\partial \theta_i}{\partial \mathbf{\Omega}_1} f_i\left(\theta_i\right) - \frac{\partial \theta_{i-1}}{\partial \mathbf{\Omega}_1} f_i\left(\theta_{i-1}\right)}{F_i\left(\theta_i\right) - F_i\left(\theta_{i-1}\right)} - \frac{\partial a_m}{\partial \mathbf{\Omega}_1}\log\left[1 - F_m\left(\theta_{m-1}\right)\right] + a_m \frac{\frac{\partial \theta_{m-1}}{\partial \mathbf{\Omega}_1} f_m\left(\theta_{m-1}\right)}{1 - F_m\left(\theta_{m-1}\right)} \\
&+ \frac{\partial \theta_1}{\partial \mathbf{\Omega}_1}\sum_{i=1}^{n}\Delta\left(x_i - \theta_1\right)\log f_1\left(x_i\right) + \sum_{x_i \le \theta_1}\frac{\frac{\partial f_1(x_i)}{\partial \mathbf{\Omega}_1}}{f_1\left(x_i\right)} \\
&+ \sum_{i=1}^{m-1}\left[\frac{\partial a_i}{\partial \mathbf{\Omega}_1}\log\phi_i + a_i\frac{\frac{\partial \phi_i}{\partial \mathbf{\Omega}_1}}{\phi_i} - \frac{\partial \theta_{i-1}}{\partial \mathbf{\Omega}_1}\sum_{x_i \le \theta_i}\Delta\left(x_i - \theta_{i-1}\right)\log f_i\left(x_i\right)\right] \\
&+ \sum_{i=1}^{m-1}\frac{\partial \theta_i}{\partial \mathbf{\Omega}_1}\sum_{\theta_{i-1} \le x_i}\Delta\left(x_i - \theta_i\right)\log f_i\left(x_i\right) \\
&+ \frac{\partial a_m}{\partial \mathbf{\Omega}_1}\log\phi_m + a_m\frac{\frac{\partial \phi_m}{\partial \mathbf{\Omega}_1}}{\phi_m} - \frac{\partial \theta_{m-1}}{\partial \mathbf{\Omega}_1}\sum_{i=1}^{n}\Delta\left(x_i - \theta_{m-1}\right)\log f_m\left(x_i\right),
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \log L}{\partial \mathbf{\Omega}_i} &= \frac{n}{\zeta}\frac{\partial \zeta}{\partial \mathbf{\Omega}_i} - \frac{\partial a_1}{\partial \mathbf{\Omega}_i}\log\left[F_1\left(\theta_1\right)\right] - \frac{\frac{\partial \theta_i}{\partial \mathbf{\Omega}_i} a_1 f_1\left(\theta_1\right)}{F_1\left(\theta_1\right)} - \sum_{i=2}^{m-1}\frac{\partial a_i}{\partial \mathbf{\Omega}_i}\log\left[F_i\left(\theta_i\right) - F_i\left(\theta_{i-1}\right)\right] \\
&- \sum_{i=2}^{m-1} a_i \frac{\frac{\partial \theta_i}{\partial \mathbf{\Omega}_i} f_i\left(\theta_i\right) - \frac{\partial \theta_{i-1}}{\partial \mathbf{\Omega}_i} f_i\left(\theta_{i-1}\right)}{F_i\left(\theta_i\right) - F_i\left(\theta_{i-1}\right)} - \frac{\partial a_m}{\partial \mathbf{\Omega}_i}\log\left[1 - F_m\left(\theta_{m-1}\right)\right] + a_m \frac{\frac{\partial \theta_{m-1}}{\partial \mathbf{\Omega}_i} f_m\left(\theta_{m-1}\right)}{1 - F_m\left(\theta_{m-1}\right)} \\
&+ \frac{\partial \theta_1}{\partial \mathbf{\Omega}_i}\sum_{i=1}^{n}\Delta\left(x_i - \theta_1\right)\log f_1\left(x_i\right) + \sum_{i=2}^{m-1}\sum_{\theta_{i-1} \le x_i \le \theta_i}\frac{\frac{\partial f_i(x_i)}{\partial \mathbf{\Omega}_i}}{f_i\left(x_i\right)} \\
&+ \sum_{i=1}^{m-1}\left[\frac{\partial a_i}{\partial \mathbf{\Omega}_i}\log\phi_i + a_i\frac{\frac{\partial \phi_i}{\partial \mathbf{\Omega}_i}}{\phi_i} - \frac{\partial \theta_{i-1}}{\partial \mathbf{\Omega}_i}\sum_{x_i \le \theta_i}\Delta\left(x_i - \theta_{i-1}\right)\log f_i\left(x_i\right)\right] \\
&+ \sum_{i=1}^{m-1}\frac{\partial \theta_i}{\partial \mathbf{\Omega}_i}\sum_{\theta_{i-1} \le x_i}\Delta\left(x_i - \theta_i\right)\log f_i\left(x_i\right) \\
&+ \frac{\partial a_m}{\partial \mathbf{\Omega}_i}\log\phi_m + a_m\frac{\frac{\partial \phi_m}{\partial \mathbf{\Omega}_i}}{\phi_m} - \frac{\partial \theta_{m-1}}{\partial \mathbf{\Omega}_i}\sum_{i=1}^{n}\Delta\left(x_i - \theta_{m-1}\right)\log f_m\left(x_i\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial \log L}{\partial \mathbf{\Omega}_m} &= \frac{n}{\zeta}\frac{\partial \zeta}{\partial \mathbf{\Omega}_m} - \frac{\partial a_1}{\partial \mathbf{\Omega}_m}\log\left[F_1\left(\theta_1\right)\right] - \frac{\frac{\partial \theta_1}{\partial \mathbf{\Omega}_m} a_1 f_1\left(\theta_1\right)}{F_1\left(\theta_1\right)} - \sum_{i=2}^{m-1}\frac{\partial a_i}{\partial \mathbf{\Omega}_m}\log\left[F_i\left(\theta_i\right) - F_i\left(\theta_{i-1}\right)\right] \\
&- \sum_{i=2}^{m-1} a_i \frac{\frac{\partial \theta_i}{\partial \mathbf{\Omega}_m} f_i\left(\theta_i\right) - \frac{\partial \theta_{i-1}}{\partial \mathbf{\Omega}_m} f_i\left(\theta_{i-1}\right)}{F_i\left(\theta_i\right) - F_i\left(\theta_{i-1}\right)} - \frac{\partial a_m}{\partial \mathbf{\Omega}_m}\log\left[1 - F_m\left(\theta_{m-1}\right)\right] + a_m \frac{\frac{\partial \theta_{m-1}}{\partial \mathbf{\Omega}_m} f_m\left(\theta_{m-1}\right)}{1 - F_m\left(\theta_{m-1}\right)} \\
&+ \frac{\partial \theta_1}{\partial \mathbf{\Omega}_m}\sum_{i=1}^{n}\Delta\left(x_i - \theta_1\right)\log f_1\left(x_i\right) \\
&+ \sum_{i=1}^{m-1}\left[\frac{\partial a_i}{\partial \mathbf{\Omega}_m}\log\phi_i + a_i\frac{\frac{\partial \phi_i}{\partial \mathbf{\Omega}_m}}{\phi_i} - \frac{\partial \theta_{i-1}}{\partial \mathbf{\Omega}_m}\sum_{x_i \le \theta_i}\Delta\left(x_i - \theta_{i-1}\right)\log f_i\left(x_i\right)\right] \\
&+ \sum_{i=1}^{m-1}\frac{\partial \theta_i}{\partial \mathbf{\Omega}_m}\sum_{\theta_{i-1} \le x_i}\Delta\left(x_i - \theta_i\right)\log f_i\left(x_i\right) \\
&+ \frac{\partial a_m}{\partial \mathbf{\Omega}_m}\log\phi_m + a_m\frac{\frac{\partial \phi_m}{\partial \mathbf{\Omega}_m}}{\phi_m} - \frac{\partial \theta_{m-1}}{\partial \mathbf{\Omega}_m}\sum_{i=1}^{n}\Delta\left(x_i - \theta_{m-1}\right)\log f_m\left(x_i\right) + \sum_{\theta_{m-1} < x_i}\frac{\frac{\partial f_m(x_i)}{\partial \mathbf{\Omega}_m}}{f_m\left(x_i\right)}
\end{aligned}
$$

for $i = 2, \ldots, m-1$. Hence, the observed information matrix is

$$\mathbf{J} = -\begin{pmatrix} \frac{\partial^2 \log L}{\partial \mathbf{\Omega}_1^2} & \frac{\partial^2 \log L}{\partial \mathbf{\Omega}_1 \partial \mathbf{\Omega}_2} & \cdots & \frac{\partial^2 \log L}{\partial \mathbf{\Omega}_1 \partial \mathbf{\Omega}_m} \\ \frac{\partial^2 \log L}{\partial \mathbf{\Omega}_1 \partial \mathbf{\Omega}_2} & \frac{\partial^2 \log L}{\partial \mathbf{\Omega}_2^2} & \cdots & \frac{\partial^2 \log L}{\partial \mathbf{\Omega}_2 \partial \mathbf{\Omega}_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 \log L}{\partial \mathbf{\Omega}_1 \partial \mathbf{\Omega}_m} & \frac{\partial^2 \log L}{\partial \mathbf{\Omega}_2 \partial \mathbf{\Omega}_m} & \cdots & \frac{\partial^2 \log L}{\partial \mathbf{\Omega}_m^2} \end{pmatrix},$$

where

$$\begin{aligned}
\frac{\partial^2 \log L}{\partial \mathbf{\Omega}_1^2} =& \frac{n}{\zeta} \frac{\partial^2 \zeta}{\partial \mathbf{\Omega}_1^2} - \frac{n}{\zeta^2} \left(\frac{\partial \zeta}{\partial \mathbf{\Omega}_1} \right)^2 - \frac{\partial^2 a_1}{\partial \mathbf{\Omega}_1^2} \log F_1 \left(\theta_1 \right) - \frac{\partial a_1}{\partial \mathbf{\Omega}_1} \frac{\partial \theta_1}{\partial \mathbf{\Omega}_1} \frac{f_1 \left(\theta_1 \right)}{F_1 \left(\theta_1 \right)} + \frac{\left(\frac{\partial \theta_1}{\partial \mathbf{\Omega}_1} \right)^2 a_1 \left[f_1 \left(\theta_1 \right) \right]^2}{\left[F_1 \left(\theta_1 \right) \right]^2} \\
& - \frac{\frac{\partial^2 \theta_1}{\partial \mathbf{\Omega}_1^2} a_1 f_1 \left(\theta_1 \right) + \frac{\partial \theta_1}{\partial \mathbf{\Omega}_1} \frac{\partial a_1}{\partial \mathbf{\Omega}_1} f_1 \left(\theta_1 \right) + \frac{\partial \theta_1}{\partial \mathbf{\Omega}_1} a_1 \frac{\partial f_1 \left(\theta_1 \right)}{\partial \mathbf{\Omega}_1}}{F_1 \left(\theta_1 \right)} - \sum_{i=2}^{m-1} \frac{\partial^2 a_i}{\partial \mathbf{\Omega}_1^2} \log \left[F_i \left(\theta_i \right) - F_i \left(\theta_{i-1} \right) \right] \\
& -2 \sum_{i=2}^{m-1} \frac{\partial a_i}{\partial \mathbf{\Omega}_1} \frac{f_i \left(\theta_i \right) \frac{\partial \theta_i}{\partial \mathbf{\Omega}_1} - f_i \left(\theta_{i-1} \right) \frac{\partial \theta_{i-1}}{\partial \mathbf{\Omega}_1}}{F_i \left(\theta_i \right) - F_i \left(\theta_{i-1} \right)} \\
& - \sum_{i=2}^{m-1} a_i \frac{\left[\frac{\partial^2 \theta_i}{\partial \mathbf{\Omega}_1^2} f_i \left(\theta_i \right) + \frac{\partial \theta_i}{\partial \mathbf{\Omega}_1} \frac{\partial f_i \left(\theta_i \right)}{\partial \mathbf{\Omega}_1} - \frac{\partial^2 \theta_{i-1}}{\partial \mathbf{\Omega}_1^2} f_i \left(\theta_{i-1} \right) - \frac{\partial \theta_{i-1}}{\partial \mathbf{\Omega}_1} \frac{\partial f_i \left(\theta_{i-1} \right)}{\partial \mathbf{\Omega}_1} \right]}{F_i \left(\theta_i \right) - F_i \left(\theta_{i-1} \right)} \\
& + \sum_{i=2}^{m-1} a_i \frac{\left[\frac{\partial \theta_i}{\partial \mathbf{\Omega}_1} f_i \left(\theta_i \right) - \frac{\partial \theta_{i-1}}{\partial \mathbf{\Omega}_1} f_i \left(\theta_{i-1} \right) \right]^2}{\left[F_i \left(\theta_i \right) - F_i \left(\theta_{i-1} \right) \right]^2} - \frac{\partial^2 a_m}{\partial \mathbf{\Omega}_1^2} \log \left[1 - F_m \left(\theta_{m-1} \right) \right] + 2 \frac{\partial a_m}{\partial \mathbf{\Omega}_1} \frac{\frac{\partial \theta_{m-1}}{\partial \mathbf{\Omega}_1} f_m \left(\theta_{m-1} \right)}{1 - F_m \left(\theta_{m-1} \right)} \\
& + a_m \frac{\frac{\partial^2 \theta_{m-1}}{\partial \mathbf{\Omega}_1^2} f_m \left(\theta_{m-1} \right) + \frac{\partial \theta_{m-1}}{\partial \mathbf{\Omega}_1} \frac{\partial f_m \left(\theta_{m-1} \right)}{\partial \mathbf{\Omega}_1}}{1 - F_m \left(\theta_{m-1} \right)} + a_m \frac{\left[\frac{\partial \theta_{m-1}}{\partial \mathbf{\Omega}_1} f_m \left(\theta_{m-1} \right) \right]^2}{\left[1 - F_m \left(\theta_{m-1} \right) \right]^2} + \frac{\partial^2 \theta_1}{\partial \mathbf{\Omega}_1^2} \sum_{i=1}^{n} \Delta \left(x_i - \theta_1 \right) \log f_1 \left(x_i \right) \\
& + 2 \frac{\partial \theta_1}{\partial \mathbf{\Omega}_1} \sum_{i=1}^{n} \Delta \left(x_i - \theta_1 \right) \frac{\frac{\partial f_1 \left(x_i \right)}{\partial \mathbf{\Omega}_1}}{f_1 \left(x_i \right)} + \sum_{x_i \leq \theta_1} \frac{\frac{\partial^2 f_1 \left(x_i \right)}{\partial \mathbf{\Omega}_1^2} f_1 \left(x_i \right) - \left[\frac{\partial f_1 \left(x_i \right)}{\partial \mathbf{\Omega}_1} \right]^2}{\left[f_1 \left(x_i \right) \right]^2} \\
& + \sum_{i=2}^{m-1} \left[\frac{\partial^2 a_i}{\partial \mathbf{\Omega}_1^2} \log \phi_i + 2 \frac{\partial a_i}{\partial \mathbf{\Omega}_1} \frac{\frac{\partial \phi_i}{\partial \mathbf{\Omega}_1}}{\phi_i} + a_i \frac{\frac{\partial^2 \phi_i}{\partial \mathbf{\Omega}_1^2} \phi_i - \left(\frac{\partial \phi_i}{\partial \mathbf{\Omega}_1} \right)^2}{\phi_i^2} - \frac{\partial^2 \theta_{i-1}}{\partial \mathbf{\Omega}_1^2} \sum_{x_i \leq \theta_i} \Delta \left(x_i - \theta_{i-1} \right) \log f_i \left(x_i \right) \right] \\
& + \sum_{i=2}^{m-1} \left[\frac{\partial^2 \theta_i}{\partial \mathbf{\Omega}_1^2} \sum_{\theta_{i-1} < x_i} \Delta \left(x_i - \theta_i \right) \log f_i \left(x_i \right) - 2 \frac{\partial \theta_i}{\partial \mathbf{\Omega}_1} \frac{\partial \theta_{i-1}}{\partial \mathbf{\Omega}_1} \sum_{i=1}^{n} \Delta \left(x_i - \theta_{i-1} \right) \Delta \left(x_i - \theta_i \right) \log f_i \left(x_i \right) \right] \\
& + \frac{\partial^2 a_m}{\partial \mathbf{\Omega}_1^2} \log \phi_m + 2 \frac{\partial a_m}{\partial \mathbf{\Omega}_1} \frac{\frac{\partial \phi_m}{\partial \mathbf{\Omega}_1}}{\phi_m} + a_m \frac{\frac{\partial^2 \phi_m}{\partial \mathbf{\Omega}_1^2} \phi_m - \left(\frac{\partial \phi_m}{\partial \mathbf{\Omega}_1} \right)^2}{\phi_m^2} - \frac{\partial^2 \theta_{m-1}}{\partial \mathbf{\Omega}_1^2} \sum_{i=1}^{n} \Delta \left(x_i - \theta_{m-1} \right) \log f_m \left(x_i \right),
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial^2 \log L}{\partial \boldsymbol{\Omega}_1 \boldsymbol{\Omega}_i} &= \frac{n}{\zeta}\frac{\partial^2 \zeta}{\partial \boldsymbol{\Omega}_1 \partial \boldsymbol{\Omega}_i} - \frac{n}{\zeta^2}\frac{\partial \zeta}{\partial \boldsymbol{\Omega}_1}\frac{\partial \zeta}{\partial \boldsymbol{\Omega}_i} - \frac{\partial^2 a_1}{\partial \boldsymbol{\Omega}_1 \partial \boldsymbol{\Omega}_i}\log\left[F_1\left(\theta_1\right)\right] - \frac{\partial a_1}{\partial \boldsymbol{\Omega}_1}\frac{\partial \theta_1}{\partial \boldsymbol{\Omega}_i}\frac{f_1\left(\theta_1\right)}{F_1\left(\theta_1\right)} \\
&+ \frac{\frac{\partial \theta_1}{\partial \boldsymbol{\Omega}_1}\frac{\partial \theta_1}{\partial \boldsymbol{\Omega}_i} a_1\left[f_1\left(\theta_1\right)\right]^2}{\left[F_1\left(\theta_1\right)\right]^2} \\
&- \frac{\frac{\partial^2 \theta_1}{\partial \boldsymbol{\Omega}_1 \partial \boldsymbol{\Omega}_i} a_1 f_1\left(\theta_1\right) + \frac{\partial \theta_1}{\partial \boldsymbol{\Omega}_1}\frac{\partial a_1}{\partial \boldsymbol{\Omega}_i} f_1\left(\theta_1\right) + \frac{\partial \theta_1}{\partial \boldsymbol{\Omega}_1} a_1 \frac{\partial f_1\left(\theta_1\right)}{\partial \boldsymbol{\Omega}_i}}{F_1\left(\theta_1\right)} - \sum_{i=2}^{m-1}\frac{\partial^2 a_i}{\partial \boldsymbol{\Omega}_1 \partial \boldsymbol{\Omega}_i}\log\left[F_i\left(\theta_i\right) - F_i\left(\theta_{i-1}\right)\right] \\
&- \sum_{i=2}^{m-1}\frac{\partial a_i}{\partial \boldsymbol{\Omega}_1}\frac{f_i\left(\theta_i\right)\frac{\partial \theta_i}{\partial \boldsymbol{\Omega}_i} - f_i\left(\theta_{i-1}\right)\frac{\partial \theta_{i-1}}{\partial \boldsymbol{\Omega}_i}}{F_i\left(\theta_i\right) - F_i\left(\theta_{i-1}\right)} - \sum_{i=2}^{m-1}\frac{\partial a_i}{\partial \boldsymbol{\Omega}_i}\frac{f_i\left(\theta_i\right)\frac{\partial \theta_i}{\partial \boldsymbol{\Omega}_1} - f_i\left(\theta_{i-1}\right)\frac{\partial \theta_{i-1}}{\partial \boldsymbol{\Omega}_1}}{F_i\left(\theta_i\right) - F_i\left(\theta_{i-1}\right)} \\
&- \sum_{i=2}^{m-1} a_i \frac{\left[\frac{\partial^2 \theta_i}{\partial \boldsymbol{\Omega}_1 \partial \boldsymbol{\Omega}_i} f_i\left(\theta_i\right) + \frac{\partial \theta_i}{\partial \boldsymbol{\Omega}_1}\frac{\partial f_i\left(\theta_i\right)}{\partial \boldsymbol{\Omega}_i} - \frac{\partial^2 \theta_{i-1}}{\partial \boldsymbol{\Omega}_1 \partial \boldsymbol{\Omega}_i} f_i\left(\theta_{i-1}\right) - \frac{\partial \theta_{i-1}}{\partial \boldsymbol{\Omega}_1}\frac{\partial f_i\left(\theta_{i-1}\right)}{\partial \boldsymbol{\Omega}_i}\right]}{F_i\left(\theta_i\right) - F_i\left(\theta_{i-1}\right)} \\
&+ \sum_{i=2}^{m-1} a_i \frac{\left[\frac{\partial \theta_i}{\partial \boldsymbol{\Omega}_1} f_i\left(\theta_i\right) - \frac{\partial \theta_{i-1}}{\partial \boldsymbol{\Omega}_1} f_i\left(\theta_{i-1}\right)\right]\left[\frac{\partial \theta_i}{\partial \boldsymbol{\Omega}_i} f_i\left(\theta_i\right) - \frac{\partial \theta_{i-1}}{\partial \boldsymbol{\Omega}_i} f_i\left(\theta_{i-1}\right)\right]}{\left[F_i\left(\theta_i\right) - F_i\left(\theta_{i-1}\right)\right]^2} - \frac{\partial^2 a_m}{\partial \boldsymbol{\Omega}_1 \partial \boldsymbol{\Omega}_i}\log\left[1 - F_m\left(\theta_{m-1}\right)\right] \\
&+ \frac{\partial a_m}{\partial \boldsymbol{\Omega}_1}\frac{\frac{\partial \theta_{m-1}}{\partial \boldsymbol{\Omega}_i} f_m\left(\theta_{m-1}\right)}{1 - F_m\left(\theta_{m-1}\right)} + \frac{\partial a_m}{\partial \boldsymbol{\Omega}_i}\frac{\frac{\partial \theta_{m-1}}{\partial \boldsymbol{\Omega}_1} f_m\left(\theta_{m-1}\right)}{1 - F_m\left(\theta_{m-1}\right)} + a_m \frac{\frac{\partial^2 \theta_{m-1}}{\partial \boldsymbol{\Omega}_1 \partial \boldsymbol{\Omega}_i} f_m\left(\theta_{m-1}\right) + \frac{\partial \theta_{m-1}}{\partial \boldsymbol{\Omega}_1}\frac{\partial f_m\left(\theta_{m-1}\right)}{\partial \boldsymbol{\Omega}_i}}{1 - F_m\left(\theta_{m-1}\right)} \\
&+ a_m \frac{\left[\frac{\partial \theta_{m-1}}{\partial \boldsymbol{\Omega}_1} f_m\left(\theta_{m-1}\right)\right]\left[\frac{\partial \theta_{m-1}}{\partial \boldsymbol{\Omega}_i} f_m\left(\theta_{m-1}\right)\right]}{\left[1 - F_m\left(\theta_{m-1}\right)\right]^2} + \frac{\partial^2 \theta_1}{\partial \boldsymbol{\Omega}_1 \partial \boldsymbol{\Omega}_i}\sum_{i=1}^{n}\Delta\left(x_i - \theta_1\right)\log f_1\left(x_i\right) \\
&+ \frac{\partial \theta_1}{\partial \boldsymbol{\Omega}_i}\sum_{i=1}^{n}\Delta\left(x_i - \theta_1\right)\frac{\frac{\partial f_1\left(x_i\right)}{\partial \boldsymbol{\Omega}_1}}{f_1\left(x_i\right)} \\
&+ \sum_{i=2}^{m-1}\left[\frac{\partial^2 a_i}{\partial \boldsymbol{\Omega}_1 \partial \boldsymbol{\Omega}_i}\log \phi_i + \frac{\partial a_i}{\partial \boldsymbol{\Omega}_1}\frac{\frac{\partial \phi_i}{\partial \boldsymbol{\Omega}_i}}{\phi_i} + \frac{\partial a_i}{\partial \boldsymbol{\Omega}_i}\frac{\frac{\partial \phi_i}{\partial \boldsymbol{\Omega}_1}}{\phi_i} + a_i \frac{\frac{\partial^2 \phi_i}{\partial \boldsymbol{\Omega}_1 \partial \boldsymbol{\Omega}_i}\phi_i - \frac{\partial \phi_i}{\partial \boldsymbol{\Omega}_1}\frac{\partial \phi_i}{\partial \boldsymbol{\Omega}_i}}{\phi_i^2}\right] \\
&+ \sum_{i=2}^{m-1}\left[\frac{\partial^2 \theta_i}{\partial \boldsymbol{\Omega}_1 \partial \boldsymbol{\Omega}_i}\sum_{\theta_{i-1} < x_i}\Delta\left(x_i - \theta_i\right)\log f_i\left(x_i\right) - \frac{\partial^2 \theta_{i-1}}{\partial \boldsymbol{\Omega}_1 \partial \boldsymbol{\Omega}_i}\sum_{x_i \le \theta_i}\Delta\left(x_i - \theta_{i-1}\right)\log f_i\left(x_i\right)\right] \\
&- \sum_{i=2}^{m-1}\left[\left(\frac{\partial \theta_i}{\partial \boldsymbol{\Omega}_1}\frac{\partial \theta_{i-1}}{\partial \boldsymbol{\Omega}_i} + \frac{\partial \theta_i}{\partial \boldsymbol{\Omega}_i}\frac{\partial \theta_{i-1}}{\partial \boldsymbol{\Omega}_1}\right)\sum_{i=1}^{n}\Delta\left(x_i - \theta_{i-1}\right)\Delta\left(x_i - \theta_i\right)\log f_i\left(x_i\right)\right] \\
&+ \sum_{i=2}^{m-1}\left[\frac{\partial \theta_i}{\partial \boldsymbol{\Omega}_1}\sum_{\theta_{i-1} \le x_i}\Delta\left(x_i - \theta_i\right)\frac{\frac{\partial f_i\left(x_i\right)}{\partial \boldsymbol{\Omega}_i}}{f_i\left(x_i\right)} - \frac{\partial \theta_{i-1}}{\partial \boldsymbol{\Omega}_1}\sum_{x_i \le \theta_i}\Delta\left(x_i - \theta_{i-1}\right)\frac{\frac{\partial f_i\left(x_i\right)}{\partial \boldsymbol{\Omega}_i}}{f_i\left(x_i\right)}\right] \\
&+ \frac{\partial^2 a_m}{\partial \boldsymbol{\Omega}_1 \partial \boldsymbol{\Omega}_i}\log \phi_m + \frac{\partial a_m}{\partial \boldsymbol{\Omega}_1}\frac{\frac{\partial \phi_m}{\partial \boldsymbol{\Omega}_i}}{\phi_m} + \frac{\partial a_m}{\partial \boldsymbol{\Omega}_i}\frac{\frac{\partial \phi_m}{\partial \boldsymbol{\Omega}_1}}{\phi_m} + a_m \frac{\frac{\partial^2 \phi_m}{\partial \boldsymbol{\Omega}_1 \partial \boldsymbol{\Omega}_i}\phi_m - \frac{\partial \phi_m}{\partial \boldsymbol{\Omega}_1}\frac{\partial \phi_m}{\partial \boldsymbol{\Omega}_i}}{\phi_m^2} \\
&- \frac{\partial^2 \theta_{m-1}}{\partial \boldsymbol{\Omega}_1 \partial \boldsymbol{\Omega}_i}\sum_{i=1}^{n}\Delta\left(x_i - \theta_{m-1}\right)\log f_m\left(x_i\right)
\end{aligned}
$$

for $2 \leq i \leq m-1$,

$$
\begin{aligned}
\frac{\partial^2 \log L}{\partial \mathbf{\Omega}_1 \mathbf{\Omega}_m} = & \frac{n}{\zeta} \frac{\partial^2 \zeta}{\partial \mathbf{\Omega}_1 \partial \mathbf{\Omega}_m} - \frac{n}{\zeta^2} \frac{\partial \zeta}{\partial \mathbf{\Omega}_1} \frac{\partial \zeta}{\partial \mathbf{\Omega}_m} - \frac{\partial^2 a_1}{\partial \mathbf{\Omega}_1 \partial \mathbf{\Omega}_m} \log \left[F_1 \left(\theta_1 \right) \right] - \frac{\partial a_1}{\partial \mathbf{\Omega}_1} \frac{\partial \theta_1}{\partial \mathbf{\Omega}_m} \frac{f_1 \left(\theta_1 \right)}{F_1 \left(\theta_1 \right)} \\
& + \frac{\frac{\partial \theta_1}{\partial \mathbf{\Omega}_1} \frac{\partial \theta_1}{\partial \mathbf{\Omega}_m} a_1 \left[f_1 \left(\theta_1 \right) \right]^2}{\left[F_1 \left(\theta_1 \right) \right]^2} \\
& - \frac{\frac{\partial^2 \theta_1}{\partial \mathbf{\Omega}_1 \partial \mathbf{\Omega}_m} a_1 f_1 \left(\theta_1 \right) + \frac{\partial \theta_1}{\partial \mathbf{\Omega}_1} \frac{\partial a_1}{\partial \mathbf{\Omega}_m} f_1 \left(\theta_1 \right) + \frac{\partial \theta_1}{\partial \mathbf{\Omega}_1} a_1 \frac{\partial f_1 \left(\theta_1 \right)}{\partial \mathbf{\Omega}_m}}{F_1 \left(\theta_1 \right)} - \sum_{i=2}^{m-1} \frac{\partial^2 a_i}{\partial \mathbf{\Omega}_1 \partial \mathbf{\Omega}_m} \log \left[F_i \left(\theta_i \right) - F_i \left(\theta_{i-1} \right) \right] \\
& - \sum_{i=2}^{m-1} \frac{\partial a_i}{\partial \mathbf{\Omega}_1} \frac{f_i \left(\theta_i \right) \frac{\partial \theta_i}{\partial \mathbf{\Omega}_m} - f_i \left(\theta_{i-1} \right) \frac{\partial \theta_{i-1}}{\partial \mathbf{\Omega}_m}}{F_i \left(\theta_i \right) - F_i \left(\theta_{i-1} \right)} - \sum_{i=2}^{m-1} \frac{\partial a_i}{\partial \mathbf{\Omega}_m} \frac{f_i \left(\theta_i \right) \frac{\partial \theta_i}{\partial \mathbf{\Omega}_1} - f_i \left(\theta_{i-1} \right) \frac{\partial \theta_{i-1}}{\partial \mathbf{\Omega}_1}}{F_i \left(\theta_i \right) - F_i \left(\theta_{i-1} \right)} \\
& - \sum_{i=2}^{m-1} a_i \frac{\left[\frac{\partial^2 \theta_i}{\partial \mathbf{\Omega}_1 \partial \mathbf{\Omega}_m} f_i \left(\theta_i \right) + \frac{\partial \theta_i}{\partial \mathbf{\Omega}_1} \frac{\partial f_i \left(\theta_i \right)}{\partial \mathbf{\Omega}_m} - \frac{\partial^2 \theta_{i-1}}{\partial \mathbf{\Omega}_1 \partial \mathbf{\Omega}_m} f_i \left(\theta_{i-1} \right) - \frac{\partial \theta_{i-1}}{\partial \mathbf{\Omega}_1} \frac{\partial f_i \left(\theta_{i-1} \right)}{\partial \mathbf{\Omega}_m} \right]}{F_i \left(\theta_i \right) - F_i \left(\theta_{i-1} \right)} \\
& + \sum_{i=2}^{m-1} a_i \frac{\left[\frac{\partial \theta_i}{\partial \mathbf{\Omega}_1} f_i \left(\theta_i \right) - \frac{\partial \theta_{i-1}}{\partial \mathbf{\Omega}_1} f_i \left(\theta_{i-1} \right) \right] \left[\frac{\partial \theta_i}{\partial \mathbf{\Omega}_m} f_i \left(\theta_i \right) - \frac{\partial \theta_{i-1}}{\partial \mathbf{\Omega}_m} f_i \left(\theta_{i-1} \right) \right]}{\left[F_i \left(\theta_i \right) - F_i \left(\theta_{i-1} \right) \right]^2} - \frac{\partial^2 a_m}{\partial \mathbf{\Omega}_1 \partial \mathbf{\Omega}_m} \log \left[1 - F_m \left(\theta_{m-1} \right) \right] \\
& + \frac{\partial a_m}{\partial \mathbf{\Omega}_1} \frac{\frac{\partial \theta_{m-1}}{\partial \mathbf{\Omega}_m} f_m \left(\theta_{m-1} \right)}{1 - F_m \left(\theta_{m-1} \right)} + \frac{\partial a_m}{\partial \mathbf{\Omega}_m} \frac{\frac{\partial \theta_{m-1}}{\partial \mathbf{\Omega}_1} f_m \left(\theta_{m-1} \right)}{1 - F_m \left(\theta_{m-1} \right)} + a_m \frac{\frac{\partial^2 \theta_{m-1}}{\partial \mathbf{\Omega}_1 \partial \mathbf{\Omega}_m} f_m \left(\theta_{m-1} \right) + \frac{\partial \theta_{m-1}}{\partial \mathbf{\Omega}_1} \frac{\partial f_m \left(\theta_{m-1} \right)}{\partial \mathbf{\Omega}_m}}{1 - F_m \left(\theta_{m-1} \right)} \\
& + a_m \frac{\left[\frac{\partial \theta_{m-1}}{\partial \mathbf{\Omega}_1} f_m \left(\theta_{m-1} \right) \right] \left[\frac{\partial \theta_{m-1}}{\partial \mathbf{\Omega}_m} f_m \left(\theta_{m-1} \right) \right]}{\left[1 - F_m \left(\theta_{m-1} \right) \right]^2} + \frac{\partial^2 \theta_1}{\partial \mathbf{\Omega}_1 \partial \mathbf{\Omega}_m} \sum_{i=1}^{n} \Delta \left(x_i - \theta_1 \right) \log f_1 \left(x_i \right) \\
& + \frac{\partial \theta_1}{\partial \mathbf{\Omega}_m} \sum_{i=1}^{n} \Delta \left(x_i - \theta_1 \right) \frac{\frac{\partial f_1 \left(x_i \right)}{\partial \mathbf{\Omega}_1}}{f_1 \left(x_i \right)} \\
& + \sum_{i=2}^{m-1} \left[\frac{\partial^2 a_i}{\partial \mathbf{\Omega}_1 \partial \mathbf{\Omega}_m} \log \phi_i + \frac{\partial a_i}{\partial \mathbf{\Omega}_1} \frac{\frac{\partial \phi_i}{\partial \mathbf{\Omega}_m}}{\phi_i} + \frac{\partial a_i}{\partial \mathbf{\Omega}_m} \frac{\frac{\partial \phi_i}{\partial \mathbf{\Omega}_1}}{\phi_i} + a_i \frac{\frac{\partial^2 \phi_i}{\partial \mathbf{\Omega}_1 \partial \mathbf{\Omega}_m} \phi_i - \frac{\partial \phi_i}{\partial \mathbf{\Omega}_1} \frac{\partial \phi_i}{\partial \mathbf{\Omega}_m}}{\phi_i^2} \right] \\
& + \sum_{i=2}^{m-1} \left[\frac{\partial^2 \theta_i}{\partial \mathbf{\Omega}_1 \partial \mathbf{\Omega}_m} \sum_{\theta_{i-1} < x_i} \Delta \left(x_i - \theta_i \right) \log f_i \left(x_i \right) - \frac{\partial^2 \theta_{i-1}}{\partial \mathbf{\Omega}_1 \partial \mathbf{\Omega}_m} \sum_{x_i \leq \theta_i} \Delta \left(x_i - \theta_{i-1} \right) \log f_i \left(x_i \right) \right] \\
& - \sum_{i=2}^{m-1} \left[\left(\frac{\partial \theta_i}{\partial \mathbf{\Omega}_1} \frac{\partial \theta_{i-1}}{\partial \mathbf{\Omega}_m} + \frac{\partial \theta_i}{\partial \mathbf{\Omega}_m} \frac{\partial \theta_{i-1}}{\partial \mathbf{\Omega}_1} \right) \sum_{i=1}^{n} \Delta \left(x_i - \theta_{i-1} \right) \Delta \left(x_i - \theta_i \right) \log f_i \left(x_i \right) \right] \\
& + \frac{\partial^2 a_m}{\partial \mathbf{\Omega}_1 \partial \mathbf{\Omega}_m} \log \phi_m + \frac{\partial a_m}{\partial \mathbf{\Omega}_1} \frac{\frac{\partial \phi_m}{\partial \mathbf{\Omega}_m}}{\phi_m} + \frac{\partial a_m}{\partial \mathbf{\Omega}_m} \frac{\frac{\partial \phi_m}{\partial \mathbf{\Omega}_1}}{\phi_m} + a_m \frac{\frac{\partial^2 \phi_m}{\partial \mathbf{\Omega}_1 \partial \mathbf{\Omega}_m} \phi_m - \frac{\partial \phi_m}{\partial \mathbf{\Omega}_1} \frac{\partial \phi_m}{\partial \mathbf{\Omega}_m}}{\phi_m^2} \\
& - \frac{\partial^2 \theta_{m-1}}{\partial \mathbf{\Omega}_1 \partial \mathbf{\Omega}_m} \sum_{i=1}^{n} \Delta \left(x_i - \theta_{m-1} \right) \log f_m \left(x_i \right) - \frac{\partial \theta_{m-1}}{\partial \mathbf{\Omega}_1} \sum_{i=1}^{n} \Delta \left(x_i - \theta_{m-1} \right) \frac{\frac{\partial f_m \left(x_i \right)}{\partial \mathbf{\Omega}_m}}{f_m \left(x_i \right)}
\end{aligned}
$$

for $2 \le i \le m-1$,

$$\begin{aligned}
\frac{\partial^2 \log L}{\partial \mathbf{\Omega}_i^2} &= \frac{n}{\zeta}\frac{\partial^2 \zeta}{\partial \mathbf{\Omega}_i^2} - \frac{n}{\zeta^2}\left(\frac{\partial \zeta}{\partial \mathbf{\Omega}_i}\right)^2 - \frac{\partial^2 a_i}{\partial \mathbf{\Omega}_i^2}\log F_1(\theta_1) - \frac{\partial a_1}{\partial \mathbf{\Omega}_i}\frac{\partial \theta_1}{\partial \mathbf{\Omega}_i}\frac{f_1(\theta_1)}{F_1(\theta_1)} + \frac{\left(\frac{\partial \theta_1}{\partial \mathbf{\Omega}_i}\right)^2 a_1 \left[f_1(\theta_1)\right]^2}{\left[F_1(\theta_1)\right]^2} \\
&- \frac{\frac{\partial^2 \theta_1}{\partial \mathbf{\Omega}_i^2} a_1 f_1(\theta_1) + \frac{\partial \theta_1}{\partial \mathbf{\Omega}_i}\frac{\partial a_1}{\partial \mathbf{\Omega}_i} f_1(\theta_1) + \frac{\partial \theta_1}{\partial \mathbf{\Omega}_i} a_1 \frac{\partial f_1(\theta_1)}{\partial \mathbf{\Omega}_i}}{F_1(\theta_1)} - \sum_{i=2}^{m-1}\frac{\partial^2 a_i}{\partial \mathbf{\Omega}_i^2}\log\left[F_i(\theta_i) - F_i(\theta_{i-1})\right] \\
&-2\sum_{i=2}^{m-1}\frac{\partial a_i}{\partial \mathbf{\Omega}_i}\frac{f_i(\theta_i)\frac{\partial \theta_i}{\partial \mathbf{\Omega}_i} - f_i(\theta_{i-1})\frac{\partial \theta_{i-1}}{\partial \mathbf{\Omega}_i}}{F_i(\theta_i) - F_i(\theta_{i-1})} \\
&- \sum_{i=2}^{m-1} a_i \frac{\left[\frac{\partial^2 \theta_i}{\partial \mathbf{\Omega}_i^2} f_i(\theta_i) + \frac{\partial \theta_i}{\partial \mathbf{\Omega}_i}\frac{\partial f_i(\theta_i)}{\partial \mathbf{\Omega}_i} - \frac{\partial^2 \theta_{i-1}}{\partial \mathbf{\Omega}_i^2} f_i(\theta_{i-1}) - \frac{\partial \theta_{i-1}}{\partial \mathbf{\Omega}_i}\frac{\partial f_i(\theta_{i-1})}{\partial \mathbf{\Omega}_i}\right]}{F_i(\theta_i) - F_i(\theta_{i-1})} \\
&+ \sum_{i=2}^{m-1} a_i \frac{\left[\frac{\partial \theta_i}{\partial \mathbf{\Omega}_i} f_i(\theta_i) - \frac{\partial \theta_{i-1}}{\partial \mathbf{\Omega}_i} f_i(\theta_{i-1})\right]^2}{\left[F_i(\theta_i) - F_i(\theta_{i-1})\right]^2} - \frac{\partial^2 a_m}{\partial \mathbf{\Omega}_i^2}\log\left[1 - F_m(\theta_{m-1})\right] + 2\frac{\partial a_m}{\partial \mathbf{\Omega}_i}\frac{\frac{\partial \theta_{m-1}}{\partial \mathbf{\Omega}_i} f_m(\theta_{m-1})}{1 - F_m(\theta_{m-1})} \\
&+ a_m \frac{\frac{\partial^2 \theta_{m-1}}{\partial \mathbf{\Omega}_i^2} f_m(\theta_{m-1}) + \frac{\partial \theta_{m-1}}{\partial \mathbf{\Omega}_i}\frac{\partial f_m(\theta_{m-1})}{\partial \mathbf{\Omega}_i}}{1 - F_m(\theta_{m-1})} + a_m \frac{\left[\frac{\partial \theta_{m-1}}{\partial \mathbf{\Omega}_i} f_m(\theta_{m-1})\right]^2}{\left[1 - F_m(\theta_{m-1})\right]^2} \\
&+ \frac{\partial^2 \theta_1}{\partial \mathbf{\Omega}_i^2}\sum_{i=1}^{n}\Delta(x_i - \theta_1)\log f_1(x_i) \\
&-2\frac{\partial \theta_{i-1}}{\partial \mathbf{\Omega}_i}\sum_{i=2}^{m-1}\sum_{x_i \le \theta_i}\Delta(x_i - \theta_{i-1})\frac{\frac{\partial f_i(x_i)}{\partial \mathbf{\Omega}_i}}{f_i(x_i)} + 2\frac{\partial \theta_i}{\partial \mathbf{\Omega}_i}\sum_{i=2}^{m-1}\sum_{\theta_{i-1} \le x_i}\Delta(x_i - \theta_i)\frac{\frac{\partial f_i(x_i)}{\partial \mathbf{\Omega}_i}}{f_i(x_i)} \\
&+ \sum_{i=2}^{m-1}\sum_{\theta_{i-1} \le x_i \le \theta_i}\frac{\frac{\partial^2 f_i(x_i)}{\partial \mathbf{\Omega}_i^2} f_i(x_i) - \left[\frac{\partial f_i(x_i)}{\partial \mathbf{\Omega}_i}\right]^2}{\left[f_i(x_i)\right]^2} \\
&+ \sum_{i=2}^{m-1}\left[\frac{\partial^2 a_i}{\partial \mathbf{\Omega}_i^2}\log \phi_i + 2\frac{\partial a_i}{\partial \mathbf{\Omega}_i}\frac{\frac{\partial \phi_i}{\partial \mathbf{\Omega}_i}}{\phi_i} + a_i \frac{\frac{\partial^2 \phi_i}{\partial \mathbf{\Omega}_i^2}\phi_i - \left(\frac{\partial \phi_i}{\partial \mathbf{\Omega}_i}\right)^2}{\phi_i^2} - \frac{\partial^2 \theta_{i-1}}{\partial \mathbf{\Omega}_i^2}\sum_{x_i \le \theta_i}\Delta(x_i - \theta_{i-1})\log f_i(x_i)\right] \\
&+ \sum_{i=2}^{m-1}\left[\frac{\partial^2 \theta_i}{\partial \mathbf{\Omega}_i^2}\sum_{\theta_{i-1} < x_i}\Delta(x_i - \theta_i)\log f_i(x_i) - 2\frac{\partial \theta_i}{\partial \mathbf{\Omega}_i}\frac{\partial \theta_{i-1}}{\partial \mathbf{\Omega}_i}\sum_{i=1}^{n}\Delta(x_i - \theta_{i-1})\Delta(x_i - \theta_i)\log f_i(x_i)\right] \\
&+ \frac{\partial^2 a_m}{\partial \mathbf{\Omega}_i^2}\log \phi_m + 2\frac{\partial a_m}{\partial \mathbf{\Omega}_i}\frac{\frac{\partial \phi_m}{\partial \mathbf{\Omega}_i}}{\phi_m} + a_m \frac{\frac{\partial^2 \phi_m}{\partial \mathbf{\Omega}_i^2}\phi_m - \left(\frac{\partial \phi_m}{\partial \mathbf{\Omega}_i}\right)^2}{\phi_m^2} - \frac{\partial^2 \theta_{m-1}}{\partial \mathbf{\Omega}_i^2}\sum_{i=1}^{n}\Delta(x_i - \theta_{m-1})\log f_m(x_i)
\end{aligned}$$

for $2 \leq i \leq m-1$,

$$\begin{aligned}
\frac{\partial^2 \log L}{\partial \boldsymbol{\Omega}_i \boldsymbol{\Omega}_j} &= \frac{n}{\zeta} \frac{\partial^2 \zeta}{\partial \boldsymbol{\Omega}_i \partial \boldsymbol{\Omega}_j} - \frac{n}{\zeta^2} \frac{\partial \zeta}{\partial \boldsymbol{\Omega}_i} \frac{\partial \zeta}{\partial \boldsymbol{\Omega}_j} - \frac{\partial^2 a_1}{\partial \boldsymbol{\Omega}_i \partial \boldsymbol{\Omega}_j} \log \left[F_1\left(\theta_1\right)\right] - \frac{\partial a_1}{\partial \boldsymbol{\Omega}_i} \frac{\partial \theta_1}{\partial \boldsymbol{\Omega}_j} \frac{f_1\left(\theta_1\right)}{F_1\left(\theta_1\right)} \\
&+ \frac{\frac{\partial \theta_1}{\partial \boldsymbol{\Omega}_i} \frac{\partial \theta_1}{\partial \boldsymbol{\Omega}_j} a_1 \left[f_1\left(\theta_1\right)\right]^2}{\left[F_1\left(\theta_1\right)\right]^2} \\
&- \frac{\frac{\partial^2 \theta_1}{\partial \boldsymbol{\Omega}_i \partial \boldsymbol{\Omega}_j} a_1 f_1\left(\theta_1\right) + \frac{\partial \theta_1}{\partial \boldsymbol{\Omega}_i} \frac{\partial a_1}{\partial \boldsymbol{\Omega}_j} f_1\left(\theta_1\right) + \frac{\partial \theta_1}{\partial \boldsymbol{\Omega}_i} a_1 \frac{\partial f_1\left(\theta_1\right)}{\partial \boldsymbol{\Omega}_j}}{F_1\left(\theta_1\right)} - \sum_{i=2}^{m-1} \frac{\partial^2 a_i}{\partial \boldsymbol{\Omega}_i \partial \boldsymbol{\Omega}_j} \log \left[F_i\left(\theta_i\right) - F_i\left(\theta_{i-1}\right)\right] \\
&- \sum_{i=2}^{m-1} \frac{\partial a_i}{\partial \boldsymbol{\Omega}_j} \frac{f_i\left(\theta_i\right) \frac{\partial \theta_i}{\partial \boldsymbol{\Omega}_i} - f_i\left(\theta_{i-1}\right) \frac{\partial \theta_{i-1}}{\partial \boldsymbol{\Omega}_i}}{F_i\left(\theta_i\right) - F_i\left(\theta_{i-1}\right)} - \sum_{i=2}^{m-1} \frac{\partial a_i}{\partial \boldsymbol{\Omega}_i} \frac{f_i\left(\theta_i\right) \frac{\partial \theta_i}{\partial \boldsymbol{\Omega}_j} - f_i\left(\theta_{i-1}\right) \frac{\partial \theta_{i-1}}{\partial \boldsymbol{\Omega}_j}}{F_i\left(\theta_i\right) - F_i\left(\theta_{i-1}\right)} \\
&- \sum_{i=2}^{m-1} a_i \frac{\left[\frac{\partial^2 \theta_i}{\partial \boldsymbol{\Omega}_i \partial \boldsymbol{\Omega}_j} f_i\left(\theta_i\right) + \frac{\partial \theta_i}{\partial \boldsymbol{\Omega}_i} \frac{\partial f_i\left(\theta_i\right)}{\partial \boldsymbol{\Omega}_j} - \frac{\partial^2 \theta_{i-1}}{\partial \boldsymbol{\Omega}_i \partial \boldsymbol{\Omega}_j} f_i\left(\theta_{i-1}\right) - \frac{\partial \theta_{i-1}}{\partial \boldsymbol{\Omega}_i} \frac{\partial f_i\left(\theta_{i-1}\right)}{\partial \boldsymbol{\Omega}_j}\right]}{F_i\left(\theta_i\right) - F_i\left(\theta_{i-1}\right)} \\
&+ \sum_{i=2}^{m-1} a_i \frac{\left[\frac{\partial \theta_i}{\partial \boldsymbol{\Omega}_i} f_i\left(\theta_i\right) - \frac{\partial \theta_{i-1}}{\partial \boldsymbol{\Omega}_i} f_i\left(\theta_{i-1}\right)\right] \left[\frac{\partial \theta_i}{\partial \boldsymbol{\Omega}_j} f_i\left(\theta_i\right) - \frac{\partial \theta_{i-1}}{\partial \boldsymbol{\Omega}_j} f_i\left(\theta_{i-1}\right)\right]}{\left[F_i\left(\theta_i\right) - F_i\left(\theta_{i-1}\right)\right]^2} - \frac{\partial^2 a_m}{\partial \boldsymbol{\Omega}_i \partial \boldsymbol{\Omega}_j} \log \left[1 - F_m\left(\theta_{m-1}\right)\right] \\
&+ \frac{\partial a_m}{\partial \boldsymbol{\Omega}_i} \frac{\frac{\partial \theta_{m-1}}{\partial \boldsymbol{\Omega}_j} f_m\left(\theta_{m-1}\right)}{1 - F_m\left(\theta_{m-1}\right)} + \frac{\partial a_m}{\partial \boldsymbol{\Omega}_j} \frac{\frac{\partial \theta_{m-1}}{\partial \boldsymbol{\Omega}_i} f_m\left(\theta_{m-1}\right)}{1 - F_m\left(\theta_{m-1}\right)} + a_m \frac{\frac{\partial^2 \theta_{m-1}}{\partial \boldsymbol{\Omega}_i \partial \boldsymbol{\Omega}_j} f_m\left(\theta_{m-1}\right) + \frac{\partial \theta_{m-1}}{\partial \boldsymbol{\Omega}_i} \frac{\partial f_m\left(\theta_{m-1}\right)}{\partial \boldsymbol{\Omega}_j}}{1 - F_m\left(\theta_{m-1}\right)} \\
&+ a_m \frac{\left[\frac{\partial \theta_{m-1}}{\partial \boldsymbol{\Omega}_i} f_m\left(\theta_{m-1}\right)\right] \left[\frac{\partial \theta_{m-1}}{\partial \boldsymbol{\Omega}_j} f_m\left(\theta_{m-1}\right)\right]}{\left[1 - F_m\left(\theta_{m-1}\right)\right]^2} + \frac{\partial^2 \theta_1}{\partial \boldsymbol{\Omega}_i \partial \boldsymbol{\Omega}_j} \sum_{i=1}^{n} \Delta\left(x_i - \theta_1\right) \log f_1\left(x_i\right) \\
&+ \sum_{i=2}^{m-1} \left[\frac{\partial \theta_i}{\partial \boldsymbol{\Omega}_j} \sum_{\theta_{i-1} \leq x_i} \Delta\left(x_i - \theta_i\right) \frac{\frac{\partial f_i\left(x_i\right)}{\partial \boldsymbol{\Omega}_i}}{f_i\left(x_i\right)} - \frac{\partial \theta_{i-1}}{\partial \boldsymbol{\Omega}_j} \sum_{x_i \leq \theta_i} \Delta\left(x_i - \theta_{i-1}\right) \frac{\frac{\partial f_i\left(x_i\right)}{\partial \boldsymbol{\Omega}_i}}{f_i\left(x_i\right)}\right] \\
&+ \sum_{i=2}^{m-1} \sum_{\theta_{i-1} \leq x_i \leq \theta_i} \frac{\frac{\partial^2 f_i\left(x_i\right)}{\partial \boldsymbol{\Omega}_i \partial \boldsymbol{\Omega}_j} f_i\left(x_i\right) - \frac{\partial f_i\left(x_i\right)}{\partial \boldsymbol{\Omega}_i} \frac{\partial f_i\left(x_i\right)}{\partial \boldsymbol{\Omega}_j}}{\left[f_i\left(x_i\right)\right]^2} \\
&+ \sum_{i=2}^{m-1} \left[\frac{\partial^2 a_i}{\partial \boldsymbol{\Omega}_i \partial \boldsymbol{\Omega}_j} \log \phi_i + \frac{\partial a_i}{\partial \boldsymbol{\Omega}_i} \frac{\frac{\partial \phi_i}{\partial \boldsymbol{\Omega}_j}}{\phi_i} + \frac{\partial a_i}{\partial \boldsymbol{\Omega}_j} \frac{\frac{\partial \phi_i}{\partial \boldsymbol{\Omega}_i}}{\phi_i} + a_i \frac{\frac{\partial^2 \phi_i}{\partial \boldsymbol{\Omega}_i \partial \boldsymbol{\Omega}_j} \phi_i - \frac{\partial \phi_i}{\partial \boldsymbol{\Omega}_i} \frac{\partial \phi_i}{\partial \boldsymbol{\Omega}_j}}{\phi_i^2}\right] \\
&+ \sum_{i=2}^{m-1} \left[\frac{\partial^2 \theta_i}{\partial \boldsymbol{\Omega}_i \partial \boldsymbol{\Omega}_j} \sum_{\theta_{i-1} \leq x_i} \Delta\left(x_i - \theta_i\right) \log f_i\left(x_i\right) - \frac{\partial^2 \theta_{i-1}}{\partial \boldsymbol{\Omega}_i \partial \boldsymbol{\Omega}_j} \sum_{x_i \leq \theta_i} \Delta\left(x_i - \theta_{i-1}\right) \log f_i\left(x_i\right)\right] \\
&- \sum_{i=2}^{m-1} \left[\left(\frac{\partial \theta_i}{\partial \boldsymbol{\Omega}_i} \frac{\partial \theta_{i-1}}{\partial \boldsymbol{\Omega}_j} + \frac{\partial \theta_i}{\partial \boldsymbol{\Omega}_j} \frac{\partial \theta_{i-1}}{\partial \boldsymbol{\Omega}_i}\right) \sum_{i=1}^{n} \Delta\left(x_i - \theta_{i-1}\right) \Delta\left(x_i - \theta_i\right) \log f_i\left(x_i\right)\right] \\
&+ \sum_{i=2}^{m-1} \left[\frac{\partial \theta_i}{\partial \boldsymbol{\Omega}_i} \sum_{\theta_{i-1} \leq x_i} \Delta\left(x_i - \theta_i\right) \frac{\frac{\partial f_i\left(x_i\right)}{\partial \boldsymbol{\Omega}_j}}{f_i\left(x_i\right)} - \frac{\partial \theta_{i-1}}{\partial \boldsymbol{\Omega}_i} \sum_{x_i \leq \theta_i} \Delta\left(x_i - \theta_{i-1}\right) \frac{\frac{\partial f_i\left(x_i\right)}{\partial \boldsymbol{\Omega}_j}}{f_i\left(x_i\right)}\right] \\
&+ \frac{\partial^2 a_m}{\partial \boldsymbol{\Omega}_i \partial \boldsymbol{\Omega}_j} \log \phi_m + \frac{\partial a_m}{\partial \boldsymbol{\Omega}_i} \frac{\frac{\partial \phi_m}{\partial \boldsymbol{\Omega}_j}}{\phi_m} + \frac{\partial a_m}{\partial \boldsymbol{\Omega}_j} \frac{\frac{\partial \phi_m}{\partial \boldsymbol{\Omega}_i}}{\phi_m} + a_m \frac{\frac{\partial^2 \phi_m}{\partial \boldsymbol{\Omega}_i \partial \boldsymbol{\Omega}_j} \phi_m - \frac{\partial \phi_m}{\partial \boldsymbol{\Omega}_i} \frac{\partial \phi_m}{\partial \boldsymbol{\Omega}_j}}{\phi_m^2} \\
&- \frac{\partial^2 \theta_{m-1}}{\partial \boldsymbol{\Omega}_i \partial \boldsymbol{\Omega}_j} \sum_{i=1}^{n} \Delta\left(x_i - \theta_{m-1}\right) \log f_m\left(x_i\right)
\end{aligned}$$

for $2 \leq i < j \leq m-1$,

$$
\begin{aligned}
\frac{\partial^2 \log L}{\partial \boldsymbol{\Omega}_i \boldsymbol{\Omega}_m} &= \frac{n}{\zeta}\frac{\partial^2 \zeta}{\partial \boldsymbol{\Omega}_i \partial \boldsymbol{\Omega}_m} - \frac{n}{\zeta^2}\frac{\partial \zeta}{\partial \boldsymbol{\Omega}_i}\frac{\partial \zeta}{\partial \boldsymbol{\Omega}_m} - \frac{\partial^2 a_1}{\partial \boldsymbol{\Omega}_i \partial \boldsymbol{\Omega}_m}\log\left[F_1\left(\theta_1\right)\right] - \frac{\partial a_1}{\partial \boldsymbol{\Omega}_i}\frac{\partial \theta_1}{\partial \boldsymbol{\Omega}_m}\frac{f_1\left(\theta_1\right)}{F_1\left(\theta_1\right)} \\
&+ \frac{\frac{\partial \theta_1}{\partial \boldsymbol{\Omega}_i}\frac{\partial \theta_1}{\partial \boldsymbol{\Omega}_m} a_1 \left[f_1\left(\theta_1\right)\right]^2}{\left[F_1\left(\theta_1\right)\right]^2} \\
&- \frac{\frac{\partial^2 \theta_1}{\partial \boldsymbol{\Omega}_i \partial \boldsymbol{\Omega}_m} a_1 f_1\left(\theta_1\right) + \frac{\partial \theta_1}{\partial \boldsymbol{\Omega}_i}\frac{\partial a_1}{\partial \boldsymbol{\Omega}_m} f_1\left(\theta_1\right) + \frac{\partial \theta_1}{\partial \boldsymbol{\Omega}_i} a_1 \frac{\partial f_1\left(\theta_1\right)}{\partial \boldsymbol{\Omega}_m}}{F_1\left(\theta_1\right)} - \sum_{i=2}^{m-1}\frac{\partial^2 a_i}{\partial \boldsymbol{\Omega}_i \partial \boldsymbol{\Omega}_m}\log\left[F_i\left(\theta_i\right) - F_i\left(\theta_{i-1}\right)\right] \\
&- \sum_{i=2}^{m-1}\frac{\partial a_i}{\partial \boldsymbol{\Omega}_m}\frac{f_i\left(\theta_i\right)\frac{\partial \theta_i}{\partial \boldsymbol{\Omega}_i} - f_i\left(\theta_{i-1}\right)\frac{\partial \theta_{i-1}}{\partial \boldsymbol{\Omega}_i}}{F_i\left(\theta_i\right) - F_i\left(\theta_{i-1}\right)} - \sum_{i=2}^{m-1}\frac{\partial a_i}{\partial \boldsymbol{\Omega}_i}\frac{f_i\left(\theta_i\right)\frac{\partial \theta_i}{\partial \boldsymbol{\Omega}_m} - f_i\left(\theta_{i-1}\right)\frac{\partial \theta_{i-1}}{\partial \boldsymbol{\Omega}_m}}{F_i\left(\theta_i\right) - F_i\left(\theta_{i-1}\right)} \\
&- \sum_{i=2}^{m-1} a_i \frac{\left[\frac{\partial^2 \theta_i}{\partial \boldsymbol{\Omega}_i \partial \boldsymbol{\Omega}_m} f_i\left(\theta_i\right) + \frac{\partial \theta_i}{\partial \boldsymbol{\Omega}_i}\frac{\partial f_i\left(\theta_i\right)}{\partial \boldsymbol{\Omega}_m} - \frac{\partial^2 \theta_{i-1}}{\partial \boldsymbol{\Omega}_i \partial \boldsymbol{\Omega}_m} f_i\left(\theta_{i-1}\right) - \frac{\partial \theta_{i-1}}{\partial \boldsymbol{\Omega}_i}\frac{\partial f_i\left(\theta_{i-1}\right)}{\partial \boldsymbol{\Omega}_m}\right]}{F_i\left(\theta_i\right) - F_i\left(\theta_{i-1}\right)} \\
&+ \sum_{i=2}^{m-1} a_i \frac{\left[\frac{\partial \theta_i}{\partial \boldsymbol{\Omega}_i} f_i\left(\theta_i\right) - \frac{\partial \theta_{i-1}}{\partial \boldsymbol{\Omega}_i} f_i\left(\theta_{i-1}\right)\right]\left[\frac{\partial \theta_i}{\partial \boldsymbol{\Omega}_m} f_i\left(\theta_i\right) - \frac{\partial \theta_{i-1}}{\partial \boldsymbol{\Omega}_m} f_i\left(\theta_{i-1}\right)\right]}{\left[F_i\left(\theta_i\right) - F_i\left(\theta_{i-1}\right)\right]^2} - \frac{\partial^2 a_m}{\partial \boldsymbol{\Omega}_i \partial \boldsymbol{\Omega}_m}\log\left[1 - F_m\left(\theta_{m-1}\right)\right] \\
&+ \frac{\partial a_m}{\partial \boldsymbol{\Omega}_i}\frac{\frac{\partial \theta_{m-1}}{\partial \boldsymbol{\Omega}_m} f_m\left(\theta_{m-1}\right)}{1 - F_m\left(\theta_{m-1}\right)} + \frac{\partial a_m}{\partial \boldsymbol{\Omega}_m}\frac{\frac{\partial \theta_{m-1}}{\partial \boldsymbol{\Omega}_i} f_m\left(\theta_{m-1}\right)}{1 - F_m\left(\theta_{m-1}\right)} + a_m \frac{\frac{\partial^2 \theta_{m-1}}{\partial \boldsymbol{\Omega}_i \partial \boldsymbol{\Omega}_m} f_m\left(\theta_{m-1}\right) + \frac{\partial \theta_{m-1}}{\partial \boldsymbol{\Omega}_i}\frac{\partial f_m\left(\theta_{m-1}\right)}{\partial \boldsymbol{\Omega}_m}}{1 - F_m\left(\theta_{m-1}\right)} \\
&+ a_m \frac{\left[\frac{\partial \theta_{m-1}}{\partial \boldsymbol{\Omega}_i} f_m\left(\theta_{m-1}\right)\right]\left[\frac{\partial \theta_{m-1}}{\partial \boldsymbol{\Omega}_m} f_m\left(\theta_{m-1}\right)\right]}{\left[1 - F_m\left(\theta_{m-1}\right)\right]^2} + \frac{\partial^2 \theta_1}{\partial \boldsymbol{\Omega}_i \partial \boldsymbol{\Omega}_m}\sum_{i=1}^{n}\Delta\left(x_i - \theta_1\right)\log f_1\left(x_i\right) \\
&+ \sum_{i=2}^{m-1}\left[\frac{\partial \theta_i}{\partial \boldsymbol{\Omega}_m}\sum_{\theta_{i-1} \leq x_i}\Delta\left(x_i - \theta_i\right)\frac{\frac{\partial f_i\left(x_i\right)}{\partial \boldsymbol{\Omega}_i}}{f_i\left(x_i\right)} - \frac{\partial \theta_{i-1}}{\partial \boldsymbol{\Omega}_m}\sum_{x_i \leq \theta_i}\Delta\left(x_i - \theta_{i-1}\right)\frac{\frac{\partial f_i\left(x_i\right)}{\partial \boldsymbol{\Omega}_i}}{f_i\left(x_i\right)}\right] \\
&+ \sum_{i=2}^{m-1}\left(\frac{\partial^2 a_i}{\partial \boldsymbol{\Omega}_i \partial \boldsymbol{\Omega}_m}\log \phi_i + \frac{\partial a_i}{\partial \boldsymbol{\Omega}_i}\frac{\frac{\partial \phi_i}{\partial \boldsymbol{\Omega}_m}}{\phi_i} + \frac{\partial a_i}{\partial \boldsymbol{\Omega}_m}\frac{\frac{\partial \phi_i}{\partial \boldsymbol{\Omega}_i}}{\phi_i} + a_i \frac{\frac{\partial^2 \phi_i}{\partial \boldsymbol{\Omega}_i \partial \boldsymbol{\Omega}_m}\phi_i - \frac{\partial \phi_i}{\partial \boldsymbol{\Omega}_i}\frac{\partial \phi_i}{\partial \boldsymbol{\Omega}_m}}{\phi_i^2}\right) \\
&+ \sum_{i=2}^{m-1}\left[\frac{\partial^2 \theta_i}{\partial \boldsymbol{\Omega}_i \partial \boldsymbol{\Omega}_m}\sum_{\theta_{i-1} \leq x_i}\Delta\left(x_i - \theta_i\right)\log f_i\left(x_i\right) - \frac{\partial^2 \theta_{i-1}}{\partial \boldsymbol{\Omega}_i \partial \boldsymbol{\Omega}_m}\sum_{x_i \leq \theta_i}\Delta\left(x_i - \theta_{i-1}\right)\log f_i\left(x_i\right)\right] \\
&- \sum_{i=2}^{m-1}\left[\left(\frac{\partial \theta_i}{\partial \boldsymbol{\Omega}_i}\frac{\partial \theta_{i-1}}{\partial \boldsymbol{\Omega}_m} + \frac{\partial \theta_i}{\partial \boldsymbol{\Omega}_m}\frac{\partial \theta_{i-1}}{\partial \boldsymbol{\Omega}_i}\right)\sum_{i=1}^{n}\Delta\left(x_i - \theta_{i-1}\right)\Delta\left(x_i - \theta_i\right)\log f_i\left(x_i\right)\right] \\
&+ \frac{\partial^2 a_m}{\partial \boldsymbol{\Omega}_i \partial \boldsymbol{\Omega}_m}\log \phi_m + \frac{\partial a_m}{\partial \boldsymbol{\Omega}_i}\frac{\frac{\partial \phi_m}{\partial \boldsymbol{\Omega}_m}}{\phi_m} + \frac{\partial a_m}{\partial \boldsymbol{\Omega}_m}\frac{\frac{\partial \phi_m}{\partial \boldsymbol{\Omega}_i}}{\phi_m} + a_m \frac{\frac{\partial^2 \phi_m}{\partial \boldsymbol{\Omega}_i \partial \boldsymbol{\Omega}_m}\phi_m - \frac{\partial \phi_m}{\partial \boldsymbol{\Omega}_i}\frac{\partial \phi_m}{\partial \boldsymbol{\Omega}_m}}{\phi_m^2} \\
&- \frac{\partial^2 \theta_{m-1}}{\partial \boldsymbol{\Omega}_i \partial \boldsymbol{\Omega}_m}\sum_{i=1}^{n}\Delta\left(x_i - \theta_{m-1}\right)\log f_m\left(x_i\right) - \frac{\partial \theta_{m-1}}{\partial \boldsymbol{\Omega}_i}\sum_{i=1}^{n}\Delta\left(x_i - \theta_{m-1}\right)\frac{\frac{\partial f_m\left(x_i\right)}{\partial \boldsymbol{\Omega}_m}}{f_m\left(x_i\right)}
\end{aligned}
$$

for $2 \leq i \leq m-1$, and

$$
\begin{aligned}
\frac{\partial^2 \log L}{\partial \boldsymbol{\Omega}_m^2} &= \frac{n}{\zeta} \frac{\partial^2 \zeta}{\partial \boldsymbol{\Omega}_m^2} - \frac{n}{\zeta^2} \left(\frac{\partial \zeta}{\partial \boldsymbol{\Omega}_m} \right)^2 - \frac{\partial^2 a_1}{\partial \boldsymbol{\Omega}_m^2} \log F_1\left(\theta_1\right) - \frac{\partial a_1}{\partial \boldsymbol{\Omega}_m} \frac{\partial \theta_1}{\partial \boldsymbol{\Omega}_m} \frac{f_1\left(\theta_1\right)}{F_1\left(\theta_1\right)} \\
&+ \frac{\left(\frac{\partial \theta_1}{\partial \boldsymbol{\Omega}_m} \right)^2 a_1 \left[f_1\left(\theta_1\right)\right]^2}{\left[F_1\left(\theta_1\right)\right]^2} \\
&- \frac{\frac{\partial^2 \theta_1}{\partial \boldsymbol{\Omega}_m^2} a_1 f_1\left(\theta_1\right) + \frac{\partial \theta_1}{\partial \boldsymbol{\Omega}_m} \frac{\partial a_1}{\partial \boldsymbol{\Omega}_m} f_1\left(\theta_1\right) + \frac{\partial \theta_1}{\partial \boldsymbol{\Omega}_m} a_1 \frac{\partial f_1\left(\theta_1\right)}{\partial \boldsymbol{\Omega}_m}}{F_1\left(\theta_1\right)} - \sum_{i=2}^{m-1} \frac{\partial^2 a_i}{\partial \boldsymbol{\Omega}_m^2} \log \left[F_i\left(\theta_i\right) - F_i\left(\theta_{i-1}\right)\right] \\
&- 2 \sum_{i=2}^{m-1} \frac{\partial a_i}{\partial \boldsymbol{\Omega}_m} \frac{f_i\left(\theta_i\right) \frac{\partial \theta_i}{\partial \boldsymbol{\Omega}_m} - f_i\left(\theta_{i-1}\right) \frac{\partial \theta_{i-1}}{\partial \boldsymbol{\Omega}_m}}{F_i\left(\theta_i\right) - F_i\left(\theta_{i-1}\right)} \\
&- \sum_{i=2}^{m-1} a_i \frac{\left[\frac{\partial^2 \theta_i}{\partial \boldsymbol{\Omega}_m^2} f_i\left(\theta_i\right) + \frac{\partial \theta_i}{\partial \boldsymbol{\Omega}_m} \frac{\partial f_i\left(\theta_i\right)}{\partial \boldsymbol{\Omega}_m} - \frac{\partial^2 \theta_{i-1}}{\partial \boldsymbol{\Omega}_m^2} f_i\left(\theta_{i-1}\right) - \frac{\partial \theta_{i-1}}{\partial \boldsymbol{\Omega}_m} \frac{\partial f_i\left(\theta_{i-1}\right)}{\partial \boldsymbol{\Omega}_m} \right]}{F_i\left(\theta_i\right) - F_i\left(\theta_{i-1}\right)} \\
&+ \sum_{i=2}^{m-1} a_i \frac{\left[\frac{\partial \theta_i}{\partial \boldsymbol{\Omega}_m} f_i\left(\theta_i\right) - \frac{\partial \theta_{i-1}}{\partial \boldsymbol{\Omega}_m} f_i\left(\theta_{i-1}\right) \right]^2}{\left[F_i\left(\theta_i\right) - F_i\left(\theta_{i-1}\right)\right]^2} - \frac{\partial^2 a_m}{\partial \boldsymbol{\Omega}_m^2} \log \left[1 - F_m\left(\theta_{m-1}\right)\right] + 2 \frac{\partial a_m}{\partial \boldsymbol{\Omega}_m} \frac{\frac{\partial \theta_{m-1}}{\partial \boldsymbol{\Omega}_m} f_m\left(\theta_{m-1}\right)}{1 - F_m\left(\theta_{m-1}\right)} \\
&+ a_m \frac{\frac{\partial^2 \theta_{m-1}}{\partial \boldsymbol{\Omega}_m^2} f_m\left(\theta_{m-1}\right) + \frac{\partial \theta_{m-1}}{\partial \boldsymbol{\Omega}_m} \frac{\partial f_m\left(\theta_{m-1}\right)}{\partial \boldsymbol{\Omega}_m}}{1 - F_m\left(\theta_{m-1}\right)} + a_m \frac{\left[\frac{\partial \theta_{m-1}}{\partial \boldsymbol{\Omega}_m} f_m\left(\theta_{m-1}\right) \right]^2}{\left[1 - F_m\left(\theta_{m-1}\right)\right]^2} + \frac{\partial^2 \theta_1}{\partial \boldsymbol{\Omega}_m^2} \sum_{i=1}^{n} \Delta\left(x_i - \theta_1\right) \log f_1\left(x_i\right) \\
&+ \sum_{i=2}^{m-1} \left[\frac{\partial^2 a_i}{\partial \boldsymbol{\Omega}_m^2} \log \phi_i + 2 \frac{\partial a_i}{\partial \boldsymbol{\Omega}_m} \frac{\frac{\partial \phi_i}{\partial \boldsymbol{\Omega}_m}}{\phi_i} + a_i \frac{\frac{\partial^2 \phi_i}{\partial \boldsymbol{\Omega}_m^2} \phi_i - \left(\frac{\partial \phi_i}{\partial \boldsymbol{\Omega}_m} \right)^2}{\phi_i^2} - \frac{\partial^2 \theta_{i-1}}{\partial \boldsymbol{\Omega}_m^2} \sum_{x_i \leq \theta_i} \Delta\left(x_i - \theta_{i-1}\right) \log f_i\left(x_i\right) \right] \\
&+ \sum_{i=2}^{m-1} \left[\frac{\partial^2 \theta_i}{\partial \boldsymbol{\Omega}_m^2} \sum_{\theta_{i-1} < x_i} \Delta\left(x_i - \theta_i\right) \log f_i\left(x_i\right) - 2 \frac{\partial \theta_i}{\partial \boldsymbol{\Omega}_m} \frac{\partial \theta_{i-1}}{\partial \boldsymbol{\Omega}_m} \sum_{i=1}^{n} \Delta\left(x_i - \theta_{i-1}\right) \Delta\left(x_i - \theta_i\right) \log f_i\left(x_i\right) \right] \\
&+ \frac{\partial^2 a_m}{\partial \boldsymbol{\Omega}_m^2} \log \phi_m + 2 \frac{\partial a_m}{\partial \boldsymbol{\Omega}_m} \frac{\frac{\partial \phi_m}{\partial \boldsymbol{\Omega}_m}}{\phi_m} + a_m \frac{\frac{\partial^2 \phi_m}{\partial \boldsymbol{\Omega}_m^2} \phi_m - \left(\frac{\partial \phi_m}{\partial \boldsymbol{\Omega}_m} \right)^2}{\phi_m^2} - \frac{\partial^2 \theta_{m-1}}{\partial \boldsymbol{\Omega}_m^2} \sum_{i=1}^{n} \Delta\left(x_i - \theta_{m-1}\right) \log f_m\left(x_i\right) \\
&- 2 \frac{\partial \theta_{m-1}}{\partial \boldsymbol{\Omega}_m} \sum_{i=1}^{n} \Delta\left(x_i - \theta_{m-1}\right) \frac{\frac{\partial f_m\left(x_i\right)}{\partial \boldsymbol{\Omega}_m}}{f_m\left(x_i\right)} + \sum_{\theta_{m-1} < x_i} \frac{\frac{\partial^2 f_m\left(x_i\right)}{\partial \boldsymbol{\Omega}_m^2} f_m\left(x_i\right) - \left[\frac{\partial f_m\left(x_i\right)}{\partial \boldsymbol{\Omega}_m} \right]^2}{\left[f_m\left(x_i\right)\right]^2}.
\end{aligned}
$$

5. A simulation study

In this section, we perform a simulation study to show the practical value of the expressions in the paper. We used the following scheme:

(i) simulate a sample of size 1000 from the m-piece composite distribution in Section 4 with each piece described by an exponential distribution and $\theta_j = j$, $j = 1, 2, \ldots, m-1$;

(ii) estimate the maximum likelihood estimates for the distribution;

(iii) compute the information matrix given in Section 4;

(iv) compute the information matrix numerically using the package numDeriv in R (R Development Core Team [32]);

(v) take the squared sum of elementwise differences of the two matrices in steps (iii) and (iv);

(vi) repeat steps (i) to (v) 100 times;

(vii) draw a boxplot of the 100 squared sums of elementwise differences of the two matrices;

(viii) repeat steps (i) to (vii) for $m = 2, 5, 10, 50$.

Figure 1 shows the boxplots. Figure 2 shows the boxplots when the lognormal distribution was used to model each piece in step (i). Figure 3 shows the boxplots when the Pareto distribution was used to model each piece in step (i).

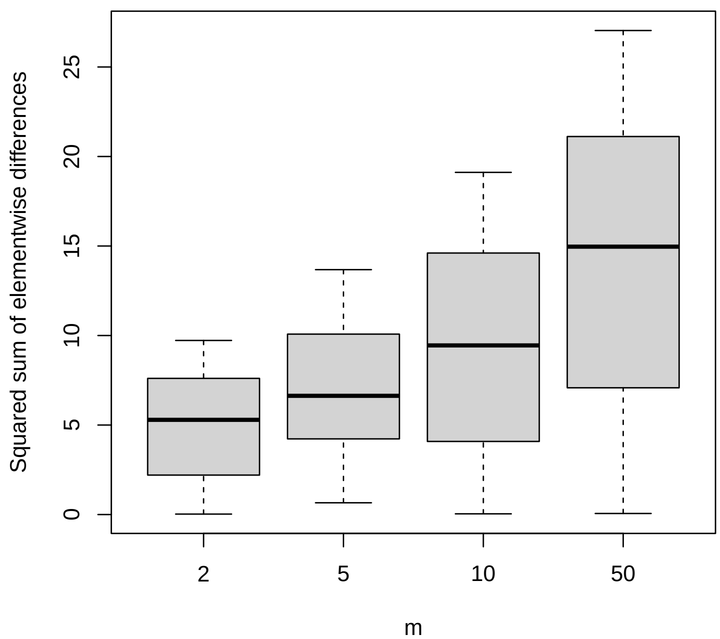

Figure 1: The squared sum of elementwise differences versus m when each piece was modeled by the exponential distribution.

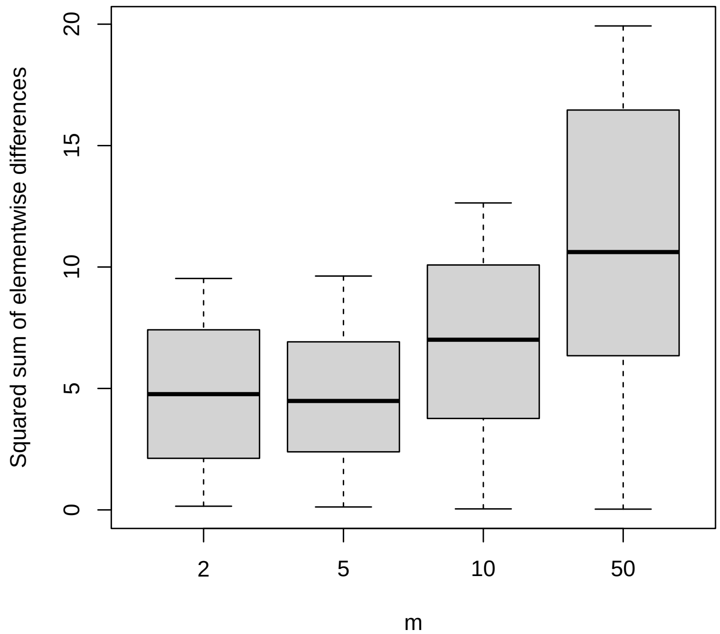

Figure 2: The squared sum of elementwise differences versus m when each piece was modeled by the lognormal distribution.

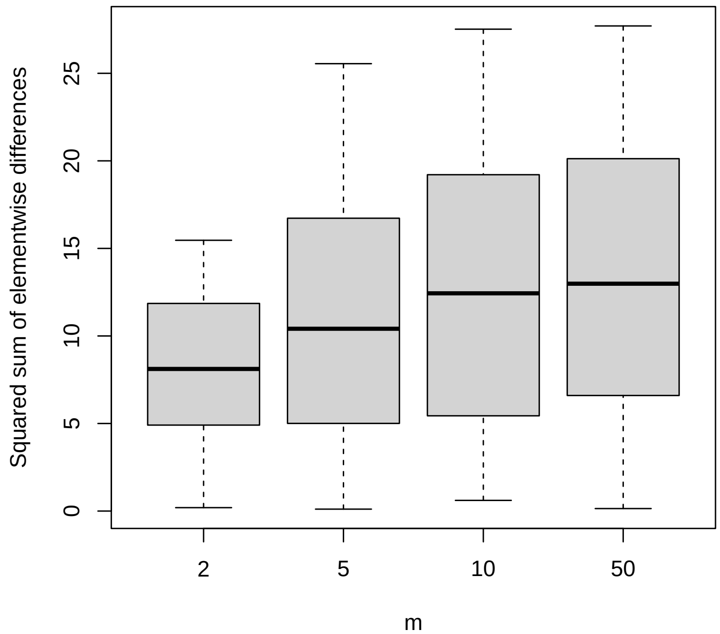

Figure 3: The squared sum of elementwise differences versus m when each piece was modeled by the Pareto distribution.

We can see that the squared sum of elementwise differences is significant whatever m is. The magnitude of the squared sum increases with increasing values of m. This is expected because of: i) numerical errors; ii) the increasing number of boundaries in the composite distribution as m increases; iii) the increasing number of conditions on continuity; iv) the increasing number of conditions on differentiability. The differences appear as large as 25 when $m = 50$. Hence, the expressions in the paper should be trusted (over numerical derivatives) for accurate computation of the information matrices and hence for accurate modeling of insurance data especially when m is large.

For simplicity, we have chosen the exponential, lognormal and Pareto distributions to describe each piece of the composite distribution. But the results were similar when other distributions were considered. In particular, the squared sum of elementwise differences were significant for any m and always increased with increasing values of m.

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