



On general weighted cumulative past extropy

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Abstract:

- In this paper, we study some properties and characterization of the general weighted cumulative past extropy (n -WCPJ). Many results including some bounds, inequalities, and effects of linear transformations are obtained. We study the characterization of n -WCPJ based on the largest order statistics. Conditional WCPJ and some of its properties are discussed. We study weighted cumulative past extropy (WCPJ) and weighted negative cumulative extropy (WNCEX) and its application for testing symmetry of dataset.

Keywords:

- *Cumulative residual entropy; Cumulative residual extropy; Weighted cumulative past extropy; Conditional weighted past extropy; Weighted negative cumulative extropy.*

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1. INTRODUCTION

We encounter phenomena or events which are associated with uncertainty. Uncertainty emerges since we have less information than the total information required to describe a system and its environment. Uncertainty and information are closely associated. Since in an experiment the information provided is equal to the amount of uncertainty removed. Entropy which is a measure of uncertainty was first introduced by Shannon [34] in communication theory. It is useful to estimate the probabilities of rare events (large deviation theory) and in the study of likelihood-based inference principles. Shannon entropy is defined as the average amount of information that we receive per event and was the first defined entropy. For continuous case, it is given by

$$\mathcal{H}(X) = - \int_0^{\infty} f_X(x) \log f_X(x) dx,$$

where X is a non-negative absolutely continuous random variable with probability density function (pdf) f_X , cumulative distribution function (cdf) F_X and survival function (sf) $\bar{F}_X = 1 - F_X$. Shannon entropy has various applications in communication theory, mathematical, physical, engineering, biological and social sciences as well. For further details on entropy one may refer Ash [2] and Cover and Thomas [7].

Rao et al. [30] introduced the notion of cumulative residual entropy (CRE) as :

$$\mathcal{E}(X) = - \int_0^{\infty} \bar{F}_X(x) \log \bar{F}_X(x) dx.$$

Some general results regarding this measure have been studied by Drissi et al [10] and Rao [31]. CRE finds applications in image alignment and in the measurement of similarity between images. Di Crescenzo and Longobardi [8] proposed an entropy called cumulative past entropy (or cumulative entropy) i.e. CPE (or CE) as :

$$\mathcal{CE}(X) = - \int_0^{\infty} F_X(x) \log F_X(x) dx.$$

Asadi and Zohrevand [1], Di Crescenzo and Longobardi [9], Khorashadizadeh et al. [16] and Navarro et al. [23] investigated many aspects of CRE (CPE).

Lad et al. [19] defined an alternative measure of the uncertainty of a random variable called extropy. For continuous non-negative random variable X the extropy is defined as :

$$(1.1) \quad J(X) = -\frac{1}{2} \int_0^{\infty} f_X^2(x) dx = -\frac{1}{2} E(f_X(X)).$$

Some results and properties of the extropy of order statistics and record values are given by Qiu [26]. Qiu and Jia [27] derived some of the results of the residual extropy of order statistics. Yang et al. [38] studied the bounds on extropy

with a variational distance constraint. Qiu et al. [29] examined certain extropy properties of mixed systems. To find out more about extropy, one may refer to Krishnan et al. [17], Noughabi and Jarrahiferiz [24] and Raqab and Qiu [32].

Jahanshahi et al. [14] introduced the cumulative residual extropy (CRJ). For continuous non-negative random variable X the cumulative residual extropy (CRJ) as :

$$(1.2) \quad \xi J(X) = -\frac{1}{2} \int_0^{\infty} \bar{F}_X^2(x) dx.$$

Kundu [18] proposed an extropy called cumulative past extropy (CPJ). For continuous non-negative random variable X the cumulative past extropy (CPJ) is defined as :

$$(1.3) \quad \bar{\xi} J(X) = -\frac{1}{2} \int_0^{\infty} F_X^2(x) dx.$$

The idea behind this is to replace the density function by distribution function in extropy (1.1). One can only define the cumulative past extropy for random variables that have a limited range of possible values, as this measure would be infinitely negative for any random variable with an unlimited range of values. Kundu [18] studied extreme order statistics on cumulative residual (past) extropy.

Hashempour et al. [13] proposed a new measure called weighted cumulative residual extropy. Mohammadi and Hashempour [21] studied a new measure called interval weighted cumulative residual extropy. For continuous non-negative random variable X the weighted cumulative residual extropy (WCRJ) is defined as :

$$(1.4) \quad \xi^1 J(X) = -\frac{1}{2} \int_0^{\infty} x \bar{F}_X^2(x) dx.$$

Tahmasebi and Toomaj [35] introduced negative cumulative extropy. For continuous non-negative random variable X the negative cumulative extropy (NCEX) can be defined as :

$$(1.5) \quad \mathcal{C}J(X) = \frac{1}{2} \int_0^{\infty} (1 - F_X^2(x)) dx.$$

Chaudhary et al. [6] proposed a new measure on general weighted negative cumulative extropy. For absolutely continuous non-negative random variable X , with weight $w(x) > 0$, the general weighted negative cumulative extropy (GWNCJ) defined as :

$$(1.6) \quad \bar{\eta}^w J(X) = \frac{1}{2} \int_0^{\infty} w(x) (1 - F_X^2(x)) dx.$$

This paper is organized in the following manner. In Section 2 we introduce the generalised weighted cumulative past extropy (m -WCPJ) and study some of

its properties. In Section 3 some bounds and inequalities are derived. In Section 4, we study the characterization of m -WCPJ based on the largest-order statistic. Conditional WCPJ and some of its properties are discussed in Section 5. Section 6, provides the empirical estimator of GWCPJ. Application of empirical estimator was discussed in Section 7. In Section 8 a statistical hypothesis testing procedure is performed to assess the goodness of fit of a sample dataset to a standard uniform distribution. Power of test is observed in Section 9. In Section 10 we introduce the weighted neagative cumulative extropy and relationship of symmetric distribution with weighted negative cumulative extropy (WNCEX) and weighted cumulative past extropy (WCPJ) was derived. Application of estimator for symmetric test of dataset was discussed in Section 11. In the end, Section 12 concludes this paper.

2. WEIGHTED CUMULATIVE PAST EXTROPY

Balakrishnan et al. [3] and Bansal and Gupta [5] independently introduced the weighted extropy. Mohammadi and Hashempour [21] studied a new measure called interval weighted cumulative past extropy. Weighted cumulative past extropy (WCPJ) is an information measure, which is a generalization of cumulative past extropy. In this section, we study the properties of WCPJ.

Definition 2.1. Let X be a non-negative absolutely continuous random variable with cdf F_X . We define the m -WCPJ of X by

$$(2.1) \quad \bar{\xi}^m J(X) = \frac{-1}{2} \int_0^\infty x^m F_X^2(x) dx.$$

The introduction explains that, like the cumulative past extropy (CPJ), the weighted cumulative past extropy (WCPJ) also has a value of negative infinity for any random variable with an unbounded range of possible values. Therefore, we must limit the definition of WCPJ to random variables with a limited range of values. If we consider a non-negative random variable X with a bounded range of possible values denoted by B , then the WCPJ of X is defined as follows:

$$(2.2) \quad \bar{\xi}^m J(X) = \frac{-1}{2} \int_0^{\sup B} x^m F_X^2(x) dx.$$

Let us consider some examples.

Example 2.1. Let X has $U[a, b]$ distribution. Then CRJ and WCRJ of the uniform distribution are

$$\xi J(X) = -\frac{b-a}{6}, \text{ and, } \xi^1 J(X) = \frac{a-b}{24}(3a+b),$$

respectively. Then CPJ and m -WCPJ of the uniform distribution are

$$\bar{\xi} J(X) = -\frac{b-a}{6}, \text{ and,}$$

$$\begin{aligned}\bar{\xi}^m J(X) &= - \left(b^{m+1} \cdot (b-a)^2 m^2 + b^{m+1} \cdot (b-a) (3b-5a) m \right. \\ &\quad \left. + 2b^{m+1} \cdot (b^2 - 3ab + 3a^2) - 2a^{m+3} \right) / \left(2(b-a)^2 (m+1)(m+2)(m+3) \right),\end{aligned}$$

respectively. Note that for $m = 1$,

$$\bar{\xi}^1 J(X) = \left(\frac{a+3b}{4} \right) \bar{\xi} J(X) = \left(\frac{E(X)+b}{2} \right) \bar{\xi} J(X).$$

If $\frac{E(X)+b}{2} > 1$, then $\bar{\xi}^1 J(X) < \bar{\xi} J(X)$, and if $\frac{E(X)+b}{2} < 1$, then $\bar{\xi}^1 J(X) > \bar{\xi} J(X)$.

Example 2.2. Let X follow power-law distribution with pdf $f_X(x) = \lambda x^{\lambda-1}$, $x \in (0, 1)$, $\lambda > 1$. The *CRJ* and *WCRJ* of the distribution are

$$\xi J(X) = -\frac{\lambda^2}{(\lambda+1)(2\lambda+1)}, \text{ and, } \xi^1 J(X) = -\frac{\lambda^2}{4(\lambda+1)(\lambda+2)},$$

respectively. Note that

$$\xi^1 J(X) = \left(\frac{2\lambda+1}{4(\lambda+2)} \right) \xi J(X) = \left(\frac{2(\lambda+1)E(X)+1}{4(\lambda+2)} \right) \xi J(X).$$

For $\lambda = -\frac{7}{2}$, $\xi^1 J(X) = \xi J(X)$. The *CPJ* and *WCPJ* of the distribution are

$$\bar{\xi} J(X) = -\frac{1}{2(2\lambda+1)}, \text{ and, } \bar{\xi}^m J(X) = -\frac{1}{4\lambda+2m+2}.$$

We conclude that for $m = 1$, $\bar{\xi}^1 J(X) = \frac{2\lambda+1}{2\lambda+2} \bar{\xi} J(X)$. Also, $\bar{\xi}^1 J(X) = -\frac{\lambda+2}{4\lambda(\lambda+1)} E(X^2)$.

Theorem 2.1. Let X be a non-negative continuous random variable with bounded support B for m -*WCPJ*, $\bar{\xi}^m J(X)$. Then we have

$$\bar{\xi}^m J(X) = \frac{-1}{2} E(G_F(X)),$$

where $G_F(t) = \int_t^{\sup B} x^m F_X(x) dx$.

Proof Using equation (2.2) and Fubini's theorem, we have

$$\begin{aligned}\bar{\xi}^m J(X) &= \frac{-1}{2} \int_0^{\sup B} x^m F_X^2(x) dx = \frac{-1}{2} \int_0^{\sup B} x^m F_X(x) \left(\int_0^x f_X(t) dt \right) dx \\ &= \frac{-1}{2} \int_0^{\sup B} f_X(t) \left(\int_t^{\sup B} x^m F_X(x) dx \right) dt = \frac{-1}{2} \int_0^{\sup B} f_X(t) G_F(t) dt \\ &= \frac{-1}{2} E(G_F(X)).\end{aligned}$$

■

Now we see the effect of linear transformation on *WCPJ* in the following proposition

Proposition 2.1. *Let X be a non-negative random variable. If $Y = aX + b$, $a > 0$, $b \geq 0$, then*

$$\bar{\xi}^m J(Y) = \sum_{i=0}^m \binom{m}{i} a^i b^{m-i} \bar{\xi}^{m-i} J(X)$$

Proof The proof holds using (2.2) and noting that $F_Y(y) = F_X\left(\frac{y-b}{a}\right)$, $y > b$. ■

Here we provide an upper bound for WCPJ in terms of extropy.

Theorem 2.2. *Let X be a random variable with pdf $f_X(\cdot)$ and extropy $J(X)$, with bounded support B then*

$$(2.3) \quad \bar{\xi}^m J(X) \leq C^* \exp\{2J(X)\},$$

where $C^* = \frac{-1}{2} \exp\{E(\log(X^m F_X^2(X)))\}$, and $\exp\{x\} := e^x$.

Proof Using the log-sum inequality, we have

$$\int_0^{\sup B} f_X(x) \log\left(\frac{f_X(x)}{x^m F_X^2(x)}\right) dx \geq -\log\left(\int_0^{\sup B} x^m F_X^2(x) dx\right).$$

Then it follows that

$$\begin{aligned} \int_0^{\sup B} f_X(x) \log f_X(x) dx - \int_0^{\sup B} f_X(x) \log(x^m F_X^2(x)) dx \\ \geq -\log\left(\int_0^{\sup B} x^m F_X^2(x) dx\right). \end{aligned}$$

Note that $\log f < f$, hence

$$(2.4) \quad \begin{aligned} -\int_0^{\sup B} f_X^2(x) dx + \int_0^{\sup B} f_X(x) \log(x^m F_X^2(x)) dx \\ = 2J(X) + E(\log(X^m F_X^2(X))) \\ \leq \log(-2\bar{\xi}^m J(X)). \end{aligned}$$

Exponentiating both sides of (2.4), we have

$$\bar{\xi}^m J(X) \leq \frac{-1}{2} \exp\{2J(X) + E(\log(X^m F_X^2(X)))\},$$

Hence the result. ■

Theorem 2.3. *X is degenerate, if and only if, $\bar{\xi}^m J(X) = 0$.*

Proof Suppose X be degenerate at point c , then by using the definition of degenerate function and $\bar{\xi}^m J(X)$, we have $\bar{\xi}^m J(X) = 0$. Now consider $\bar{\xi}^m J(X) = 0$, i.e., $\int_0^\infty x^m F_X^2(x) dx = 0$. Noting that the integrand in the above integral is non-negative, we have $F_X(x) = 0$, for almost all $x \in S$, where S denotes the support of random variable X , i.e., it is 0 in $\inf S$ and then 1.

3. SOME INEQUALITIES

This section deals with obtaining the lower and upper bounds for WCPJ.

Remark 3.1. Consider X be a non-negative random variable. then

$$(3.1) \quad \bar{\xi}^m J(X) \geq \frac{-1}{2} \int_0^{\sup B} x^m F_X(x) dx.$$

Proposition 3.1. Consider a non-negative continuous random variable X having cdf $F_X(\cdot)$ and support $[a, \sup B)$, $a > 0$. Then

$$(3.2) \quad \bar{\xi}^m J(X) \leq a^m \bar{\xi} J(X).$$

Proof Note that

$$\begin{aligned} \int_a^{\sup B} x^m F_X^2(x) dx &\geq a^m \int_a^{\sup B} F_X^2(x) dx \\ \frac{-1}{2} \int_a^{\sup B} x^m F_X^2(x) dx &\leq \frac{-a^m}{2} \int_a^{\sup B} F_X^2(x) dx \\ \bar{\xi}^m J(X) &\leq a^m \bar{\xi} J(X). \end{aligned}$$

■

Corollary 3.1. Let X be a continuous random variable with cdf F that takes values on $[0, b]$ where b is finite. Then,

1. $\bar{\xi}^m J(X) \leq \frac{-1}{2(m+1)} (b^{m+1} - E(X^{m+1})) \left[\log \left(\frac{b^{m+1} - E(X^{m+1})}{b^{m+1}} \right) - 1 \right],$
2. $\bar{\xi}^m J(X) \geq b^m \bar{\xi} J(X).$

Proof Using log-sum inequality, we have

$$\begin{aligned} \int_0^b F_X(t) t^m \log(F_X(t)) dt &\geq \int_0^b F_X(t) t^m dt \log \left(\frac{\int_0^b F_X(t) t^m dt}{\int_0^b t^m dt} \right) \\ &= \left(\frac{b^{m+1} - E(X^{m+1})}{m+1} \right) \log \left(\frac{b^{m+1} - E(X^{m+1})}{b^{m+1}} \right) \end{aligned}$$

Also note that $\log F_X(t) \leq F_X(t) - 1$, then

$$\begin{aligned} \int_0^b F_X(t) t^m \log F_X(t) dt &\leq -2\bar{\xi}^m J(X) - \int_0^b t^m F_X(t) dt \\ &= -2\bar{\xi}^m J(X) - \left(\frac{b^{m+1} - E(X^{m+1})}{m+1} \right), \end{aligned}$$

Now using the above two inequalities, the first part follows. The second part can be verified easily. ■

Consider two random variables X and Y having cdfs F and G , respectively. Then $X \leq_{st} Y$ whenever $F(x) \geq G(x)$, $\forall x \in \mathbb{R}$; where the notation $X \leq_{st} Y$ means that X is less than or equal to Y in usual stochastic order. One may refer Shaked and Shanthikumar [33] for detail of stochastic ordering. In the following proposition, we show the ordering of WCPJ is implied by the usual stochastic order.

Proposition 3.2. *Let X_1 and X_2 be non-negative continuous random variables. If $X_1 \leq_{st} X_2$, then $\bar{\xi}^m J(X_1) \leq \bar{\xi}^m J(X_2)$.*

Proof Using $X_1 \leq_{st} X_2$ and (2.2), the result follows. ■

4. WCPJ BASED ON LARGEST-ORDER STATISTIC

Let X_1, \dots, X_n be a random sample from a random variable X having absolutely continuous cdf $F_X(x)$ and pdf $f_X(x)$. Then $X_{1:n} \leq X_{2:n} \leq \dots \leq X_{n:n}$ be the ordered statistics to random sample X_1, \dots, X_n . In the following, we obtain the WCPJ of the largest-order statistic. The WCPJ of the n th-order statistic is

$$(4.1) \quad \bar{\xi}^m J(X_{n:n}) = -\frac{1}{2} \int_0^{\sup B} x^m F_{X_{n:n}}^2(x) dx,$$

where $F_{X_{n:n}}^2(x) = F_X^{2n}(x)$. Using transformation $u = F_X(x)$ in (4.1),

$$(4.2) \quad \bar{\xi}^m J(X_{n:n}) = -\frac{1}{2} \int_0^1 \frac{u^{2n} [F_X^{-1}(u)]^m}{f_X(F_X^{-1}(u))} du,$$

where $F_X^{-1}(x)$ is the inverse function of $F_X(x)$.

Example 4.1. Let X have the uniform distribution on $(0,1)$ with pdf $f_X(x) = 1$, $x \in (0,1)$. Then $F_X^{-1}(u) = u$, $u \in (0,1)$ and $f_X(F_X^{-1}(u)) = 1$, $u \in (0,1)$, hence $\bar{\xi}^m J(X_{n:n}) = \frac{-1}{2(2n+m+1)}$.

Example 4.2. let X follow power-law distribution with pdf $f_X(x) = \lambda x^{\lambda-1}$, $\lambda > 1$, $x \in (0,1)$. Then $F_X^{-1}(u) = u^{\frac{1}{\lambda}}$, $u \in (0,1)$ and $f_X(F_X^{-1}(u)) = \lambda u^{\frac{\lambda-1}{\lambda}}$, $u \in (0,1)$, hence $\bar{\xi}^m J(X_{n:n}) = \frac{-1}{2(2n\lambda+m+1)}$.

Remark 4.1. Consider $\Lambda = \bar{\xi}^m J(X_{n:n}) - \bar{\xi}^m J(X)$. Since $F_X^{2n}(x) \leq F_X^2(x)$, hence $\Lambda \geq 0$.

For the proof of Theorem 4.1, we need the following lemma.

Lemma 4.1. [Lemma 4.1 of Hashempour et al. [13]] Let g be a continuous function with support $[0, 1]$, such that $\int_0^1 g(y)y^m dy = 0$, for $m \geq 0$, then $g(y) = 0$, $\forall y \in [0, 1]$

Theorem 4.1. Let X_1, \dots, X_n and Y_1, \dots, Y_n be two non negative random samples from continuous cdfs $F(x)$ and $G(x)$, respectively with common bounded support. Then $F(x) = G(x)$ if and only if $\bar{\xi}^m J(X_{n:n}) = \bar{\xi}^m J(Y_{n:n})$, for all n

Proof The necessary condition is trivial. Hence, it remains to prove the sufficient part. If $\bar{\xi}^m J(X_{n:n}) = \bar{\xi}^m J(Y_{n:n})$, then we have

$$-\frac{1}{2} \int_0^1 u^{2n} \left(\frac{[F^{-1}(u)]^m}{f(F^{-1}(u))} - \frac{[G^{-1}(u)]^m}{g(G^{-1}(u))} \right) du = 0,$$

By using Lemma 4.1, it follows that

$$\begin{aligned} \frac{[F^{-1}(u)]^m}{f(F^{-1}(u))} &= \frac{[G^{-1}(u)]^m}{g(G^{-1}(u))} \\ \implies [F^{-1}(u)]^m \frac{dF^{-1}(u)}{du} &= [G^{-1}(u)]^m \frac{dG^{-1}(u)}{du}, u \in [0, 1], \end{aligned}$$

since $\frac{dF^{-1}(u)}{du} = \frac{1}{f(F^{-1}(u))}$. Hence it follows $F^{-1}(u) = G^{-1}(u)$, $u \in [0, 1]$. Thus the proof is completed. \blacksquare

5. CONDITIONAL WEIGHTED PAST EXTROPY

Now we consider the conditional weighted cumulative past extropy (CWCPJ). Consider a random variable Z on probability space (Ω, \mathbb{A}, P) such that $E|Z| < \infty$. The conditional expectation of Z given sub σ -field \mathbb{G} , where $\mathbb{G} \subseteq \mathbb{A}$, is denoted by $E(Z|\mathbb{G})$. For the random variable $I_{(Z \leq z)}$, we denote $E(I_{(Z \leq z)}|\mathbb{G})$ by $F_Z(z|\mathbb{G})$.

Definition 5.1. For a non-negative random variable X with bounded support B , given σ -field \mathbb{G} , the CWCPJ $\bar{\xi}^m J(X|\mathbb{G})$ is defined as

$$(5.1) \quad \bar{\xi}^m J(X|\mathbb{G}) = \frac{-1}{2} \int_0^{\sup B} x^m F_X^2(x|\mathbb{G}) dx.$$

Now we assume that the random variables are continuous and non-negative.

Lemma 5.1. *If \mathbb{G} is a trivial σ -field, then $\bar{\xi}^m J(X|\mathbb{G}) = \bar{\xi}^m J(X)$.*

Proof Since here $F_X(x|\mathbb{G}) = F_X(x)$, then the proof follows. \blacksquare

Proposition 5.1. *If $X \in L^p$ for some $p > 2$, then $E[\bar{\xi}^m J(X|\mathbb{G})|\mathbb{G}^*] \leq \bar{\xi}^m J(X|\mathbb{G}^*)$, provided that $\mathbb{G}^* \subseteq \mathbb{G}$.*

Proof Consider

$$\begin{aligned} E[\bar{\xi}^m J(X|\mathbb{G})|\mathbb{G}^*] &= \frac{-1}{2} \int_0^{\sup B} x^m E\left([P(X \leq x|\mathbb{G})]^2|\mathbb{G}^*\right) dx \\ &\leq \frac{-1}{2} \int_0^{\sup B} x^m [E(P(X \leq x|\mathbb{G})|\mathbb{G}^*)]^2 dx \\ &= \frac{-1}{2} \int_0^{\sup B} x^m [E(E(I_{(X \leq x)}|\mathbb{G})|\mathbb{G}^*)]^2 dx \\ &= \frac{-1}{2} \int_0^{\sup B} x^m [E(I_{(X \leq x)}|\mathbb{G}^*)]^2 dx \\ &= \frac{-1}{2} \int_0^{\sup B} x^m F_X^2(x|\mathbb{G}^*) dx \\ &= \bar{\xi}^m J(X|\mathbb{G}^*), \end{aligned}$$

where the second step follows using Jensen's inequality for convex function $\phi(x) = x^2$. Hence the result. \blacksquare

In the following theorem, we investigate the relationship between conditional entropy and $\bar{\xi}^m J(X|\mathbb{G})$.

Theorem 5.1. *Let $\bar{\xi}^m J(X|\mathbb{G})$ is conditional past entropy. Then we have*

$$(5.2) \quad \bar{\xi}^m J(X|\mathbb{G}) \leq B^* \exp\{2J(X|\mathbb{G})\},$$

where $B^* = \frac{-1}{2} \exp\{E(\log(X^m F_X^2(X))|\mathbb{G})\}$

Proof The proof is on the similar lines as of Theorem 2.2, hence omitted. \blacksquare

Theorem 5.2. *For a random variable X with bounded support B and σ -field \mathbb{G} , we have*

$$(5.3) \quad E(\bar{\xi}^m J(X|\mathbb{G})) \leq \bar{\xi}^m J(X),$$

and the equality holds if and only if X is independent of \mathbb{G} .

Proof If in Proposition 5.1, \mathbb{G}^* is trivial σ -field, then (5.3) can be easily obtained. Now assume that X is independent of \mathbb{G} , then

$$(5.4) \quad \begin{aligned} F_X(x|\mathbb{G}) &= F_X(x) \\ \implies \bar{\xi}^m J(X|\mathbb{G}) &= \bar{\xi}^m J(X). \end{aligned}$$

On taking expectation to both sides of (5.4), we get equality in (5.3). Conversely, assume that equality in (5.3) holds. It is sufficient to show that $F_X(x|\mathbb{G}) = F_X(x)$, to prove independence between X and σ -field \mathbb{G} . Take $U = F_X(x|\mathbb{G})$, and since the function $\phi(u) = u^2$ is convex hence $E(U^2) \geq E^2(U) = F_X^2(x)$, and also due to equality in (5.3), we have

$$\int_0^{\sup B} x^m E(U^2) dx = \int_0^{\sup B} x^m F_X^2(x) dx = \int_0^{\sup B} x^m E^2(U) dx.$$

Hence $E(U^2) = E^2(U)$. Now using the Corollary 8.1 of Hashempour et al. [13], we have $F_X(x|\mathbb{G}) = F_X(x)$. Hence the proof. \blacksquare

For the Markov property for non-negative random variables X , Y and Z , we have the following proposition.

Proposition 5.2. *Let $X \rightarrow Y \rightarrow Z$ is a Markov chain, then*

$$(5.5) \quad \bar{\xi}^m J(Z|X, Y) = \bar{\xi}^m J(Z|Y)$$

and

$$(5.6) \quad E(\bar{\xi}^m J(Z|Y)) \leq E(\bar{\xi}^m J(Z|X)).$$

Proof By the definition of $\bar{\xi}^m J(Z|X, Y)$ and using the Markov property, (5.5) holds.

Now letting $\mathbb{G}^* = \sigma(X)$, $\mathbb{G} = \sigma(X, Y)$ and $X = Z$ in Proposition 5.1, we have

$$(5.7) \quad \bar{\xi}^m J(Z|X) \geq E(\bar{\xi}^m J(Z|X, Y)|X)$$

Taking expectation on both sides of (5.7), we have

$$\begin{aligned} E(\bar{\xi}^m J(Z|X)) &\geq E(E(\bar{\xi}^m J(Z|X, Y)|X)) \\ &= E(\bar{\xi}^m J(Z|X, Y)) \\ &= E(\bar{\xi}^m J(Z|Y)), \end{aligned}$$

where the last equality holds using (5.5). Hence the result (5.6) holds. \blacksquare

6. EMPIRICAL GWCPJ

We want to create an estimator for the general Weighted Cumulative Past Entropy (WCPJ) using the empirical m -WCPJ. To do this, we assume that we

have a random sample of n non-negative, continuous, independent, and identically distributed random variables X_1, \dots, X_n from a population with a cumulative distribution function (cdf) $F_X(x)$. We will use the "plug-in" method to define the estimator by replacing the unknown quantities in the formula with their empirical counterparts based on the sample. Specifically, we will define the empirical general weighted cumulative past extropy as

$$(6.1) \quad \bar{\xi}_n^m J(X) = \frac{-1}{2} \int_0^\infty x^m F_n^2(x) dx.$$

where $F_n(x)$ is the empirical distribution function. Suppose we have a random sample of n observations, and let $X_{1:n} \leq X_{2:n} \leq \dots \leq X_{n:n}$ be the ordered statistics to random sample X_1, \dots, X_n . We can express the value of the empirical general weighted cumulative past extropy $\bar{\xi}_n^m J(X)$ in terms of these ordered statistics. In other words, we can rewrite the formula for $\bar{\xi}_n^m J(X)$ in a way that involves the ordered statistics of the sample.

The empirical measure of $\bar{\xi}_n^m J(X) = \frac{-1}{2} \int_0^\infty x^m F_n^2(x) dx$ is obtained as

$$(6.2) \quad \begin{aligned} \bar{\xi}_n^m J(X) &= -\frac{1}{2} \int_0^\infty x^m F_n^2(x) dx \\ &= -\frac{1}{2} \sum_{i=1}^{n-1} \int_{X_{i:n}}^{X_{i+1:n}} \left(\frac{i}{n}\right)^2 x^m dx \\ &= -\frac{1}{2(m+1)} \sum_{i=1}^{n-1} (X_{i+1:n}^{m+1} - X_{i:n}^{m+1}) \left(\frac{i}{n}\right)^2. \end{aligned}$$

7. APPLICATION

To test the uniformity of a random sample X_1, X_2, \dots, X_n , we use the empirical weighted cumulative past extropy $\bar{\xi}_n^m J(X)$ obtained in equation (6.2) as a test statistic. Before discussing this test statistic, we first need to understand a property of the uniform distribution, which is defined on the interval $(0,1)$. For any random variable X with a cumulative distribution function F , and for any probability value p in the interval $(0,1)$, we define the function $\phi_p^m J(F)$ as

$$(7.1) \quad \phi_p^m J(F) = -\frac{1}{2} \int_0^p x^m F^2(x) dx, \quad m \geq 0.$$

Example 7.1. If F_0 is cdf of uniform random variable on interval $(0,1)$, then $\phi_p^m J(F_0) = -\frac{p^{m+3}}{2(m+3)}$ and in particular, when $m = 1$, $\phi_p^1 J(F_0) = -\frac{p^4}{8}$ and when $m = 0$, $\phi_p^0 J(F_0) = -\frac{p^3}{6}$.

In view of Lemma 4.1 and Example 7.1, the following proposition is obtained.

Proposition 7.1. *Let F_0 is cdf of uniform random variable on interval $(0, 1)$. Suppose that for a cdf F in the class of cdfs defined on interval $(0, 1)$, $\phi_p^m J(F) = \phi_p^m J(F_0)$. Then $F(x) = F_0(x)$, almost everywhere.*

Proof Let $\phi_p^m J(F) = \phi_p^m J(F_0)$, using (7.1) we get

$$\int_0^p x^m (F^2(x) - F_0^2(x)) dx = 0, \quad \forall p \in (0, 1).$$

It is known that the $(0, p)$ generate the Borel σ -algebra of $\Omega = (0, 1]$. Therefore, we can write

$$\int_B x^m (F^2(x) - F_0^2(x)) dx = 0, \quad \forall B \subseteq (0, 1].$$

By Lemma 4.1, we conclude that $F(x) = F_0(x)$, almost everywhere.

Thus, $\phi_p^m J(F)$ for $p \in (0, 1)$ is uniquely determined by the uniform distribution in the sense that for some cdfs defined on $(0, 1)$, they take a value less than $-\frac{p^{m+3}}{2(m+3)}$ and for some of them, they take more than $-\frac{p^{m+3}}{2(m+3)}$, and only for the standard uniform distribution, we have $\phi_p^m J(F_0) = -\frac{p^{m+3}}{2(m+3)}$. ■

8. THE TEST OF UNIFORMITY

We aim to develop a non-parametric test for the uniform goodness of fit problem based on the GWCPJ measure. Specifically, we want to test if a random sample of size n from an unknown distribution F is uniformly distributed on the interval $(0, 1)$. Our test statistic is $\bar{\xi}_n^m J(X)$, which is computed using the empirical quantiles of the sample. We choose $m = 1$ for computational simplicity, but the procedure is the same for other values of m . To determine the critical region, we need to calculate $G_1(\alpha)$ and $G_2(\alpha)$, which are the lower and upper bounds for the test statistic, respectively, based on the empirical distribution of $\bar{\xi}_n^m J(X)$ under the standard uniform distribution. If $\bar{\xi}_n^m J(X)$ falls outside this interval, we reject the null hypothesis $H_0 : F = F_0$ in favor of the alternative hypothesis $H_1 : F \neq F_0$. We estimate $G_1(\alpha)$ and $G_2(\alpha)$ using the 0.025-th and 0.975-th quantiles of the empirical distribution of $\bar{\xi}_n^m J(X)$, respectively, based on 100,000 replications. We report the values of $G_1(\alpha)$ and $G_2(\alpha)$ for different sample sizes in Table 1 at a significance level of 5% ($\alpha = 0.05$).

Table 1. Values of $G_1(\alpha)$ and $G_2(\alpha)$, for $\alpha = 0.05$

n	20	30	40	50
$G_1(\alpha)$	-0.144891	-0.144001	-0.143339	-0.142433
$G_2(\alpha)$	-0.066024	-0.078560	-0.085851	-0.090792

Simillary we can obtain for $m = 2$. We report the values of $G_1(\alpha)$ and $G_2(\alpha)$ for different sample sizes in Table 2 at a significance level of 5% ($\alpha = 0.05$).

Table 2. Values of $G_1(\alpha)$ and $G_2(\alpha)$, for $\alpha = 0.05$

n	20	30	40	50
$G_1(\alpha)$	-0.109538	-0.110579	-0.110441	-0.110317
$G_2(\alpha)$	-0.047356	-0.059339	-0.066270	-0.070851

9. POWER OF THE TEST

This section discusses the power of the proposed test statistic against alternative Beta(1.5,1.5) and Beta(1.0,1.0) distributions at the significance level of $\alpha = 0.05$ and $m = 1$, for different sample sizes. The Beta distribution is chosen as an alternative because it has support (0,1) like the standard uniform distribution. Previous researchers has also examined the power of the WCPJ test against the Beta(1.5,1.5) distribution for testing uniformity. The proposed test is found to be more powerful for large sample sizes, and the power increases as the sample size increases. As Beta(1,1) is equivalent to a standard uniform distribution, the power of the test is equivalent to the size of the test. The power against Beta (1.5,1.5) and Beta(1.0,1.0) for sample sizes of 40, 50, 100 and 150 are presented in Table 3.

Table 3. Power of test against different alternative for $\alpha = 0.05$ and $m = 1$

n	40	50	100	150
Beta(1.5,1.5)	0.08527	0.08509	0.08363	0.09927
Beta(1.0,1.0)	0.05046	0.05131	0.051	0.04934

10. WEIGHTED NEGATIVE CUMULATIVE EXTROPY

Tahmasebi and Toomaj [35] defined negative cumulative extropy and Chaudhary et al. [6] defined general weighted negative cumulative extropy.

Definition 10.1. Let X be a non-negative absolutely continuous random variable with cdf F_X . We define the WNCEX of X by

$$(10.1) \quad \xi^w J(X) = \frac{1}{2} \int_0^\infty w(x)(1 - F_X^2(x))dx.$$

If $w(x) = x^m$, $m > 0$, then we represent $\xi^w J(X)$ as $\xi^m J(X)$.

The Lemma by Fashandi and Ahmadi from [11] serves as a foundational tool for creating various descriptions of symmetric distributions.

Lemma 10.1. (Fashandi and Ahmadi, [11]) *Let X be a continuous random variable with pdf f_X and cdf F_X with support S_X . Then, the identity,*

$$f_X(F_X^{-1}(u)) = f_X(F_X^{-1}(1-u)),$$

for almost all $u \in (0, \frac{1}{2})$ if and only if that there exists a constant k such that $F_X(k-x) + F_X(k+x) = 1$ for all $x \in S_X$.

Let \mathbb{C} denote the class of all continuous pdf f_X , having cdf F_X such that

$$f_X(F_X^{-1}(1-u)) \geq (\leq) f_X(F_X^{-1}(u)),$$

for all $u \in (0, \frac{1}{2})$. It can be observed using Lemma 10.1 that F is symmetric if and only if $f_X(F_X^{-1}(u)) = f_X(F_X^{-1}(1-u))$ for almost all $u \in (0, \frac{1}{2})$. The class \mathbb{C} is non-empty and includes various distributions.

For any random variable X with an absolute continuous distribution function (df) $F(\cdot)$ and the Lebesgue probability density function (pdf) $f(\cdot)$. Let $l_X = \inf\{x \in R : F(x) > 0\}$, $u_X = \sup\{x \in R : F(x) < 1\}$ and $S_X = (l_X, u_X)$, we can define WNCEX and WCPJ with support S_X as

$$(10.2) \quad \xi^{w_1} J(X) = \frac{1}{2} \int_{S_X} w_1(x)(1 - F_X^2(x))dx,$$

and

$$(10.3) \quad \tilde{\xi}^{w_2} J(X) = -\frac{1}{2} \int_{S_X} w_2(x)F_X^2(x)dx,$$

respectively. The following theorem gives a relationship of symmetric distribution with WNCEX and WCPJ.

Theorem 10.1. *Let $w_1(x) = F_X(x)$ in (10.2) and $w_2(x) = \frac{\bar{F}_X(x)(1+\bar{F}_X(x))}{F_X(x)}$ in (10.3), then the following two statements are equivalent for any $F_X \in \mathbb{C}$:*

- (i) *random variable X has a symmetric distribution;*
- (ii) $\xi^{w_1} J(X) = -\tilde{\xi}^{w_2} J(X)$.

Proof From (10.2), the WNCEX can be written as

$$(10.4) \quad \xi^{w_1} J(X) = \frac{1}{2} \int_{S_X} w_1(x)(1 - F_X^2(x))dx = \frac{1}{2} \int_0^1 \frac{u(1-u^2)}{f_X(F_X^{-1}(u))} du$$

and from (10.3) the WCPJ can be written as

$$(10.5) \quad \tilde{\xi}^{w_2} J(X) = -\frac{1}{2} \int_0^1 \frac{u(1-u^2)}{f_X(F_X^{-1}(1-u))} du.$$

When $\xi^{w_1} J(X) = -\tilde{\xi}^{w_2} J(X)$ holds then from (10.4) and (10.5), we have

$$(10.6) \quad \int_0^1 u(1-u^2)\eta(u)du = 0,$$

$$(10.7) \quad \text{where } \eta(u) = \frac{1}{f_X(F_X^{-1}(u))} - \frac{1}{f_X(F_X^{-1}(1-u))}.$$

Then as $\eta(1-u) = -\eta(u)$, we can write (10.6) as

$$(10.8) \quad \int_0^{\frac{1}{2}} \eta(u)u(1-u)(2u-1)du = 0.$$

Note that $u(1-u)(2u-1) < 0$ for all $0 \leq u \leq \frac{1}{2}$. Since by assumption $F_X \in \mathbb{C}$, hence from (10.8) we have $f_X(F_X^{-1}(u)) = f_X(F_X^{-1}(1-u))$, so Lemma 10.1 completes the proof. ■

Example 10.1. Let X have the uniform distribution on (a, b) with pdf $f_X(x) = \frac{1}{b-a}$, $x \in (a, b)$. For $u \in (a, b)$ $F^{-1}(u) = (b-a)u + a$, $F_X^{-1}(1-u) = (b-a)(1-u) + a$, and $f_X(F_X^{-1}(u)) = \frac{1}{b-a}$, $f_X(F_X^{-1}(1-u)) = \frac{1}{b-a}$, Then $f_X(F_X^{-1}(u)) = f_X(F_X^{-1}(1-u))$. The equality is true in \mathbb{C} . It is symmetric.

Example 10.2. let X follow power-law distribution with pdf $f_X(x) = \lambda x^{\lambda-1}$, $\lambda > 1$, $x \in (0, 1)$. Then $F_X^{-1}(u) = u^{\frac{1}{\lambda}}$, $u \in (0, 1)$ and $f_X(F_X^{-1}(u)) = \lambda u^{\frac{\lambda-1}{\lambda}}$, $f_X(F_X^{-1}(1-u)) = \lambda(1-u)^{\frac{\lambda-1}{\lambda}}$, $u \in (0, 1)$.
 If $\lambda > 1$ then $f_X(F_X^{-1}(u)) \leq f_X(F_X^{-1}(1-u))$ for $u \in (0, \frac{1}{2})$.
 If $\lambda < 1$ then $f_X(F_X^{-1}(u)) \leq f_X(F_X^{-1}(1-u))$ for $u \in (0, \frac{1}{2})$.
 If $\lambda = 1$ then $f_X(F_X^{-1}(u)) = f_X(F_X^{-1}(1-u))$ for $u \in (0, \frac{1}{2})$.
 Thus power distribution belongs to class \mathbb{C} and for $u \in (0, \frac{1}{2})$, it is symmetric for $\lambda = 1$.

11. APPLICATION

Assume that X_1, X_2, \dots, X_N is a random sample of size N taken from a population with pdf f_X and cdf F_X . As discussed in section 10, for $w_1(x) = F_X(x)$ in (10.2) and $w_2(x) = \frac{\bar{F}_X(x)(1+\bar{F}_X(x))}{F_X(x)}$ in (10.3), the $\xi^{w_1} J(X)$ and $\tilde{\xi}^{w_2} J(X)$ can be written as

$$\xi^{w_1} J(X) = \frac{1}{2} \int_0^1 \frac{u(1-u^2)}{f_X(F_X^{-1}(u))} du, \text{ and } \tilde{\xi}^{w_2} J(X) = -\frac{1}{2} \int_0^1 \frac{u(1-u)(2-u)}{f_X(F_X^{-1}(u))} du.$$

We note from Theorem 10.1 that $\xi^{w_1}J(X) + \tilde{\xi}^{w_2}J(X) = 0$ if and only if X is symmetric distribution. This motivates us that

$$\Delta = \xi^{w_1}J(X) + \tilde{\xi}^{w_2}J(X)$$

can be used to test whether X is symmetric distribution. We can write,

$$\begin{aligned}\Delta &= \frac{1}{2} \int_0^1 \left[\frac{u(1-u^2) - u(1-u)(2-u)}{f_X(F_X^{-1}(u))} \right] du \\ &= \frac{1}{2} \int_0^1 (-2u^3 + 3u^2 - u) \left[\frac{d}{du} F_X^{-1}(u) \right] du.\end{aligned}$$

Small or large values of Δ can be indicative of asymmetry, prompting the rejection of the null hypothesis of symmetry. Therefore, we introduce a symmetry test relying on the sample estimator of Δ . Drawing inspiration from Vasicek [36] and adopting a methodology akin to Vasicek [36], Park [25], Xiong et al. [37], and Jose and Sathar [15], we formulate an estimator for Δ in the following manner:

$$\hat{\Delta} = -\frac{1}{2N} \sum_{i=1}^N \left[2 \left(\frac{i}{N+1} \right)^3 - 3 \left(\frac{i}{N+1} \right)^2 + \left(\frac{i}{N+1} \right) \right] \frac{X_{i+m:N} - X_{i-m:N}}{2m/N}.$$

Here window size m is a positive integer less than $\frac{N}{2}$ and $X_{r:N}$ denotes r th order statistics from sample X_1, X_2, \dots, X_N . If $i+m > N$ then we consider $X_{i+m:N} = X_{N:N}$ and if $i-m < 1$ then we consider $X_{i-m:N} = X_{1:N}$.

Dataset 1, sourced from Montgomery et al. [22], conforms to a normal distribution, representing a symmetric model. The dataset, consisting of values such as :

15.5, 23.75, 8.0, 17.0, 5.5, 19.0, 24.0, 2.5, 7.5, 11.0, 13.0, 3.75, 25.0, 9.75, 22.0, 18.0, 6.0, 12.5, 2.0, 21.5.

The data set was also utilized by Jose and Sathar [15] and Gupta and Chaudhary [12] in their proposed symmetry test, affirming its normal distribution and symmetry. This fact is verified by our test. The absolute value of the test statistics $\hat{\Delta}$ is 0.03246 with an estimated p-value 0.1054 when window size $m = 2$ and sample size $N = 20$. Remarkably, our symmetry test based on $\hat{\Delta}$ does not lead to the rejection of the null hypothesis. This outcome aligns with the dataset's adherence to a normal distribution, establishing it as a suitable model for the data.

Dataset 2, sourced from Qiu and Jia [28], represents active repair times (in hours) for an airborne communication transceiver. The dataset comprises values such as:

0.2, 0.3, 0.5, 0.5, 0.5, 0.5, 0.6, 0.6, 0.7, 0.7, 0.7, 0.8, 0.8, 1.0, 1.0, 1.0, 1.0, 1.1, 1.3, 1.5, 1.5, 1.5, 1.5, 2.0, 2.0, 2.2, 2.5, 3.0, 3.0, 3.3, 3.3, 4.0, 4.0, 4.5, 4.7, 5.0, 5.4,

5.4, 7.0, 7.5, 8.8, 9.0, 10.3, 22.0, 24.5.

and has been employed by Xiong et al. [37] in their proposed test of symmetry. As per Qiu and Jia [28], Dataset 2 can be adequately modeled by the inverse Gaussian (IG) distribution. It's worth noting that the IG distribution is inherently asymmetric, as highlighted by Xiong et al. [37] and Qiu and Jia [28]. Our conducted test affirms this asymmetry, revealing an absolute value of the test statistic $\hat{\Delta}$ as 0.5679, along with an estimated p-value of 0.0. These computations were conducted under the conditions of a window size $m = 2$ and a sample size N of 45. Remarkably, our symmetry test based on $\hat{\Delta}$ successfully rejects the null hypothesis. This rejection aligns with the inherent asymmetry of the dataset, as indicated by its fit to the inverse Gaussian distribution.

Now consider the dataset 3 such as :

0.014, 0.034, 0.059, 0.061, 0.069, 0.080, 0.123, 0.142, 0.165, 0.210, 0.381, 0.464, 0.479, 0.556, 0.574, 0.839, 0.917, 0.969, 0.991, 1.064, 1.088, 1.091, 1.174, 1.270, 1.275, 1.355, 1.397, 1.477, 1.578, 1.649, 1.702, 1.893, 1.932, 2.001, 2.161, 2.292, 2.326, 2.337, 2.628, 2.785, 2.811, 2.886, 2.993, 3.122, 3.248, 3.715, 3.790, 3.857, 3.912, 4.100.

from Lawless [20] which represents the quantity of 1000 cycles to failure for electrical appliances in a life test. Xiong et al. [37] used this dataset for their proposed test of symmetry. The absolute value of the test statistics $\hat{\Delta}$ is 0.0486 with an estimated p -value 0.0034 when window size $m = 2$ and sample size $N = 50$. Our test based on $\hat{\Delta}$ succeeds in rejecting the null hypothesis that the dataset does not follow symmetry.

See Table 4 for the value of test statistics and p-value for different datasets based on the specific window size and sample size of each dataset.

Table 4. Description of models fitted

Dataset	N	m	$ \hat{\Delta} $	p-value
Dataset 1	20	2	0.0343	0.1054
Dataset 2	45	2	0.5679	0.0
Dataset 3	50	2	0.0486	0.0034

When conducting tests at a 5% significance level, a p-value below 0.05 signals an asymmetric nature, while a p-value above 0.05 suggests symmetry in the data. Table 4 confirms that the newly proposed test effectively discerns the symmetry or asymmetry of the distribution within random samples. The observed p-values, at the 5% significance level, indicate that Datasets 2 and 3 exhibit asymmetries in the distribution of their random samples. Conversely, Dataset 1 presents moderate p-values, implying acceptance of symmetry in their distributions. Consequently, we can affirm that the test statistic accurately identifies the presence or absence of symmetry in the distribution of the random variable.

12. CONCLUSION

The introduction section of the paper highlights the various studies and generalizations of the concept of extropy. The paper then proceeds to define and analyze GWCPJ, and conditional WCPJ. To estimate GWCPJ, non-parametric estimators and empirical distribution functions have been proposed. Additionally, upper and lower bounds for WCPJ have been derived, and several results pertaining to WCPJ have been presented .

Furthermore, a new test of uniformity has also been proposed based on GWCPJ, which can be used to determine whether a given dataset follows a uniform distribution. Also we have studied the relationship of symmetric distribution with weighted negative cumulative extropy (WNCEX) and weighted cumulative past extropy (WCPJ) and formulated an estimator for symmetry test of data set.

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REFERENCES

- [1] Asadi, M. and Zohrevand, Y. (2007). On the dynamic cumulative residual entropy. *Journal of Statistical Planning and Inference* 137 (6): 1931–41.
- [2] Ash, R. (1990). *Information Theory*. Dover Publications Inc, New York.
- [3] Bagnoli, M. and Bergstrom, T. (2005). Log-concave probability and its applications. *Economic Theory*, 26, 455-469.
- [4] Balakrishnan, N., Buono, F. and Longobardi, M. (2020). On weighted extropies, *Communications in Statistics-Theory and Methods* 1–31.
- [5] Bansal, S. and Gupta, N. (2021). Weighted extropies and past extropy of order statistics and k-record values. *Communications in Statistics-Theory and Methods* 1-24.
- [6] Chaudhary, S. K., Gupta, N. and Sahu, P. K. (2023) On general weighted cumulative residual extropy and general weighted negative cumulative extropy, *Statistics*, 57:5, 1117-1141.

- [7] Cover T.M. and Thomas, J.A. (2006). Elements of information theory. 2nd ed. Hoboken (NJ): Wiley.
- [8] Di Crescenzo A. and Longobardi, M. (2009). On cumulative entropies. *J Statist Plann Inference*. 139: 4072–4087.
- [9] Di Crescenzo, A. and Longobardi, M. (2013). Stochastic comparisons of cumulative entropies. In *Stochastic orders in reliability and risk*, eds. H. Li and X. Li, *Lecture Notes in Statistics* 208, 167–82. New York: Springer.
- [10] Drissi, N., Chonavel, T. and Boucher, J.M. (2008). Generalized cumulative residual entropy for distributions with unrestricted supports. *J Electr Comput Eng* ; Article ID 790607:5.
- [11] Fashandi, M. and Ahmadi, J. (2012) Characterizations of symmetric distributions based on Rényi entropy. *Stat Probab Lett* 82(4): 798–804 .
- [12] Gupta, N., and Chaudhary, S.K. (2023). Some characterizations of continuous symmetric distributions based on extropy of record values. *Stat Papers* <https://doi.org/10.1007/s00362-022-01392-y>.
- [13] Hashempour, M., Kazemi, M. R. and Tahmasebi, S. (2022). On weighted cumulative residual extropy: characterization, estimation and testing, *Statistics*, 56:3, 681-698.
- [14] Jahanshahi, S. M. A., Zarei, H. and Khammar, A. H. (2020). On cumulative residual extropy. *Probability in the Engineering and Informational Sciences* 34 (4): 605–21.
- [15] Jose, J. and Sathar, E.I.A. (2022). Symmetry being tested through simultaneous application of upper and lower k-records in extropy, *Journal of Statistical Computation and Simulation*, 92 (4): 830-846.
- [16] Khorashadizadeh, M., Rezaei Roknabadi, A. H. and Mohtashami Borzadaran, G. R. (2013). Doubly truncated (interval) cumulative residual and past entropy. *Statistics and Probability Letters* 83 (5): 1464–71.
- [17] Krishnan, A. S., Sunoj, S. M. and Nair, N. U. (2020). Some reliability properties of extropy for residual and past lifetime random variables. *Journal of the Korean Statistical Society* 49 (2): 457–74.
- [18] Kundu, C. (2021). On cumulative residual (past) extropy of extreme order statistics, *Communications in Statistics - Theory and Methods*, <https://doi.org/10.1080/03610926.2021.2021238>.
- [19] Lad, F., Sanfilippo, G. and Agro, G. (2015). Extropy: Complementary dual of entropy. *Statistical Science* 30 (1): 40–58.
- [20] Lawless, J.F. (2011). *Statistical Models and Methods for Lifetime Data*, vol. 362, Wiley, Hoboken.
- [21] Mohammadi, M. and Hashempour, M. (2022). On interval weighted cumulative residual and past extropies, *Statistics*.
- [22] Montgomery, D.C., Peck, E.A. and G.G. Vining. (2021). *Introduction to Linear Regression Analysis*, 6th Edition. New York: John Wiley and Sons.
- [23] Navarro, J., del Aguila, Y. and Asadi M. (2010). Some new results on the cumulative residual entropy. *Journal of Statistical Planning and Inference* 140 (1): 310–22.

- [24] Noughabi, H. A. and Jarrahiferiz, J. (2019). On the estimation of extropy. *Journal of Nonparametric Statistics* 31 (1): 88–99.
- [25] Park, S. (1999). A goodness-of-fit test for normality based on the sample entropy of order statistics. *Stat Prob Lett* 44(4): 359–363 .
- [26] Qiu, G. (2017). The extropy of order statistics and record values. *Statistics and Probability Letters* 120: 52–60.
- [27] Qiu, G. and Jia, K. (2018). The residual extropy of order statistics. *Statistics and Probability Letters* 133: 15–22.
- [28] Qiu, G. and Jia, K. (2018). Extropy estimators with applications in testing uniformity, *Journal of Nonparametric Statistics*, 30 (1): 182–96.
- [29] Qiu, G., Wang, L. and Wang, X. (2019). On extropy properties of mixed systems. *Probability in the Engineering and Informational Sciences* 33 (3): 471–86.
- [30] Rao, M., Chen, Y. Vemuri, B. C. and Wang, Fei. (2004). Cumulative residual entropy: a new measure of information. *IEEE Transactions on Information Theory*. 50(6): 1220-1228.
- [31] Rao M. (2005). More on a new concept of entropy and information. *J Theor Probab.* 18: 967–981.
- [32] Raqab, M. Z. and Qiu, G. (2019). On extropy properties of ranked set sampling. *Statistics* 53 (1): 210–26.
- [33] Shaked M. and Shanthikumar, J.G. (2007). *Stochastic orders*. New York (NY): Springer.
- [34] Shannon, C. E. (1948). A mathematical theory of communication. *Bell System Technical Journal* 27 (3): 379–423.
- [35] Tahmasebi, S. and Toomaj, A. (2022). On negative cumulative extropy with applications. *Communications in Statistics-Theory and Methods*, 51(15), 5025-5047.
- [36] Vasicek, O. (1976). A test for normality based on sample entropy, *J R Stat Soc Ser B Methodol*, 38: 54–59.
- [37] Xiong P., Zhuang, W. and Qiu, G. (2021). Testing symmetry based on the extropy of record values, *Journal of Nonparametric Statistics*, 33 (1): 134-155.
- [38] Yang J., Xia, W. and Hu, T. (2018). Bounds on extropy with variational distance constraint. *Probab Eng Inf Sci.* 33: 1–19.