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The Gauss Hypergeometric Beta Distribution

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Abstract:

- The Gauss hypergeometric beta distribution, introduced by Gordy (1998), has been recognized since the 1990s, yet its mathematical properties remain relatively unexplored. This paper aims to address this gap by presenting a comprehensive analysis of the mathematical properties of the Gauss hypergeometric beta distribution. We derive the shape properties of its probability density function, along with expressions for its cumulative distribution function, moment generating function, characteristic function, moments, conditional moments, entropies, and stochastic orderings. Additionally, we develop procedures for maximum likelihood estimation and evaluate their performance in finite samples. The majority of the derived properties are novel contributions. Finally, we demonstrate a real-world application of the Gauss hypergeometric beta distribution.

Keywords:

- *Gauss hypergeometric function; Maximum likelihood estimation; Moments*

AMS Subject Classification:

- Primary 62E15.

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1. INTRODUCTION

Armero and Bayarri (1994) and Gordy (1998) independently introduced the Gauss hypergeometric beta distribution given by the probability density function

$$f_X(x) = \frac{K x^{a-1}(1-x)^{b-1}}{(1+dx)^c} \tag{1.1}$$

for $0 < x < 1$, $a > 0$, $b > 0$, $-\infty < c < \infty$ and $d > -1$, where

$$\frac{1}{K} = B(a,b)\ {}_2F_1\left(c, a; a+b; -d\right),$$

where $B(a,b)$ and ${}_2F_1\left(a, b; c; x\right)$ are the beta and Gauss hypergeometric functions defined by

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt$$

and

$${}_2F_1\left(a, b; c; x\right) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!},$$

respectively, where $(a)_k = a(a+1)\cdots(a+k-1)$ is the ascending factorial. The beta function can also be expressed as $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$, where $\Gamma(a)$ denotes the gamma function defined by

$$\Gamma(a) = \int_0^{\infty} t^{a-1} \exp(-t) dt.$$

The standard beta distribution is the particular case of (1.1) for $c = 0$ or $d = 0$. Libby and Novick (1982)'s beta distribution is the particular case of (1.1) for $c = a + b$. If X is a Gauss hypergeometric beta random variable with parameters (a, b, c, d) then $1 - X$ is also a Gauss hypergeometric beta random variable with parameters $(b, a, c, -d/(1+d))$. Hence, X is symmetric if and only if $a = b$ and $d = 0$.

There have been only a few papers studying the Gauss hypergeometric beta distribution. Nadarajah (2008) proposed a bivariate extension of (1.1). Nagar and Valencia (2011) derived the distributions of the product and quotient of two independent Gauss hypergeometric beta random variables. Nagar and Valencia (2011) also derived expressions for the cumulative distribution function, moments, Shannon entropy (Shannon, 1948a,b) and Rényi entropy (Rényi, 1960) corresponding to (1.1). Gupta and Nagar (2012) proposed a matrix variate extension of (1.1). Nagar et al. (2015) proposed a multivariate extension of (1.1). The main applications of the Gauss hypergeometric beta distribution have been to queue inference.

The objective of this paper is to offer a comprehensive examination of the mathematical properties of (1.1). The derived properties encompass various aspects, including the shape properties of the probability density function (Section 2), cumulative distribution function (Section 3), moment generating and characteristic functions (Section 4), moments (Section 5), conditional moments (Section 6), entropies (Section 7), and stochastic orderings (Section 8). Additionally, we present procedures for maximum likelihood estimation (Section 9) and evaluate their performance in finite samples (Section 10). Furthermore, we provide a real-world application of the Gauss hypergeometric beta distribution in Section 11. Finally, Section 12 outlines some conclusions drawn from the study and suggests directions for future research.

Apart from the specified special functions, the calculations in this paper entail the use of the degenerate hypergeometric series of two variables and the degenerate hypergeometric function of two variables, defined as

$$\Phi_1\left(a, b, c, x, y\right) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_n x^m y^n}{(c)_{m+n} m! n!}$$

and

$$F_1\left(a, b, c; d; x, y\right) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_m(c)_n x^m y^n}{(d)_{m+n} m! n!},$$

respectively. The properties of these special functions are documented in Prudnikov et al. (1986) and Gradshteyn and Ryzhik (2000).

2. SHAPE

The critical points of $f_X(x)$ are the roots of

$$\frac{d \log f_X(x)}{dx} = \frac{a-1}{x} - \frac{b-1}{1-x} - \frac{cd}{1+dx} = 0$$

or equivalently the roots of

$$A + Bx + Cx^2 = 0, \tag{2.1}$$

where $A = a - 1$, $B = ad - cd - a - b - d + 2$ and $C = cd - ad - bd + 2d$. Let $\delta_1 = \left(-B - \sqrt{B^2 - 4AC}\right)/(2C)$ and $\delta_2 = \left(-B + \sqrt{B^2 - 4AC}\right)/(2C)$ denote the roots of (2.1). If $c = 0$ or $d = 0$ then $A + Bx + Cx^2$ factorizes as $\left[a - 1 - (a + b - 2)x\right](1 + dx)$, so the only root is $\frac{a-1}{a+b-2}$.

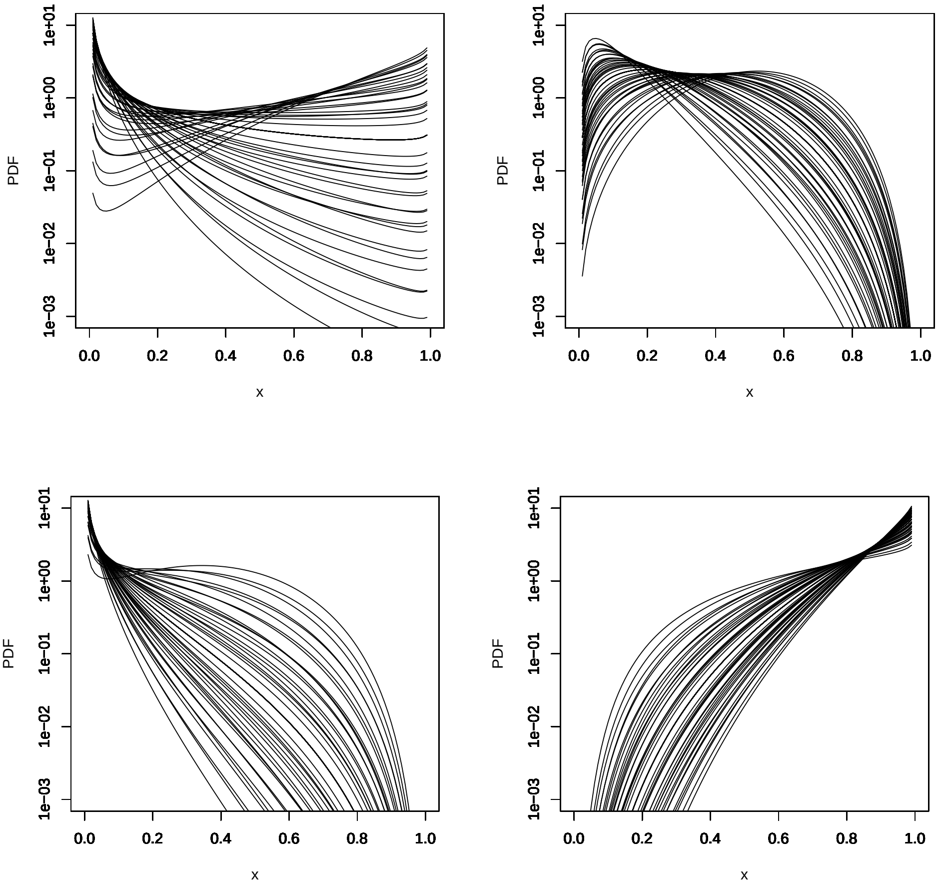

Figure 1: Shapes of (1.1) for $a = 0.2$, $b = 0.9$, $c = -5, -4, \ldots, 5$ and $d = 0, 1, \ldots, 4$ (top left); $a = 2$, $b = 5$, $c = -5, -4, \ldots, 5$ and $d = 0, 1, \ldots, 4$ (top right); $a = 0.2$, $b = 5$, $c = -5, -4, \ldots, 5$ and $d = 0, 1, \ldots, 4$ (bottom left); $a = 6$, $b = 0.9$, $c = -5, -4, \ldots, 5$ and $d = 0, 1, \ldots, 4$ (bottom right).

The probability density function can exhibit the following shapes

a) if $0 \leq \delta_1 < \delta_2 \leq 1$,

$$-\frac{a-1}{\delta_1^2} - \frac{b-1}{(1-\delta_1)^2} + \frac{cd^2}{(1+d\delta_1)^2} < 0$$

and

$$-\frac{a-1}{\delta_2^2} - \frac{b-1}{(1-\delta_2)^2} + \frac{cd^2}{(1+d\delta_2)^2} < 0$$

then $f(x)$ has one mode followed by another mode.

b) if $0 \leq \delta_1 < \delta_2 \leq 1$,

$$-\frac{a-1}{\delta_1^2} - \frac{b-1}{(1-\delta_1)^2} + \frac{cd^2}{(1+d\delta_1)^2} > 0$$

and

$$-\frac{a-1}{\delta_2^2}-\frac{b-1}{(1-\delta_2)^2}+\frac{cd^2}{(1+d\delta_2)^2}>0$$

then $f(x)$ has one anti mode followed by another anti mode.

c) if $0\le\delta_1<\delta_2\le 1$,

$$-\frac{a-1}{\delta_1^2}-\frac{b-1}{(1-\delta_1)^2}+\frac{cd^2}{(1+d\delta_1)^2}<0$$

and

$$-\frac{a-1}{\delta_2^2}-\frac{b-1}{(1-\delta_2)^2}+\frac{cd^2}{(1+d\delta_2)^2}>0$$

then $f(x)$ has one mode followed by an anti mode.

d) if $0\le\delta_1<\delta_2\le 1$,

$$-\frac{a-1}{\delta_1^2}-\frac{b-1}{(1-\delta_1)^2}+\frac{cd^2}{(1+d\delta_1)^2}>0$$

and

$$-\frac{a-1}{\delta_2^2}-\frac{b-1}{(1-\delta_2)^2}+\frac{cd^2}{(1+d\delta_2)^2}<0$$

then $f(x)$ has an anti mode followed by a mode.

e) if $\delta_1<0\le\delta_2\le 1$ and

$$-\frac{a-1}{\delta_2^2}-\frac{b-1}{(1-\delta_2)^2}+\frac{cd^2}{(1+d\delta_2)^2}<0$$

then $f(x)$ has a single a mode.

f) if $\delta_1<0\le\delta_2\le 1$ and

$$-\frac{a-1}{\delta_2^2}-\frac{b-1}{(1-\delta_2)^2}+\frac{cd^2}{(1+d\delta_2)^2}>0$$

then $f(x)$ has a single anti mode.

g) if $0\le\delta_1\le 1<\delta_2$ and

$$-\frac{a-1}{\delta_1^2}-\frac{b-1}{(1-\delta_1)^2}+\frac{cd^2}{(1+d\delta_1)^2}<0$$

then $f(x)$ has a single a mode.

h) if $0\le\delta_1\le 1<\delta_2$ and

$$-\frac{a-1}{\delta_1^2}-\frac{b-1}{(1-\delta_1)^2}+\frac{cd^2}{(1+d\delta_1)^2}>0$$

then $f(x)$ has a single anti mode.

i) if

$$\frac{a-1}{x} - \frac{b-1}{1-x} - \frac{cd}{1+dx} < 0$$

for all $0 \leq x \leq 1$ then $f(x)$ is monotonically decreasing.

j) if

$$\frac{a-1}{x} - \frac{b-1}{1-x} - \frac{cd}{1+dx} > 0$$

for all $0 \leq x \leq 1$ then $f(x)$ is monotonically increasing.

Figure 1 illustrates various possible shapes of $f(x)$. Certain shapes depicted, such as the plot in the bottom left of Figure 1, cannot be represented by a standard beta distribution.

3. CUMULATIVE DISTRIBUTION FUNCTION

By equation (2.2.8.5) in volume 1 of Prudnikov et al. (1986), the cumulative distribution function of X is

$$F_X(x) = K \int_0^x \frac{y^{a-1}(1-y)^{b-1}}{(1+dy)^c} dy = \frac{Kx^a}{a} F_1 \left(a, c, 1-b, a+1; -dx, x\right).$$

The quantile function defined by $F_X\left(Q(p)\right) = p$ is the root of

$$\frac{K\left[Q(p)\right]^a}{a} F_1 \left(a, c, 1-b, a+1; -dQ(p), Q(p)\right) = p.$$

In particular, the median of X, $Q(1/2)$, is the root of $\frac{K[Q(1/2)]^a}{a} F_1 \left(a, c, 1-b, a+1; -dQ(1/2), Q(1/2)\right) = 1/2$. The expression for the cumulative distribution function has been previously given in Nagar and Valencia (2011).

4. MOMENT GENERATING AND CHARACTERISTIC FUNCTIONS

By equation (2.3.8.1) in volume 1 of Prudnikov et al. (1986), the moment generating function of X is

$$M_X(t) = E\left[\exp(tX)\right] = K \int_0^1 \frac{\exp(tx)x^{a-1}(1-x)^{b-1}}{(1+dx)^c} dx = KB(a,b)\Phi_1 \left(a, -c, a+b; -d, -t\right).$$

The corresponding characteristic function is

$$\phi_X(t) = E\left[\exp(\mathrm{i}tX)\right] = KB(a,b)\Phi_1 \left(a, -c, a+b; -d, -\mathrm{i}t\right),$$

where $\mathrm{i} = \sqrt{-1}$.

5. MOMENTS

The nth moment of X is

$$E\left(X^n\right)=K\int_0^1\frac{x^{n+a-1}(1-x)^{b-1}}{(1+dx)^c}dx=\frac{\Gamma(a+b)}{\Gamma(a)}\frac{\Gamma(n+a)}{\Gamma(n+a+b)}\frac{{}_2F_1\left(c,n+a;n+a+b;-d\right)}{{}_2F_1\left(c,a;a+b;-d\right)}.$$

In particular, the first four moments are

$$E\left(X\right)=\frac{a}{a+b}\frac{{}_2F_1\left(c,1+a;1+a+b;-d\right)}{{}_2F_1\left(c,a;a+b;-d\right)},$$

$$E\left(X^2\right)=\frac{a(a+1)}{(a+b)(a+b+1)}\frac{{}_2F_1\left(c,2+a;2+a+b;-d\right)}{{}_2F_1\left(c,a;a+b;-d\right)},$$

$$E\left(X^3\right)=\frac{a(a+1)(a+2)}{(a+b)(a+b+1)(a+b+2)}\frac{{}_2F_1\left(c,3+a;3+a+b;-d\right)}{{}_2F_1\left(c,a;a+b;-d\right)}$$

and

$$E\left(X^4\right)=\frac{a(a+1)(a+2)(a+3)}{(a+b)(a+b+1)(a+b+2)(a+b+3)}\frac{{}_2F_1\left(c,4+a;4+a+b;-d\right)}{{}_2F_1\left(c,a;a+b;-d\right)}.$$

The harmonic means of X are

$$\begin{aligned}\left[E\left(\frac{1}{X}\right)\right]^{-1} &= \left[K\int_0^1\frac{x^{a-2}(1-x)^{b-1}}{(1+dx)^c}dx\right]^{-1}\\ &= \frac{B(a,b)}{B(a-1,b)}\frac{{}_2F_1\left(c,a;a+b;-d\right)}{{}_2F_1\left(c,a-1;a+b-1;-d\right)}\\ &= \frac{a}{a+b-1}\frac{{}_2F_1\left(c,a;a+b;-d\right)}{{}_2F_1\left(c,a-1;a+b-1;-d\right)}\end{aligned}$$

and

$$\begin{aligned}\left[E\left(\frac{1}{1-X}\right)\right]^{-1} &= \left[K\int_0^1\frac{x^{a-1}(1-x)^{b-2}}{(1+dx)^c}dx\right]^{-1}\\ &= \frac{B(a,b)}{B(a,b-1)}\frac{{}_2F_1\left(c,a;a+b;-d\right)}{{}_2F_1\left(c,a;a+b-1;-d\right)}\\ &= \frac{b}{a+b-1}\frac{{}_2F_1\left(c,a;a+b;-d\right)}{{}_2F_1\left(c,a;a+b-1;-d\right)}.\end{aligned}$$

Further,

$$\begin{aligned}\mathrm{Var}\left(\frac{1}{X}\right) &= K\int_0^1\frac{x^{a-3}(1-x)^{b-1}}{(1+dx)^c}dx-\frac{(a+b-1)^2}{a^2}\left[\frac{{}_2F_1\left(c,a-1;a+b-1;-d\right)}{{}_2F_1\left(c,a;a+b;-d\right)}\right]^2\\ &= \frac{B(a-2,b)}{B(a,b)}\frac{{}_2F_1\left(c,a-2;a+b-2;-d\right)}{{}_2F_1\left(c,a;a+b;-d\right)}\\ &\qquad -\frac{(a+b-1)^2}{a^2}\left[\frac{{}_2F_1\left(c,a-1;a+b-1;-d\right)}{{}_2F_1\left(c,a;a+b;-d\right)}\right]^2\\ &= \frac{(a+b-1)(a+b-2)}{a(a-1)}\frac{{}_2F_1\left(c,a-2;a+b-2;-d\right)}{{}_2F_1\left(c,a;a+b;-d\right)}\\ &\qquad -\frac{(a+b-1)^2}{a^2}\left[\frac{{}_2F_1\left(c,a-1;a+b-1;-d\right)}{{}_2F_1\left(c,a;a+b;-d\right)}\right]^2\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Var}\left(\frac{1}{1-X}\right) &= K\int_0^1 \frac{x^{a-1}(1-x)^{b-3}}{(1+dx)^c}dx - \frac{(a+b-1)^2}{b^2}\left[\frac{{}_2F_1\left(c,a;a+b-1;-d\right)}{{}_2F_1\left(c,a;a+b;-d\right)}\right]^2 \\
&= \frac{B(a,b-2)}{B(a,b)}\frac{{}_2F_1\left(c,a;a+b-2;-d\right)}{{}_2F_1\left(c,a;a+b;-d\right)} \\
&\qquad -\frac{(a+b-1)^2}{b^2}\left[\frac{{}_2F_1\left(c,a;a+b-1;-d\right)}{{}_2F_1\left(c,a;a+b;-d\right)}\right]^2 \\
&= \frac{(a+b-1)(a+b-2)}{b(b-1)}\frac{{}_2F_1\left(c,a;a+b-2;-d\right)}{{}_2F_1\left(c,a;a+b;-d\right)} \\
&\qquad -\frac{(a+b-1)^2}{b^2}\left[\frac{{}_2F_1\left(c,a;a+b-1;-d\right)}{{}_2F_1\left(c,a;a+b;-d\right)}\right]^2.
\end{aligned}$$

The expressions for the nth moment have been previously provided in Armero and Bayarri (1994) and Nagar and Valencia (2011).

6. CONDITIONAL MOMENTS

By equation (2.2.8.5) in volume 1 of Prudnikov et al. (1986), the nth conditional moment of X is

$$\begin{aligned}
E\left(X^n \mid X>x\right) &= \frac{1}{1-F_X(x)}\left[E\left(X^n\right) - K\int_0^x \frac{y^{n+a-1}(1-y)^{b-1}}{(1+dy)^c}dy\right] \\
&= \frac{1}{1-F_X(x)}\left[\frac{\Gamma(a+b)}{\Gamma(a)}\frac{\Gamma(n+a)}{\Gamma(n+a+b)}\frac{{}_2F_1\left(c,n+a;n+a+b;-d\right)}{{}_2F_1\left(c,a;a+b;-d\right)}\right. \\
&\qquad \left. -\frac{Kx^{a+n}}{a+n}F_1\left(a+n,c,1-b,a+n+1;-dx,x\right)\right].
\end{aligned}$$

In particular, the first four conditional moments are

$$\begin{aligned}
E\left(X \mid X>x\right) = \frac{1}{1-F_X(x)}&\left[\frac{a}{a+b}\frac{{}_2F_1\left(c,1+a;1+a+b;-d\right)}{{}_2F_1\left(c,a;a+b;-d\right)}\right. \\
&\left. -\frac{Kx^{a+1}}{a+1}F_1\left(a+1,c,1-b,a+2;-dx,x\right)\right],
\end{aligned}$$

$$\begin{aligned}
E\left(X^2 \mid X>x\right) = \frac{1}{1-F_X(x)}&\left[\frac{a(a+1)}{(a+b)(a+b+1)}\frac{{}_2F_1\left(c,2+a;2+a+b;-d\right)}{{}_2F_1\left(c,a;a+b;-d\right)}\right. \\
&\left. -\frac{Kx^{a+2}}{a+2}F_1\left(a+2,c,1-b,a+3;-dx,x\right)\right],
\end{aligned}$$

$$\begin{aligned}
E\left(X^3 \mid X>x\right) = \frac{1}{1-F_X(x)}&\left[\frac{a(a+1)(a+2)}{(a+b)(a+b+1)(a+b+2)}\frac{{}_2F_1\left(c,3+a;3+a+b;-d\right)}{{}_2F_1\left(c,a;a+b;-d\right)}\right. \\
&\left. -\frac{Kx^{a+3}}{a+3}F_1\left(a+3,c,1-b,a+4;-dx,x\right)\right]
\end{aligned}$$

and

$$E\left(X^4 \mid X > x\right) = \frac{1}{1 - F_X(x)} \left[\frac{a(a+1)(a+2)(a+3)}{(a+b)(a+b+1)(a+b+2)(a+b+3)} \frac{{}_2F_1\left(c, 4+a; 4+a+b; -d\right)}{{}_2F_1\left(c, a; a+b; -d\right)} - \frac{Kx^{a+4}}{a+4} F_1\left(a+4, c, 1-b, a+5; -dx, x\right) \right].$$

Note that $E(X \mid X > x) - x$ and $E\left(X^2 \mid X > x\right) - \left[E(X \mid X > x)\right]^2$ are the mean residual life and the variance residual functions, respectively.

7. ENTROPIES

Three of the most widely recognized entropies are the geometric mean (Feng et al., 2013; Vogel, 2022), Shannon entropy (Shannon, 1948a,b), and Rényi entropy (Rényi, 1960), as defined by

$$GM(X) = \exp\left[\int \log x f_X(x) dx\right], \tag{7.1}$$

$$S(X) = -\int \log f_X(x) f_X(x) dx \tag{7.2}$$

and

$$R(X) = \frac{1}{1-\alpha} \log\left\{\int \left[f_X(x)\right]^\alpha dx\right\}, \tag{7.3}$$

respectively, for $\alpha \geq 0$ and $\alpha \neq 1$.

For X having the Gauss hypergeometric beta distribution,

$$\begin{aligned}
\int_0^1 \log x f_X(x) dx &= \int_0^1 \left.\frac{dx^\alpha}{d\alpha}\right|_{\alpha=0} f_X(x) dx \\
&= \frac{\partial}{\partial \alpha} \int_0^1 \left.\frac{Kx^{\alpha+a-1}(1-x)^{b-1}}{(1+dx)^c} dx\right|_{\alpha=0} \\
&= \frac{\partial}{\partial \alpha} \frac{\Gamma(a+b)}{\Gamma(a)} \frac{\Gamma(\alpha+a)}{\Gamma(\alpha+a+b)} \left.\frac{{}_2F_1\left(c, \alpha+a; \alpha+a+b; -d\right)}{{}_2F_1\left(c, a; a+b; -d\right)}\right|_{\alpha=0} \\
&= \frac{\Gamma'(a)}{\Gamma(a)} + \frac{1}{{}_2F_1\left(c, a; a+b; -d\right)} \frac{\partial}{\partial \alpha} \left. {}_2F_1\left(c, \alpha+a; \alpha+a+b; -d\right)\right|_{\alpha=0} \\
&\quad - \frac{\Gamma'(a+b)}{\Gamma(a+b)},
\end{aligned} \tag{7.4}$$

$$
\begin{aligned}
\int_0^1 \log(1-x) f_X(x) dx &= \int_0^1 \left. \frac{d(1-x)^\alpha}{d\alpha} \right|_{\alpha=0} f_X(x) dx \\
&= \frac{\partial}{\partial \alpha} \int_0^1 \left. \frac{K x^{a-1}(1-x)^{\alpha+b-1}}{(1+dx)^c} dx \right|_{\alpha=0} \\
&= \left. \frac{\partial}{\partial \alpha} \frac{\Gamma(a+b)}{\Gamma(b)} \frac{\Gamma(\alpha+b)}{\Gamma(\alpha+a+b)} \frac{{}_2F_1(c, a; \alpha+a+b; -d)}{{}_2F_1(c, a; a+b; -d)} \right|_{\alpha=0} \\
&= \frac{\Gamma'(b)}{\Gamma(b)} + \frac{1}{{}_2F_1(c, a; a+b; -d)} \frac{\partial}{\partial \alpha} \left. {}_2F_1(c, a; \alpha+a+b; -d) \right|_{\alpha=0} \\
&\quad - \frac{\Gamma'(a+b)}{\Gamma(a+b)},
\end{aligned} \tag{7.5}
$$

$$
\begin{aligned}
\int_0^1 \log(1+dx) f_X(x) dx &= \int_0^1 \left. \frac{d(1+dx)^\alpha}{d\alpha} \right|_{\alpha=0} f_X(x) dx \\
&= \frac{\partial}{\partial \alpha} \int_0^1 \left. \frac{K x^{a-1}(1-x)^{\alpha+b-1}}{(1+dx)^{c-\alpha}} dx \right|_{\alpha=0} \\
&= \frac{1}{{}_2F_1(c, a; a+b; -d)} \frac{\partial}{\partial \alpha} {}_2F_1(c-\alpha, a; a+b; -d)
\end{aligned} \tag{7.6}
$$

and

$$
\begin{aligned}
\int \left[f_X(x) \right]^\alpha dx &= \int_0^1 \frac{K^\alpha x^{a\alpha-\alpha}(1-x)^{b\alpha-\alpha}}{(1+dx)^c \alpha} dx \\
&= \frac{B(a\alpha-\alpha+1, b\alpha-\alpha+1)\ {}_2F_1(c\alpha, a\alpha-\alpha+1; a\alpha+b\alpha-2\alpha+2; -d)}{[B(a,b)]^\alpha \left[{}_2F_1(c, a; a+b; -d) \right]^\alpha}.
\end{aligned}
$$

Hence, (7.1), (7.2) and (7.3) can be expressed as

$$
GM(X) = \exp \left[\frac{\Gamma'(a)}{\Gamma(a)} + \frac{1}{{}_2F_1(c, a; a+b; -d)} \frac{\partial}{\partial \alpha} \left. {}_2F_1(c, \alpha+a; \alpha+a+b; -d) \right|_{\alpha=0} - \frac{\Gamma'(a+b)}{\Gamma(a+b)} \right],
$$

$$
\begin{aligned}
S(X) = & \frac{(1-a)\Gamma'(a)}{\Gamma(a)} + \frac{1-a}{{}_2F_1(c, a; a+b; -d)} \frac{\partial}{\partial \alpha} \left. {}_2F_1(c, \alpha+a; \alpha+a+b; -d) \right|_{\alpha=0} \\
& - \frac{(2-a-b)\Gamma'(a+b)}{\Gamma(a+b)} + \frac{(1-b)\Gamma'(b)}{\Gamma(b)} \\
& + \frac{1-b}{{}_2F_1(c, a; a+b; -d)} \frac{\partial}{\partial \alpha} \left. {}_2F_1(c, a; \alpha+a+b; -d) \right|_{\alpha=0} \\
& + \frac{c}{{}_2F_1(c, a; a+b; -d)} \frac{\partial}{\partial \alpha} {}_2F_1(c-\alpha, a; a+b; -d) - \log B(a,b) \\
& - \log\ {}_2F_1(c-\alpha, a; a+b; -d)
\end{aligned}
$$

and

$$
\begin{aligned}
R(X) = & \frac{1}{1-\alpha} \log B(a\alpha-\alpha+1, b\alpha-\alpha+1) \\
& + \frac{1}{1-\alpha} \log\ {}_2F_1(c\alpha, a\alpha-\alpha+1; a\alpha+b\alpha-2\alpha+2; -d) \\
& - \frac{\alpha}{1-\alpha} \log B(a,b) - \frac{\alpha}{1-\alpha} \log\ {}_2F_1(c, a; a+b; -d),
\end{aligned}
$$

respectively.

Another popular entropy is the relative entropy (Kullback and Leibler, 1951). If X_1 and X_2 are Gauss hypergeometric beta random variables with parameters (a_1, b_1, c_1, d_1) and (a_2, b_2, c_2, d_2), respectively, the relative entropy is defined by

$$\begin{aligned} D(X_1, X_2) &= \int_0^1 f_{X_1}(x) \frac{f_{X_1}(x)}{f_{X_2}(x)} dx \\ &= \log \frac{B(a_2, b_2)\ {}_2F_1(c_2, a_2; a_2 + b_2; -d_2)}{B(a_1, b_1)\ {}_2F_1(c_1, a_1; a_1 + b_1; -d_1)} + (a_1 - a_2) \int_0^1 \log x f_{X_1}(x) dx \\ &\qquad + (b_1 - b_2) \int_0^1 \log(1 - x) f_{X_1}(x) dx + c_2 \int_0^1 \log(1 + d_2 x) f_{X_1}(x) dx \\ &\qquad - c_1 \int_0^1 \log(1 + d_1 x) f_{X_1}(x) dx. \end{aligned} \tag{7.7}$$

Using (7.4), (7.5) and (7.6), we can express (7.7) as

$$\begin{aligned} D(X_1, X_2) &= \log \frac{B(a_2, b_2)\ {}_2F_1(c_2, a_2; a_2 + b_2; -d_2)}{B(a_1, b_1)\ {}_2F_1(c_1, a_1; a_1 + b_1; -d_1)} + (a_1 - a_2) \frac{\Gamma'(a_1)}{\Gamma(a_1)} \\ &\quad + (a_1 - a_2) \frac{1}{{}_2F_1(c_1, a_1; a_1 + b_1; -d_1)} \frac{\partial}{\partial \alpha}\ {}_2F_1(c_1, \alpha + a_1; \alpha + a_1 + b_1; -d_1) \Big|_{\alpha=0} \\ &\quad - (a_1 - a_2) \frac{\Gamma'(a_1 + b_1)}{\Gamma(a_1 + b_1)} + (b_1 - b_2) \frac{\Gamma'(b_1)}{\Gamma(b_1)} \\ &\quad + (b_1 - b_2) \frac{1}{{}_2F_1(c_1, a_1; a_1 + b_1; -d_1)} \frac{\partial}{\partial \alpha}\ {}_2F_1(c_1, a_1; \alpha + a_1 + b_1; -d_1) \Big|_{\alpha=0} \\ &\quad - (b_1 - b_2) \frac{\Gamma'(a_1 + b_1)}{\Gamma(a_1 + b_1)} \\ &\quad + \frac{c_2}{{}_2F_1(c_1, a_1; a_1 + b_1; -d_1)} \frac{\partial}{\partial \alpha} F_1(a_1, -\alpha, c_1; a_1 + b_1; -d_2, -d_1) \Big|_{\alpha=0} \\ &\quad - \frac{c_1}{{}_2F_1(c_1, a_1; a_1 + b_1; -d_1)} \frac{\partial}{\partial \alpha}\ {}_2F_1(c_1 - \alpha, a_1; a_1 + b_1; -d_1). \end{aligned}$$

Additional measures of entropy include geometric variances and geometric covariance.

Note that

$$\begin{aligned}
\int_0^1 (\log x)^2 f_X(x) dx &= \int_0^1 \left. \frac{d^2 x^\alpha}{d\alpha^2} \right|_{\alpha=0} f_X(x) dx \\
&= \left. \frac{d^2}{d\alpha^2} \int_0^1 \frac{K x^{\alpha+a-1}(1-x)^{b-1}}{(1+dx)^c} dx \right|_{\alpha=0} \\
&= \left. \frac{d^2}{d\alpha^2} \frac{\Gamma(a+b)}{\Gamma(a)} \frac{\Gamma(\alpha+a)}{\Gamma(\alpha+a+b)} \frac{{}_2F_1(c, \alpha+a; \alpha+a+b; -d)}{{}_2F_1(c, a; a+b; -d)} \right|_{\alpha=0} \\
&= \frac{1}{[\Gamma(a+b)]^2 \ {}_2F_1(c, a; a+b; -d)} \left. \frac{\partial^2}{\partial\alpha^2} \ {}_2F_1(c, \alpha+a; \alpha+a+b; -d) \right|_{\alpha=0} \\
&\quad + \frac{\Gamma'(a)}{\Gamma(a) [\Gamma(a+b)]^2 \ {}_2F_1(c, a; a+b; -d)} \left. \frac{\partial}{\partial\alpha} \ {}_2F_1(c, \alpha+a; \alpha+a+b; -d) \right|_{\alpha=0} \\
&\quad - \frac{\Gamma'(a+b)}{[\Gamma(a+b)]^3 \ {}_2F_1(c, a; a+b; -d)} \left. \frac{\partial}{\partial\alpha} \ {}_2F_1(c, \alpha+a; \alpha+a+b; -d) \right|_{\alpha=0} \\
&\quad + \frac{\Gamma'(a)}{\Gamma(a) [\Gamma(a+b)]^2 \ {}_2F_1(c, a; a+b; -d)} \left. \frac{\partial}{\partial\alpha} \ {}_2F_1(c, \alpha+a; \alpha+a+b; -d) \right|_{\alpha=0} \\
&\quad + \frac{\Gamma''(a)}{\Gamma(a) [\Gamma(a+b)]^2} - \frac{\Gamma'(a)\Gamma'(a+b)}{\Gamma(a) [\Gamma(a+b)]^3} \\
&\quad - \frac{\Gamma'(a+b)}{\Gamma(a+b) \ {}_2F_1(c, a; a+b; -d)} \left. \frac{\partial}{\partial\alpha} \ {}_2F_1(c, \alpha+a; \alpha+a+b; -d) \right|_{\alpha=0} \\
&\quad - \frac{\Gamma''(a)\Gamma'(a+b)}{\Gamma(a)\Gamma(a+b)} - \frac{\Gamma''(a+b)}{\Gamma(a+b)} + \frac{2\left[\Gamma'(a+b)\right]^2}{[\Gamma(a+b)]^2}
\end{aligned}$$

and

$$\begin{aligned}
\int_0^1 (\log(1-x))^2 f_X(x) dx &= \int_0^1 \left. \frac{d^2 (1-x)^\alpha}{d\alpha^2} \right|_{\alpha=0} f_X(x) dx \\
&= \left. \frac{d^2}{d\alpha^2} \int_0^1 \frac{K x^{a-1}(1-x)^{\alpha+b-1}}{(1+dx)^c} dx \right|_{\alpha=0} \\
&= \left. \frac{d^2}{d\alpha^2} \frac{\Gamma(a+b)}{\Gamma(b)} \frac{\Gamma(\alpha+b)}{\Gamma(\alpha+a+b)} \frac{{}_2F_1(c, a; \alpha+a+b; -d)}{{}_2F_1(c, a; a+b; -d)} \right|_{\alpha=0} \\
&= \frac{1}{[\Gamma(a+b)]^2 \ {}_2F_1(c, a; a+b; -d)} \left. \frac{\partial^2}{\partial\alpha^2} \ {}_2F_1(c, a; \alpha+a+b; -d) \right|_{\alpha=0} \\
&\quad + \frac{\Gamma'(b)}{\Gamma(b) [\Gamma(a+b)]^2 \ {}_2F_1(c, a; a+b; -d)} \left. \frac{\partial}{\partial\alpha} \ {}_2F_1(c, a; \alpha+a+b; -d) \right|_{\alpha=0} \\
&\quad - \frac{\Gamma'(a+b)}{[\Gamma(a+b)]^3 \ {}_2F_1(c, a; a+b; -d)} \left. \frac{\partial}{\partial\alpha} \ {}_2F_1(c, a; \alpha+a+b; -d) \right|_{\alpha=0} \\
&\quad + \frac{\Gamma'(b)}{\Gamma(b) [\Gamma(a+b)]^2 \ {}_2F_1(c, a; a+b; -d)} \left. \frac{\partial}{\partial\alpha} \ {}_2F_1(c, a; \alpha+a+b; -d) \right|_{\alpha=0} \\
&\quad + \frac{\Gamma''(b)}{\Gamma(b) [\Gamma(a+b)]^2} - \frac{\Gamma'(b)\Gamma'(a+b)}{\Gamma(b) [\Gamma(a+b)]^3} \\
&\quad - \frac{\Gamma'(a+b)}{\Gamma(a+b) \ {}_2F_1(c, a; a+b; -d)} \left. \frac{\partial}{\partial\alpha} \ {}_2F_1(c, a; \alpha+a+b; -d) \right|_{\alpha=0} \\
&\quad - \frac{\Gamma''(b)\Gamma'(a+b)}{\Gamma(b)\Gamma(a+b)} - \frac{\Gamma''(a+b)}{\Gamma(a+b)} + \frac{2\left[\Gamma'(a+b)\right]^2}{[\Gamma(a+b)]^2}.
\end{aligned}$$

Hence, the geometric variances are

$$
\begin{aligned}
\mathrm{Var}(\log X) \;=\; & \frac{1}{[\Gamma(a+b)]^2 \; {}_2F_1(c, a; a+b; -d)} \frac{\partial^2}{\partial \alpha^2} \; {}_2F_1(c, \alpha+a; \alpha+a+b; -d)\Big|_{\alpha=0} \\
& + \frac{\Gamma^{'}(a)}{\Gamma(a)\,[\Gamma(a+b)]^2 \; {}_2F_1(c, a; a+b; -d)} \frac{\partial}{\partial \alpha} \; {}_2F_1(c, \alpha+a; \alpha+a+b; -d)\Big|_{\alpha=0} \\
& - \frac{\Gamma^{'}(a+b)}{[\Gamma(a+b)]^3 \; {}_2F_1(c, a; a+b; -d)} \frac{\partial}{\partial \alpha} \; {}_2F_1(c, \alpha+a; \alpha+a+b; -d)\Big|_{\alpha=0} \\
& + \frac{\Gamma^{'}(a)}{\Gamma(a)\,[\Gamma(a+b)]^2 \; {}_2F_1(c, a; a+b; -d)} \frac{\partial}{\partial \alpha} \; {}_2F_1(c, \alpha+a; \alpha+a+b; -d)\Big|_{\alpha=0} \\
& + \frac{\Gamma^{''}(a)}{\Gamma(a)\,[\Gamma(a+b)]^2} - \frac{\Gamma^{'}(a)\Gamma^{'}(a+b)}{\Gamma(a)\,[\Gamma(a+b)]^3} \\
& - \frac{\Gamma^{'}(a+b)}{\Gamma(a+b) \; {}_2F_1(c, a; a+b; -d)} \frac{\partial}{\partial \alpha} \; {}_2F_1(c, \alpha+a; \alpha+a+b; -d)\Big|_{\alpha=0} \\
& - \frac{\Gamma^{''}(a)\Gamma^{'}(a+b)}{\Gamma(a)\Gamma(a+b)} - \frac{\Gamma^{''}(a+b)}{\Gamma(a+b)} + \frac{2\left[\Gamma^{'}(a+b)\right]^2}{[\Gamma(a+b)]^2} - \{E[\log X]\}^2
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{Var}(\log(1-X)) \;=\; & \frac{1}{[\Gamma(a+b)]^2 \; {}_2F_1(c, a; a+b; -d)} \frac{\partial^2}{\partial \alpha^2} \; {}_2F_1(c, a; \alpha+a+b; -d)\Big|_{\alpha=0} \\
& + \frac{\Gamma^{'}(b)}{\Gamma(b)\,[\Gamma(a+b)]^2 \; {}_2F_1(c, a; a+b; -d)} \frac{\partial}{\partial \alpha} \; {}_2F_1(c, a; \alpha+a+b; -d)\Big|_{\alpha=0} \\
& - \frac{\Gamma^{'}(a+b)}{[\Gamma(a+b)]^3 \; {}_2F_1(c, a; a+b; -d)} \frac{\partial}{\partial \alpha} \; {}_2F_1(c, a; \alpha+a+b; -d)\Big|_{\alpha=0} \\
& + \frac{\Gamma^{'}(b)}{\Gamma(b)\,[\Gamma(a+b)]^2 \; {}_2F_1(c, a; a+b; -d)} \frac{\partial}{\partial \alpha} \; {}_2F_1(c, a; \alpha+a+b; -d)\Big|_{\alpha=0} \\
& + \frac{\Gamma^{''}(b)}{\Gamma(b)\,[\Gamma(a+b)]^2} - \frac{\Gamma^{'}(b)\Gamma^{'}(a+b)}{\Gamma(b)\,[\Gamma(a+b)]^3} \\
& - \frac{\Gamma^{'}(a+b)}{\Gamma(a+b) \; {}_2F_1(c, a; a+b; -d)} \frac{\partial}{\partial \alpha} \; {}_2F_1(c, a; \alpha+a+b; -d)\Big|_{\alpha=0} \\
& - \frac{\Gamma^{''}(b)\Gamma^{'}(a+b)}{\Gamma(b)\Gamma(a+b)} - \frac{\Gamma^{''}(a+b)}{\Gamma(a+b)} + \frac{2\left[\Gamma^{'}(a+b)\right]^2}{[\Gamma(a+b)]^2} - \{E[\log(1-X)]\}^2 .
\end{aligned}
$$

Note also that

$$
\begin{aligned}
(7.8) \qquad & \int_0^1 \log x \log(1-x) f_X(x) dx \\
= & \int_0^1 \left. \frac{d^2 x^\alpha (1-x)^\beta}{d\alpha d\beta} \right|_{\alpha=0,\beta=0} f_X(x) dx \\
= & \left. \frac{d^2}{d\alpha d\beta} \int_0^1 \frac{K x^{\alpha+a-1}(1-x)^{\beta+b-1}}{(1+dx)^c} dx \right|_{\alpha=0,\beta=0} \\
= & \left. \frac{d^2}{d\alpha d\beta} \frac{B(a+\alpha, b+\beta)}{B(a,b)} \frac{{}_2F_1(c, \alpha+a; \alpha+a+b; -d)}{{}_2F_1(c, a; a+b; -d)} \right|_{\alpha=0,\beta=0} \\
= & \frac{\Gamma'(a)\Gamma'(b)}{\Gamma(a)\Gamma(b)} \\
& + \frac{\Gamma'(b)}{\Gamma(b)\ {}_2F_1(c, a; a+b; -d)} \left. \frac{\partial}{\partial \alpha}\ {}_2F_1(c, \alpha+a; \alpha+\beta+a+b; -d) \right|_{\alpha=0,\beta=0} \\
& - \frac{\Gamma'(b)\Gamma'(a+b)}{\Gamma(a+b)\Gamma(b)} \\
& + \frac{\Gamma'(a)}{\Gamma(a)\ {}_2F_1(c, a; a+b; -d)} \left. \frac{\partial}{\partial \beta}\ {}_2F_1(c, \alpha+a; \alpha+\beta+a+b; -d) \right|_{\alpha=0,\beta=0} \\
& + {}_2F_1(c, a; a+b; -d) \left. \frac{\partial^2}{\partial \alpha \partial \beta}\ {}_2F_1(c, \alpha+a; \alpha+\beta+a+b; -d) \right|_{\alpha=0,\beta=0} \\
& - \frac{\Gamma'(a+b)}{\Gamma(a+b)\ {}_2F_1(c, a; a+b; -d)} \left. \frac{\partial}{\partial \beta}\ {}_2F_1(c, \alpha+a; \alpha+\beta+a+b; -d) \right|_{\alpha=0,\beta=0} \\
& - \frac{\Gamma'(a)\Gamma'(a+b)}{\Gamma(a)\Gamma(a+b)} - \frac{\Gamma'(a)\Gamma''(a+b)}{\Gamma(a)\Gamma(a+b)} \\
& - \frac{\Gamma'(a+b)}{\Gamma(a+b)\ {}_2F_1(c, a; a+b; -d)} \left. \frac{\partial}{\partial \alpha}\ {}_2F_1(c, \alpha+a; \alpha+\beta+a+b; -d) \right|_{\alpha=0,\beta=0} \\
& + \frac{2\left[\Gamma'(a+b)\right]^2}{\left[\Gamma(a+b)\right]^2}.
\end{aligned}
$$

Hence, the geometric covariance is

$$
\begin{aligned}
(7.9)\qquad & \operatorname{Cov}\left[\log X, \log(1-X)\right] \\
= & \frac{\Gamma'(a)\Gamma'(b)}{\Gamma(a)\Gamma(b)} \\
& + \frac{\Gamma'(b)}{\Gamma(b)\ {}_2F_1\left(c, a; a+b; -d\right)} \frac{\partial}{\partial\alpha}\ {}_2F_1\left(c, \alpha+a; \alpha+\beta+a+b; -d\right)\Big|_{\alpha=0,\beta=0} \\
& - \frac{\Gamma'(b)\Gamma'(a+b)}{\Gamma(a+b)\Gamma(b)} \\
& + \frac{\Gamma'(a)}{\Gamma(a)\ {}_2F_1\left(c, a; a+b; -d\right)} \frac{\partial}{\partial\beta}\ {}_2F_1\left(c, \alpha+a; \alpha+\beta+a+b; -d\right)\Big|_{\alpha=0,\beta=0} \\
& + {}_2F_1\left(c, a; a+b; -d\right) \frac{\partial^2}{\partial\alpha\partial\beta}\ {}_2F_1\left(c, \alpha+a; \alpha+\beta+a+b; -d\right)\Big|_{\alpha=0,\beta=0} \\
& - \frac{\Gamma'(a+b)}{\Gamma(a+b)\ {}_2F_1\left(c, a; a+b; -d\right)} \frac{\partial}{\partial\beta}\ {}_2F_1\left(c, \alpha+a; \alpha+\beta+a+b; -d\right)\Big|_{\alpha=0,\beta=0} \\
& - \frac{\Gamma'(a)\Gamma'(a+b)}{\Gamma(a)\Gamma(a+b)} - \frac{\Gamma'(a)\Gamma''(a+b)}{\Gamma(a)\Gamma(a+b)} \\
& - \frac{\Gamma'(a+b)}{\Gamma(a+b)\ {}_2F_1\left(c, a; a+b; -d\right)} \frac{\partial}{\partial\alpha}\ {}_2F_1\left(c, \alpha+a; \alpha+\beta+a+b; -d\right)\Big|_{\alpha=0,\beta=0} \\
& + \frac{2\left[\Gamma'(a+b)\right]^2}{\left[\Gamma(a+b)\right]^2} - E\left[\log X\right] E\left[\log(1-X)\right].
\end{aligned}
$$

The expressions for Shannon and Rényi entropies have been previously provided in Nagar and Valencia (2011).

8. ORDERING

Let X_1 and X_2 be Gauss hypergeometric beta random variables with parameters (a_1, b_1, c_1, d_1) and (a_2, b_2, c_2, d_2), respectively. We say that X_1 is smaller than X_2 with respect to likelihood ratio ordering if $f_{X_2}(x)/f_{X_1}(x)$ is an increasing function of x. Note that

$$
\frac{f_{X_2}(x)}{f_{X_1}(x)} = \frac{B\left(a_1, b_1\right)\ {}_2F_1\left(c_1, a_1; a_1+b_1; -d_1\right)}{B\left(a_2, b_2\right)\ {}_2F_1\left(c_2, a_2; a_2+b_2; -d_2\right)} x^{a_2-a_1}(1-x)^{b_2-b_1}\left(1+d_2 x\right)^{c_2}\left(1+d_1 x\right)^{-c_1}
$$

and

$$
\begin{aligned}
\log \frac{f_{X_2}(x)}{f_{X_1}(x)} = & \log \frac{B\left(a_1, b_1\right)\ {}_2F_1\left(c_1, a_1; a_1+b_1; -d_1\right)}{B\left(a_2, b_2\right)\ {}_2F_1\left(c_2, a_2; a_2+b_2; -d_2\right)} + \left(a_2-a_1\right)\log x + \left(b_2-b_1\right)\log(1-x) \\
& + c_2 \log\left(1+d_2 x\right) - c_1 \log\left(1+d_1 x\right).
\end{aligned}
$$

So,

$$
\frac{d}{dx} \log \frac{f_{X_2}(x)}{f_{X_1}(x)} = \frac{a_2-a_1}{x} - \frac{b_2-b_1}{1-x} + \frac{c_2 d_2}{1+d_2 x} - \frac{c_1 d_1}{1+d_1 x}
$$

Hence, X_1 is smaller than X_2 with respect to likelihood ratio ordering if $a_2 > a_1$, $b_1 > b_2$, $c_2 d_2 > c_1 d_1$ and $c_2 d_1 d_2 > c_1 d_1 d_2$.

9. MAXIMUM LIKELIHOOD ESTIMATION

Suppose $x_1, x_2, \ldots, x_n$ is a random sample from (1.1) with a, b, c and d unknown. The joint loglikelihood function of a, b, c and d is

$$\log L(a,b,c,d) = n \log K + (a-1)\sum_{i=1}^{n} \log x_i + (b-1)\sum_{i=1}^{n} \log\left(1-x_i\right) - c\sum_{i=1}^{n} \log\left(1+dx_i\right).$$

The partial derivatives of $\log L$ with respect to a, b, c and d are

$$\frac{\partial \log L}{\partial a} = \frac{n}{K}\frac{\partial K}{\partial a} + \sum_{i=1}^{n} \log x_i,$$

$$\frac{\partial \log L}{\partial b} = \frac{n}{K}\frac{\partial K}{\partial b} + \sum_{i=1}^{n} \log\left(1-x_i\right),$$

$$\frac{\partial \log L}{\partial c} = \frac{n}{K}\frac{\partial K}{\partial c} - \sum_{i=1}^{n} \log\left(1+dx_i\right)$$

and

$$\frac{\partial \log L}{\partial d} = \frac{n}{K}\frac{\partial K}{\partial d} - c\sum_{i=1}^{n} \frac{x_i}{1+dx_i},$$

where

$$\frac{\partial K}{\partial a} = -K^2 \frac{\Gamma'(a)\Gamma(b)}{\Gamma(a+b)}\, {}_2F_1\left(c,a;a+b;-d\right) - K^2 \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \frac{\partial\, {}_2F_1\left(c,a;a+b;-d\right)}{\partial a} + K^2 \frac{\Gamma(a)\Gamma(b)\Gamma'(a+b)}{\left[\Gamma(a+b)\right]^2}\, {}_2F_1\left(c,a;a+b;-d\right),$$

$$\frac{\partial K}{\partial b} = -K^2 \frac{\Gamma'(b)\Gamma(a)}{\Gamma(a+b)}\, {}_2F_1\left(c,a;a+b;-d\right) - K^2 \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \frac{\partial\, {}_2F_1\left(c,a;a+b;-d\right)}{\partial b} + K^2 \frac{\Gamma(a)\Gamma(b)\Gamma'(a+b)}{\left[\Gamma(a+b)\right]^2}\, {}_2F_1\left(c,a;a+b;-d\right),$$

$$\frac{\partial K}{\partial c} = -K^2 \frac{\partial\, {}_2F_1\left(c,a;a+b;-d\right)}{\partial c}$$

and

$$\frac{\partial K}{\partial d} = -K^2 \frac{\partial\, {}_2F_1\left(c,a;a+b;-d\right)}{\partial d}.$$

The maximum likelihood estimates of a, b, c and d, say $\widehat{a}$, $\widehat{b}$, $\widehat{c}$ and $\widehat{d}$, are the simultaneous solutions of $\frac{\partial \log L}{\partial a} = 0$, $\frac{\partial \log L}{\partial b} = 0$, $\frac{\partial \log L}{\partial c} = 0$ and $\frac{\partial \log L}{\partial d} = 0$.

The second order partial derivatives of the loglikelihood are

$$\frac{\partial^2 \log L}{\partial a^2} = \frac{n}{K}\frac{\partial^2 K}{\partial a^2} - \frac{n}{K^2}\left(\frac{\partial K}{\partial a}\right)^2,$$

$$\frac{\partial^2 \log L}{\partial a \partial b} = \frac{n}{K}\frac{\partial^2 K}{\partial a \partial b} - \frac{n}{K^2}\frac{\partial K}{\partial a}\frac{\partial K}{\partial b},$$

$$\frac{\partial^2 \log L}{\partial a \partial c} = \frac{n}{K}\frac{\partial^2 K}{\partial a \partial c} - \frac{n}{K^2}\frac{\partial K}{\partial a}\frac{\partial K}{\partial c},$$

$$\frac{\partial^2 \log L}{\partial a \partial d} = \frac{n}{K}\frac{\partial^2 K}{\partial a \partial d} - \frac{n}{K^2}\frac{\partial K}{\partial a}\frac{\partial K}{\partial d},$$

$$\frac{\partial^2 \log L}{\partial b^2} = \frac{n}{K}\frac{\partial^2 K}{\partial b^2} - \frac{n}{K^2}\left(\frac{\partial K}{\partial b}\right)^2,$$

$$\frac{\partial^2 \log L}{\partial b \partial c} = \frac{n}{K}\frac{\partial^2 K}{\partial b \partial c} - \frac{n}{K^2}\frac{\partial K}{\partial b}\frac{\partial K}{\partial c},$$

$$\frac{\partial^2 \log L}{\partial b \partial d} = \frac{n}{K}\frac{\partial^2 K}{\partial b \partial d} - \frac{n}{K^2}\frac{\partial K}{\partial b}\frac{\partial K}{\partial d},$$

$$\frac{\partial^2 \log L}{\partial c^2} = \frac{n}{K}\frac{\partial^2 K}{\partial c^2} - \frac{n}{K^2}\left(\frac{\partial K}{\partial c}\right)^2,$$

$$\frac{\partial^2 \log L}{\partial c \partial d} = \frac{n}{K}\frac{\partial^2 K}{\partial c \partial d} - \frac{n}{K^2}\frac{\partial K}{\partial c}\frac{\partial K}{\partial d} - \sum_{i=1}^{n}\frac{x_i}{1 + d x_i}$$

and

$$\frac{\partial^2 \log L}{\partial d^2} = \frac{n}{K}\frac{\partial^2 K}{\partial d^2} - \frac{n}{K^2}\left(\frac{\partial K}{\partial d}\right)^2 + c\sum_{i=1}^{n}\frac{x_i^2}{(1 + d x_i)^2},$$

where

$$\begin{aligned}\frac{\partial^2 K}{\partial a^2} =& -2K\frac{\partial K}{\partial a}\frac{\Gamma'(a)\Gamma(b)}{\Gamma(a+b)}\ {}_2F_1\left(c,a;a+b;-d\right) - K^2\frac{\Gamma''(a)\Gamma(b)}{\Gamma(a+b)}\ {}_2F_1\left(c,a;a+b;-d\right)\\
&-K^2\frac{\Gamma'(a)\Gamma(b)}{\Gamma(a+b)}\frac{\partial\ {}_2F_1\left(c,a;a+b;-d\right)}{\partial a} + K^2\frac{\Gamma'(a)\Gamma(b)\Gamma'(a+b)}{\left[\Gamma(a+b)\right]^2}\ {}_2F_1\left(c,a;a+b;-d\right)\\
&-2K\frac{\partial K}{\partial a}\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\ \frac{\partial_2F_1\left(c,a;a+b;-d\right)}{\partial a} - K^2\frac{\Gamma'(a)\Gamma(b)}{\Gamma(a+b)}\ \frac{\partial_2F_1\left(c,a;a+b;-d\right)}{\partial a}\\
&-K^2\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\frac{\partial^2\ {}_2F_1\left(c,a;a+b;-d\right)}{\partial a^2} + K^2\frac{\Gamma(a)\Gamma(b)\Gamma'(a+b)}{\left[\Gamma(a+b)\right]^2}\ \frac{\partial_2F_1\left(c,a;a+b;-d\right)}{\partial a}\\
&+2K\frac{\partial K}{\partial a}\frac{\Gamma(a)\Gamma(b)\Gamma'(a+b)}{\left[\Gamma(a+b)\right]^2}\ {}_2F_1\left(c,a;a+b;-d\right)\\
&+K^2\frac{\Gamma'(a)\Gamma(b)\Gamma'(a+b)}{\left[\Gamma(a+b)\right]^2}\ {}_2F_1\left(c,a;a+b;-d\right)\\
&+K^2\frac{\Gamma(a)\Gamma(b)\Gamma''(a+b)}{\left[\Gamma(a+b)\right]^2}\ {}_2F_1\left(c,a;a+b;-d\right)\\
&+K^2\frac{\Gamma(a)\Gamma(b)\Gamma'(a+b)}{\left[\Gamma(a+b)\right]^2}\ \frac{{}_2F_1\left(c,a;a+b;-d\right)}{\partial a}\\
&-2K^2\frac{\Gamma(a)\Gamma(b)\left[\Gamma'(a+b)\right]^2}{\left[\Gamma(a+b)\right]^3}\ {}_2F_1\left(c,a;a+b;-d\right),\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 K}{\partial a\partial b} =& -2K\frac{\partial K}{\partial b}\frac{\Gamma'(a)\Gamma(b)}{\Gamma(a+b)}\ {}_2F_1\left(c,a;a+b;-d\right) - K^2\frac{\Gamma'(a)\Gamma'(b)}{\Gamma(a+b)}\ {}_2F_1\left(c,a;a+b;-d\right)\\
&-K^2\frac{\Gamma'(a)\Gamma(b)}{\Gamma(a+b)}\frac{\partial\ {}_2F_1\left(c,a;a+b;-d\right)}{\partial b} + K^2\frac{\Gamma'(a)\Gamma(b)\Gamma'(a+b)}{\left[\Gamma(a+b)\right]^2}\ {}_2F_1\left(c,a;a+b;-d\right)\\
&-2K\frac{\partial K}{\partial b}\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\ \frac{\partial_2F_1\left(c,a;a+b;-d\right)}{\partial a} - K^2\frac{\Gamma(a)\Gamma'(b)}{\Gamma(a+b)}\ \frac{\partial_2F_1\left(c,a;a+b;-d\right)}{\partial a}\\
&-K^2\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\frac{\partial^2\ {}_2F_1\left(c,a;a+b;-d\right)}{\partial a\partial b} + K^2\frac{\Gamma(a)\Gamma(b)\Gamma'(a+b)}{\left[\Gamma(a+b)\right]^2}\ \frac{\partial_2F_1\left(c,a;a+b;-d\right)}{\partial a}\\
&+2K\frac{\partial K}{\partial a}\frac{\Gamma(a)\Gamma(b)\Gamma'(a+b)}{\left[\Gamma(a+b)\right]^2}\ {}_2F_1\left(c,a;a+b;-d\right)\\
&+K^2\frac{\Gamma(a)\Gamma'(b)\Gamma'(a+b)}{\left[\Gamma(a+b)\right]^2}\ {}_2F_1\left(c,a;a+b;-d\right)\\
&+K^2\frac{\Gamma(a)\Gamma(b)\Gamma''(a+b)}{\left[\Gamma(a+b)\right]^2}\ {}_2F_1\left(c,a;a+b;-d\right)\\
&+K^2\frac{\Gamma(a)\Gamma(b)\Gamma'(a+b)}{\left[\Gamma(a+b)\right]^2}\ \frac{{}_2F_1\left(c,a;a+b;-d\right)}{\partial b}\\
&-2K^2\frac{\Gamma(a)\Gamma(b)\left[\Gamma'(a+b)\right]^2}{\left[\Gamma(a+b)\right]^3}\ {}_2F_1\left(c,a;a+b;-d\right),\end{aligned}$$

$$
\begin{aligned}
\frac{\partial^2 K}{\partial a \partial c} = & -2K \frac{\partial K}{\partial c} \frac{\Gamma^{'}(a)\Gamma(b)}{\Gamma(a+b)} \, {}_2F_1\left(c, a; a+b; -d\right) - K^2 \frac{\Gamma^{'}(a)\Gamma(b)}{\Gamma(a+b)} \frac{\partial \, {}_2F_1\left(c, a; a+b; -d\right)}{\partial c} \\
& -2K \frac{\partial K}{\partial c} \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \frac{\partial \, {}_2F_1\left(c, a; a+b; -d\right)}{\partial a} - K^2 \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \frac{\partial^2 \, {}_2F_1\left(c, a; a+b; -d\right)}{\partial a \partial c} \\
& +2K \frac{\partial K}{\partial c} \frac{\Gamma(a)\Gamma(b)\Gamma^{'}(a+b)}{\left[\Gamma(a+b)\right]^2} \, {}_2F_1\left(c, a; a+b; -d\right) \\
& +K^2 \frac{\Gamma(a)\Gamma(b)\Gamma^{'}(a+b)}{\left[\Gamma(a+b)\right]^2} \frac{\partial \, {}_2F_1\left(c, a; a+b; -d\right)}{\partial c},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 K}{\partial a \partial d} = & -2K \frac{\partial K}{\partial d} \frac{\Gamma^{'}(a)\Gamma(b)}{\Gamma(a+b)} \, {}_2F_1\left(c, a; a+b; -d\right) - K^2 \frac{\Gamma^{'}(a)\Gamma(b)}{\Gamma(a+b)} \frac{\partial \, {}_2F_1\left(c, a; a+b; -d\right)}{\partial d} \\
& -2K \frac{\partial K}{\partial d} \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \frac{\partial \, {}_2F_1\left(c, a; a+b; -d\right)}{\partial a} - K^2 \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \frac{\partial^2 \, {}_2F_1\left(c, a; a+b; -d\right)}{\partial a \partial d} \\
& +2K \frac{\partial K}{\partial d} \frac{\Gamma(a)\Gamma(b)\Gamma^{'}(a+b)}{\left[\Gamma(a+b)\right]^2} \, {}_2F_1\left(c, a; a+b; -d\right) \\
& +K^2 \frac{\Gamma(a)\Gamma(b)\Gamma^{'}(a+b)}{\left[\Gamma(a+b)\right]^2} \frac{\partial \, {}_2F_1\left(c, a; a+b; -d\right)}{\partial d},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 K}{\partial b^2} = & -2K \frac{\partial K}{\partial b} \frac{\Gamma^{'}(b)\Gamma(a)}{\Gamma(a+b)} \, {}_2F_1\left(c, a; a+b; -d\right) - K^2 \frac{\Gamma^{''}(b)\Gamma(a)}{\Gamma(a+b)} \, {}_2F_1\left(c, a; a+b; -d\right) \\
& -K^2 \frac{\Gamma^{'}(b)\Gamma(a)}{\Gamma(a+b)} \frac{\partial \, {}_2F_1\left(c, a; a+b; -d\right)}{\partial b} + K^2 \frac{\Gamma^{'}(b)\Gamma(a)\Gamma^{'}(a+b)}{\left[\Gamma(a+b)\right]^2} \, {}_2F_1\left(c, a; a+b; -d\right) \\
& -2K \frac{\partial K}{\partial b} \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \frac{\partial_2 F_1\left(c, a; a+b; -d\right)}{\partial b} - K^2 \frac{\Gamma^{'}(b)\Gamma(a)}{\Gamma(a+b)} \frac{\partial_2 F_1\left(c, a; a+b; -d\right)}{\partial b} \\
& -K^2 \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \frac{\partial^2 \, {}_2F_1\left(c, a; a+b; -d\right)}{\partial b^2} + K^2 \frac{\Gamma(a)\Gamma(b)\Gamma^{'}(a+b)}{\left[\Gamma(a+b)\right]^2} \frac{\partial_2 F_1\left(c, a; a+b; -d\right)}{\partial b} \\
& +2K \frac{\partial K}{\partial b} \frac{\Gamma(a)\Gamma(b)\Gamma^{'}(a+b)}{\left[\Gamma(a+b)\right]^2} \, {}_2F_1\left(c, a; a+b; -d\right) \\
& +K^2 \frac{\Gamma^{'}(b)\Gamma(a)\Gamma^{'}(a+b)}{\left[\Gamma(a+b)\right]^2} \, {}_2F_1\left(c, a; a+b; -d\right) \\
& +K^2 \frac{\Gamma(a)\Gamma(b)\Gamma^{''}(a+b)}{\left[\Gamma(a+b)\right]^2} \, {}_2F_1\left(c, a; a+b; -d\right) \\
& +K^2 \frac{\Gamma(a)\Gamma(b)\Gamma^{'}(a+b)}{\left[\Gamma(a+b)\right]^2} \frac{{}_2F_1\left(c, a; a+b; -d\right)}{\partial b} \\
& -2K^2 \frac{\Gamma(a)\Gamma(b)\left[\Gamma^{'}(a+b)\right]^2}{\left[\Gamma(a+b)\right]^3} \, {}_2F_1\left(c, a; a+b; -d\right),
\end{aligned}
$$

$$\begin{aligned}\frac{\partial^2 K}{\partial b \partial c} =& -2K \frac{\partial K}{\partial c} \frac{\Gamma^{'}(b)\Gamma(a)}{\Gamma(a+b)} \ {}_2F_1\left(c, a; a+b; -d\right) - K^2 \frac{\Gamma^{'}(b)\Gamma(a)}{\Gamma(a+b)} \frac{\partial \ {}_2F_1\left(c, a; a+b; -d\right)}{\partial c} \\ & -2K \frac{\partial K}{\partial c} \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \frac{\partial \ {}_2F_1\left(c, a; a+b; -d\right)}{\partial b} - K^2 \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \frac{\partial^2 \ {}_2F_1\left(c, a; a+b; -d\right)}{\partial b \partial c} \\ & +2K \frac{\partial K}{\partial c} \frac{\Gamma(a)\Gamma(b)\Gamma^{'}(a+b)}{\left[\Gamma(a+b)\right]^2} \ {}_2F_1\left(c, a; a+b; -d\right) \\ & +K^2 \frac{\Gamma(a)\Gamma(b)\Gamma^{'}(a+b)}{\left[\Gamma(a+b)\right]^2} \frac{\partial \ {}_2F_1\left(c, a; a+b; -d\right)}{\partial c},\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 K}{\partial b \partial d} =& -2K \frac{\partial K}{\partial d} \frac{\Gamma^{'}(b)\Gamma(a)}{\Gamma(a+b)} \ {}_2F_1\left(c, a; a+b; -d\right) - K^2 \frac{\Gamma^{'}(b)\Gamma(a)}{\Gamma(a+b)} \frac{\partial \ {}_2F_1\left(c, a; a+b; -d\right)}{\partial d} \\ & -2K \frac{\partial K}{\partial d} \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \frac{\partial \ {}_2F_1\left(c, a; a+b; -d\right)}{\partial b} - K^2 \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \frac{\partial^2 \ {}_2F_1\left(c, a; a+b; -d\right)}{\partial b \partial d} \\ & +2K \frac{\partial K}{\partial d} \frac{\Gamma(a)\Gamma(b)\Gamma^{'}(a+b)}{\left[\Gamma(a+b)\right]^2} \ {}_2F_1\left(c, a; a+b; -d\right) \\ & +K^2 \frac{\Gamma(a)\Gamma(b)\Gamma^{'}(a+b)}{\left[\Gamma(a+b)\right]^2} \frac{\partial \ {}_2F_1\left(c, a; a+b; -d\right)}{\partial d},\end{aligned}$$

$$\frac{\partial^2 K}{\partial c^2} = -2K \frac{\partial K}{\partial c} \frac{\partial \ {}_2F_1\left(c, a; a+b; -d\right)}{\partial c} - K^2 \frac{\partial^2 \ {}_2F_1\left(c, a; a+b; -d\right)}{\partial c^2},$$

$$\frac{\partial^2 K}{\partial c \partial d} = -2K \frac{\partial K}{\partial d} \frac{\partial \ {}_2F_1\left(c, a; a+b; -d\right)}{\partial c} - K^2 \frac{\partial^2 \ {}_2F_1\left(c, a; a+b; -d\right)}{\partial c \partial d}$$

and

$$\frac{\partial^2 K}{\partial d^2} = -2K \frac{\partial K}{\partial d} \frac{\partial \ {}_2F_1\left(c, a; a+b; -d\right)}{\partial d} - K^2 \frac{\partial^2 \ {}_2F_1\left(c, a; a+b; -d\right)}{\partial d^2}.$$

Therefore, subject to certain regularity conditions, $\sqrt{n}\left(\widehat{a}-a, \widehat{b}-b, \widehat{c}-c, \widehat{d}-d\right)$ converges to a quadrivariate normal distribution with zero means. Its variance-covariance matrix can be approximated by

$$\mathbf{I} = \begin{pmatrix} I_{1,1} & I_{1,2} & I_{1,3} & I_{1,4} \\ I_{1,2} & I_{2,2} & I_{2,3} & I_{2,4} \\ I_{1,3} & I_{2,3} & I_{3,3} & I_{3,4} \\ I_{1,4} & I_{2,4} & I_{3,4} & I_{4,4} \end{pmatrix},$$

where

$$I_{1,1} = -\frac{n}{K} \frac{\partial^2 K}{\partial a^2} + \frac{n}{K^2} \left(\frac{\partial K}{\partial a}\right)^2,$$

$$I_{1,2} = -\frac{n}{K} \frac{\partial^2 K}{\partial a \partial b} + \frac{n}{K^2} \frac{\partial K}{\partial a} \frac{\partial K}{\partial b},$$

$$I_{1,3} = -\frac{n}{K} \frac{\partial^2 K}{\partial a \partial c} + \frac{n}{K^2} \frac{\partial K}{\partial a} \frac{\partial K}{\partial c},$$

$$I_{1,4} = -\frac{n}{K}\frac{\partial^2 K}{\partial a \partial d} + \frac{n}{K^2}\frac{\partial K}{\partial a}\frac{\partial K}{\partial d},$$

$$I_{2,2} = -\frac{n}{K}\frac{\partial^2 K}{\partial b^2} + \frac{n}{K^2}\left(\frac{\partial K}{\partial b}\right)^2,$$

$$I_{2,3} = -\frac{n}{K}\frac{\partial^2 K}{\partial b \partial c} + \frac{n}{K^2}\frac{\partial K}{\partial b}\frac{\partial K}{\partial c},$$

$$I_{2,4} = -\frac{n}{K}\frac{\partial^2 K}{\partial b \partial d} + \frac{n}{K^2}\frac{\partial K}{\partial b}\frac{\partial K}{\partial d},$$

$$I_{3,3} = -\frac{n}{K}\frac{\partial^2 K}{\partial c^2} + \frac{n}{K^2}\left(\frac{\partial K}{\partial c}\right)^2,$$

$$I_{3,4} = -\frac{n}{K}\frac{\partial^2 K}{\partial c \partial d} + \frac{n}{K^2}\frac{\partial K}{\partial c}\frac{\partial K}{\partial d} + \frac{na}{a+b}\frac{{}_2F_1\left(c+1, a+1; a+b+1; -d\right)}{{}_2F_1\left(c, a; a+b; -d\right)}$$

and

$$I_{4,4} = -\frac{n}{K}\frac{\partial^2 K}{\partial d^2} + \frac{n}{K^2}\left(\frac{\partial K}{\partial d}\right)^2 - \frac{nca(a+1)}{(a+b)(a+b+1)}\frac{{}_2F_1\left(c+2, a+2; a+b+2; -d\right)}{{}_2F_1\left(c, a; a+b; -d\right)}.$$

This result is applicable for conducting hypothesis tests and constructing confidence intervals for (a, b, c, d). For instance, approximate $100(1-\alpha)$ percent confidence intervals for a, b, c, and d can be expressed as

$$\widehat{a} \pm z_{\alpha/2}\sqrt{\widehat{I}^{1,1}},$$

$$\widehat{b} \pm z_{\alpha/2}\sqrt{\widehat{I}^{2,2}},$$

$$\widehat{c} \pm z_{\alpha/2}\sqrt{\widehat{I}^{3,3}}$$

and

$$\widehat{d} \pm z_{\alpha/2}\sqrt{\widehat{I}^{4,4}},$$

respectively, where

$$\begin{aligned}\widehat{I}^{1,1} = & \left[\widehat{I}_{2,2}\widehat{I}_{3,3}\widehat{I}_{4,4} - \widehat{I}_{2,2}\widehat{I}_{3,4}^2 - \widehat{I}_{2,3}^2\widehat{I}_{4,4} + 2\widehat{I}_{2,3}\widehat{I}_{2,4}\widehat{I}_{3,4} - \widehat{I}_{2,4}^2\widehat{I}_{3,3}\right] \\ & \Big/ \Big[\widehat{I}_{1,1}\widehat{I}_{2,2}\widehat{I}_{3,3}\widehat{I}_{4,4} - \widehat{I}_{1,1}\widehat{I}_{2,2}\widehat{I}_{3,4}^2 - \widehat{I}_{1,1}\widehat{I}_{2,3}^2\widehat{I}_{4,4} + 2\widehat{I}_{1,1}\widehat{I}_{2,3}\widehat{I}_{2,4}\widehat{I}_{3,4} - \widehat{I}_{1,1}\widehat{I}_{2,4}^2\widehat{I}_{3,3} \\ & -\widehat{I}_{1,2}^2\widehat{I}_{3,3}\widehat{I}_{4,4} + \widehat{I}_{1,2}^2\widehat{I}_{3,4}^2 + 2\widehat{I}_{1,2}\widehat{I}_{1,3}\widehat{I}_{2,3}\widehat{I}_{4,4} - 2\widehat{I}_{1,2}\widehat{I}_{1,3}\widehat{I}_{2,4}\widehat{I}_{3,4} - 2\widehat{I}_{1,2}\widehat{I}_{1,4}\widehat{I}_{2,3}\widehat{I}_{3,4} \\ & +2\widehat{I}_{1,2}\widehat{I}_{1,4}\widehat{I}_{2,4}\widehat{I}_{3,3} - \widehat{I}_{1,3}^2\widehat{I}_{2,2}\widehat{I}_{4,4} + \widehat{I}_{1,3}^2\widehat{I}_{2,4}^2 + 2\widehat{I}_{1,3}\widehat{I}_{1,4}\widehat{I}_{2,2}\widehat{I}_{3,4} \\ & -2\widehat{I}_{1,3}\widehat{I}_{1,4}\widehat{I}_{2,3}\widehat{I}_{2,4} - \widehat{I}_{1,4}^2\widehat{I}_{2,2}\widehat{I}_{3,3} + \widehat{I}_{1,4}^2\widehat{I}_{2,3}^2\Big],\end{aligned}$$

$$\widehat{I}^{2,2} = \left[\widehat{I}_{1,1}\widehat{I}_{3,3}\widehat{I}_{4,4} - \widehat{I}_{1,1}\widehat{I}_{3,4}^2 - \widehat{I}_{1,3}^2\widehat{I}_{4,4} + 2\widehat{I}_{1,3}\widehat{I}_{1,4}\widehat{I}_{3,4} - \widehat{I}_{1,4}^2\widehat{I}_{3,3} \right]$$
$$\Big/ \left[\widehat{I}_{1,1}\widehat{I}_{2,2}\widehat{I}_{3,3}\widehat{I}_{4,4} - \widehat{I}_{1,1}\widehat{I}_{2,2}\widehat{I}_{3,4}^2 - \widehat{I}_{1,1}\widehat{I}_{2,3}^2\widehat{I}_{4,4} + 2\widehat{I}_{1,1}\widehat{I}_{2,3}\widehat{I}_{2,4}\widehat{I}_{3,4} \right.$$
$$- \widehat{I}_{1,1}\widehat{I}_{2,4}^2\widehat{I}_{3,3} - \widehat{I}_{1,2}^2\widehat{I}_{3,3}\widehat{I}_{4,4} + \widehat{I}_{1,2}^2\widehat{I}_{3,4}^2 + 2\widehat{I}_{1,2}\widehat{I}_{1,3}\widehat{I}_{2,3}\widehat{I}_{4,4}$$
$$- 2\widehat{I}_{1,2}\widehat{I}_{1,3}\widehat{I}_{2,4}\widehat{I}_{3,4} - 2\widehat{I}_{1,2}\widehat{I}_{1,4}\widehat{I}_{2,3}\widehat{I}_{3,4} + 2\widehat{I}_{1,2}\widehat{I}_{1,4}\widehat{I}_{2,4}\widehat{I}_{3,3}$$
$$- \widehat{I}_{1,3}^2\widehat{I}_{2,2}\widehat{I}_{4,4} + \widehat{I}_{1,3}^2\widehat{I}_{2,4}^2 + 2\widehat{I}_{1,3}\widehat{I}_{1,4}\widehat{I}_{2,2}\widehat{I}_{3,4}$$
$$\left. - 2\widehat{I}_{1,3}\widehat{I}_{1,4}\widehat{I}_{2,3}\widehat{I}_{2,4} - \widehat{I}_{1,4}^2\widehat{I}_{2,2}\widehat{I}_{3,3} + \widehat{I}_{1,4}^2\widehat{I}_{2,3}^2 \right],$$

$$\widehat{I}^{3,3} = \left[\widehat{I}_{1,1}\widehat{I}_{2,2}\widehat{I}_{4,4} - \widehat{I}_{1,1}\widehat{I}_{2,4}^2 - \widehat{I}_{1,2}^2\widehat{I}_{4,4} + 2\widehat{I}_{1,2}\widehat{I}_{1,4}\widehat{I}_{2,4} - \widehat{I}_{1,4}^2\widehat{I}_{2,2} \right]$$
$$\Big/ \left[\widehat{I}_{1,1}\widehat{I}_{2,2}\widehat{I}_{3,3}\widehat{I}_{4,4} - \widehat{I}_{1,1}\widehat{I}_{2,2}\widehat{I}_{3,4}^2 - \widehat{I}_{1,1}\widehat{I}_{2,3}^2\widehat{I}_{4,4} + 2\widehat{I}_{1,1}\widehat{I}_{2,3}\widehat{I}_{2,4}\widehat{I}_{3,4} \right.$$
$$- \widehat{I}_{1,1}\widehat{I}_{2,4}^2\widehat{I}_{3,3} - \widehat{I}_{1,2}^2\widehat{I}_{3,3}\widehat{I}_{4,4} + \widehat{I}_{1,2}^2\widehat{I}_{3,4}^2 + 2\widehat{I}_{1,2}\widehat{I}_{1,3}\widehat{I}_{2,3}\widehat{I}_{4,4}$$
$$- 2\widehat{I}_{1,2}\widehat{I}_{1,3}\widehat{I}_{2,4}\widehat{I}_{3,4} - 2\widehat{I}_{1,2}\widehat{I}_{1,4}\widehat{I}_{2,3}\widehat{I}_{3,4} + 2\widehat{I}_{1,2}\widehat{I}_{1,4}\widehat{I}_{2,4}\widehat{I}_{3,3}$$
$$- \widehat{I}_{1,3}^2\widehat{I}_{2,2}\widehat{I}_{4,4} + \widehat{I}_{1,3}^2\widehat{I}_{2,4}^2 + 2\widehat{I}_{1,3}\widehat{I}_{1,4}\widehat{I}_{2,2}\widehat{I}_{3,4}$$
$$\left. - 2\widehat{I}_{1,3}\widehat{I}_{1,4}\widehat{I}_{2,3}\widehat{I}_{2,4} - \widehat{I}_{1,4}^2\widehat{I}_{2,2}\widehat{I}_{3,3} + \widehat{I}_{1,4}^2\widehat{I}_{2,3}^2 \right]$$

and

$$\widehat{I}^{4,4} = \left[\widehat{I}_{1,1}\widehat{I}_{2,2}\widehat{I}_{3,3} - \widehat{I}_{1,1}\widehat{I}_{2,3}^2 - \widehat{I}_{1,2}^2\widehat{I}_{3,3} + 2\widehat{I}_{1,2}\widehat{I}_{1,3}\widehat{I}_{2,3} - \widehat{I}_{1,3}^2\widehat{I}_{2,2} \right]$$
$$\Big/ \left[\widehat{I}_{1,1}\widehat{I}_{2,2}\widehat{I}_{3,3}\widehat{I}_{4,4} - \widehat{I}_{1,1}\widehat{I}_{2,2}\widehat{I}_{3,4}^2 - \widehat{I}_{1,1}\widehat{I}_{2,3}^2\widehat{I}_{4,4} + 2\widehat{I}_{1,1}\widehat{I}_{2,3}\widehat{I}_{2,4}\widehat{I}_{3,4} \right.$$
$$- \widehat{I}_{1,1}\widehat{I}_{2,4}^2\widehat{I}_{3,3} - \widehat{I}_{1,2}^2\widehat{I}_{3,3}\widehat{I}_{4,4} + \widehat{I}_{1,2}^2\widehat{I}_{3,4}^2 + 2\widehat{I}_{1,2}\widehat{I}_{1,3}\widehat{I}_{2,3}\widehat{I}_{4,4}$$
$$- 2\widehat{I}_{1,2}\widehat{I}_{1,3}\widehat{I}_{2,4}\widehat{I}_{3,4} - 2\widehat{I}_{1,2}\widehat{I}_{1,4}\widehat{I}_{2,3}\widehat{I}_{3,4} + 2\widehat{I}_{1,2}\widehat{I}_{1,4}\widehat{I}_{2,4}\widehat{I}_{3,3}$$
$$- \widehat{I}_{1,3}^2\widehat{I}_{2,2}\widehat{I}_{4,4} + \widehat{I}_{1,3}^2\widehat{I}_{2,4}^2 + 2\widehat{I}_{1,3}\widehat{I}_{1,4}\widehat{I}_{2,2}\widehat{I}_{3,4}$$
$$\left. - 2\widehat{I}_{1,3}\widehat{I}_{1,4}\widehat{I}_{2,3}\widehat{I}_{2,4} - \widehat{I}_{1,4}^2\widehat{I}_{2,2}\widehat{I}_{3,3} + \widehat{I}_{1,4}^2\widehat{I}_{2,3}^2 \right],$$

where $\widehat{I}_{1,1}$, $\widehat{I}_{1,2}$, $\widehat{I}_{1,3}$, $\widehat{I}_{1,4}$, $\widehat{I}_{2,2}$, $\widehat{I}_{2,3}$, $\widehat{I}_{2,4}$, $\widehat{I}_{3,3}$, $\widehat{I}_{3,4}$ and $\widehat{I}_{4,4}$ are the same as $I_{1,1}$, $I_{1,2}$, $I_{1,3}$, $I_{1,4}$, $I_{2,2}$, $I_{2,3}$, $I_{2,4}$, $I_{3,3}$, $I_{3,4}$ and $I_{4,4}$, respectively, with a, b, c and d replaced by $\widehat{a}$, $\widehat{b}$, $\widehat{c}$ and $\widehat{d}$, respectively.

10. SIMULATION STUDY

In this section, we evaluate the finite sample performance of the maximum likelihood estimators introduced in Section 9, focusing on biases, mean squared errors, coverage probabilities, and coverage lengths. We conducted the following simulation study to achieve this assessment.

a) set initial values for a, b, c and d;

b) simulate a random sample of size n from (1.1) by the inversion method;

c) compute the maximum likelihood estimates of a, b, c and d as well as their standard errors for the sample in step b);

d) repeat steps b) and c) a thousand times, giving the estimates $\widehat{a}_i$, $\widehat{b}_i$, $\widehat{c}_i$ and $\widehat{d}_i$ as well as their standard errors $\widehat{SE}\left(\widehat{a}_i\right)$, $\widehat{SE}\left(\widehat{b}_i\right)$, $\widehat{SE}\left(\widehat{c}_i\right)$ and $\widehat{SE}\left(\widehat{d}_i\right)$ for $i = 1, 2, \ldots, 1000$;

e) compute the biases of the estimators as

$$\text{bias}\left(\widehat{a}\right) = \frac{1}{1000} \sum_{i=1}^{1000} \left(\widehat{a}_i - a\right),$$

$$\text{bias}\left(\widehat{b}\right) = \frac{1}{1000} \sum_{i=1}^{1000} \left(\widehat{b}_i - b\right),$$

$$\text{bias}\left(\widehat{c}\right) = \frac{1}{1000} \sum_{i=1}^{1000} \left(\widehat{c}_i - c\right)$$

and

$$\text{bias}\left(\widehat{d}\right) = \frac{1}{1000} \sum_{i=1}^{1000} \left(\widehat{d}_i - d\right);$$

f) compute the mean squared errors of the estimators as

$$\text{MSE}\left(\widehat{a}\right) = \frac{1}{1000} \sum_{i=1}^{1000} \left(\widehat{a}_i - a\right)^2,$$

$$\text{MSE}\left(\widehat{b}\right) = \frac{1}{1000} \sum_{i=1}^{1000} \left(\widehat{b}_i - b\right)^2,$$

$$\text{MSE}\left(\widehat{c}\right) = \frac{1}{1000} \sum_{i=1}^{1000} \left(\widehat{c}_i - c\right)^2$$

and

$$\text{MSE}\left(\widehat{d}\right) = \frac{1}{1000} \sum_{i=1}^{1000} \left(\widehat{d}_i - d\right)^2;$$

g) compute the 95 percent coverage probabilities of the estimators as

$$\text{CP}\left(\widehat{a}\right) = \frac{1}{1000} \sum_{i=1}^{1000} I\left\{\widehat{a}_i - z_{0.975}\widehat{SE}\left(\widehat{a}_i\right) < a < \widehat{a}_i + z_{0.975}\widehat{SE}\left(\widehat{a}_i\right)\right\},$$

$$\mathrm{CP}\left(\widehat{b}\right) = \frac{1}{1000} \sum_{i=1}^{1000} I\left\{\widehat{b}_i - z_{0.975}\widehat{SE}\left(\widehat{b}_i\right) < b < \widehat{b}_i + z_{0.975}\widehat{SE}\left(\widehat{b}_i\right)\right\},$$

$$\mathrm{CP}\left(\widehat{c}\right) = \frac{1}{1000} \sum_{i=1}^{1000} I\left\{\widehat{c}_i - z_{0.975}\widehat{SE}\left(\widehat{c}_i\right) < c < \widehat{c}_i + z_{0.975}\widehat{SE}\left(\widehat{c}_i\right)\right\}$$

and

$$\mathrm{CP}\left(\widehat{d}\right) = \frac{1}{1000} \sum_{i=1}^{1000} I\left\{\widehat{d}_i - z_{0.975}\widehat{SE}\left(\widehat{d}_i\right) < d < \widehat{d}_i + z_{0.975}\widehat{SE}\left(\widehat{d}_i\right)\right\},$$

where $I\{\cdot\}$ denotes the indicator function;

h) compute the 95 percent coverage lengths of the estimators as

$$\mathrm{CL}\left(\widehat{a}\right) = \frac{z_{0.975}}{500} \sum_{i=1}^{1000} \widehat{SE}\left(\widehat{a}_i\right),$$

$$\mathrm{CL}\left(\widehat{b}\right) = \frac{z_{0.975}}{500} \sum_{i=1}^{1000} \widehat{SE}\left(\widehat{b}_i\right),$$

$$\mathrm{CL}\left(\widehat{c}\right) = \frac{z_{0.975}}{500} \sum_{i=1}^{1000} \widehat{SE}\left(\widehat{c}_i\right)$$

and

$$\mathrm{CL}\left(\widehat{d}\right) = \frac{z_{0.975}}{500} \sum_{i=1}^{1000} \widehat{SE}\left(\widehat{d}_i\right);$$

i) Repeat steps b) - h) for $n = 100, 101, \ldots, 1000$.

We initialized the parameters as $a = 2$, $b = 2$, $c = 1$, and $d = 0$. Figure 2 displays plots illustrating biases against n. In this figure, the horizontal lines represent zero biases. Figure 3 showcases plots depicting mean squared errors against n. Figure 4 exhibits plots demonstrating coverage probabilities against n, with horizontal lines indicating probabilities equal to 0.95. Finally, Figure 5 displays plots illustrating coverage lengths against n.

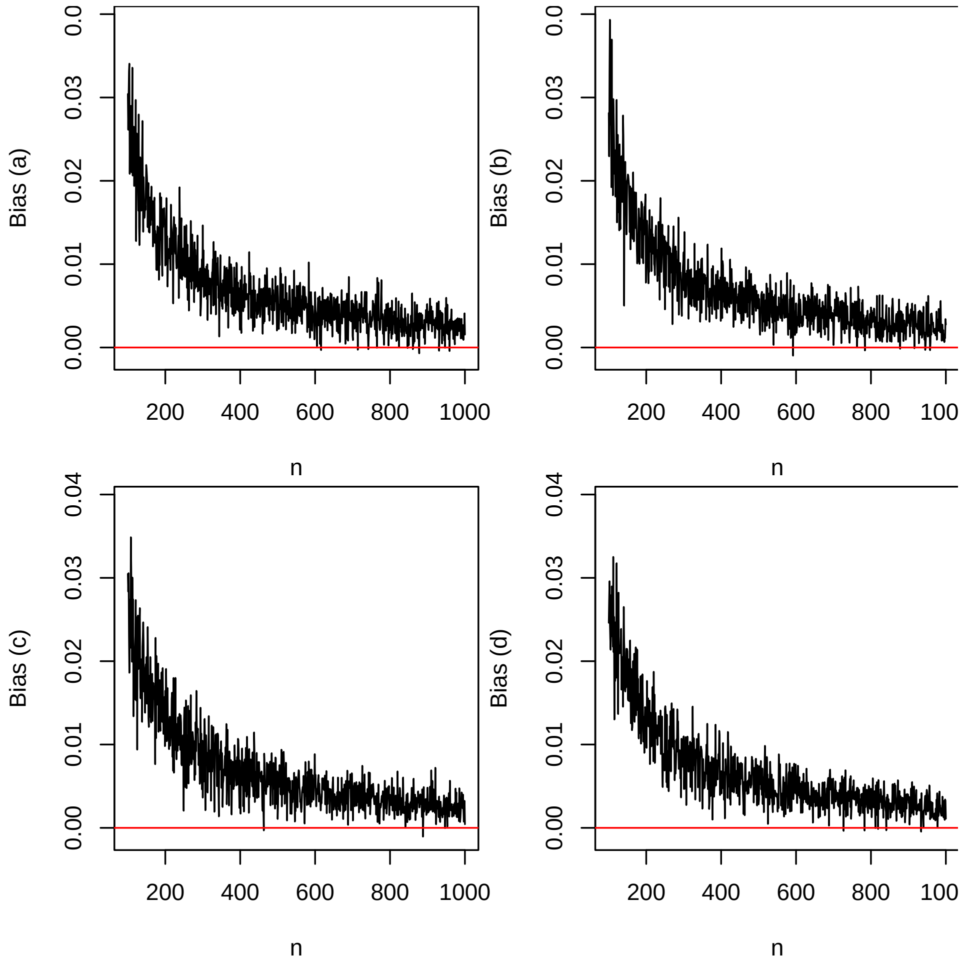

Figure 2: Biases of $\widehat{a}$ (top left), $\widehat{b}$ (top right), $\widehat{c}$ (bottom left) and $\widehat{d}$ (bottom right) versus n. The horizontal lines correspond to the biases being zero.

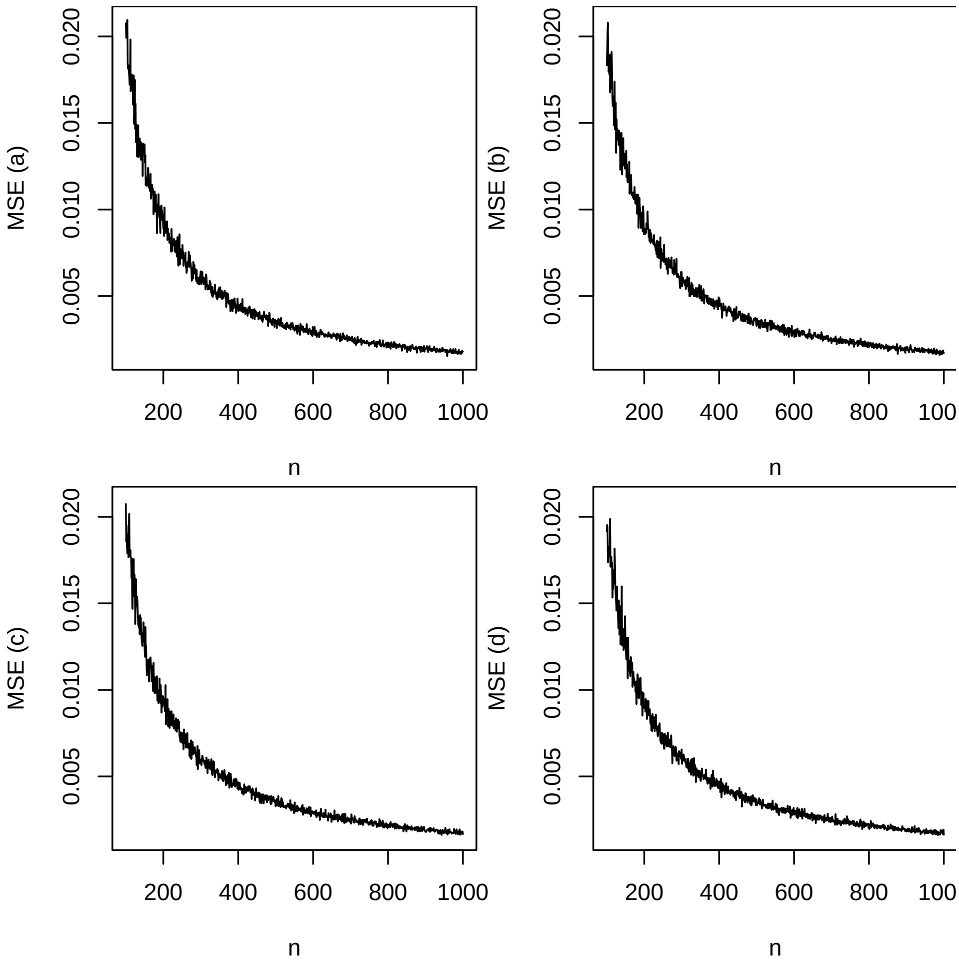

Figure 3: Mean squared errors of $\widehat{a}$ (top left), $\widehat{b}$ (top right), $\widehat{c}$ (bottom left) and $\widehat{d}$ (bottom right) versus n.

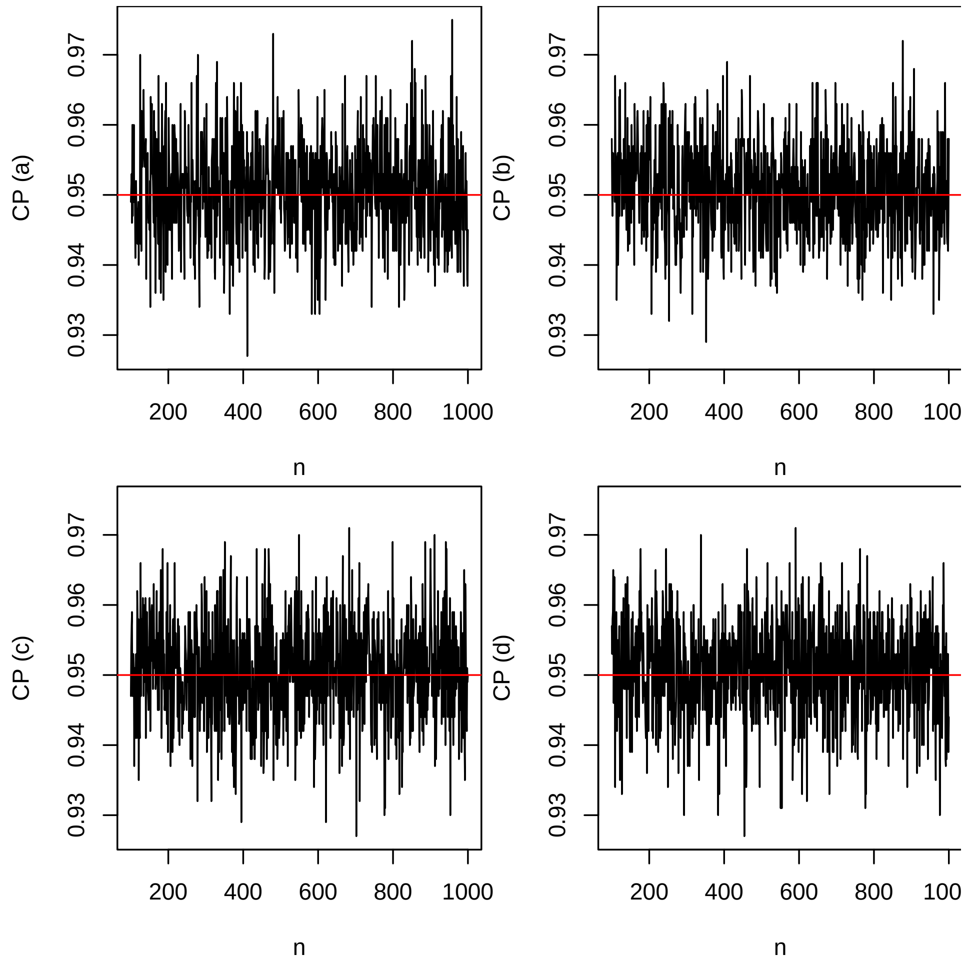

Figure 4: Coverage probabilities of $\widehat{a}$ (top left), $\widehat{b}$ (top right), $\widehat{c}$ (bottom left) and $\widehat{d}$ (bottom right) versus n. The horizontal lines correspond to the coverage probabilities being equal to 0.95.

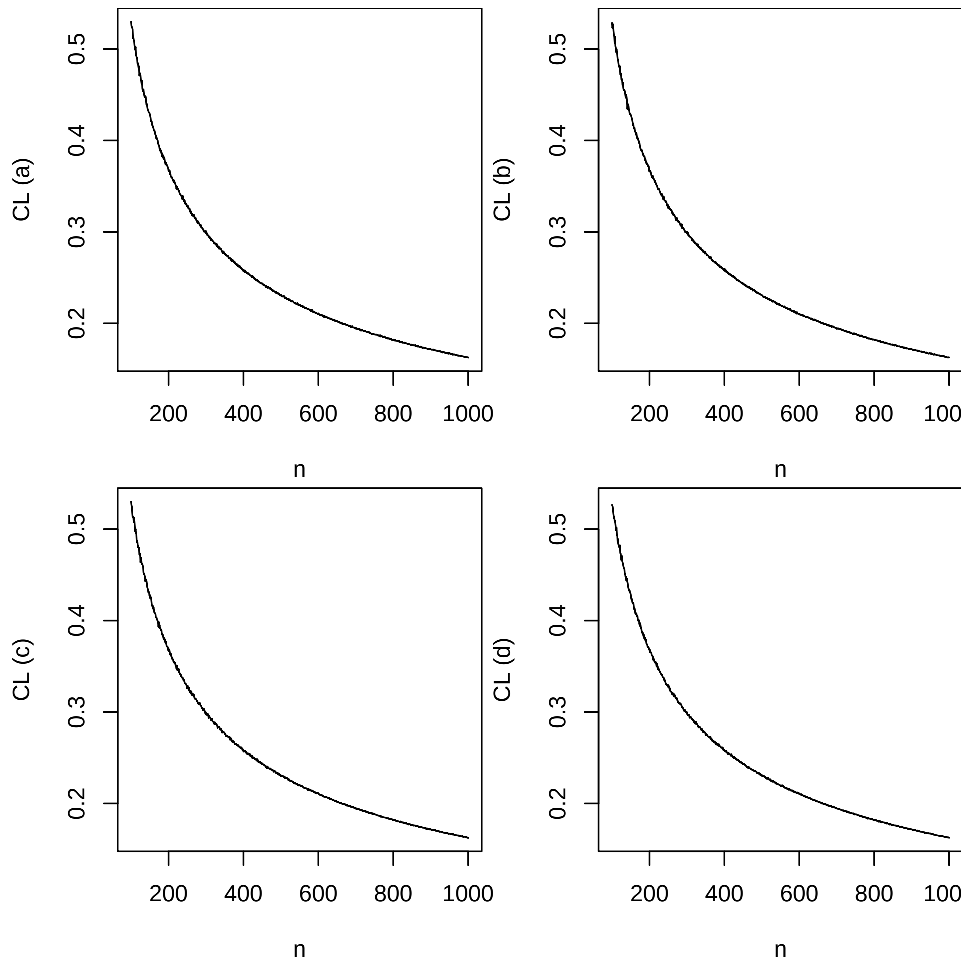

Figure 5: Coverage lengths of $\widehat{a}$ (top left), $\widehat{b}$ (top right), $\widehat{c}$ (bottom left) and $\widehat{d}$ (bottom right) versus n.

We can observe the following from the figures:

a) The biases are generally positive and decrease to zero with increasing n;

b) The biases appear larger for $\widehat{b}$ and smallest for $\widehat{d}$;

c) The mean squared errors generally decrease to zero with increasing n;

d) The coverage probabilities are around the nominal level even for n as small as 100;

e) The coverage lengths generally decrease to zero with increasing n.

The observations mentioned pertain to specific initial values of a, b, c, and d. However, these observations remain consistent across a broad spectrum of alternative values for a, b, c, and d. Notably, biases consistently diminished to zero as n increases, mean squared errors consistently decreased with increasing n, coverage probabilities consistently hovered around the nominal level even for relatively small n values such as 100, and coverage lengths consistently decreased to zero with increasing n.

11. REAL DATA APPLICATION

In this section, we conduct a performance comparison among the Gauss hypergeometric beta distribution, the standard beta distribution, the beta distributions proposed by Libby and Novick (1982) and the confluent hypergeometric beta distribution studied by Nadarajah and Kebe (2023), utilizing a real dataset. The dataset comprises the proportions of individuals voting "Remain" in the Brexit (EU referendum) polls, encompassing 126 polls conducted from January 2016 until the referendum date in June 2016. The specific values of the actual dataset are 0.52, 0.55, 0.49, 0.44, 0.54, 0.48, 0.41, 0.45, 0.42, 0.53, 0.45, 0.44, 0.44, 0.42, 0.42 0.37, 0.46, 0.43, 0.39, 0.45, 0.44, 0.46, 0.40, 0.48, 0.42, 0.44, 0.45, 0.43, 0.43, 0.48 0.41, 0.43, 0.40, 0.41, 0.42, 0.44, 0.51, 0.44, 0.44, 0.41, 0.41, 0.45, 0.55, 0.44, 0.44 0.52, 0.55, 0.47, 0.43, 0.55, 0.38, 0.36, 0.38, 0.44, 0.42, 0.44, 0.43, 0.42, 0.49, 0.39 0.41, 0.45, 0.43, 0.44, 0.51, 0.51, 0.49, 0.48, 0.43, 0.53, 0.38, 0.40, 0.39, 0.35, 0.45 0.42, 0.40, 0.39, 0.44, 0.51, 0.39, 0.35, 0.41, 0.51, 0.45, 0.49, 0.40, 0.48, 0.41, 0.46 0.47, 0.43, 0.45, 0.48, 0.49, 0.40, 0.40, 0.40, 0.39, 0.41, 0.39, 0.48, 0.48, 0.37, 0.38 0.42, 0.51, 0.45, 0.40, 0.54, 0.36, 0.43, 0.49, 0.41, 0.36, 0.42, 0.38, 0.55, 0.44, 0.54 0.41, 0.52, 0.42, 0.38, 0.42, 0.44.

The method of maximum likelihood was used. The fit of the standard beta distribution gave $\widehat{a} = 45.589(5.726)$ and $\widehat{b} = 57.576(7.240)$ with $\log L = 202.134$, AIC $= -400.269$ and BIC $= -394.596$, where the numbers within parentheses are standard errors. The fit of Libby and Novick (1982)'s beta distribution gave $\widehat{a} = 3.866 \times 10^6(2.048)$, $\widehat{b} = 27.119(0.462)$ and $\widehat{d} = 183838.1(1.192)$ with $\log L = 204.054$, AIC $= -402.109$ and BIC $= -393.600$. The fit of the confluent hypergeometric distribution due to Nadarajah and Kebe (2023) gave $\widehat{a} = 8.443(2.013)$, $\widehat{b} = 2.132 \cdot 10^{-5}(0.001)$ and $\widehat{c} = 1.929 \cdot 10^2(5.934)$ with $\log L = 203.635$, AIC $= -401.270$ and BIC $= -392.761$. The fit of (1.1) gave $\widehat{a} = 70.949(1.032)$, $\widehat{b} = 33.987(0.103)$, $\widehat{c} = 165.032(3.062)$ and $\widehat{d} = 83175.8(0.972)$ with $\log L = 213.481$, AIC $= -418.961$ and BIC $= -407.616$. We see that the fit of (1.1) gives the largest $\log L$, the smallest AIC and the smallest BIC. The use of likelihood ratio test shows that Libby and Novick (1982)'s beta distribution does not provide a significantly better than the standard beta distribution. But (1.1) provides significantly better fits than the standard beta and Libby and Novick (1982)'s beta distributions.

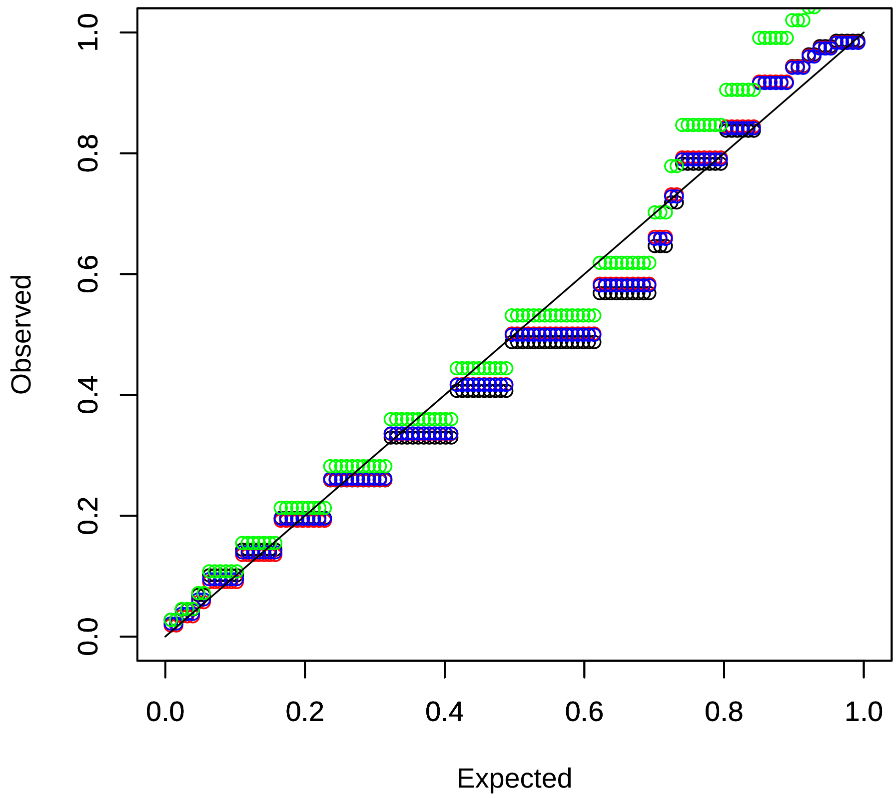

Figure 6: Probability plots of the fits of the standard beta (black), Libby and Novick (1982)'s beta (red), the confluent hypergeometric beta (blue) and the Gauss hypergeometric beta (green) distributions.

The superior fit of (1.1) is substantiated by the probability plots depicted in Figure 6. Notably, the plotted points for (1.1) closely align with the diagonal line, demonstrating a closer proximity overall, with slight deviations observed primarily in the upper tail.

12. CONCLUSIONS

This paper has revisited the Gauss hypergeometric beta distribution originally introduced by Gordy (1998) and explored its various mathematical properties. While some of these properties are already established, the majority are novel discoveries. Notably, we have delved into the shape properties of the probability density function, derived expressions for moment generating and characteristic functions, established formulas for harmonic means, investigated variances of inverse Gauss hypergeometric beta random variables, determined conditional moments, computed geometric mean, relative entropy, geometric variances, and geometric covariance, analyzed stochastic ordering properties, and proposed procedures for maximum likelihood estimation, all of which are original contributions. Furthermore, we have evaluated the finite sample performance of maximum likelihood estimators and provided a practical application using real data.

Future research avenues include exploring estimation methods beyond maximum likelihood, such as Bayesian methods, method of moments, generalized method of moments, method of probability weighted moments, least squares methods (including weighted least squares and method of maximum entropy), pseudo maximum likelihood, minimum chi-square method, minimum L_p norm method, minmax method, M-estimation, quantile-based methods, ranked set sampling method, and bootstrap methods.

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