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A New General Class of Ridge-Type Estimator in Linear Regression Models

Authors: Kadri Ulaş AKAY ✉
– Department of Mathematics, Science Faculty, University of Istanbul
Istanbul, Türkiye
kulas@istanbul.edu.tr

Esra ERTAN
– Department of Mathematics, Science Faculty, University of Istanbul
Istanbul, Türkiye
eertan@istanbul.edu.tr

Ali ERKOÇ
– Department of Statistics, Faculty of Arts and Sciences,
Mimar Sinan Fine Arts University, Istanbul, Türkiye
ali.erkoc@msgsu.edu.tr

Ferhat TAŞ
– Department of Mathematics, Science Faculty, University of Istanbul
Istanbul, Türkiye
tasf@istanbul.edu.tr

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Abstract:

- In linear regression models, researchers have developed new biased estimators to mitigate the effects of multicollinearity instead of using the Ordinary Least Squares (OLS) estimator, which is affected by multicollinearity. In this study, we define a general class of estimators called Ridge-type estimators (RTE). The superiority of RTE over other biased estimators is investigated under the matrix mean square error criterion. In addition, two separate Monte Carlo simulation studies are conducted to compare the performance of the considered biased estimators. A numerical example is given to demonstrate the superiority of the proposed estimator over other biased estimators.

Keywords:

- *Biased regression; Liu Estimator; Liu-type estimator; Multicollinearity; Ridge Estimator.*

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✉ Corresponding author

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1. INTRODUCTION

Regression analysis is widely used in many disciplines, including business, engineering, agriculture, and economics, to describe the statistical relationship between explanatory and response variables by using a model. The linear regression model, which assumes that the response variable is normally distributed, is one of the most commonly used statistical models. Let us consider the following linear regression model:

$$Y = X\beta + \varepsilon \tag{1.1}$$

where Y is an $n \times 1$ vector of dependent variables, X is an $n \times p$ full column rank matrix of n observations on p independent explanatory variables, β is a $p \times 1$ vector of unknown parameters, and ε is an $n \times 1$ vector of random errors which are distributed as Normal with the mean vector 0 and the covariance matrix $\sigma^2 I$. The Ordinary Least Squares (OLS) estimator of β is given by

$$\hat{\beta}_{OLS} = (X'X)^{-1} X'Y. \tag{1.2}$$

In addition, the covariance matrix of $\hat{\beta}_{OLS}$ is obtained as $cov\left(\hat{\beta}_{OLS}\right) = \sigma^2 (X'X)^{-1}$. In linear regression models, computational difficulties arise when the independent variables are collinear. The problem of multicollinearity occurs when one or more variables can be expressed as an exact or almost linear combination of the others in the data set. Multicollinearity will also provide statistical challenges if the problem aims to estimate parameters. There are many criteria to determine multicollinearity. Multicollinearity causes the diagonal elements of $(X'X)^{-1}$ to inflate, which implies that the estimated variance of $\hat{\beta}_{OLS}$ will be large. In addition, the coefficients of the OLS estimator may have wrong signs and large variances and be statistically insignificant. For such cases, alternative biased estimators have been proposed by many researchers to overcome the problems caused by the presence of multicollinearity. Issues related to these proposed biased estimators in linear regression models have been investigated and discussed in the literature by many researchers (Stein [30]; Hoerl and Kennard [12]; Liu [19]; Liu [20]; Kibria [15]; Özkale and Kaçıranlar [24]; Sakallıoğlu and Kaçıranlar [27]; Yang and Chang [34]; Kurnaz and Akay [17]; Kurnaz and Akay [18]; Qasim *et al.* [25]; Lukman *et al.* [21]; Lukman *et al.* [22]; Aslam and Ahmad [4]; Zeinal and Azmoun [35]; Üstündağ *et al.* [33]; Ahmad and Aslam [1]; Babar and Chand [5]; Dawoud *et al.* [6]; Qasim *et al.* [26]; Shewa and Ugwuowo [28]; Idowu *et al.* [14]). The Ridge Estimator (RE), proposed by Hoerl and Kennard [12], is the most significant of these estimators. The RE is defined by

$$\hat{\beta}_{RE} = (X'X + kI)^{-1} X'Y, \qquad k > 0 \tag{1.3}$$

where k is a biasing parameter. On the other hand, Liu [19] proposed the Liu Estimator (LE) combining the advantages of RE and Stein estimator. The Stein estimator was defined by Stein [30] as follows $\hat{\beta}_S = c\hat{\beta}_{OLS}$ where $0 < c < 1$. The LE is defined as follows:

$$\hat{\beta}_{LE} = (X'X + I)^{-1}\left(X'Y + d\hat{\beta}_{OLS}\right), \quad 0 < d < 1 \tag{1.4}$$

where d is a biasing parameter. On the other hand, Lukman *et al.* [22] noted that the estimates of the parameter d in LE are usually negative. To overcome this, model (1.1) is augmented with $-d_{ML}\hat{\beta}_{OLS} = \beta + \varepsilon'$ and then the OLS method is used. The resulting estimator is called the Modified One-Parameter Liu (ML) Estimator and is defined as follows:

$$\hat{\beta}_{ML} = (X'X + I)^{-1}(X'X - d_{ML}I)\hat{\beta}_{OLS}, \quad 0 < d_{ML} < 1 \tag{1.5}$$

where d_{ML} is a biasing parameter. According to Lukman *et al.* [22], this modification provides a positive value of the biasing parameter d_{ML}. However, although RE and LE are often preferred in the presence of collinearity in linear regression models, these estimators have some drawbacks. Researchers have developed estimators with two biasing parameters k and d to cover both RE and LE. For example, Liu [20] introduced an estimator that is based on k and d as follows:

$$\hat{\beta}_{LTE} = (X'X + kI)^{-1}\left(X'Y - d\hat{\beta}^*\right), \quad k > 0, \quad -\infty < d < \infty \tag{1.6}$$

where $\hat{\beta}^*$ can be any estimator of β. This estimator, which is called the Liu-type estimator, is obtained by augmenting $\left(-d/k^{1/2}\right)\beta^* = k^{1/2}\beta + \varepsilon'$ to (1.1) and then using the OLS method (Liu [20]). As an alternative, Özkale and Kaçıranlar [24] introduced a Two-Parameter Estimator (TPE) as follows:

$$\hat{\beta}_{TPE} = (X'X + kI)^{-1}\left(X'Y + kd\hat{\beta}_{OLS}\right), \quad k > 0, \quad 0 < d < 1, \tag{1.7}$$

where k and d are two biasing parameters. The TPE is a general estimator that includes the OLS, RE, and LE as special cases. As an alternative to the estimators introduced so far, Kurnaz and Akay [17] proposed a general Liu-type estimator that includes estimators given by(1.2), (1.3),(1.4), (1.5), (1.6), and (1.7) estimators as special cases. The new Liu-type estimator is defined as follows:

$$\hat{\beta}_{NLTE} = (X'X + kI)^{-1}\left(X'Y + f(k)\hat{\beta}^*\right), \quad k > 0 \tag{1.8}$$

where $\hat{\beta}^*$ is any estimator of β, and $f(k)$ is a continuous function of the biasing parameter k. Similarly, NLTE is obtained by augmenting $\frac{f(k)}{k^{1/2}}\hat{\beta}^* = k^{1/2}\beta + \varepsilon'$ to (1.1) and then using the OLS method. For example, if $f(k) = -k$ and $\hat{\beta}^* = \hat{\beta}_{OLS}$, the KL estimator given by Kibria and Lukman [16] is obtained. The KL estimator, which is a special case of the estimator (1.8), is defined as follows:

$$\hat{\beta}_{KL} = (X'X + kI)^{-1}(X'X - kI)\hat{\beta}_{OLS}, \quad k > 0 \tag{1.9}$$

where k is the biasing parameter. On the other hand, Qasim et. al. [26] proposed a Two-Step Shrinkage (TSS) estimator in the presence of multicollinearity as follows:

$$\hat{\beta}_{TSS} = (X'X + kI)^{-1}(X'X - kdI)\hat{\beta}_{OLS}, \quad k > 0, \ 0 \leq d < 1 \tag{1.10}$$

where k and d are two biasing parameters. Note that this estimator given in (1.10) can be obtained by taking $f(k) = -kd$ and $\hat{\beta}^* = \hat{\beta}_{OLS}$ in (1.8). Furthermore, Sakallıoğlu and Kaçıranlar [27] proposed another biased estimator based on RE which is given by

$$\hat{\beta}_{SK}(k,d) = (X'X+I)^{-1}\left(X'Y + d\hat{\beta}_{RE}\right), \qquad k>0, \ -\infty < d < \infty \tag{1.11}$$

where k and d are two biasing parameters. The estimator given in (1.11) is called a k-d class estimator and is a general estimator that includes the OLS, RE, and LEs as special cases (Sakallıoğlu and Kaçıranlar [27]). The k-d class estimator is obtained by augmenting the equation $d\hat{\beta}_{RE} = \beta + \varepsilon'$ to (1.1) and using the OLS method, too. Also, Yang and Chang [34] proposed a new biased estimator based on RE as follows:

$$\hat{\beta}_{YC}(k,d) = (X'X+I)^{-1}(X'X+dI)\hat{\beta}_{RE}, \qquad k>0, \ 0<d<1 \tag{1.12}$$

where k and d are two biasing parameters. The estimator given in (1.12) is obtained by augmenting $(d-k)\hat{\beta}_{RE} = \beta + \varepsilon'$ to (1.1) and using the OLS method. In addition, the YC estimator is a general estimator that includes OLS, RE, and LE as special cases. Ahmad and Aslam [1] proposed another biased estimator similar to the YC estimator. Instead of $\hat{\beta}_{RE}$ in (1.12), they used the estimator proposed by Dorugade [7] . This estimator, called Modified New Two-Parameter Estimator (MNTPE), is given as follows:

$$\hat{\beta}_{MNTP} = (X'X+I)^{-1}(X'X+dI)(X'X+kdI)^{-1}X'Y, \qquad k>0, \ 0<d<1 \tag{1.13}$$

where k and d are two biasing parameters. Dawoud *et al.* [6] proposed another biased estimator with the biasing parameters k and d similar to the YC estimator. They also used a similar approach applied by Ahmad and Aslam [1]. Instead of OLS in (1.7), defined by Özkale and Kaçıranlar [24], they preferred to use the KL estimator. They defined this estimator, called the NBR estimator, as follows:

$$\hat{\beta}_{NBR} = (X'X+kI)^{-1}(X'X+kdI)(X'X+kI)^{-1}(X'X-kI)\hat{\beta}_{OLS}, \tag{1.14}$$

where $k>0$ and $0<d<1$ are two biasing parameters. On the other hand, Shewa and Ugwuowo [28] proposed another biased estimator based on the KL estimator. Following the modification by Aladeitan *et al.* [3], they proposed a new estimator called KL-MRT as follows:

$$\hat{\beta}_{KLMRT} = (X'X+kI)^{-1}(X'X-kI)(X'X+k(1+d)I)^{-1}X'Y, \quad k\geq 0, d\geq 0 \tag{1.15}$$

where k and d are two biasing parameters. On the other hand, Idowu *et al.* [14] made a modification to the LE given by (1.4). Instead of the OLS estimator utilized in LE, they used the KL estimator given by (1.9). Their estimator called LKL is defined as follows:

$$\hat{\beta}_{LKL} = (X'X+I)^{-1}(X'X+dI)(X'X+kI)^{-1}(X'X-kI)\hat{\beta}_{OLS}, \quad k>0, \ 0<d<1 \tag{1.16}$$

where k and d are two biasing parameters. The estimators with two biasing parameters k and d have been generally developed based on RE, LE, and LTE. In particular, these estimators depend on the OLS estimator, and a more powerful estimator is preferred over the OLS estimator to minimize the effects of multicollinearity. In addition to these modifications to reduce the effects of multicollinearity, it is also necessary to consider the optimal performance of the proposed estimator. From another point of view, as the number of biasing parameters included in the estimator increases, it becomes more difficult to assess the optimal performance of the estimator because the performance of biased estimators is affected by the selection of the biasing parameter. In general, the estimates of the biasing parameters are obtained in such a way that the scalar mean square error function is minimized. Since the mean square error function is a nonlinear function of the biasing parameters, the estimates of these biasing parameters can be approximately obtained. There are many studies focusing on this issue in the literature (Hoerl and Kennard [12]; Liu [20]; Kibria [15]; Yang and Chang [34]; Sakallıoğlu and Kaçıranlar [27]; Shukur, Månsson, and Sjölander [29]; Lukman *et al.* [22]; Ahmad and Aslam [1]; Dawoud et al. [6]; Qasim *et al.* [25]; Shewa and Ugwuowo [28]; Idowu *et al.* [14]).

On the other hand, estimators with two biasing parameters k and d have attracted the attention of many researchers in recent years. However, the most important problem for these estimators is that the number of these biasing parameters is large and it is also very difficult to find their optimal estimates. Although many iterative techniques have been proposed to find the optimal estimates of these biasing parameters, it is a complex process to obtain these estimates. In these cases, one of the biasing parameters can be estimated depending on the other biasing parameters or vice versa (Liu [20]; Özkale and Kaçıranlar [24]; Sakallıoğlu and Kaçıranlar [27]; Yang and Chang [34]; Ahmad and Aslam [1]; Dawoud *et al.* [6]; Qasim et al. [26]; Shewa and Ugwuowo [28]; Idowu *et al.* [14]). Therefore, it can be considered that there is an unknown functional relationship between these two biasing parameters k and d. In the literature, there are some studies that examine the applications of this consideration in various other statistical models(Ertan and Akay [9], Akay and Ertan [2], and Erkoç *et al.* [8])

The purpose of this paper is to examine the performance of the estimator to be obtained under the hypothesis of an unknown functional relationship between these two biasing parameters k and d. In this context, we first develop a new hybrid estimator that combines the advantages of LE and RE. Then, we try to find the optimal functional relationship between the biasing parameters. With the help of the functional structure used in this hybrid estimator, it is expected that the estimated model parameter values will not be affected at large biasing parameter values k. In addition to this feature, the proposed hybrid estimator can be defined to include the estimators given by (1.2), (1.3), (1.4), (1.5), (1.11), and (1.12) estimators as special cases. In other words, it can be said that the proposed estimator forms a general class of estimators like the estimator given in (1.8). In addition, a comprehensive comparison of these two proposed classes of estimators was carried out using simulation studies.

The article is organized as follows: In Section 2, the proposed biased estimator is introduced and some properties are given. In Section 3, a general theorem is given to compare RTE and NLTE in the sense of the matrix mean square error. In Section 4, alternative approaches to determine the functional relationship between the biasing parameters are presented. Two Monte Carlo simulation studies are designed to evaluate the performances of the considered estimators in Section 5. In Section 6, the performance evaluation of all considered estimators is given in the Portland cement data. Finally, the conclusion of the study is given in Section 7.

2. A new general Ridge-type estimator

To mitigate the effect of multicollinearity, researchers have made efforts to develop alternative estimators for linear regression models instead of the OLS, which are affected by collinearity between variables. Especially when the estimators given by (1.11) and (1.12) are examined, it is observed that RE, which is more resistant to collinearity effects, is used instead of the OLS estimator. However, a major disadvantage of RE is that it can result in small parameter estimates at large values of the biasing parameter k. To overcome this problem, researchers have developed hybrid estimators that combine the advantages of RE and Liu Estimators (Sakallıoğlu and Kaçıranlar [27]; Yang and Chang [34]). In order to collect these estimators under a general class with the help of an unknown functional relationship that can be among the biasing parameters, we can define the new Ridge-type estimator (RTE) for β as follows:

$$\hat{\beta}_{RTE}(k) = (X'X+I)^{-1}(X'X+g(k)I)(X'X+kI)^{-1}X'Y, \quad k>0 \tag{2.1}$$

where $g(k)$ is a continuous function of the biasing parameter k. We can obtain the estimator given in (2.1) by augmenting $(g(k)-k)\hat{\beta}_{RE} = \beta + \varepsilon'$ to model (1.1) and using the OLS method. The advantage of RTE over other estimators is that the $g(k)$ function helps us determine the optimal estimator. When we select $g(k)$ as a linear function of the biasing parameter, such as $g(k) = ak+b$ where $a,b \in R$, RTE is a general estimator that includes other biased estimators as follows: $\hat{\beta}_{RTE} = \hat{\beta}_{OLS}$ for $g(0)=1$ where $k=0$ and $b=1$. $\hat{\beta}_{RTE} = \hat{\beta}_{RE}$ for $g(k)=1$ where $a=0$ and $b=1$. $\hat{\beta}_{RTE} = \hat{\beta}_{LE}$ for $g(0)=b$ where b corresponds to the biasing parameter d. $\hat{\beta}_{RTE} = \hat{\beta}_{ML}$ for $g(0) = -b$ where b corresponds to the biasing parameter d_{ML}. $\hat{\beta}_{RTE} = \hat{\beta}_{YC}(k,d)$ for $g(k)=b$ where $a=0$ and the b corresponds to the biasing parameter d. $\hat{\beta}_{RTE} = \hat{\beta}_{SK}(k,d)$ for $g(k)=k+b$ where $a=1$ and b corresponds to the biasing parameter d. Note that the proposed estimator given in (2.1) is different from the biased estimator given in (1.7). That is, when we use $\hat{\beta}_{RE}$ instead of $\hat{\beta}^*$ in (1.8), $\hat{\beta}_{NLTE}$ does not correspond to the estimator in (2.1). Also, if $\hat{\beta}_{RE}$ is used instead of $\hat{\beta}^*$ in (1.8), the obtained estimator does not exactly correspond to the estimators proposed by Yang and Chang [34] and

Sakallıoğlu and Kaçıranlar [27], respectively. We rewrite the model given in (1.1) in canonical form:

$$Y = Z\alpha + \varepsilon \tag{2.2}$$

where $Z = XQ$, $\alpha = Q'\beta$, and Q is the orthogonal matrix whose columns constitute the eigenvectors of $X'X$. Then $Z'Z = Q'X'XQ = \Lambda = diag\left(\lambda_1, \lambda_2, ..., \lambda_p\right)$ where $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_p \geq 0$ are the ordered eigenvalues of $X'X$. For the model (2.2), we can rewrite the above estimators in canonical form as follows:

(2.3)

$$\hat{\alpha}_{NLTE} = (\Lambda + kI)^{-1} (\Lambda + f(k) I) \hat{\alpha}_{OLS} = (\Lambda + kI)^{-1} (\Lambda + f(k) I) \Lambda^{-1} Z'Y \ = A_1 Y$$

$$\hat{\alpha}_{RTE} = (\Lambda + I)^{-1} (\Lambda + g(k) I) (\Lambda + kI)^{-1} Z'Y = A_2 Y \tag{2.4}$$

where the other existing estimators can be obtained based on the appropriate selection of $f(k)$ and $g(k)$. As known, Matrix Mean Squared Error (MMSE) and Scalar Mean Squared Error (SMSE) are the two most common methods used to detect the superiority of the estimators to each other. The MMSE and SMSE of an estimator $\tilde{\beta}$ is defined as:

$$\begin{aligned} &MMSE\left(\tilde{\beta}\right) = var\left(\tilde{\beta}\right) + \left[bias\left(\tilde{\beta}\right)\right]\left[bias\left(\tilde{\beta}\right)\right]' \\ &SMSE\left(\tilde{\beta}\right) = tr\left(MMSE\left(\tilde{\beta}\right)\right) = tr\left(var\left(\tilde{\beta}\right)\right) + bias\left(\tilde{\beta}\right)' bias\left(\tilde{\beta}\right). \end{aligned} \tag{2.5}$$

where $var\left(\tilde{\beta}\right)$ is the variance-covariance matrix and $bias\left(\tilde{\beta}\right) = E\left(\tilde{\beta}\right) - \beta$ is the biasing vector. Let $\tilde{\beta}_1$ and $\tilde{\beta}_2$be any two estimators of parameter β. Then, $\tilde{\beta}_2$ is superior to $\tilde{\beta}_1$ with respect to the MMSE criterion if and only if $MMSE\left(\tilde{\beta}_1\right) - MMSE\left(\tilde{\beta}_2\right)$ is a positive definite (pd) matrix. If $MMSE\left(\tilde{\beta}_1\right) - MMSE\left(\tilde{\beta}_2\right)$ is a non-negative definite matrix, then $SMSE\left(\tilde{\beta}_1\right) - SMSE\left(\tilde{\beta}_2\right) \geq 0$. But, the reverse is not always true (Theobald, 1974). Because of the relation of $\alpha = Q'\beta$; $\hat{\beta}_{OLS}$, $\hat{\beta}_{RE}, \hat{\beta}_{LE}$, $\hat{\beta}_{NLTE}, \hat{\beta}_{SK}(k,d), \hat{\beta}_{YC}(k,d)$ and $\hat{\beta}_{RTE}(k)$ have the same mean squared error values as $\hat{\alpha}_{OLS}$, $\hat{\alpha}_{RE}, \hat{\alpha}_{LE}$, $\hat{\alpha}_{NLTE}$, $\hat{\alpha}_{SK}(k,d)$, $\hat{\alpha}_{YC}(k,d)$ and $\hat{\alpha}_{RTE}(k)$, respectively. To compare the biased estimators mentioned above in terms of MMSE, we use the following theorems:

Theorem 2.1. *(Farebrother [10]): Let A be a positive definite matrix, namely $A > 0$, and c be a nonzero vector. Then, $A - cc'$ is a positive definite matrix iff $c'A^{-1}c \leq 1$.*

Theorem 2.2. *(Trenkler and Toutenburg [32]): Let $\tilde{\beta}_l = B_l Y$, $l = 1,2$ be two homogeneous linear estimators of β and C be a positive definite matrix, where $C = B_1 B_1' - B_2 B_2'$. Then $MMSE\left(\tilde{\beta}_1\right) - MMSE\left(\tilde{\beta}_2\right) > 0$ if and only if $bias\left(\tilde{\beta}_2\right)' \left(\sigma^2 C + bias\left(\tilde{\beta}_1\right) bias\left(\tilde{\beta}_1\right)'\right)^{-1} bias\left(\tilde{\beta}_2\right) < 1$.*

3. The superiority of the proposed Ridge-Type estimator

In this section, we give a general theorem to compare RTE and NLTE in the sense of MMSE. With this general theorem, it is possible to compare the above-mentioned estimators obtained by choosing different $g(k)$ and $f(k)$ functions in terms of MMSE sense. As a result of this comparison, the superiority of RTE over OLS, RE, LE, LTE, TPE, ML, TSS, and KL estimators is determined. Similarly, to determine the superiority of the RTE over the MNTP, NBR, KLMRT, and LKL estimators, the constraints on the function $g(k)$ are given.

3.1. The comparison between the RTE and the NLTE estimator

Firstly, we can compute *the MMSE* of $\hat{\alpha}_{NLTE} = A_1 Y$ and $\hat{\alpha}_{RTE} = A_2 Y$ as follows:

$$\begin{aligned}
\mathrm{MMSE}(\hat{\alpha}_{NLTE}) =& \sigma^2 A_1 A_1' + (A_1 Z - I)\alpha\alpha'(A_1 Z - I)\\
=& \sigma^2 (\Lambda + kI)^{-1}(\Lambda + f(k)I)\Lambda^{-1}(\Lambda + f(k)I)(\Lambda + kI)^{-1}\\
&+ (f(k) - k)^2(\Lambda + kI)^{-1}\alpha\alpha'(\Lambda + kI)^{-1}\\
\mathrm{MMSE}(\hat{\alpha}_{RTE}) =& \sigma^2 A_2 A_2' + (A_2 Z - I)\alpha\alpha'(A_2 Z - I)\\
=& \sigma^2 (\Lambda + I)^{-1}(\Lambda + g(k)I)(\Lambda + kI)^{-1}\Lambda(\Lambda + kI)^{-1}(\Lambda + g(k)I)(\Lambda + I)^{-1}\\
&+ ((g(k) - k - 1)\Lambda - kI)(\Lambda + I)^{-1}(\Lambda + kI)^{-1}\alpha\alpha'(\Lambda + kI)^{-1}(\Lambda + I)^{-1}((g(k) - k - 1)\Lambda - kI)
\end{aligned}$$

Then, we can give the following theorem:

Theorem 3.1. *Let be $k > 0$ and $-\lambda_j - \frac{|\lambda_j + f(k)|(\lambda_j + 1)}{\lambda_j} < g(k) < -\lambda_j + \frac{|\lambda_j + f(k)|(\lambda_j + 1)}{\lambda_j}$ where $j = 1, 2, ..., p+1$. Then, $MMSE(\hat{\alpha}_{NLTE}) - MMSE(\hat{\alpha}_{RTE}) > 0$ if and only if*

(3.1)
$$bias(\hat{\alpha}_{RTE})' \left[\sigma^2 (A_1 A_1' - A_2 A_2') + bias(\hat{\alpha}_{NLTE}) bias(\hat{\alpha}_{NLTE})'\right]^{-1} bias(\hat{\alpha}_{RTE}) < 1$$

where $\hat{\alpha}_{RTE}$ and $\hat{\alpha}_{NLTE}$ are two estimators for α and $bias(\hat{\alpha}_{NLTE}) = (f(k) - k)(\Lambda + kI)^{-1}\alpha$.

Proof: Using (2.3) and (2.4), we obtain

$$\begin{aligned}
\mathrm{cov}(\hat{\alpha}_{NLTE}) - \mathrm{cov}(\hat{\alpha}_{RTE}) &= \sigma^2 \left[A_1 A_1' - A_2 A_2'\right]\\
&= \sigma^2 \Big[(\Lambda + kI)^{-1}(\Lambda + f(k)I)\Lambda^{-1}(\Lambda + f(k)I)(\Lambda + kI)^{-1}\\
&\quad -(\Lambda + I)^{-1}(\Lambda + g(k)I)(\Lambda + kI)^{-1}\Lambda(\Lambda + kI)^{-1}(\Lambda + g(k)I)(\Lambda + I)^{-1}\Big]\\
&= \sigma^2 \,\mathrm{diag}\left\{\frac{\left(\lambda_j + f(k)\right)^2}{\lambda_j\left(\lambda_j + k\right)^2} - \frac{\left(\lambda_j + g(k)\right)^2 \lambda_j}{\left(\lambda_j + 1\right)^2\left(\lambda_j + k\right)^2}\right\}_{j=1}^{p+1}.
\end{aligned}$$

We observe that $A_1 A_1' - A_2 A_2' > 0$ if and only if $\left(\lambda_j + 1\right)^2\left(\lambda_j + f(k)\right)^2 - \lambda_j^2\left(\lambda_j + g(k)\right)^2 > 0$. If this inequality is rearranged for $g(k)$ function, we can obtain $-\lambda_j - \frac{|\lambda_j + f(k)|(\lambda_j + 1)}{\lambda_j} <$

$g(k) < -\lambda_j + \frac{|\lambda_j+f(k)|(\lambda_j+1)}{\lambda_j}$ where $j = 1,2,...,p+1$. That is, the RTE is superior to NLTE when $g(k)$ function is selected as $-\lambda_j - \frac{|\lambda_j+f(k)|(\lambda_j+1)}{\lambda_j} < g(k) < -\lambda_j + \frac{|\lambda_j+f(k)|(\lambda_j+1)}{\lambda_j}$ where $j = 1,2,...,p+1$. Therefore, $A_1A_1' - A_2A_2'$ is the pd matrix. By Theorem 2.2, the proof is complete. □

3.2. The comparison between the RTE and the MNTP estimator

The $MMSE$ of $\hat{\alpha}_{MNTP} = (\Lambda + I)^{-1}(\Lambda + dI)(\Lambda + kdI)^{-1} Z'Y = A_3Y$ estimator is

$$MMSE(\hat{\alpha}_{MNTP}) = \sigma^2 A_3A_3' + (A_3Z - I)\alpha\alpha'(A_3Z - I).$$

We use the MMSE difference given below to compare the MNTP and the RTE: $MMSE(\hat{\alpha}_{MNTP}) - MMSE(\hat{\alpha}_{RTE}) = \sigma^2\left[A_3A_3' - A_2A_2'\right] + (A_3Z - I)\alpha\alpha'(A_3Z - I) - (A_2Z - I)\alpha\alpha'(A_2Z - I)$

Then, we give the following theorem:

Theorem 3.2. *Let be $k > 0$, $0 < d < 1$ and $-\lambda_j - \frac{(\lambda_j+d)(\lambda_j+k)}{(\lambda_j+kd)} < g(k) < -\lambda_j + \frac{(\lambda_j+d)(\lambda_j+k)}{(\lambda_j+kd)}$ where $j = 1,2,...,p+1$. Then, $MMSE(\hat{\alpha}_{MNTP}) - MMSE(\hat{\alpha}_{RTE}) > 0$ if and only if*

$$bias(\hat{\alpha}_{RTE})'\left[\sigma^2(A_3A_3' - A_2A_2') + (A_3Z - I)\alpha\alpha'(A_3Z - I)\right]^{-1} bias(\hat{\alpha}_{RTE}) < 1 \tag{3.2}$$

where $\hat{\alpha}_{RTE}$ and $\hat{\alpha}_{MNTP}$ are two linear estimators for the parameter α.

Proof: We can obtain

$$\begin{aligned}
\operatorname{cov}(\hat{\alpha}_{MNTP}) - \operatorname{cov}(\hat{\alpha}_{RTE}) &= \sigma^2\left[A_3A_3' - A_2A_2'\right] \\
&= \sigma^2\Big[(\Lambda+I)^{-1}(\Lambda+dI)(\Lambda+kdI)^{-1}\Lambda(\Lambda+kdI)^{-1}(\Lambda+dI)(\Lambda+I)^{-1} \\
&\quad -(\Lambda+I)^{-1}(\Lambda+g(k)I)(\Lambda+kI)^{-1}\Lambda(\Lambda+kI)^{-1}(\Lambda+g(k)I)(\Lambda+I)^{-1}\Big] \\
&= \sigma^2 \operatorname{diag}\left\{\frac{(\lambda_j+d)^2\lambda_i}{(\lambda_j+1)^2(\lambda_j+kd)^2} - \frac{(\lambda_j+g(k))^2\lambda_j}{(\lambda_j+1)^2(\lambda_j+k)^2}\right\}_{j=1}^{p}.
\end{aligned}$$

We can observe that $A_3A_3' - A_2A_2' > 0$ if and only if $(\lambda_j+d)^2(\lambda_j+k)^2 - (\lambda_j+g(k))^2(\lambda_j+kd)^2 > 0$. The RTE is superior to the MNTP estimator when $g(k)$ function is selected as $-\lambda_j - \frac{(\lambda_j+d)(\lambda_j+k)}{(\lambda_j+kd)} < g(k) < -\lambda_j + \frac{(\lambda_j+d)(\lambda_j+k)}{(\lambda_j+kd)}$ where $j = 1,2,...,p+1$. Therefore, $A_3A_3' - A_2A_2'$ is the pd matrix. By Theorem 2.2, the proof is complete. □

3.3. The comparison between the RTE and the NBR estimator

The *MMSE* of $\hat{\alpha}_{NBR} = (\Lambda + kI)^{-1}(\Lambda + kdI)(\Lambda + kI)^{-1}(\Lambda - kI)\Lambda^{-1}Z'Y = A_4Y$ estimator is

$$MMSE(\hat{\alpha}_{NBR}) = \sigma^2 A_4A_4' + (A_4Z - I)\alpha\alpha'(A_4Z - I).$$

We use *the MMSE* difference given below to compare the NBR and the RTE: $MMSE(\hat{\alpha}_{NBR}) - MMSE(\hat{\alpha}_{RTE}) = \sigma^2\left[A_4A_4' - A_2A_2'\right] + (A_4Z - I)\alpha\alpha'(A_4Z - I) - (A_2Z - I)\alpha\alpha'(A_2Z - I)$. Then, we can give the following theorem:

Theorem 3.3. *Let be $k > 0$, $0 < d < 1$ and $-\frac{(\lambda_j+kd)(\lambda_j+1)|\lambda_j-k|}{\lambda_j(\lambda_j+k)} - \lambda_j < g(k) < \frac{(\lambda_j+kd)(\lambda_j+1)|\lambda_j-k|}{\lambda_j(\lambda_j+k)} - \lambda_j$, where $j = 1,2,\ldots,p+1$. Then, $MMSE(\hat{\alpha}_{NBR}) - MMSE(\hat{\alpha}_{RTE}) > 0$ if and only if*

$$bias(\hat{\alpha}_{RTE})'\left[\sigma^2\left(A_4A_4' - A_2A_2'\right) + (A_4Z - I)\alpha\alpha'(A_4Z - I)\right]^{-1} bias(\hat{\alpha}_{RTE}) < 1 \tag{3.3}$$

where $\hat{\alpha}_{RTE}$ and $\hat{\alpha}_{NBR}$ are two linear estimators for α parameter.

Proof: We can obtain

$$\begin{aligned}
\operatorname{cov}(\hat{\alpha}_{NBR}) - \operatorname{cov}(\hat{\alpha}_{RTE}) &= \sigma^2\left[A_4A_4' - A_2A_2'\right] \\
&= \sigma^2\Big[(\Lambda + kI)^{-1}(\Lambda + kdI)(\Lambda + kI)^{-1}(\Lambda - kI)\Lambda^{-1}(\Lambda - kI)(\Lambda + kI)^{-1}(\Lambda + kdI)(\Lambda + kI)^{-1} \\
&\quad - (\Lambda + I)^{-1}(\Lambda + g(k)I)(\Lambda + kI)^{-1}\Lambda(\Lambda + kI)^{-1}(\Lambda + g(k)I)(\Lambda + I)^{-1}\Big] \\
&= \sigma^2 \operatorname{diag}\left\{\frac{\left(\lambda_j + kd\right)^2\left(\lambda_j - k\right)^2}{\lambda_j\left(\lambda_j + k\right)^4} - \frac{\left(\lambda_j + g(k)\right)^2\lambda_j}{\left(\lambda_j + 1\right)^2\left(\lambda_j + k\right)^2}\right\}_{j=1}^{p+1}.
\end{aligned}$$

We can observe that $A_4A_4' - A_2A_2' > 0$ if and only if $\left(\lambda_j + kd\right)^2\left(\lambda_j - k\right)^2\left(\lambda_j + 1\right)^2 - \left(\lambda_j + k\right)^2\left(\lambda_j + g(k)\right)^2\lambda_j^2 > 0$. From the solution of this inequality with respect to the function $g(k)$ we can derive the following condition: $-\frac{(\lambda_j+kd)(\lambda_j+1)|\lambda_j-k|}{\lambda_j(\lambda_j+k)} - \lambda_j < g(k) < \frac{(\lambda_j+kd)(\lambda_j+1)|\lambda_j-k|}{\lambda_j(\lambda_j+k)} - \lambda_j$, where $j = 1,2,\ldots,p+1$, $k > 0$, $0 < d < 1$. RTE outperforms the NBR estimator in terms of MMSE if the function $g(k)$ is determined in a way that satisfies the condition given above. Therefore, $A_4A_4' - A_2A_2'$ is the pd matrix. By Theorem 2.2, the proof is complete. □

3.4. The comparison between the RTE and the KLMRT estimator

The *MMSE* of $\hat{\alpha}_{KLMRT} = (\Lambda + kI)^{-1}(\Lambda - kI)(\Lambda + k(1 + d)I)^{-1}Z'Y = A_5Y$ is

$$MMSE(\hat{\alpha}_{KLMRT}) = \sigma^2 A_5A_5' + (A_5Z - I)\alpha\alpha'(A_5Z - I).$$

We get attention to the MMSE difference given below to compare the KLMRT and the RTE:

$MMSE(\hat{\alpha}_{KLMRT}) - MMSE(\hat{\alpha}_{RTE}) = \sigma^2\left[A_5A_5' - A_2A_2'\right] + (A_5Z - I)\alpha\alpha'(A_5Z - I) - (A_2Z - I)\alpha\alpha'(A_2Z - I)$. Then, we can give the following theorem:

Theorem 3.4. *Let be* $k > 0$, $d > 0$ *and* $-\lambda_j - \sqrt{\frac{(\lambda_j-k)^2(\lambda_j+1)^2}{\lambda_j(\lambda_j+k(1+d))^2}} < g(k) < -\lambda_j + \sqrt{\frac{(\lambda_j-k)^2(\lambda_j+1)^2}{\lambda_j(\lambda_j+k(1+d))^2}}$, $j = 1, 2, ..., p+1$. *Then,* $MMSE(\hat{\alpha}_{KLMRT}) - MMSE(\hat{\alpha}_{RTE}) > 0$ *if and only if*

$$bias(\hat{\alpha}_{RTE})'\left[\sigma^2(A_5A_5' - A_1A_1') + (A_5Z - I)\alpha\alpha'(A_5Z - I)\right]^{-1} bias(\hat{\alpha}_{RTE}) < 1 \tag{3.4}$$

where $\hat{\alpha}_{RTE}$ *and* $\hat{\alpha}_{KLMRT}$ *are two linear estimators for the parameter* α.

Proof: We can obtain

$$\begin{aligned} cov(\hat{\alpha}_{KLMRT}) - cov(\hat{\alpha}_{RTE}) &= \sigma^2\left[A_5A_5' - A_2A_2'\right] \\ &= \sigma^2\Big[(\Lambda + kI)^{-1}(\Lambda - kI)(\Lambda + k(1+d)I)^{-1}\Lambda(\Lambda + k(1+d)I)^{-1}(\Lambda - kI)(\Lambda + kI)^{-1} \\ &\quad - (\Lambda + I)^{-1}(\Lambda + g(k)I)(\Lambda + kI)^{-1}\Lambda(\Lambda + kI)^{-1}(\Lambda + g(k)I)(\Lambda + I)^{-1}\Big] \\ &= \sigma^2\, diag\left\{\frac{(\lambda_j-k)^2}{(\lambda_j+k)^2(\lambda_j+k(1+d))^2} - \frac{(\lambda_j+g(k))^2\lambda_j}{(\lambda_j+1)^2(\lambda_j+k)^2}\right\}_{j=1}^{p+1}. \end{aligned}$$

We observe that $A_5A_5' - A_2A_2' > 0$ if and only if $(\lambda_j - k)^2(\lambda_j + 1)^2 - (\lambda_j + g(k))^2(\lambda_j + k(1+d))^2\lambda_j > 0$. So, the RTE is superior to the KLMRT estimator when $g(k)$ function is selected as $-\lambda_j - \sqrt{\frac{(\lambda_j-k)^2(\lambda_j+1)^2}{(\lambda_j+k(1+d))^2\lambda_j}} < g(k) < -\lambda_j + \sqrt{\frac{(\lambda_j-k)^2(\lambda_j+1)^2}{(\lambda_j+k(1+d))^2\lambda_j}}$, $j = 1, 2, ..., p+1$. Therefore, $A_5A_5' - A_2A_2'$ is the pd matrix. By Theorem 2.2, the proof is complete. □

3.5. The comparison between the RTE and the LKL estimator

The *MMSE* of $\hat{\alpha}_{LKL} = (\Lambda + kI)^{-1}(\Lambda + dI)(\Lambda + kI)^{-1}(\Lambda - kI)\Lambda^{-1}Z'Y = A_6Y$ estimator is

$$MMSE(\hat{\alpha}_{LKL}) = \sigma^2A_6A_6' + (A_6Z - I)\alpha\alpha'(A_6Z - I).$$

We use *the MMSE* difference given below to compare the LKL estimator and RTE,

$MMSE(\hat{\alpha}_{LKL}) - MMSE(\hat{\alpha}_{RTE}) = \sigma^2\left[A_6A_6' - A_2A_2'\right] + (A_6Z - I)\alpha\alpha'(A_6Z - I) - (A_2Z - I)\alpha\alpha'(A_2Z - I)$. Then, we give the following theorem:

Theorem 3.5. *Let be $k > 0, 0 < d < 1$ and $-\lambda_j - \frac{|\lambda_j - k|(\lambda_j + d)}{\lambda_j} < g(k) < -\lambda_j + \frac{|\lambda_j - k|(\lambda_j + d)}{\lambda_j}$ where $j = 1, 2, ..., p+1$. Then, $MMSE(\hat{\alpha}_{LKL}) - MMSE(\hat{\alpha}_{RTE}) > 0$ if and only if*

$$\text{(3.5)} \quad bias(\hat{\alpha}_{RTE})'\left[\sigma^2(A_6A_6' - A_2A_2') + (A_6Z - I)\alpha\alpha'(A_6Z - I)\right]^{-1} bias(\hat{\alpha}_{RTE}) < 1$$

where $\hat{\alpha}_{RTE}$ and $\hat{\alpha}_{LKL}$ are two linear estimators for the parameter α.

Proof: We can obtain

$$\begin{aligned} cov(\hat{\alpha}_{LKL}) - cov(\hat{\alpha}_{RTE}) &= \sigma^2\left[A_6A_6' - A_2A_2'\right] \\ &= \sigma^2\Big[(\Lambda + I)^{-1}(\Lambda + dI)(\Lambda + kI)^{-1}(\Lambda - kI)\Lambda^{-1}(\Lambda - kI)(\Lambda + kI)^{-1}(\Lambda + dI)(\Lambda + I)^{-1} \\ &\quad - (\Lambda + I)^{-1}(\Lambda + g(k)I)(\Lambda + kI)^{-1}\Lambda(\Lambda + kI)^{-1}(\Lambda + g(k)I)(\Lambda + I)^{-1}\Big] \\ &= \sigma^2\, diag\left\{\frac{(\lambda_j + d)^2(\lambda_j - k)^2}{\lambda_j(\lambda_j + 1)^2(\lambda_j + k)^2} - \frac{(\lambda_j + g(k))^2\lambda_j}{(\lambda_j + 1)^2(\lambda_j + k)^2}\right\}_{j=1}^{p+1}. \end{aligned}$$

We can observe that $A_6A_6' - A_2A_2' > 0$ if and only if $(\lambda_j + d)^2(\lambda_j - k)^2 - (\lambda_j + g(k))^2\lambda_j^2 > 0$ where $j = 1, 2, ..., p+1$. The RTE is superior to the MNTP estimator when $g(k)$ function is selected as $-\lambda_j - \frac{|\lambda_j - k|(\lambda_j + d)}{\lambda_j} < g(k) < -\lambda_j + \frac{|\lambda_j - k|(\lambda_j + d)}{\lambda_j}$ where $j = 1, 2, ..., p+1$. Therefore, $A_6A_6' - A_2A_2'$ is the pd matrix. By Theorem 2.2, the proof is complete. □

4. Determination of $g(k)$ function

Determining the optimal estimate of the biasing parameter is very important because it is associated with the performance of the biased estimator. For practitioners, this is a complex process. This process becomes even more complicated for a biased estimator with biasing parameters k and d. Many different techniques have been proposed by many researchers to estimate the biasing parameter(s) (Hoerl and Kennard [12]; Liu [19][20]; Kibria [15]; Yang and Chang [34]; Sakallıoğlu and Kaçıranlar [27]; Shukur, et al. [29]; Ahmad and Aslam [1]; Dawoud et al. [6]; Qasim et al. [26]; Shewa and Ugwuowo [28]).

The main advantage of RTE over the estimators with two biasing parameters k and d is that there is a functional relationship between the biasing parameters. The performance of the proposed RTE is based on $g(k)$, and therefore the single biasing parameter is k. Note that different choices of the $g(k)$ function lead to different estimators. To find the optimal $g(k)$ function, let's take the derivative of $SMSE(\hat{\alpha}_{RTE})$ depending on k. The $SMSE(\hat{\alpha}_{RTE})$ is calculated using (2.4) as follows:

$$SMSE(\hat{\alpha}_{RTE}) = \sigma^2 \sum_{j=1}^{p+1} \frac{\left(\lambda_j + g(k)\right)^2 \lambda_i}{\left(\lambda_j+1\right)^2\left(\lambda_j+k\right)^2} + \sum_{j=1}^{p+1} \frac{\left((g(k)-k-1)\lambda_j - k\right)^2 \alpha_j^2}{\left(\lambda_j+1\right)^2\left(\lambda_j+k\right)^2}. \tag{4.1}$$

Note that Equation (4.1) is a function of the k parameter; that is, $h(k) = SMSE(\hat{\alpha}_{RTE})$. We can find $h'(k)$ as follows:

$$h'(k) = \sum_{j=1}^{p+1} \frac{2\left[\lambda_j\left(\lambda_j - g'(k)\lambda_j - g'(k)k + g(k)\right)\right]\left[\alpha_j^2\left((k+1-g(k))\lambda_j + k\right) - \sigma^2\left(\lambda_j + g(k)\right)\right]}{\left(\lambda_j+1\right)^2\left(\lambda_j+k\right)^3}.$$

In case $h'(k) = 0$, there are two scenarios:

Fact 1. $\lambda_j\left(\lambda_j - g'(k)\lambda_j - g'(k)k + g(k)\right) = 0$ differential equation is found. Then, we have

$$g(k) = ck + (c-1)\lambda_j \tag{4.2}$$

where c is the constant of integration.

Fact 2. $\alpha_j^2\left((k+1-g(k))\lambda_j + k\right) - \sigma^2\left(\lambda_j + g(k)\right) = 0$ equation is found. Here, $g(k)$ is obtained as follows:

$$g(k) = \frac{\left(1+\lambda_j\right)\alpha_j^2}{\sigma^2 + \lambda_j\alpha_j^2}k + \frac{\left(\alpha_j^2 - \sigma^2\right)}{\sigma^2 + \lambda_j\alpha_j^2}\lambda_j \quad or \quad g(k) = \frac{\left(1+\lambda_j\right)\alpha_j^2}{\sigma^2 + \lambda_j\alpha_j^2}k + \left(\frac{\left(1+\lambda_j\right)\alpha_j^2}{\sigma^2 + \lambda_j\alpha_j^2} - 1\right)\lambda_j \tag{4.3}$$

where $j = 1, 2, ..., p+1$. Based on the first and second facts, it can be said that the selection of $g(k)$ as a linear function of the biasing parameter k is appropriate. Also, $g(k)$ which is obtained in Fact 2, is a solution of the differential equation, which is obtained in Fact 1. Here, depending on the functions obtained in Fact 1 and Fact 2, we can observe the following results: Firstly, note that $g(k)$ given in (4.2) and (4.3) makes $SMSE(\hat{\alpha}_{RTE})$ function approximately minimum for a given j value. So, the determination of $g(k)$ depends on the eigenvalues of $X'X$, the unknown α parameter, and the estimate of the biasing parameter k. In other words, many $g(k)$ functions can be determined depending on the functional relationship given in (4.2) and (4.3). For example, the following functional relationships can be given to determine $g(k)$ in this sense:

$$g(k) = \frac{(1+\lambda_{\min})\alpha_{\min}^2}{\hat{\sigma}^2 + \lambda_{\max}\alpha_{\max}^2}k + \left(\frac{(1+\lambda_{\min})\alpha_{\min}^2}{\hat{\sigma}^2 + \lambda_{\max}\alpha_{\max}^2} - 1\right)\lambda_{\min} \tag{4.4}$$

$$g(k) = \frac{(1+\lambda_{\min})\alpha_{\min}^2}{\hat{\sigma}^2 + \lambda_{\max}\alpha_{\max}^2}k + \min\left(\frac{\alpha^2 - \hat{\sigma}^2}{\hat{\sigma}^2 + \lambda_{\max}\alpha_{\max}^2}\right)\lambda_{\min} \tag{4.5}$$

$$g(k) = \frac{(1+\lambda_{\min})\,\alpha_{\min}^2}{\hat{\sigma}^2 + \lambda_{\max}\alpha_{\max}^2} k + \min\left(\frac{\alpha^2 - \hat{\sigma}^2}{\hat{\sigma}^2 + \lambda\alpha^2}\right)\lambda_{\min} \tag{4.6}$$

where $\alpha_{\min}^2$ and $\alpha_{\max}^2$ are defined as the minimum and maximum value of α_j^2, $j = 1,2,...,p+1$, respectively. Similarly, $\lambda_{\min}$ and $\lambda_{\max}$ indicate the minimum and maximum values of the eigenvalues of $X'\hat{W}X$, respectively.

In this study, to determine the optimal $g(k)$ function, we examined only the first-degree polynomial functions such as those given in equations from (4.4) to (4.6). Note that it is clear that $g(k)$ can be selected as any continuous function of k. However, the proposed estimator depends on a single biasing parameter k. In this case, we should use an appropriate estimator of k to control the conditioning of the $X'X$ matrix. Since the proposed estimator depends on a parameter k, a suitable estimator of k can be used, as given in Kibria [15]. In addition to the previously proposed estimators, we can use the following estimators to estimate k: $\hat{k}_{RTE} = \frac{p\sigma^2\alpha_{\min}^2}{n}$, $\hat{k}_{RTE} = \frac{p\sigma^2\lambda_{\min}}{n\lambda_{\max}}$, $\hat{k}_{RTE} = \frac{\sigma^2}{n\sum_{j=1}^{p+1}\lambda_j\alpha_j^2}$, $\hat{k}_{RTE} = \frac{p\sigma^2}{n\alpha_{\max}^2}$, $\hat{k}_{RTE} = \frac{p\sigma^2\min\left(\lambda_j\alpha_j^2\right)}{n\max\left(\lambda_j\alpha_j^2\right)}$, $\hat{k}_{RTE} = \frac{\lambda_{\max}+\lambda_{\min}}{p}$, $\hat{k}_{RTE} = \frac{\lambda_{\max}-\lambda_{\min}}{p}$ where $\hat{\sigma}^2 = \frac{\sum_{i=1}^n(y_i-\hat{y}_i)^2}{n-p-1}$.

5. The Monte Carlo simulation studies

In this section, we designed two separate Monte Carlo simulations to examine the performance of the proposed biased estimator relative to other existing estimators in linear regression models. In the first design, we investigated the effects of sample size (n), the degree of the collinearity (ρ), the number of explanatory variables (p), and the variance $\left(\sigma^2\right)$ on the performances of OLS, RE, LE, LTE, SK, YC, MNTP, NBR, ML, TSS, KLMRT, LKL estimators and RTEs. In the second simulation design, we examined RTE and NLTE performances for each of n, p, ρ, and σ^2 values at certain values of k. For both simulation designs, we generate the explanatory variables by following McDonald and Galarneau [23] and Kibria [15] as

$$x_{ij} = \left(1-\rho^2\right)^{1/2} w_{ij} + \rho w_{ip+1}, \quad i = 1,2,..,n, \quad j = 1,2,...,p \tag{5.1}$$

where w_{ij} are independent standard normal pseudo-random numbers and ρ is specified in such a way that the correlation between any two variables is given by ρ^2. These variables are standardized such that $X'X$ is a correlation matrix. Four different sets of correlations are investigated corresponding to $\rho = 0.8, 0.9$ and 0.99. The response variable is generated by

$$y_i = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + ... + \beta_p x_{pi} + \varepsilon_i, \quad i = 1,2,\ldots,n$$

where $\varepsilon_i \sim N\left(0,\sigma^2\right)$ and β_0 considered to be zero. For different comparisons of the error term, the value of σ^2 is considered to be 0.5, 1, 5, and 10. For each set of explanatory variables, the real parameter vector β is chosen as the normalized eigenvector corresponding to the largest eigenvalue of $X'X$ so that $\beta'\beta = 1$. The sample size n is taken to be 50, 100, and 200. The number of explanatory variables is chosen as $p = 2, 4, 8$, and 12.

In the simulation and application sections, the estimates of the biasing parameters for RE, LE, LTE, SK, YC, MNTP, NBR, ML, TSS, KLMRT, and LKL are chosen based on the best estimators suggested in the literature (Kibria [15]; Liu [20]; Qasim *et al.* [25]; Sakallıoğlu and Kaçıranlar [27]; Yang and Chang [34]; Ahmad and Aslam [1]; Dawoud *et al.* [6]; Idowu *et al.* [14]; Lukman *et al.* [22]; Qasim *et al.* [26]; Shewa and Ugwuowo [28]).

To estimate the biasing parameter k in RE, Kibria [15] proposed the best estimates of k as follows, $\hat{k}_{RE} = \frac{\hat{\sigma}^2}{\left(\prod_{j=1}^{p+1} \hat{\alpha}_j^2\right)^{\frac{1}{p+1}}}$ where $\hat{\sigma}^2 = \frac{\sum_{i=1}^{n}(y_i-\hat{y}_i)^2}{n-p-1}$. Based on the results given by Qasim *et al.* [25], we use the best estimation of d in LE as $\hat{d}_{LE} = \max\left(0, \min\left(\frac{\hat{\alpha}_j^2-\hat{\sigma}^2}{\max\left(\frac{\hat{\sigma}^2}{\lambda_j}\right)+\hat{\alpha}_{\max}^2}\right)\right)$. On the other hand, k_{LTE} and d_{LTE} in LTE are estimated by using the methods given by Liu [20]. Sakallıoğlu and Kaçıranlar [27] and Yang and Chang [34] did not provide a specific technique for estimating the biasing parameters k and d for SK and YC estimators, respectively. Therefore, we used $\hat{k}_{RE}$ as an estimate of k for the SK estimator. Also, the estimate of the biasing parameter d was determined in such a way that $SMSE(\hat{\alpha}_{SK})$ was minimized. Moreover, we used two methods proposed by Huang and Yang [13] to estimate the biasing parameters of the YC estimator. Huang and Yang [13] proposed two methods. We referred to these methods as (K1, D1) and (K2, D2) (Huang and Yang [13]). We used these methods by adapting them for the YC estimator in linear regression models. As a result, the estimator obtained with (K1, D1) indicated YC I, and the estimator obtained with (K2, D2) indicated YC II. Moreover, for the MNTP, NBR, ML, TSS, KLMRT, and LKL estimators, the iterative techniques from the relevant papers are used together with the optimal biasing parameters. Since there are many combinations to determine k and $g(k)$ functions in RTE, we only report the simulation results for the following k estimates and $g(k)$ functions:

RTE I: $\hat{k}_{RTE\,\mathrm{I}} = \frac{p\sigma^2\alpha_{\min}^2}{n}$ and $g(k) = \frac{(1+\lambda_{\min})\alpha_{\min}^2}{\hat{\sigma}^2+\lambda_{\max}\alpha_{\max}^2}k + \left(\frac{(1+\lambda_{\min})\alpha_{\min}^2}{\hat{\sigma}^2+\lambda_{\max}\alpha_{\max}^2} - 1\right)\lambda_{\min}$

RTE II: $\hat{k}_{RTE\,\mathrm{II}} = \frac{p\sigma^2\min\left(\lambda_j\, \alpha_j^2\right)}{n}$ and $g(k) = \frac{(1+\lambda_{\min})\alpha_{\min}^2}{\hat{\sigma}^2+\lambda_{\max}\alpha_{\max}^2}k + \min\left(\frac{\alpha^2-\hat{\sigma}^2}{\hat{\sigma}^2+\lambda_{\max}\alpha_{\max}^2}\right)\lambda_{\min}$

RTE III: $\hat{k}_{RTE\,\mathrm{III}} = \frac{p\sigma^2\min\left(\lambda_j\, \alpha_j^2\right)}{n\max\left(\lambda_j\, \alpha_j^2\right)}$ and $g(k) = \frac{(1+\lambda_{\min})\alpha_{\min}^2}{\hat{\sigma}^2+\lambda_{\max}\alpha_{\max}^2}k + \min\left(\frac{\alpha_j^2-\hat{\sigma}^2}{\hat{\sigma}^2+\lambda_j\, \alpha_j^2}\right)\lambda_{\min}$

RTE IV: $\hat{k}_{RTE\,\mathrm{IV}} = \frac{\lambda_{\max}+\lambda_{\min}}{p}$ and $g(k) = \frac{(1+\lambda_{\min})\alpha_{\min}^2}{n(\hat{\sigma}^2+\lambda_{\max}\alpha_{\max}^2)}k + \left(\frac{(1+\lambda_{\min})\alpha_{\min}^2}{n(\hat{\sigma}^2+\lambda_{\max}\alpha_{\max}^2)} - 1\right)\lambda_{\min}$

The performance of the estimated MSEs (EMSEs) is used as the basis for comparison of the proposed estimators calculated for an estimator $\hat{\beta}$ of β as follows:

$$EMSE\left(\hat{\beta}\right) = \frac{1}{N}\sum_{r=1}^{N}\left(\hat{\beta}_r - \beta\right)'\left(\hat{\beta}_r - \beta\right) \tag{5.2}$$

where $\left(\hat{\beta}_r - \beta\right)$ is the difference between the estimated and true parameter vectors at rth replication, and N is the number of replications. For each case of n, p, σ^2, and ρ, the experiment was replicated 2000 times by generating response variables using R programming. The results are given in Tables 1-4 where the bold numbers show the smallest EMSE values. In addition, the signs (*), (**), and (***) show the first, second, and third smallest EMSE values in each row, respectively. Based on Tables 1-4, we can conclude that the degree of correlation, number of explanatory variables, sample size, and variance have different effects on all estimators in the simulation. Several observations can be obtained as follows:

1. When the number of observations n and σ^2 are kept constant, it is observed that as the number of variables increased, generally, the EMSE values of the estimators tend to increase for models with low correlation variables and to decrease for models with high correlation. However, it is seen that in the increasing trend of EMSE values, the slopes of the proposed estimators RTE I, RTE II, RTE III, and RTE IV are much lower than the other existing estimators.

2. It is observed that when the number of variables p, n, and σ^2 are kept constant, as the correlations of the variables increase, the EMSE values of all estimators tend to decrease in general. However, the RTE I is not as dramatically affected by the increase in the correlation between the independent variables compared to the other existing estimators. Based on this situation, it can be concluded that RTE I has a robust structure depending on low or high correlation.

3. When the correlations ρ, n, and p are kept constant, the increase in the variance leads to an increase in the EMSE values of all estimators. However, in terms of EMSE values, the increases in all proposed estimators are smaller compared to the increases in other available estimators.

4. It is seen that when the number of variables p, ρ, and σ^2 are kept constant, the EMSE values of the proposed estimators are lower than the EMSE values of the existing estimators in $n = 50, 100, 200$. However, it is observed that there is no significant systematic change in the EMSE values of all estimators as the number of observations increases. As a result, it can be said that compared to ρ and σ^2, the number of observations has a relatively small effect on EMSE values.

In all scenarios examined, it is observed that all our proposed estimators are significantly superior to existing estimators: OLS, RE, LE, LTE, SK, YC I, YC II, MNTP, NBR, ML, TSS, KLMRT, and LKL. However, even if the estimators RTE I and RTE IV are better than other estimators accessible in all cases, they behave differently in each scenario. In general, RTE I has the best EMSE value in models with few variables and low variance. In contrast, RTE IV has a smaller EMSE value in models with large variance. When the number of variables increased, RTE IV generally gave better results in all scenarios.

In the second simulation scheme, we only investigated the performances of RTE and NLTE for each n, p, ρ, and σ^2. The purpose of this simulation is to examine the performances of NLTE and RTE at various values of the biasing parameter k depending on EMSE values given in (5.2). There are many $f(k)$ and $g(k)$ functions that can be considered to evaluate the performances of these two classes of estimators. The biasing parameter k is not estimated in the second simulation scheme. Only the EMSE values obtained by increasing k values in the range $[0, 1]$ by 0.05 are compared. In order to compare the performances of these two estimators under some situations as an example, the following estimators with $f(k)$ and $g(k)$ functions are taken:

$\hat{\beta}_{NLTE} = (X'X + kI)^{-1}(X'X + f(k)I)\hat{\beta}_{OLS}$
where $f(k) = \frac{\lambda_{\min}\alpha^2_{\min}}{1+\lambda_{\max}\alpha^2_{\max}}k + \left(\frac{\lambda_{\min}\alpha^2_{\min}}{1+\lambda_{\max}\alpha^2_{\max}} - 1\right)\lambda_{\min}$
$\hat{\beta}_{NLTE(RE)} = (X'X + kI)^{-1}(X'X + (k + f(k))I)\hat{\beta}_{RE}$
where $f(k) = \frac{\alpha^2_{\min}(k+\lambda_{\min})^2}{1+\lambda_{\max}\alpha^2_{\max}} - (k + \lambda_{\min})$
$\hat{\beta}_{RTE} = (X'X + I)^{-1}(X'X + g(k)I)\hat{\beta}_{RE}$
where $g(k) = \frac{(1+\lambda_{\min})\alpha^2_{\min}}{\hat{\sigma}^2+\lambda_{\max}\alpha^2_{\max}}k + \left(\frac{(1+\lambda_{\min})\alpha^2_{\min}}{\hat{\sigma}^2+\lambda_{\max}\alpha^2_{\max}} - 1\right)\lambda_{\min}.$

Note that, when we use $\hat{\beta}_{RE}$ instead of $\hat{\beta}^*$ in $\hat{\beta}_{NLTE}$, the obtained estimator is shown as $\hat{\beta}_{NLTE(RE)}$. Also, $f(k)$ functions used in $\hat{\beta}_{NLTE}$ and $\hat{\beta}_{NLTE(RE)}$ were determined in accordance with the rules given by [17]. We only consider the cases $\rho = 0.9,\ 0.99$, $n = 50,\ 200$, and $p = 4,\ 8,\ 12$, and $\sigma^2 = 1,\ 10$. Depending on these n, ρ, p, and σ^2 values, the explanatory variables are generated according to equation (5.1). The simulation is repeated 2000 times for each k value. The results are collectively presented graphically in Figures 1 and 2.

Based on Figures 1-2, we can interpret the results as follows depending on each $\left(n, \rho, p, \sigma^2\right)$.

1) At small values of the biasing parameter k, $\hat{\beta}_{RTE}$ outperforms $\hat{\beta}_{NLTE}$ and $\hat{\beta}_{NLTE(RE)}$. Although both $\hat{\beta}_{RTE}$ and $\hat{\beta}_{NLTE(RE)}$ include the $\hat{\beta}_{RE}$, the performance of $\hat{\beta}_{NLTE(RE)}$ is quite poor compared to $\hat{\beta}_{RTE}$ at small values of the biasing parameter.

2) For $p = 4$ and $\rho = 0.9$, $\hat{\beta}_{NLTE(RE)}$ exhibits quite different behavior from $\hat{\beta}_{NLTE}$ and $\hat{\beta}_{RTE}$. If the value of the biasing parameter and the number of explanatory variables increases, $\hat{\beta}_{NLTE}$, $\hat{\beta}_{NLTE(RE)}$, and $\hat{\beta}_{RTE}$ show almost the same behaviors. In general, $\hat{\beta}_{RTE}$ exhibits a more consistent behavior at different values of the biasing parameter k.

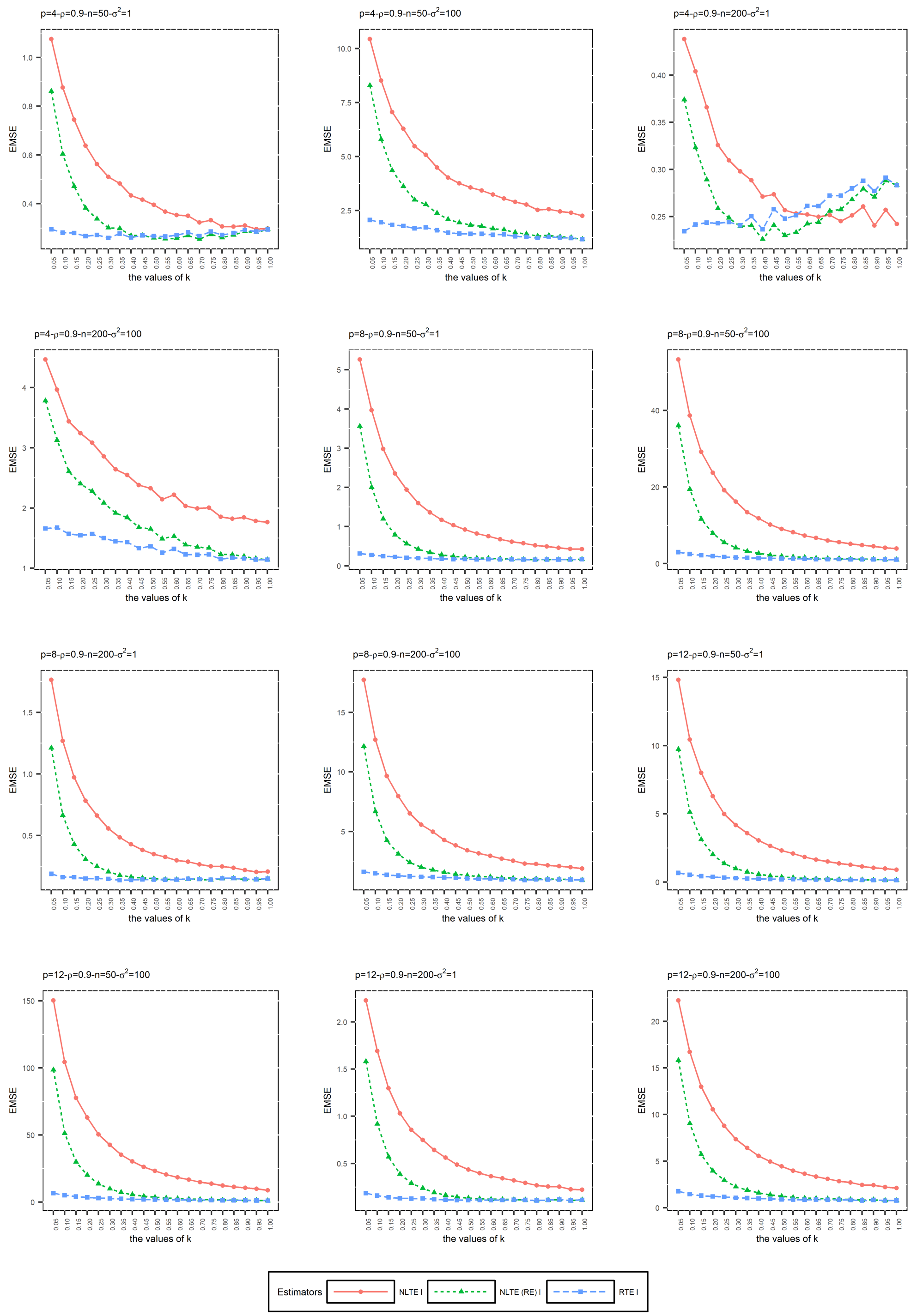

Figure 1: The EMSE values of NLTE I, NLTE(RE) I, RTE I as a function of k where $\rho = 0.9$

Based on the results of the second simulation design, we can recommend $\hat{\beta}_{RTE}$ to the researchers because it is a more consistent estimator than $\hat{\beta}_{NLTE}$ and $\hat{\beta}_{NLTE(RE)}$ for the considered conditions. In general, the performances of these

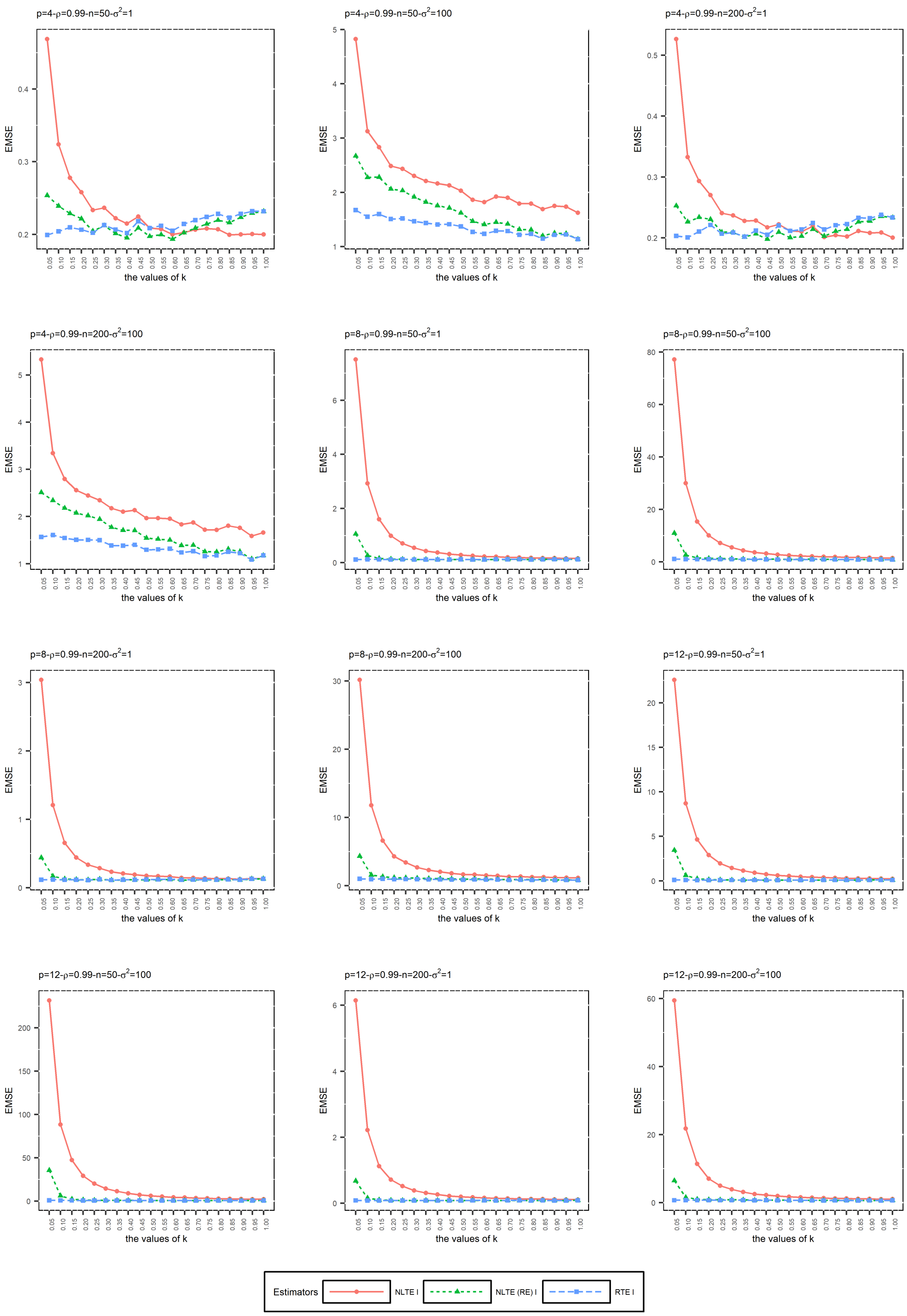

Figure 2: The EMSE values of NLTE I, NLTE(RE) I, RTE I as a function of k where $\rho = 0.99$

estimators depend on $f(k)$ and $g(k)$ functions. In practice, we need to replace these functions with functional relationships that can occur between the biasing parameters. Therefore, it should be kept in mind that the results of graphical findings may change.

6. Numerical example

In this section, we reconsider the Portland cement data that was analyzed by Hald [11], Liu [19], Sakallıoğlu and Kaçıranlar [27], Yang and Chang [34], and Kurnaz and Akay [18]. In this data set, the following four compounds are independent variables: tricalcium aluminate(x_1), tetracalcium silicate (x_2), tetracalcium alumino ferrite (x_3), and dicalcium silicate (x_4). The dependent variable y is the heat evolved in calories per gram of cement. We fit a linear regression model with intercept to the data by adding a column of ones to the matrix X. Then, the eigenvalues of $X'X$ are $\lambda_1 = 44676.2059, \lambda_2 = 5965.4221, \lambda_3 = 809.9521$, $\lambda_4 = 105.4187$ and $\lambda_5 = 0.0012$. The condition number is approximately 3.66×10^7, therefore the matrix X is quite ill-conditioned. The numerical results are summarized in Table 5 to compare RTEs with other estimators. In addition, three different $f(k)$ functions for both $\hat{\beta}_{NLTE}$ and $\hat{\beta}_{NLTE(RE)}$ are given in Table 5. Since there are many combinations to determine k and $f(k)$ functions in NLTE and NLTE(RE), we use the following k estimators and $f(k)$ functions based on the Kibria [15] and Kurnaz and Akay [17]. Note that the function $f(k)$ that minimizes $SMSE\left(\hat{\beta}_{NLTE(RE)}\right)$ is a quadratic function.

NLTE I: $\hat{k}_{\text{NLTE I}} = \frac{\sigma^2}{\sum_{j=1}^{p} \alpha_j^2}$ and $f(k) = \frac{\lambda_{\min}\alpha_{\min}^2}{\hat{\sigma}^2+\lambda_{\max}\alpha_{\max}^2}k + \left(\frac{\lambda_{\min}\alpha_{\min}^2}{\hat{\sigma}^2+\lambda_{\max}\alpha_{\max}^2} - 1\right)\lambda_{\min}$

NLTE II: $\hat{k}_{\text{NLTE II}} = \frac{p\sigma^2}{n\sum_{j=1}^{p} \alpha_j^2}$ and $f(k) = \frac{\lambda_{\min}\alpha_{\min}^2}{\hat{\sigma}^2+\lambda_{\max}\alpha_{\max}^2}k - \frac{\hat{\sigma}^2\lambda_{\min}}{\hat{\sigma}^2+\lambda_{\max}\alpha_{\max}^2}$

NLTE III: $\hat{k}_{\text{NLTE III}} = \frac{\hat{\sigma}^2}{\left(\prod_{j=1}^{p} \hat{\alpha}_j^2\right)^{\frac{1}{p}}}$ and $f(k) = \frac{\lambda_{\min}\alpha_{\min}^2}{\hat{\sigma}^2+\lambda_{\max}\alpha_{\max}^2}k - \min\left(\frac{\hat{\sigma}^2}{\hat{\sigma}^2+\lambda_j\,\alpha_j^2}\right)\lambda_{\min}$

NLTE(RE) I: $\hat{k}_{\text{NLTE(RE) I}} = \frac{\sigma^2}{\sum_{j=1}^{p} \alpha_j^2}$ and $f(k) = \frac{\alpha_{\min}^2}{\hat{\sigma}^2+\lambda_{\max}\alpha_{\max}^2}(k+\lambda_{\min})^2 - (k+\lambda_{\min})$

NLTE(RE) II: $\hat{k}_{\text{NLTE(RE) II}} = \frac{p\sigma^2}{n\sum_{j=1}^{p} \alpha_j^2}$ and $f(k) = \frac{\alpha_{\max}^2}{\hat{\sigma}^2+\lambda_{\max}\alpha_{\max}^2}(k+\lambda_{\min})^2 - (k+\lambda_{\min})$

NLTE(RE) III: $\hat{k}_{\text{NLTE(RE) II}} = \frac{\hat{\sigma}^2}{\left(\prod_{j=1}^{p} \hat{\alpha}_j^2\right)^{\frac{1}{p}}}$ and $f(k) = \frac{\alpha_{\min}^2}{\max\left(\hat{\sigma}^2+\lambda_i\alpha_i^2\right)}(k+\lambda_{\min})^2 - (k+\lambda_{\min})$

In addition, the bootstrap sampling method was used to obtain the actual parameter values to be used instead of the α parameter. Therefore, 10000 bootstrap samples were created and the parameter estimates associated with the estimators were calculated for each of the samples. The mean of the OLS estimates is considered as an estimate of α. The calculated SMSE values are given in Table 5. As seen in Table 5, the estimated variance values and the SMSE values of RTE I, RTE II, RTE III, and RTE IV under the proposed $g(k)$ functions with k estimates can yield appropriate results compared to other existing estimators. To compare the estimators under the MMSE sense, $\hat{\alpha}_{OLS}$ is used in place of the unknown parameter α. Here, the eigenvalues of the matrices obtained with the MMSE differences are taken into account. That is, if any of the eigenvalues is less than or equal to tolerance, then the MMSE difference is not pd. Otherwise, the MMSE difference is pd. The R Programming is used with tolerance 10^{-10} to find whether MMSE differences are pd or not. To illustrate Theorem 3.1, the function $f(k)$ is taken as $f(k) = 3.0866 \times 10^{-13}k - 0.0012$ by using NLTE I. Also, the $g(k)$ is obtained as $g(k) = 2.5372 \times 10^{-10}k - 0.0012$ in RTE I using (4.4). In this case, $cov\left(\hat{\beta}_{NLTE\,I}\right) - cov\left(\hat{\beta}_{RTE\,I}\right)$ is the pd matrix for $k > 0$. But, the criterion (3.1) given in Theorem 3.1 is not held. On the other hand, if functions $g(k)$ and $f(k)$ are arbitrarily taken as $f(k) = 0.5k - 0.05$ and $g(k) = 0.6k - 0.05$, $cov\left(\hat{\beta}_{NLTE}\right) - cov\left(\hat{\beta}_{RTE}\right)$ is pd matrix for $0 < k \leq 0.09754$ or $k \geq 0.09758$. Also, k values which provide (3.1) criterion are $0 < k < 0.0479$. Consequently, $MMSE\left(\hat{\beta}_{NLTE}\right) - MMSE\left(\hat{\beta}_{RTE}\right)$ is the pd matrix where $0 < k < 0.0479$.

7. Conclusion

In this study, a new general biased estimator called RTE is proposed as an alternative to other existing biased estimators used in the presence of multicollinearity. The RTE is a general estimator that includes other biased estimators, such as the OLS, RE, LE, ML, YC, and SK estimators as special cases. The RTE is based on a functional relationship $g(k)$ between the biasing parameters, which would provide an alternative method for overcoming multicollinearity. In this study, we investigated several rules for determining the optimal function $g(k)$. The performance of these functions is analyzed using different k estimators. Results revealed that the estimators obtained with these $g(k)$ functions outperformed the other existing estimators under the examined conditions. In particular, RTE I has the best EMSE value in models with few variables and low variance. On the other hand, RTE IV has a small EMSE value in high-variance models. When the number of variables increased, RTE IV generally gave better results in all scenarios. Besides, a general simulation study is performed to compare RTE and NLTE. In the cases we have considered, it has been observed that RTE performs well when the biasing parameter k is small values. Although RTE and NLTE(RE) are both dependent on RE, the main advantage of RTE over NLTE(RE) is that it can minimize the SMSE function with the help of a simpler function. Additionally, Portland data is also considered to illustrate the advantage of RTEs in the linear regression models. Since NLTE and RTE are two general classes of biased estimators, a comparison of these classes is given in Portland data from various perspectives. Finally, based on the results of the simulations and application, it can be recommended that the RTE can be used when there is multicollinearity in the linear regression models.

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Table 1: The EMSE values of the estimators for the model when $p = 2$.

σ^2	n	ρ	**OLS**	**RE**	**LE**	**YC I**	**YC II**	**SK**	**LTE**	**MNTP**	**NBR**	**ML**	**TSS**	**KLMRT**	**LKL**	**RTE I**	**RTE II**	**RTE III**	**RTE IV**
0.5	50	0.8	1.618	0.777	0.459	1.005	0.945	0.59	0.936	0.562	0.952	0.551	0.977	1.130	0.957	**0.336***	0.38**	0.417***	0.525
1	50	0.8	3.362	1.189	0.733	1.897	1.794	0.963	1.905	1.028	1.842	0.727	1.867	1.141	1.843	**0.415***	0.568***	0.592	0.546**
5	50	0.8	15.906	3.584	2.601	8.143	7.851	3.405	9.008	4.392	8.381	2.133	8.154	1.204***	8.237	0.947**	1.422	1.641	**0.811***
10	50	0.8	33.336	7.192	5.255	17.225	16.474	6.966	19.286	9.123	17.965	3.993	17.254	1.332***	17.46	1.252**	1.928	2.93	**1.058***
0.5	50	0.9	2.411	0.839	0.420	1.321	1.276	0.569	1.258	0.634	1.256	0.757	1.285	1.020	1.257	**0.284***	0.325**	0.346***	0.454
1	50	0.9	5.001	1.314	0.619	2.626	2.56	0.93	2.636	1.212	2.579	1.271	2.600	1.030	2.579	**0.372***	0.472**	0.491	0.49***
5	50	0.9	24.621	4.372	2.320	12.272	12.015	3.401	13.022	5.519	12.619	4.99	12.312	1.113***	12.478	0.882**	1.225	1.502	**0.781***
10	50	0.9	50.032	7.954	4.370	24.739	24.087	6.421	26.396	11.049	25.349	9.838	24.845	1.224***	25.236	1.161**	1.729	2.549	**1.127***
0.5	50	0.99	32.129	2.127	0.253	15.181	15.201	0.513	15.179	4.105	15.178	14.331	15.18	0.369	15.179	**0.229***	0.23**	0.232***	0.364
1	50	0.99	64.220	2.882	0.370	30.653	30.609	0.689	30.675	8.207	30.674	28.759	30.659	0.434	30.675	**0.329***	0.34**	0.344***	0.414
5	50	0.99	330.294	8.046	1.259	157.766	157.492	1.895	158.013	41.855	158.007	147.428	157.838	0.942***	158.013	0.781**	0.958	1.132	**0.779***
10	50	0.99	688.415	16.812	2.472	338.238	337.655	4.038	338.758	89.554	338.749	315.882	338.397	1.614	338.759	**0.978***	1.35***	2.115	1.259**
0.5	100	0.8	1.656	0.750	0.456	0.994	0.938	0.567	0.927	0.552	0.942	0.551	0.965	1.119	0.943	**0.324***	0.372**	0.403***	0.513
1	100	0.8	3.426	1.145	0.699	1.925	1.822	0.917	1.927	1.033	1.865	0.763	1.894	1.129	1.868	**0.412***	0.571***	0.573	0.544**
5	100	0.8	16.554	3.701	2.605	8.336	7.987	3.455	9.219	4.471	8.596	2.288	8.359	1.199***	8.441	0.992**	1.567	1.743	**0.798***
10	100	0.8	31.583	6.382	4.756	15.488	14.833	6.19	17.442	8.262	16.186	3.928	15.529	1.304**	15.755	1.312***	2.087	2.85	**1.023***
0.5	100	0.9	2.296	0.824	0.424	1.26	1.209	0.572	1.2	0.616	1.195	0.704	1.223	1.039	1.197	**0.280***	0.329**	0.35***	0.454
1	100	0.9	4.828	1.302	0.642	2.537	2.434	0.941	2.561	1.195	2.487	1.147	2.505	1.051	2.485	**0.365***	0.498***	0.509	0.477**
5	100	0.9	23.645	4.422	2.415	11.858	11.605	3.531	12.601	5.447	12.287	4.528	11.896	1.131***	12.04	0.961**	1.404	1.627	**0.792***
10	100	0.9	45.931	7.982	4.472	22.675	22.214	6.535	24.381	10.456	23.447	8.439	22.797	1.231**	23.166	1.3***	1.923	2.817	**1.149***
0.5	100	0.99	28.916	2.003	0.249	13.852	13.85	0.496	13.852	3.779	13.85	12.98	13.851	0.39	13.852	**0.225***	0.227**	0.23***	0.364
1	100	0.99	58.151	2.905	0.373	28.054	28.014	0.695	28.078	7.579	28.075	26.089	28.061	0.45	28.077	**0.332***	0.341**	0.346***	0.414
5	100	0.99	295.883	8.379	1.393	138.579	138.274	2.152	138.815	37.168	138.805	128.047	138.651	0.963***	138.812	0.908**	1.061	1.256	**0.827***
10	100	0.99	566.008	14.678	2.539	264.646	264.037	3.807	265.197	70.921	265.174	244.384	264.817	1.522	265.187	**1.092***	1.477***	2.23	1.268**
0.5	200	0.8	1.946	0.792	0.445	1.124	1.062	0.575	1.059	0.588	1.064	0.606	1.091	1.086	1.067	**0.306***	0.362**	0.385***	0.486
1	200	0.8	3.819	1.215	0.683	2.069	1.974	0.937	2.087	1.07	2.019	0.882	2.041	1.094	2.018	**0.406***	0.558***	0.563	0.525**
5	200	0.8	18.927	3.905	2.421	9.432	9.118	3.396	10.251	4.719	9.67	2.976	9.456	1.178***	9.555	0.97**	1.575	1.681	**0.747***
10	200	0.8	39.471	7.575	4.818	19.579	19.033	6.747	21.442	9.769	20.203	5.94	19.674	1.276**	19.969	1.498***	2.39	3.157	**1.123***
0.5	200	0.9	2.647	0.860	0.403	1.416	1.374	0.573	1.365	0.651	1.358	0.823	1.381	0.993	1.355	**0.264***	0.31**	0.328***	0.432
1	200	0.9	5.335	1.278	0.626	2.699	2.627	0.876	2.717	1.222	2.653	1.365	2.672	1.003	2.653	**0.372***	0.493	0.492***	0.476**
5	200	0.9	27.563	4.503	2.171	13.77	13.507	3.337	14.429	5.898	14.152	6.125	13.814	1.091***	13.98	0.986**	1.432	1.553	**0.777***
10	200	0.9	52.921	7.628	4.079	25.52	25.09	5.905	27.057	12.545	26.533	11.137	25.669	1.193**	26.081	1.443***	2.109	2.727	**1.145***
0.5	200	0.99	30.667	2.026	0.245	14.453	14.447	0.477	14.454	3.924	14.453	13.578	14.453	0.376	14.454	**0.219***	0.223**	0.226***	0.358
1	200	0.99	62.054	2.855	0.38	29.61	29.564	0.664	29.631	7.973	29.629	27.617	29.617	0.449	29.631	**0.343***	0.352**	0.356***	0.418
5	200	0.99	307.111	8.671	1.345	147.727	147.469	2.187	147.983	39.423	147.973	137.132	147.806	0.958**	147.98	0.997***	1.133	1.237	**0.810***
10	200	0.99	628.440	15.042	2.577	298.862	298.277	3.799	299.384	79.68	299.37	277.161	299.021	1.597***	299.38	1.309**	1.694	2.32	**1.294***

Table 2: The EMSE values of the estimators for the model when $p = 4$.

σ^2	n	ρ	OLS	RE	LE	YC I	YC II	SK	LTE	MNTP	NBR	ML	TSS	KLMRT	LKL	RTE I	RTE II	RTE III	RTE IV
0.5	50	0.8	5.159	0.993	0.444	2.857	2.038	0.677	3.247	0.689	2.811	0.367	2.811	0.693	2.811	**0.211***	0.402	0.313***	0.249**
1	50	0.8	10.701	1.722	0.833	5.873	4.312	1.26	6.777	1.342	5.87	0.656	5.867	0.749	5.875	0.332**	0.722	0.532***	**0.308***
5	50	0.8	53.403	5.821	3.717	29.135	21.639	5.221	33.801	6.355	29.374	2.783	29.298	1.183***	29.402	0.975**	2.158	1.801	**0.745***
10	50	0.8	110.123	11.571	7.527	60.821	45.317	10.557	69.948	13.014	61.312	5.711	61.159	1.733***	61.359	1.411**	3.006	3.29	**1.315***
0.5	50	0.9	9.815	1.608	0.365	5.404	3.867	0.767	4.329	0.761	5.378	0.546	5.379	0.441	5.379	**0.151***	0.307	0.26***	0.216**
1	50	0.9	19.185	2.462	0.629	10.326	7.448	1.244	8.3	1.42	10.329	0.967	10.329	0.544	10.333	**0.24***	0.519	0.405***	0.274**
5	50	0.9	96.371	8.474	2.754	51.882	37.645	4.814	41.907	6.887	52.004	4.367	51.988	1.341	52.025	0.732**	1.521	1.293***	**0.675***
10	50	0.9	192.27	15.905	5.521	103.568	74.457	9.336	83.282	13.76	103.826	8.783	103.79	2.362	103.867	**1.088***	2.208***	2.387	1.226**
0.5	50	0.99	129.096	8.632	0.15	69.259	50.331	1.215	47.186	3.364	69.487	9.151	69.475	0.492	69.588	**0.121***	0.132**	0.135***	0.184
1	50	0.99	257.941	13.235	0.249	138.182	100.785	1.66	93.728	6.657	138.569	16.563	138.543	0.989	138.765	**0.197***	0.219**	0.223***	0.238
5	50	0.99	1276.487	42.965	1.042	689.326	503.699	4.917	470.116	33.16	691.244	85.176	691.176	4.961	692.19	**0.591***	0.806***	0.88	0.646**
10	50	0.99	2583.039	78.338	1.945	1377.057	1020.014	8.466	941.009	66.43	1382.205	198.231	1382.19	10.056	1384.479	**0.791***	1.217***	1.578	1.096**
0.5	100	0.8	4.942	0.966	0.449	2.733	1.927	0.669	2.912	0.689	2.683	0.312	2.685	0.704	2.682	**0.196***	0.409	0.305***	0.259**
1	100	0.8	10.159	1.623	0.812	5.641	3.995	1.224	6.05	1.325	5.614	0.503	5.616	0.752	5.616	**0.285***	0.734	0.483***	0.3**
5	100	0.8	49.748	5.627	3.746	26.716	19.237	5.213	29.431	6.134	26.839	2.075	26.805	1.193***	26.848	0.947**	2.466	1.702	**0.745***
10	100	0.8	104.954	11.041	7.567	57.643	42.168	10.48	62.764	12.824	57.906	4.439	57.834	1.744***	57.924	1.402**	3.569	3.083	**1.252***
0.5	100	0.9	8.317	1.417	0.391	4.597	3.225	0.741	3.29	0.735	4.56	0.439	4.562	0.491	4.559	**0.154***	0.336	0.273***	0.228**
1	100	0.9	16.407	2.225	0.669	8.929	6.365	1.226	6.46	1.375	8.917	0.759	8.918	0.577	8.918	**0.229***	0.576	0.419***	0.276**
5	100	0.9	79.538	7.14	2.886	42.363	29.83	4.609	30.384	6.337	42.427	3.251	42.41	1.251***	42.432	0.749**	1.788	1.289	**0.657***
10	100	0.9	160.483	13.853	5.774	85.528	61.062	9.235	61.882	12.815	85.688	6.445	85.656	2.101***	85.699	**1.132***	2.667	2.344	1.137**
0.5	100	0.99	80.933	6.319	0.167	43.118	30.476	1.063	28.578	2.224	43.129	5.561	43.129	0.404	43.133	**0.123***	0.143**	0.144***	0.186
1	100	0.99	163.625	10.569	0.27	87.433	62.725	1.687	58.019	4.473	87.446	11.806	87.445	0.792	87.451	**0.199***	0.232**	0.235***	0.237
5	100	0.99	815.854	31.082	1.087	435.566	311.746	4.172	287.582	22.013	435.567	51.586	435.549	3.984	435.571	0.637**	0.816***	0.857	**0.62***
10	100	0.99	1588.408	53.5	2.193	841.228	599.104	7.264	555.067	42.624	841.296	99.113	841.281	7.959	841.328	**0.974***	1.44***	1.703	1.166**
0.5	200	0.8	4.435	0.961	0.505	2.57	1.778	0.731	2.198	0.734	2.514	0.283**	2.515	0.804	2.512	**0.205***	0.465	0.326	0.29***
1	200	0.8	8.575	1.487	0.883	4.795	3.385	1.247	4.218	1.327	4.751	0.39***	4.754	0.843	4.75	**0.272***	0.805	0.474	0.322**
5	200	0.8	41.602	4.843	3.951	22.222	15.839	5.156	20.055	5.986	22.294	1.453	22.276	1.205***	22.294	0.84**	2.932	1.49	**0.688***
10	200	0.8	83.286	9.97	7.961	45.071	32.168	10.666	40.72	12.049	45.241	2.925	45.19	1.632***	45.241	1.367**	4.429	2.924	**1.166***
0.5	200	0.9	6.861	1.194	0.401	3.71	2.575	0.686	2.793	0.681	3.671	0.354	3.673	0.542	3.67	**0.16***	0.353	0.272***	0.234**
1	200	0.9	14.134	1.993	0.707	7.616	5.371	1.203	5.819	1.325	7.596	0.615	7.597	0.607	7.597	**0.231***	0.634	0.429***	0.271**
5	200	0.9	68.251	6.538	3.071	36.202	25.71	4.705	27.842	6.137	36.266	2.665	36.252	1.192***	36.272	0.785**	2.207	1.365	**0.644***
10	200	0.9	140.725	12.843	6.247	76.495	54.611	9.643	58.804	12.788	76.662	5.423	76.628	1.992***	76.671	1.297**	3.412	2.521	**1.149***
0.5	200	0.99	74.461	5.788	0.172	39.196	27.954	0.948	26.337	2.069	39.24	4.935	39.238	0.388	39.257	**0.126***	0.15**	0.151***	0.19
1	200	0.99	144.674	8.872	0.285	75.511	53.939	1.348	50.165	3.939	75.598	9.06	75.592	0.757	75.632	**0.211***	0.253	0.251***	0.241**
5	200	0.99	739.499	26.722	1.173	390.456	277.177	3.639	259.167	20.088	390.762	47.168	390.74	3.788	390.9	0.755**	0.927***	0.943	**0.678***
10	200	0.99	1483.401	51.917	2.274	784.535	559.662	7.328	523.261	40.447	785.653	89.251	785.68	7.598	786.074	**1.093***	1.575***	1.77	1.17**

Table 3: The EMSE values of the estimators for the model when $p = 8$.

σ^2	n	ρ	**OLS**	**RE**	**LE**	**YC I**	**YC II**	**SK**	**LTE**	**MNTP**	**NBR**	**ML**	**TSS**	**KLMRT**	**LKL**	**RTE I**	**RTE II**	**RTE III**	**RTE IV**
0.5	50	0.8	10.963	1.281	0.767	6.713	3.627	1.054	5.994	1.039	6.718	0.275***	6.719	0.449	6.723	0.256**	0.745	0.335	**0.155***
1	50	0.8	21.551	2.237	1.497	13.031	6.972	1.988	11.671	1.998	13.092	0.476***	13.094	0.587	13.106	0.459**	1.412	0.62	**0.231***
5	50	0.8	109.743	8.648	7.435	66.498	35.953	9.338	59.944	9.915	66.9	2.3	66.914	1.729***	66.974	1.556**	4.921	2.653	**0.833***
10	50	0.8	221.218	16.724	14.939	133.789	73.022	18.632	120.621	19.83	134.668	4.661	134.708	3.205***	134.825	2.373**	7.15	5.021	**1.638***
0.5	50	0.9	23.039	2.231	0.534	14.013	7.557	0.987	9.93	0.913	14.085	0.344	14.089	0.36	14.103	0.191**	0.514	0.281***	**0.107***
1	50	0.9	44.473	3.946	1.024	26.587	14.128	1.832	18.779	1.737	26.74	0.669	26.746	0.611	26.777	0.326**	0.936	0.524***	**0.154***
5	50	0.9	222.946	14.449	5.068	133.78	70.271	7.788	93.708	8.649	134.549	3.008	134.584	2.744	134.737	1.047**	3.016	2.139***	**0.553***
10	50	0.9	440.11	26.692	9.859	262.906	141.258	14.853	187.16	16.969	264.814	6.374	264.928	5.308	265.268	1.524**	4.074	3.866***	**1.067***
0.5	50	0.99	179.643	13.262	0.152	107.686	56.714	1.6	57.125	1.408	108.098	1.637	108.137	1.518	108.235	**0.07***	0.128	0.116***	0.084**
1	50	0.99	347.312	20.067	0.265	205.644	105.842	2.066	106.649	2.631	206.557	2.669	206.647	2.94	206.837	**0.117***	0.212	0.199***	0.121**
5	50	0.99	1780.847	77.538	1.257	1079.538	568.707	7.628	573.19	13.93	1084.237	14.127	1084.676	15.133	1085.599	**0.428***	0.694***	0.751	0.439**
10	50	0.99	3574.818	157.409	2.559	2164.081	1139.681	16.077	1146.738	27.97	2172.414	32.092	2173.152	30.483	2174.92	**0.688***	1.154***	1.409	0.855**
0.5	100	0.8	10.603	1.201	0.769	6.449	3.368	1.021	5.482	1.018	6.442	0.247***	6.444	0.453	6.446	0.225**	0.75	0.297	**0.153***
1	100	0.8	21.435	2.203	1.519	12.894	6.802	1.992	11.05	1.991	12.952	0.46***	12.958	0.601	12.964	0.405**	1.456	0.563	**0.226***
5	100	0.8	106.783	8.475	7.429	64.153	34.314	9.276	55.277	9.84	64.524	2.115	64.543	1.718***	64.587	1.477**	5.59	2.35	**0.75***
10	100	0.8	216.062	16.232	14.804	131.28	70.792	18.483	112.891	19.669	131.983	4.288	132.023	3.122***	132.097	2.279**	8.422	4.334	**1.346***
0.5	100	0.9	19.838	1.905	0.54	11.856	6.091	0.919	7.999	0.868	11.895	0.247***	11.898	0.355	11.906	0.161**	0.524	0.247***	**0.106***
1	100	0.9	40.778	3.66	1.077	24.838	13.024	1.849	16.825	1.769	24.943	0.473	24.953	0.616	24.967	0.29**	1.022	0.472***	**0.153***
5	100	0.9	199.467	13.852	5.209	119.483	61.984	8.02	80.836	8.575	120.02	2.188	120.07	2.647	120.132	0.989**	3.57	1.921***	**0.528***
10	100	0.9	394.458	23.96	10.242	235.022	121.152	14.934	159.1	16.842	236.083	4.328	236.181	5.143	236.322	1.541**	5.221	3.46***	**0.972***
0.5	100	0.99	185.186	12.524	0.148	111.087	56.641	1.328	56.856	1.397	111.356	1.434	111.382	1.616	111.434	**0.073***	0.13	0.115***	0.088**
1	100	0.99	366.431	21.167	0.267	217.66	110.366	2.164	110.517	2.718	218.195	2.965	218.259	3.217	218.362	**0.127***	0.228	0.203***	0.132**
5	100	0.99	1830.505	81.447	1.191	1088.923	556.968	7.795	558.111	13.573	1092.375	15.336	1092.759	15.972	1093.342	0.449**	0.722***	0.731	**0.436***
10	100	0.99	3751.459	153.338	2.384	2246.778	1156.604	14.89	1159.543	28.067	2252.611	31.294	2253.289	32.708	2254.351	**0.719***	1.192***	1.316	0.814**
0.5	200	0.8	10.66	1.246	0.761	6.451	3.304	1.037	4.71	1.012	6.427	0.157***	6.427	0.441	6.427	0.146**	0.74	0.222	**0.137***
1	200	0.8	21.333	2.212	1.476	12.979	6.666	1.989	9.429	1.961	12.987	0.263***	12.99	0.576	12.991	0.235**	1.423	0.381	**0.173***
5	200	0.8	104.209	7.95	7.137	62.972	31.982	8.91	45.832	9.391	63.039	1.159***	63.042	1.785	63.052	0.935**	5.852	1.549	**0.578***
10	200	0.8	210.074	14.973	14.4	126.321	64.277	17.708	92.213	18.896	126.483	2.272***	126.495	3.333	126.513	1.627**	9.579	2.925	**1.094***
0.5	200	0.9	19.286	1.941	0.543	11.538	5.922	0.951	7.366	0.871	11.559	0.196***	11.562	0.36	11.565	0.135**	0.529	0.223	**0.104***
1	200	0.9	38.292	3.33	1.063	22.818	11.587	1.748	14.491	1.704	22.868	0.367***	22.874	0.618	22.88	0.243**	1.025	0.409	**0.151***
5	200	0.9	192.914	13.086	5.232	116.524	59.091	7.861	73.792	8.529	116.721	1.808	116.737	2.702	116.761	0.944**	4.024	1.725***	**0.537***
10	200	0.9	388.286	25.075	10.535	233.444	119.683	15.599	148.703	17.191	233.989	3.688	234.033	5.326	234.104	1.56**	6.184	3.171***	**0.987***
0.5	200	0.99	214.867	15.351	0.134	129.551	66.368	1.637	66.195	1.551	129.674	1.906	129.692	1.913	129.715	**0.064***	0.116	0.101***	0.082**
1	200	0.99	422.703	24.4	0.244	254.318	128.303	2.299	127.823	3.003	254.639	3.046	254.689	3.796	254.717	**0.122***	0.217	0.188***	0.131**
5	200	0.99	2110.968	96.261	1.094	1258.604	642.555	8.7	637.585	14.92	1260.247	19.688	1260.535	18.974	1260.777	0.465**	0.723	0.683***	**0.433***
10	200	0.99	4251.352	171.417	2.109	2543.115	1292.978	14.78	1287.173	29.982	2546.31	35.578	2546.865	38.132	2547.238	**0.731***	1.179***	1.198	0.761**

Table 4: The EMSE values of the estimators for the model when $p = 12$.

σ^2	n	ρ	**OLS**	**RE**	**LE**	**YC I**	**YC II**	**SK**	**LTE**	**MNTP**	**NBR**	**ML**	**TSS**	**KLMRT**	**LKL**	**RTE I**	**RTE II**	**RTE III**	**RTE IV**
0.5	50	0.8	16.062	1.395	1.141	10.29	4.98	1.404	7.764	1.392	10.353	0.349**	10.359	0.395	10.364	0.374***	1.12	0.419	**0.151***
1	50	0.8	32.244	2.652	2.299	20.661	10.092	2.815	15.728	2.776	20.801	0.706***	20.813	0.604**	20.822	0.726	2.217	0.833	**0.268***
5	50	0.8	161.578	10.882	11.419	103.893	50.303	13.458	78.631	13.756	104.593	3.381	104.648	2.228**	104.703	2.636***	8.36	3.948	**1.155***
10	50	0.8	326.405	21.24	23.154	209.722	101.952	27.231	158.963	28.099	211.033	6.844	211.162	4.288***	211.24	4.026**	12.52	7.808	**2.273***
0.5	50	0.9	46.094	3.415	0.671	29.253	14.036	1.202	17.017	1.062	29.495	0.406	29.513	0.51	29.545	0.296**	0.655	0.371***	**0.071***
1	50	0.9	91.234	6.28	1.334	57.947	27.806	2.321	33.651	2.113	58.388	0.8	58.418	0.994	58.487	0.528**	1.252	0.731***	**0.12***
5	50	0.9	449.125	23.934	6.508	282.055	133.922	10.008	163.335	10.289	284.821	3.602	285.028	4.819	285.392	1.557**	4.145	3.23***	**0.49***
10	50	0.9	914.791	47.537	13.168	579.627	277.278	20.252	336.158	20.96	584.434	7.618	584.874	9.782	585.492	2.174**	5.7***	6.221	**0.97***
0.5	50	0.99	393.849	22.08	0.145	248.962	118.37	1.703	118.852	1.37	251.029	0.991	251.199	3.376	251.652	0.061**	0.129	0.113***	**0.052***
1	50	0.99	785.605	41.203	0.277	497.184	233.947	3.125	234.915	2.713	501.477	1.915	501.841	6.734	502.702	0.099**	0.223	0.213***	**0.083***
5	50	0.99	4028.641	168.722	1.374	2569.958	1228.338	11.144	1234.773	14.095	2590.424	9.587	2591.946	34.304	2596.141	0.357**	0.675***	0.869	**0.34***
10	50	0.99	7958.812	316.947	2.66	5062.842	2399.464	21.361	2410.877	27.608	5101.034	21.861	5103.976	68.094	5112.183	**0.542***	0.993***	1.485	0.618**
0.5	100	0.8	18.144	1.547	1.053	11.42	5.267	1.351	7.965	1.284	11.481	0.262**	11.49	0.381	11.494	0.33***	1.039	0.383	**0.116***
1	100	0.8	36.618	2.853	2.093	23.255	10.765	2.643	16.115	2.558	23.377	0.486**	23.389	0.626	23.398	0.62***	2.042	0.731	**0.184***
5	100	0.8	185.297	12.147	10.432	118.112	55.017	12.819	82.06	12.789	118.806	2.374***	118.877	2.643	118.916	2.327**	8.265	3.457	**0.762***
10	100	0.8	362.361	21.715	20.624	230.075	105.134	24.746	158.979	25.043	231.241	4.573***	231.388	5.106	231.472	3.641**	12.85	6.683	**1.501***
0.5	100	0.9	39.29	3.1	0.701	25.17	11.743	1.229	13.902	1.073	25.29	0.247***	25.304	0.516	25.31	0.208**	0.691	0.278	**0.068***
1	100	0.9	76.287	5.474	1.369	48.491	22.418	2.285	26.615	2.084	48.74	0.439***	48.772	0.983	48.788	0.38**	1.322	0.527	**0.11***
5	100	0.9	385.046	22.913	6.852	245.131	113.562	10.703	134.636	10.458	246.162	2.336***	246.286	4.836	246.355	1.241**	4.872	2.419	**0.425***
10	100	0.9	769.431	41.92	13.707	489.421	225.579	20.456	268.219	20.896	491.508	4.471***	491.789	9.643	491.918	1.919**	7.268	4.591	**0.818***
0.5	100	0.99	259.13	16.191	0.175	165.169	75.718	1.446	76.167	1.105	165.716	0.526	165.806	2.779	165.848	0.065**	0.163	0.116***	**0.057***
1	100	0.99	512.278	28.695	0.331	324.623	146.629	2.456	147.451	2.151	325.761	0.935	325.945	5.512	326.036	0.106**	0.291	0.216***	**0.087***
5	100	0.99	2552.714	118.048	1.607	1625.383	741.581	10.129	745.285	10.783	1630.484	5.271	1631.261	27.471	1631.832	0.388**	0.898	0.861***	**0.344***
10	100	0.99	5095.394	224.208	3.216	3219.162	1464.12	18.544	1470.211	21.468	3228.083	10.415	3229.43	55.372	3230.382	**0.619***	1.324***	1.534	0.64**
0.5	200	0.8	16.075	1.49	1.133	10.235	4.72	1.432	6.839	1.358	10.249	0.197**	10.251	0.398	10.252	0.249***	1.116	0.303	**0.115***
1	200	0.8	32.374	2.655	2.25	20.598	9.554	2.777	13.78	2.697	20.667	0.368**	20.674	0.625	20.677	0.466***	2.203	0.568	**0.18***
5	200	0.8	160.754	10.983	11.162	102.411	47.202	13.345	68.11	13.364	102.741	1.726**	102.778	2.459	102.79	1.953***	9.646	2.701	**0.735***
10	200	0.8	325.34	20.966	22.659	208.12	95.943	26.76	138.599	27.102	208.834	3.547***	208.919	4.824	208.942	3.361**	16.278	5.31	**1.441***
0.5	200	0.9	33.743	2.914	0.758	21.586	9.878	1.305	11.145	1.101	21.628	0.147***	21.635	0.499	21.636	0.131**	0.75	0.2	**0.066***
1	200	0.9	65.092	4.867	1.458	41.336	18.445	2.343	20.878	2.107	41.427	0.283***	41.441	0.93	41.444	0.237**	1.427	0.372	**0.1***
5	200	0.9	332.328	20.235	7.389	213.16	96.096	10.901	108.732	10.766	213.505	1.327***	213.558	4.607	213.566	0.923**	5.97	1.634	**0.393***
10	200	0.9	657.186	37.248	14.656	416.99	187.992	21.132	212.632	21.274	417.699	2.642***	417.797	9.129	417.828	1.527**	9.192	3.11	**0.784***
0.5	200	0.99	292.039	18.556	0.155	185.59	82.558	1.583	82.718	1.126	185.791	0.571	185.83	3.303	185.837	**0.049***	0.141	0.095***	0.054**
1	200	0.99	590.23	33.715	0.287	375.064	166.974	2.759	167.187	2.266	375.468	1.2	375.545	6.68	375.565	0.083**	0.259	0.171***	**0.080***
5	200	0.99	2921.739	136.846	1.377	1859.166	827.946	10.449	829.533	11.213	1861.452	6.048	1861.882	33.06	1861.937	0.349**	0.838	0.652***	**0.323***
10	200	0.99	5901.947	255.942	2.752	3751.628	1685.386	19.081	1689.929	22.74	3756.582	12.904	3757.596	66.628	3757.709	**0.614***	1.303	1.154***	0.622**

Table 5: The estimated parameter values and the estimated variance values of the estimators

	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{\beta}_4$	$var(\hat{\beta})$	$SMSE(\hat{\beta})$
$\hat{\beta}_{OLS}$	62.4054	1.5511	0.5102	0.1019	−0.1441	4912.0902	
$\hat{\beta}_{RE}\left(\hat{k}_{RE}=1.4250\right)$	0.1003	2.1725	1.1568	0.7435	0.4882	0.067330	0.067346
$\hat{\beta}_{LE}\left(\hat{d}_{LE}=0\right)$	0.1230	2.1781	1.1552	0.7473	0.4871	0.071467	0.071479
$\hat{\beta}_{LTE}\left(\hat{k}_{LTE}=451.2736,\ \hat{d}_{LTE}=-199.5073\right)$	27.6065	1.1641	1.0097	0.0891	0.2955	960.089121	960.375122
$\hat{\beta}_{SK}\left(\hat{k}_{SK}=1.4250,\ \hat{d}_{SK}=493.7504\right)$	26.4790	8.5996	−0.6618	5.2740	−0.7883	878.099704	879.220053
$\hat{\beta}_{YC\ I}\left(\hat{K}_1=0.0015,\ \hat{D}_1=0.9992\right)$	27.6068	1.9090	0.8688	0.4680	0.2075	959.502978	959.502981
$\hat{\beta}_{YC\ II}\left(\hat{K}_2=0.0008,\ \hat{D}_2=0.7206\right)$	27.6067	1.9052	0.8697	0.4653	0.2080	959.502677	959.502680
$\hat{\beta}_{MNTP}$ $\left(\hat{k}_{MNTP}=1.3761\times 10^{-6},\ \hat{d}_{MNTP}=0.0883\right)$	5.6197	2.1227	1.0983	0.6904	0.4314	39.291526	39.291535
$\hat{\beta}_{NBR}\left(\hat{k}_{NBR}=0.3388\times 10^{-3},\ \hat{d}_{NBR}=0.0015\right)$	27.6068	1.9091	0.8688	0.468	0.2075	959.502980	959.502982
$\hat{\beta}_{ML}\left(\hat{d}_{ML}=0.4426\right)$	-27.4454	2.4556	1.4408	1.033	0.7665	954.838327	954.838351
$\hat{\beta}_{TSS}\left(\hat{k}_{TSS}=0.5509\times 10^{-3},\ \hat{d}_{TSS}=0.7920\right)$	27.6068	1.9091	0.8688	0.468	0.2075	959.502980	959.502982
$\hat{\beta}_{KLMRT}$ $\left(\hat{k}_{KLMRT}=348.2785,\ \hat{d}_{KLMRT}=0.4420\right)$	0.0244	0.2284	1.4622	0.0276	0.6772	0.001739	6.4125088
$\hat{\beta}_{LKL}\left(\hat{k}_{LKL}=0.4714\times 10^{-3},\ \hat{d}_{LKL}=1\right)$	27.6068	1.9091	0.8688	0.468	0.2075	959.502980	959.502982
$\hat{\beta}_{NLTE\ \mathrm{I}}\left(f(k)=3.0866\times 10^{-13}k-0.0012\right)$ $\hat{k}_{NLTE\ \mathrm{I}}=0.0015$	0.0473	2.1925	1.1528	0.7580	0.4858	0.065275	0.065282
$\hat{\beta}_{NLTE\ \mathrm{II}}$ $\left(f(k)=3.0866\times 10^{-13}k-4.1930\times 10^{-11}\right)$ $\hat{k}_{NLTE\ \mathrm{II}}=0.0006$	42.0456	1.7605	0.7200	0.3161	0.0616	2228.180975	2228.180976
$\hat{\beta}_{NLTE\ \mathrm{III}}$ $\left(f(k)=3.0866\times 10^{-13}\ k-6.0859\times 10^{-8}\right)$ $\hat{k}_{NLTE\ \mathrm{III}}=1.4250$	0.1003	2.1725	1.1568	0.7435	0.4882	0.067330	0.067346
$\hat{\beta}_{NLTE(RE)\ I}$ $\left(f(k)=2.5341\times 10^{-10}\ (k+0.0012)^2-(k+0.0012)\right)$ $\hat{k}_{NLTE(RE)\ I}=0.0015$	0.0473	2.1925	1.1529	0.758	0.4858	0.065273	0.065280
$\hat{\beta}_{NLTE(RE)\ II}$ $\left(f(k)=2.2383\times 10^{-5}\ (k+0.0012)^2-(k+0.0012)\right)$ $\hat{k}_{NLTE(RE)\ II}=0.0006$	0.0473	2.1925	1.1528	0.758	0.4858	0.065275	0.065282
$\hat{\beta}_{NLTE(RE)\ III}$ $\left(f(k)=3.6781\times 10^{-6}\ (k+0.0012)^2-(k+0.0012)\right)$ $\hat{k}_{NLTE(RE)\ III}=1.4250$	0.0468	2.1538	1.1618	0.7302	0.4917	0.062256	0.062300
$\hat{\beta}_{RTE\ \mathrm{I}}\left(g(k)=2.5372\times 10^{-10}k-0.0012\right)$ $\hat{k}_{RTE\ \mathrm{I}}=0.1013$	0.0471	2.1774	1.1563	0.7471	0.4881	0.064088	0.064099
$\hat{\beta}_{RTE\ \mathrm{II}}\left(g(k)=2.5372\times 10^{-10}k-4.1621\times 10^{-11}\right)$ $\hat{k}_{RTE\ \mathrm{II}}=10.9035$	0.0452	2.0425	1.1873	0.6513	0.5083	0.054048	0.054708
$\hat{\beta}_{RTE\ \mathrm{III}}\left(g(k)=2.5372\times 10^{-10}k-0.2692\times 10^{-4}\right)$ $\hat{k}_{RTE\ \mathrm{III}}=0.9106\times 10^{-4}$	0.1161	2.1781	1.1553	0.7474	0.4872	0.070216	0.070227
$\hat{\beta}_{RTE\ \mathrm{IV}}\left(g(k)=0.1952\times 10^{-10}k-0.0122\right)$ $\hat{k}_{RTE\ \mathrm{IV}}=8935.2414$	0.0231	0.2431	1.1857	0.2203	0.6003	0.000312	255.273061