REVSTAT – Statistical Journal
Volume 6, Number 2, June 2008, 101–121

# ON A CLASS OF $\mathbb{Z}_+$-VALUED AUTOREGRESSIVE MOVING AVERAGE (ARMA) PROCESSES

Authors: Emad-Eldin A. A. Aly
– Department of Statistics and Operations Research,
Kuwait University, Kuwait
emad@kuc01.kuniv.edu.kw

Nadjib Bouzar
– Department of Mathematics and Computer Science,
University of Indianapolis, U.S.A.
nbouzar@uindy.edu

Received: April 2007 Revised: November 2007 Accepted: November 2007

Abstract:

- A convolution semigroup of probability generating functions and its related operator $\odot_F$ are used to construct a class of stationary $\mathbb{Z}_+$-valued autoregressive moving average (*ARMA*) processes. Several distributional and regression properties are obtained. A number of *ARMA* processes with specific innovation sequences are presented.

Key-Words:

- *stationarity; semigroup of probability generating functions; Mittag–Leffler distribution; Linnik distribution; time-reversibility.*

AMS Subject Classification:

- 60G10, 62M10.

## 1. INTRODUCTION

Time series models for count data have been the object of growing interest in the last twenty years. Numerous articles dealing with the theoretical aspects of these models as well as their applicability have appeared in the literature. We refer to McKenzie (2003) for an overview of the recent work in this area,

Stationary time series with a given marginal distribution have been developed by several authors. Most notably, McKenzie (1986, 1988), Al-Osh and Alzaid (1988) and Pillai and Jayakumar (1995) constructed stationary integer-valued autoregressive moving average (INARMA) processes with Poisson, negative binomial and discrete Mittag–Leffler marginal distributions. These models are based on the binomial thinning operator $\odot$ of Steutel and van Harn (1979) which is defined as follows: if $X$ is a $\mathbb{Z}_+$-valued random variable (rv) and $\alpha \in (0,1)$, then

$$\alpha \odot X = \sum_{i=1}^{X} X_i \ , \tag{1.1}$$

where $(X_i, i \geq 1)$ is a sequence of iid Bernoulli($\alpha$) rv's independent of $X$. The binomial thinning operator incorporates the discrete nature of the variates and replaces the multiplication used in the definition of standard *ARMA* processes. Related models that make use of a more general operator were introduced by Aly and Bouzar (1994) and Zhu and Joe (2003). Other aspects of the analysis of INARMA processes, such as parameter estimation and the study of extremal properties, can be found in Al-Osh and Alzaid (1987), McCormick and Park (1997), Park and Oh (1997), Kim and Park (2006), and Hall and Scotto (2006).

Aly and Bouzar (2005) used a convolution semigroup of probability generating functions (pgf's) and the related operator $\odot_F$ (see definitions below) to construct a class of stationary $\mathbb{Z}_+$-valued INAR($p$) processes. They developed a number of models with specific marginals which were shown to generalize several existing INAR($p$) models. The aim of this paper is to use the semigroup approach to construct a family of stationary $F$-INMA$(1)$, $F$-INMA$(q)$, and $F$-INARMA$(1,q)$ processes. These processes can be seen as extensions of the classical branching processes of Galton–Watson–Bienaymé (Arthreya and Ney, 1972). We obtain various distributional and regression properties of $F$-INARMA$(1,q)$ processes. We establish in particular that a stationary $F$-INMA$(1)$ process has the property of linear regression if and only if its marginal distribution is (discrete) $F$-stable. $F$-INARMA$(1,q)$ processes with $F$-stable, $F$-Mittag–Leffler, and compound discrete Linnik innovation sequences are studied. Examples are developed throughout the paper.

In the remainder of this section we recall some definitions and results that will be needed throughout the paper. For proofs and further details we refer to Athreya and Ney (1972, Chapter 3), van Harn *et al.* (1982) and van Harn and Steutel (1993).

$F := (F_t; t \geq 0)$ will denote a continuous composition semigroup of pgf's such that $F_t \not\equiv 1$ and $\delta_F = -\ln F_1'(1) > 0$. For any $|z| \leq 1$,

$$F_s \circ F_t(z) = F_{s+t}(z), \quad (s,t \geq 0)\,; \qquad \lim_{t \downarrow 0} F_t(z) = z\,; \qquad \lim_{t \to \infty} F_t(z) = 1\,. \tag{1.2}$$

The infinitesimal generator $U$ of the semigroup $F$ is defined by

$$U(z) = \lim_{t \downarrow 0} \big(F_t(z) - z\big)/t \qquad (|z| \leq 1)\ , \tag{1.3}$$

and satisfies $U(z) > 0$ for $0 \leq z < 1$. There exists a constant $a > 0$ and a distribution $(h_n, n \geq 0)$ on $\mathbb{Z}_+$ with pgf $H(z)$ such that $h_1 = 0$,

$$H'(1) = \sum_{n=1}^{\infty} n\,h_n \leq 1\ , \tag{1.4}$$

and

$$U(z) = a\big\{H(z) - z\big\}\ , \qquad |z| \leq 1\ , \tag{1.5}$$

The related $A$-function is defined by

$$A(z) = \exp\left\{-\int_0^z \big(U(x)\big)^{-1}\,dx\right\}\ , \qquad z \in [0,1]\ . \tag{1.6}$$

$A(z)$ is strictly decreasing over $[0,1]$, with $A(0)=1$ and $A(1)=0$. The functions $U(z)$ and $A(z)$ satisfy

$$U\big(F_t(z)\big) = U(z)\,F_t'(z) \quad \text{and} \quad A\big(F_t(z)\big) = e^{-t}A(z) \qquad (t \geq 0;\ 0 \leq z \leq 1)\ . \tag{1.7}$$

Moreover,

$$\delta_F = a\big(1 - H'(1)\big) = -U'(1) \quad \text{and} \quad F_t'(1) = e^{-\delta_F t} \qquad (t \geq 0)\ . \tag{1.8}$$

The function $B(z)$ defined by

$$B(z) = \lim_{t \to \infty} \frac{F_t(z) - F_t(0)}{1 - F_t(0)} \tag{1.9}$$

is a pgf such that $B(0) = 0$ and takes the form

$$B(z) = 1 - A(z)^{\delta_F}\ . \tag{1.10}$$

For a $\mathbb{Z}_+$-valued rv $X$ and $\eta \in (0,1)$, the generalized multiplication $\eta \odot_F X$ is defined by

$$\eta \odot_F X = \sum_{i=1}^{X} Y_i\ , \tag{1.11}$$

where $(Y_i,\, i \geq 1)$ is a sequence of iid rv's independent of $X$, with common pgf $F_t$, $t = -\ln \eta$.

A distribution on $\mathbb{Z}_+$ with pgf $P(z)$ is said to be $F$-self-decomposable if for any $t > 0$, there exists a pgf $P_t(z)$ such that

$$(1.12)\qquad P(z) = P\big(F_t(z)\big)\, P_t(z)\ , \qquad |z| \leq 1\ .$$

$F$-self-decomposable distributions are infinitely divisible.

Throughout the paper, stationarity of a stochastic process is considered to be in the strict sense. Finally, $P_X$ will denote the pgf of the distribution of the $\mathbb{Z}_+$-valued rv $X$.

## 2. $F$-INMA(1) PROCESSES

**Definition 2.1.** A sequence $(X_n,\, n \in \mathbb{Z})$ of $\mathbb{Z}_+$-valued rv's is said to be an $F$-INMA(1) process if for any $n \in \mathbb{Z}$,

$$(2.1)\qquad X_n \;=\; \eta \odot_F \epsilon_{n-1} + \epsilon_n\ ,$$

where $0 < \eta < 1$ and $(\epsilon_n,\, n \in \mathbb{Z})$ is a sequence of iid, $\mathbb{Z}_+$-valued rv's. $(\epsilon_n,\, n \in \mathbb{Z})$ is called the innovation sequence.

The generalized multiplication $\eta \odot_F \epsilon_{n-1}$ in (2.1) is performed independently for each $n$. More precisely, we assume the existence of an array $(Y_{i,n},\, i \geq 0,\, n \in \mathbb{Z})$ of iid $\mathbb{Z}_+$-valued rv's, independent of $(\epsilon_n,\, n \in \mathbb{Z})$, such that the array's common pgf is $F_t(z)$, $t = -\ln \eta$, and

$$(2.2)\qquad \eta \odot_F \epsilon_{n-1} \;=\; \sum_{i=1}^{\epsilon_{n-1}} Y_{i,n-1}\ .$$

It is clear that model (2.1) is not a Galton–Watson process. However, one can give a branching process-like interpretation as follows. In the time interval $(n-1, n]$, each element of $\epsilon_{n-1}$ brings into the system $k$ new elements (offspring) according to the probability distribution with pgf $F_t(z)$. This gives rise to a total of $U_{n-1} = \eta \odot_F \epsilon_{n-1}$ new elements in the system by time $n$. The variable $X_n$ is then obtained by superposing $U_{n-1}$ and $\epsilon_n$. For any $n$, elements of $\epsilon_n$ can only be present at time $n$ and their offspring at time $n+1$. In other words, elements of $\epsilon_n$ and their offspring remain in the system for at most two units of time.

The transformed version of (2.1) in terms of pgf's is given by

$$(2.3)\qquad P_{X_n}(z) = P_\epsilon(z)\, P_\epsilon\big[F_t(z)\big]\ , \qquad |z| \leq 1\ ,$$

where $P_\epsilon$ is the common pgf of the $\epsilon_n$'s and $t = -\ln\eta$. Furthermore, it can be easily shown that for an $F$-INMA(1) process $(X_n,\, n\in\mathbb{Z})$ the joint pgf $\Phi_k$ of $(X_{n+1}, X_{n+2}, ..., X_{n+k})$ for any $n\in\mathbb{Z}$ and $k\geq 2$ is

$$\Phi_k(z_1, z_2, ..., z_k) \;=\; P_\epsilon(z_k)\,\prod_{i=1}^{k} P_\epsilon\big(z_{i-1}F_t(z_i)\big)\;, \tag{2.4}$$

where $z_0 = 1,\ |z_i|\leq 1,\ i = 1, 2, ..., k,$ and $t = -\ln\eta$. It follows from (2.4) that any $F$-INMA(1) process is stationary.

Further distributional and correlation properties of $F$-INMA(1) processes are gathered in the following proposition.

**Proposition 2.1.** *Let $(X_n,\, n\in\mathbb{Z})$ be an $F$-INMA(1) process with coefficient $\eta\in(0,1)$. Assume further that the mean $\mu_\epsilon$ and the variance $\sigma^2_\epsilon$ of $\epsilon_n$ are finite and that $\sum_{n=2}^{\infty} n\,(n-1)\,h_n < \infty$. Then*

- **(i)** $E(X_n) \;=\; \mu_\epsilon(\eta^{\delta_F}+1)$;
- **(ii)** $\mathrm{Var}(X_n) \;=\; \sigma^2_\epsilon(1+\eta^{2\delta_F}) + \mu_\epsilon\left(1-\dfrac{U''(1)}{U'(1)}\right)\eta^{\delta_F}(1-\eta^{\delta_F})$;
- **(iii)** *for any $n\in\mathbb{Z}$,* $\mathrm{Cov}(X_{n-1}, X_n) \;=\; \eta^{\delta_F}\,\sigma^2_\epsilon$;
- **(iv)** *the autocorrelation function (ACRF) of $(X_n,\, n\in\mathbb{Z})$ at lag $k$ is*

$$\rho(k) = \begin{cases} \eta^{\delta_F}\sigma^2_\epsilon/\big(\sigma^2_\epsilon(1+\eta^{2\delta_F})\big) + \mu_\epsilon\left(\left(1-\frac{U''(1)}{U'(1)}\right)\eta^{\delta_F}(1-\eta^{\delta_F})\right), & \text{if } k=1\,, \\ 0, & \text{if } k>1\,. \end{cases} \tag{2.5}$$

**Proof:** We first note that $\sum_{n=2}^{\infty} n\,(n-1)\,h_n < \infty$ implies $U''(1)$ exists (see (1.4) and (1.5)). By (1.8) and (2.2), we have $E\big(\eta\odot_F\epsilon_{n-1}\,|\,\epsilon_{n-1}\big) = \eta^{\delta_F}\,\epsilon_{n-1}$. Therefore, $E(X_n) = E(\epsilon_n) + E(\eta\odot_F\epsilon_{n-1}) = E(\epsilon_n) + \eta^{\delta_F}E(\epsilon_{n-1}) = \mu_\epsilon(\eta^{\delta_F}+1)$, and thus (**i**) holds. By differentiating twice the expression $U(F_t(z)) = F'_t(z)\,U(z)$ $(t = -\ln\eta)$ with respect to $z$ and letting $z\to 1$, we obtain $F''_t(1) = \eta^{\delta_F}(\eta^{\delta_F}-1)\cdot$ $\cdot\,U''(1)/U'(1)$. By (1.8) and (2.2), $E\big((\eta\odot_F\epsilon_{n-1})^2\,|\,\epsilon_{n-1}\big) = \mathrm{Var}(Y_{1,n-1})\,\epsilon_{n-1} +$ $\eta^{2\delta_F}\,\epsilon^2_{n-1}$. Noting that

$$\mathrm{Var}(Y_{1,n-1}) \;=\; F''_t(1) + F'_t(1) - F'_t(1)^2 \;=\; \eta^{\delta_F}(1-\eta^{\delta_F})\left(1-\frac{U''(1)}{U'(1)}\right),$$

(**ii**) follows by direct calculations. By (2.1) and independence,

$$E(X_{n-1}X_n) = E(X_{n-1})E(\epsilon_n) + E\big(\epsilon_{n-1}(\eta\odot_F\epsilon_{n-1})\big) + E(\eta\odot_F\epsilon_{n-2})\,E(\eta\odot_F\epsilon_{n-1})\,.$$

Since $E\big(\epsilon_{n-1}(\eta\odot_F\epsilon_{n-1})\,|\,\epsilon_{n-1}\big) = \eta^{\delta_F}\,\epsilon^2_{n-1}$, again direct calculations (and (i)) yield (**iii**). (**iv**) results from (ii) and (iii) combined with the fact that $X_{n-k}$ and $X_n$ are independent for $k>1$. □

A stochastic process $(Z_n, n \in \mathbb{Z})$ is time reversible if for all $n$, $(Z_1, Z_2, ..., Z_n)$ and $(Z_n, Z_{n-1}, ..., Z_1)$ have the same distribution. By a result of McKenzie (1988), an $F$-INMA(1) process is time reversible if and only if for any $n$, $(X_{n-1}, X_n)$ has the same distribution as $(X_n, X_{n-1})$. The following result gives a necessary condition for the time reversibility of an $F$-INMA(1) process.

**Theorem 2.1.** *Assume that $(X_n, n \in \mathbb{Z})$ is a time reversible $F$-INMA(1) process with coefficient $\eta \in (0,1)$. Then the pgf $P_\epsilon(z)$ of the marginal distribution of the innovation sequence $(\epsilon_n, n \in \mathbb{Z})$ admits the representation*

$$P_\epsilon(z) \;=\; C\, z^m \exp\left\{ -\lambda \int_0^{z/F_t(0)} \frac{F_t(x) - F_t(0)}{x}\, dx \right\}, \qquad z \in [0, F_t(0)]\;, \tag{2.6}$$

*where $t = -\ln \eta$, $m$ is a nonnegative integer, $\lambda > 0$ and $0 < C < 1$ are real numbers.*

**Proof:** Since $A(z)$ is strictly decreasing on $[0,1]$ (with $A(0) = 1$ and $A(1) = 0$), it is invertible. It follows by (1.7) that $F_t(0) = A^{-1}(e^{-t}) > 0$ for any $t > 0$. Assume $P_\epsilon(0) \neq 0$. By (2.4) (applied to $k = 2$) and the property of time reversibility, we have

$$P_\epsilon(z_2)\, P_\epsilon\big(F_t(z_1)\big)\, P_\epsilon\big(z_1 F_t(z_2)\big) \;=\; P_\epsilon(z_1)\, P_\epsilon\big(F_t(z_2)\big)\, P_\epsilon\big(z_2 F_t(z_1)\big)\;, \tag{2.7}$$

where $t = -\ln\eta$ and $|z_i| \leq 1$, $i = 1, 2$. Setting $z_1 = 0$ and $z_2 = z$ in (2.7) yields

$$P_\epsilon(z) \;=\; \frac{P_\epsilon\big(F_t(z)\big)\, P_\epsilon\big(z\, F_t(0)\big)}{P_\epsilon\big(F_t(0)\big)}\;. \tag{2.8}$$

Moreover, differentiating with respect to $z_1$ in (2.7), setting $z_1 = 0$ and $z_2 = z$ in the resulting equation, and using (2.8), we obtain

$$P_\epsilon\big(z\,F_t(0)\big) \left[ \frac{F_t'(0)\, P_\epsilon'\big(F_t(0)\big)}{P_\epsilon\big(F_t(0)\big)} + \frac{P_\epsilon'(0)}{P_\epsilon(0)} \big(F_t(z) - 1\big) \right] \;=\; F_t'(0)\; z\, P_\epsilon'\big(z\,F_t(0)\big)\;. \tag{2.9}$$

Setting $z = 0$ in (2.9) gives $\frac{F_t'(0)\, P_\epsilon'(F_t(0))}{P_\epsilon(F_t(0))} = \frac{P_\epsilon'(0)}{P_\epsilon(0)}\,(1 - F_t(0))$. Therefore,

$$F_t(0)\, \frac{P_\epsilon'\big(z\,F_t(0)\big)}{P_\epsilon\big(z\,F_t(0)\big)} \;=\; \lambda\, \frac{F_t(z) - F_t(0)}{z}\;, \tag{2.10}$$

where $\lambda = \frac{P_\epsilon'(0)}{P_\epsilon(0)} \frac{F_t(0)}{F_t'(0)}$. The solution to the differential equation (2.10) is easily seen to be

$$P_\epsilon\big(z\,F_t(0)\big) \;=\; P_\epsilon(0)\, \exp\left\{ \lambda \int_0^z \frac{F_t(x) - F_t(0)}{x}\, dx \right\}, \qquad z \in [0,1]\;. \tag{2.11}$$

If $P_\epsilon(0) = 0$, then let $m$ be the smallest positive integer such that $P_\epsilon^*(0) \neq 0$, where $P_\epsilon^*(z) = P_\epsilon(z)/z^m$. It is easily seen that $P_\epsilon^*(z)$ satisfies (2.7) and thus admits the representation (2.11). This leads to (2.6) with $C = P_\epsilon^*(0) = P_\epsilon^{(m)}(0)$. $\square$

A process $(Z_n, n\in\mathbb{Z})$ has the property of (forward) linear regression if for any $n\in\mathbb{Z}$,

$$E(Z_n\,|\,Z_{n-1}) = a + b\,Z_{n-1}\ . \tag{2.12}$$

Aly and Bouzar (2005) showed that $F$-INAR(1) processes do possess this property. This is not true in general for $F$-INMA(1) models. In fact the following result gives a characterization of those $F$-INMA(1) processes that possess the property of linear regression. Recall that $F$-stable distributions (see van Harn *et al.*, 1982) have a pgf of the form

$$P(z) = \exp\bigl\{-\lambda\,A(z)^\gamma\bigr\}\ , \qquad \lambda>0,\quad |z|\le 1\ , \tag{2.13}$$

where $\gamma$, called the exponent of the distribution, must satisfy $0<\gamma\le\delta_F$. $F$-stable distributions are $F$-self-decomposable (see (1.12)).

**Theorem 2.2.** *Assume that the distribution $(h_n, n\ge 0)$ of (1.4)–(1.5) satisfies*

$$\sum_{n=2}^{\infty} h_n\,n\ln n\ <\ \infty\ . \tag{2.14}$$

*Let $(X_n, n\in\mathbb{Z})$ be an $F$-INMA(1) process such that $0<P_\epsilon(0)<1$ and $\mu_\epsilon = P'_\epsilon(1)<\infty$. Then $(X_n, n\in\mathbb{Z})$ has the property of linear regression if and only if $\epsilon_n$ has an $F$-stable distribution with exponent $\delta_F$, and in this case*

$$E(X_n\,|\,X_{n-1}) = \mu_\epsilon + \frac{\eta^{\delta_F}}{1+\eta^{\delta_F}}\,X_{n-1}\ . \tag{2.15}$$

**Proof:** Assume that (2.12) holds for some real numbers $a$ and $b$. By (2.4) (for $k=2$), the joint pgf of $(X_{n-1}, X_n)$, $n\in\mathbb{Z}$, is

$$\Phi_2(z_1,z_2) = P_\epsilon(z_2)\,P_\epsilon\bigl(F_t(z_1)\bigr)\,P_\epsilon\bigl(z_1\,F_t(z_2)\bigr)\ , \qquad t=-\ln\eta\ .$$

Differentiating $\Phi_2$ with respect to $z_2$ and then setting $z_2=1$ and $z_1=z$, we obtain

$$E\bigl(X_n\,z^{X_{n-1}}\bigr) = P_\epsilon\bigl(F_t(z)\bigr)\Bigl[P'_\epsilon(1)\,P_\epsilon(z) + F'_t(1)\,z\,P'_\epsilon(z)\Bigr]\ , \qquad n\in\mathbb{Z}\ . \tag{2.16}$$

By (2.12), we have for any $n\in\mathbb{Z}$,

$$\begin{aligned} E(X_n\,z^{X_{n-1}}) &= E\bigl(z^{X_{n-1}}\,E(X_n\,|\,X_{n-1})\bigr) \\ &= b\,z\,E\bigl(X_{n-1}\,z^{X_{n-1}-1}\bigr) + a\,E(z^{X_{n-1}})\ . \end{aligned} \tag{2.17}$$

Note that $E(X_{n-1}\,z^{X_{n-1}-1}) = P'(z)$, where $P(z)$ is the pgf of $X_{n-1}$. It follows by (2.17) that $E(X_n\,z^{X_{n-1}}) = a\,P(z) + b\,z\,P'(z)$ which, combined with (2.3), implies

$$\begin{aligned} E(X_n z^{X_{n-1}}) &= a\,P_\epsilon\bigl(F_t(z)\bigr)\,P_\epsilon(z) \\ &\quad + b\,z\Bigl[F'_t(z)\,P'_\epsilon\bigl(F_t(z)\bigr)\,P_\epsilon(z) + P'_\epsilon(z)\,P_\epsilon\bigl(F_t(z)\bigr)\Bigr]\ . \end{aligned} \tag{2.18}$$

Letting $Q(z) = P'_\epsilon(z)/P_\epsilon(z)$ and noting, by (1.8), $F'_t(1) = \eta^{\delta_F}$, it follows by (2.16) and (2.18) that

$$P'_\epsilon(1) + \eta^{\delta_F} z\, Q(z) \;=\; a + b\, z\Big[F'_t(z)\, Q\big(F_t(z)\big) + Q(z)\Big]\,. \tag{2.19}$$

Setting $z = 0$ and $z = 1$ in (2.19) (recall $Q(1) = P'_\epsilon(1) \neq 0$), we deduce that $a = P'_\epsilon(1)$ and $b = \eta^{\delta_F}/(\eta^{\delta_F}+1)$. Therefore, (2.19) reduces to

$$F'_t(z)\, Q\big(F_t(z)\big) \;=\; \eta^{\delta_F}\, Q(z)\ ,$$

or, by (1.7),

$$Q(z) \;=\; \eta^{-\delta_F}\, \frac{U\big(F_t(z)\big)}{U(z)}\; Q\big(F_t(z)\big)\ .$$

The additivity property $F_t(F_{jt}(z)) = F_{(j+1)t}(z)$ and an induction argument yield for any $n \geq 1$,

$$Q(z) \;=\; e^{n\delta_F t}\, \frac{U\big(F_{nt}(z)\big)}{U(z)}\; Q\big(F_{nt}(z)\big)\ . \tag{2.20}$$

From the semigroup properties (1.2), (1.8), and (1.9) we have

$$\lim_{n\to\infty} F_{nt}(z) = 1\,, \quad \lim_{n\to\infty} \frac{U\big(F_{nt}(z)\big)}{F_{nt}(z)-1} = U'(1) = -\delta_F\,, \quad \lim_{n\to\infty} \frac{F_{nt}(z)-1}{F_{nt}(0)-1} = 1-B(z)\ .$$

Moreover, (2.14) implies (see van Harn *et al.*, 1982)

$$\lim_{n\to\infty} e^{n\delta_F t}\big(F_{nt}(0)-1\big) = -1\ .$$

By letting $n \to \infty$ in (2.20), we obtain

$$Q(z) \;=\; \frac{P'_\epsilon(z)}{P_\epsilon(z)} \;=\; \delta_F\, Q(1)\, \frac{1-B(z)}{U(z)}\ . \tag{2.21}$$

Since (by (1.6) and (1.10)) $1/U(z) = -A'(z)/A(z)$ and $1 - B(z) = A(z)^{\delta_F}$, it follows from (2.21)

$$\ln P_\epsilon(z) \;=\; -\delta_F\, Q(1) \int_1^z A'(x)\, A(z)^{\delta_F - 1}\, dx \;=\; -\delta_F\, Q(1)\, A(z)^{\delta_F}\ .$$

This proves the necessary part. To prove sufficiency, assume that $P_\epsilon(z) = \exp\big\{-\lambda\, A(z)^{\delta_F}\big\}$ for some $\lambda > 0$. Since by (1.9) $P_\epsilon(z) = \exp(B(z)-1)$, assumption (2.14), which is equivalent to $B'(1) < \infty$ (see Athreya and Ney (1972), Chapter 3, or van Harn *et al.* (1982), Remark 7.3), implies $\mu_\epsilon = E(\epsilon_n) < \infty$, and thus $E(X_n) < \infty$. We have for any $n \in \mathbb{Z}$

$$E(X_n\,|\,X_{n-1}) \;=\; E\Big(E\Big(X_n\,|\,\epsilon_{n-1},\, \epsilon_{n-2},\, \big(Y_{i,n-1}, Y_{i,n-2},\, i \geq 1\big)\Big)\;\Big|\; X_{n-1}\Big)$$

and, by independence and (2.2),

$$E\Big(X_n\,|\,\epsilon_{n-1},\, \epsilon_{n-2},\, \big(Y_{i,n-1}, Y_{i,n-2},\, i \geq 1\big)\Big) \;=\; \mu_\epsilon + \eta^{\delta_F}\, \epsilon_{n-1}\ .$$

Therefore,

$$E(X_n \,|\, X_{n-1}) \;=\; \mu_\epsilon + \eta^{\delta_F}\, E(\epsilon_{n-1} \,|\, X_{n-1}) \;. \tag{2.22}$$

The joint pgf $g(z_1, z_2) = E(z_1^{\epsilon_n}\, z_2^{X_n})$ of $(\epsilon_n, X_n)$ is independent of $n$ and is given by

$$g(z_1, z_2) \;=\; P_\epsilon(z_1 z_2)\, P_\epsilon\big(F_t(z_2)\big) \;. \tag{2.23}$$

By (2.3) and (1.7), the pgf $P(z)$ of $X_n$ is

$$P(z) \;=\; \exp\Big\{-\lambda\Big(A\big(F_t(z)\big)^{\delta_F} + A(z)^{\delta_F}\Big)\Big\} \;=\; \exp\Big\{-\lambda\big(1+\eta^{\delta_F}\big)\, A(z)^{\delta_F}\Big\} \;. \tag{2.24}$$

Moreover,

$$\frac{d}{dz_1} g(z_1, z_2)\bigg|_{z_1=1,\, z_2=z} \;=\; E(\epsilon_n z^{X_n}) \;=\; \sum_{k=0}^{\infty} z^k\, E\big(\epsilon_n \,|\, X_n = k\big)\, p_k \;, \tag{2.25}$$

where $(p_k,\, k \geq 0)$ is the distribution of $X_n$. By (2.23),

$$\frac{d}{dz_1} g(z_1, z_2) \;=\; z_2\, P_\epsilon\big(F_t(z_2)\big)\, P'_\epsilon(z_1 z_2) \;.$$

Direct calculations, combined with (1.7) and the equation $A'(z)/A(z) = -1/U(z)$ (from (1.6)), yield

$$\frac{d}{dz_1} g(z_1, z_2)\bigg|_{z_1=1,\, z_2=z} \;=\; \lambda\, \delta_F\, \frac{z\, A(z)^{\delta_F}}{U(z)}\; \exp\Big\{-\lambda\big(1+\eta^{\delta_F}\big)\, A(z)^{\delta_F}\Big\} \;.$$

We deduce (in view of (2.24))

$$\frac{d}{dz_1} g(z_1, z_2)\bigg|_{z_1=1,\, z_2=z} \;=\; \big(1+\eta^{\delta_F}\big)^{-1}\, z\, P'(z) \;=\; \big(1+\eta^{\delta_F}\big)^{-1} \sum_{k=0}^{\infty} k\; p_k\, z^k \;. \tag{2.26}$$

Since $P(z)$ is infinitely divisible and $p_1 = P'(0) = \lambda\, \delta_F\, e^{-\lambda} > 0$, it follows by Corollary 8.3, p. 51, in Steutel and van Harn (2004) that $p_k > 0$ for all $k \geq 0$. Uniqueness of the power series coefficients in (2.25) and (2.26) implies that for any $n \in \mathbb{Z}$

$$E(\epsilon_n \,|\, X_n) \;=\; (1 + \eta^{\delta_F})^{-1}\, X_n \;. \tag{2.27}$$

Equation (2.15) follows then from (2.22) and (2.27). □

van Harn *et al.* (1982) (see also Zhu and Joe, 2003) give some rich examples of continuous composition semigroups of pgf's from which one can generate $F$-INMA(1) processes. We mention the parameterized family of semigroups $\big(F^{(\theta)},\, \theta \in [0,1)\big)$ described by

$$F_t^{(\theta)}(z) \;=\; 1 - \frac{\overline{\theta}\, e^{-\overline{\theta} t}\, (1-z)}{\overline{\theta} + \theta\, (1 - e^{-\overline{\theta} t})\, (1-z)} \;, \qquad t \geq 0, \quad |z| \leq 1, \quad \overline{\theta} = 1 - \theta \;. \tag{2.28}$$

In this case we have $\delta_{F^{(\theta)}} = \overline{\theta}$, $U^{(\theta)}(z) = (1-z)(1-\theta z)$ and $A^{(\theta)}(z) = \left(\frac{1-z}{1-\theta z}\right)^{\frac{1}{\overline{\theta}}}$. We note that for $\theta = 0$, $F^{(\theta)}$ corresponds to the standard semigroup $F_t^{(0)}(z) = 1 - e^{-t} + e^{-t} z$ and $\odot_{F^{(0)}}$ is the binomial thinning operator of Steutel and van Harn (1979) (see (1.1)).

For the family of semigroups $\big(F^{(\theta)}, \theta \in [0,1)\big)$ of (2.28), the pgf $P_\epsilon(z)$ of (2.6) is shown to be (via analytic continuation):

$$P_\epsilon(z) = \begin{cases} z^m\, e^{-\lambda(1-z)}, & \text{if} \quad \theta = 0 \quad (\lambda > 0), \\ z^m \left(\dfrac{\overline{\theta}}{1-\theta z}\right)^r, & \text{if} \quad 0 < \theta < 1 \quad (r > 0)\;, \end{cases} \tag{2.29}$$

for some nonnegative integer $m$. Therefore, by Theorem 2.1, for time reversibility for an $F^{(0)}$-INMA(1) (resp. $F^{(\theta)}$-INMA(1), $0<\theta<1$) to hold it is necessary that $\epsilon_n \overset{d}{=} \epsilon + m$ where $\epsilon$ has a Poisson distribution with some mean $\lambda > 0$ (resp. a negative binomial distribution with probability of success $\theta$). In this case the converse holds as well, as shown by Al-Osh and Alzaid (1988), for $\theta = 0$, and by Aly and Bouzar (1994), for $0 < \theta < 1$.

The family of semigroups $\big(F^{(\theta)}, \theta \in [0,1)\big)$ of (2.28) necessarily satisfies condition (2.14) (since $h_n = 0$ for $n \geq 3$). By Theorem 2.2, an $F^{(\theta)}$-INMA(1) process has the property of (forward) linear regression if and only if its innovation sequence has a Poisson geometric distribution with pgf

$$P_\epsilon(z) = \exp\left\{-\lambda\, \frac{1-z}{1-\theta z}\right\} \qquad (\lambda > 0)\;. \tag{2.30}$$

The version of Theorem 2.2 for the semigroup $F^{(\theta)}$ was established Al-Osh and Alzaid (1988) (for $\theta = 0$) and by Aly and Bouzar (1994) (for $0 < \theta < 1$).

## 3. $F$-INMA(1) PROCESSES WITH A DISCRETE STABLE INNOVATION SEQUENCE

Aly and Bouzar (2005) introduced a stationary $F$-INAR(1) process with an $F$-stable marginal. In this section, we construct its $F$-INMA(1) counterpart.

Let $(X_n, n \in \mathbb{Z})$ be an $F$-INMA(1) process such that $\epsilon_n$ has the $F$-stable distribution with exponent $\gamma$, $0 < \gamma \leq \delta_F$, and pgf (2.13). Then by (1.3), (2.3), and (2.13), the marginal distribution of $(X_n, n \in \mathbb{Z})$ is $F$-stable with the same exponent and with pgf

$$P(z) = \exp\Big\{-\lambda(1+\eta^\gamma)\, A(z)^\gamma\Big\}\;, \qquad \lambda > 0, \quad |z| \leq 1\;. \tag{3.1}$$

The joint pgf of $(X_1, X_2, ..., X_k)$ is (by way of (2.4) and (1.3))

$$\Phi_k(z_1, z_2, ..., z_k) = \exp\left\{-\lambda\left(\eta^\gamma A(z_1)^\gamma + \sum_{i=2}^{k} A\big(z_{i-1}F_t(z_i)\big)^\gamma + A(z_k)^\gamma\right)\right\}, \tag{3.2}$$

where $t = -\ln\eta$.

By van Harn *et al.* (1982), an $F$-stable distribution with exponent $\gamma$ has a finite mean if and only if $\gamma = \delta_F$ and $B'(1) < \infty$ (or, equivalently, (2.14) holds). Therefore, a finite mean $F$-INMA(1) process with an $F$-stable marginal distribution exists only if $\gamma = \delta_F$ and $B'(1) < \infty$. In this case $\mu_\epsilon = \lambda B'(1)$. If we further assume that $B''(1) < \infty$, the variance of $\epsilon_n$ is $\sigma_\epsilon^2 = \lambda(B''(1) + B'(1))$. The mean and variance of $X_n$ as well as the correlation coefficient of $(X_n, X_{n+1})$, follow from Proposition 2.1, under the further assumption $\sum_{n=2}^\infty n(n-1)h_n < \infty$.

The branching process-like interpretation of an $F$-INMA(1) process (described in Section 2) leads naturally to consider the variable $T_k = \sum_{i=1}^k X_i$. $T_k$ represents the total number of elements that were present in the system during the time interval $[0, k]$. It can be easily seen that the pgf of $T_k$ is $P_{T_k}(z) = \Phi(z, z, ..., z)$ (see (2.4)), or

$$P_{T_k}(z) = \exp\left\{-\lambda\Big[(1+\eta^\gamma)\big(A(z)\big)^\gamma + (k-1)\,A\big(z\,F_t(z)\big)^\gamma\Big]\right\}, \quad t = -\ln\eta\,. \tag{3.3}$$

It is easily shown from (3.3) that $T_k \stackrel{d}{=} Y_1 + Z_1$, where $Y_1$ is $F$-stable with exponent $\gamma$, $Z_1$ is an $F$-stable compounding (with exponent $\gamma$) of the distribution with pgf $z\,F_t(z)$, and $Y_1$ and $Z_1$ are independent.

Considering the family of semigroups $\big(F^{(\theta)}, \theta \in [0,1)\big)$ of (2.28), we note that the Poisson INMA(1) process of McKenzie (1988) is the finite mean $F^{(0)}$-INMA(1) process with an $F^{(0)}$-stable marginal. The Poisson geometric INMA(1) process of Aly and Bouzar (1994) (with pgf (2.30)) arises as the finite mean $F^{(\theta)}$-INMA(1) process with an $F^{(\theta)}$-stable marginal.

## 4. $F$-INMA(1) PROCESSES WITH A DISCRETE MITTAG–LEFFLER INNOVATION SEQUENCE

A distribution on $\mathbb{Z}_+$ is said to have an $F$-Mittag–Leffler (or $F$-ML ) distribution with exponent $\gamma$, $0 < \gamma \le \delta_F$, if its pgf is of the form

$$P(z) = \big(1 + c\,A(z)^\gamma\big)^{-1} \qquad \text{for some} \quad c > 0\,. \tag{4.1}$$

$F$-ML distributions are $F$-self-decomposable (van Harn and Steutel, 1993). Aly and Bouzar (2005) presented a stationary $F$-INAR(1) process with an $F$-ML marginal.

If $(X_n,\, n\in\mathbb{Z})$ is an $F$-INMA(1) process such that $\epsilon_n$ has the $F$-ML distribution of (4.1), then $X_n$ admits the following representation:

$$X_n \stackrel{d}{=} \sum_{i=1}^{Y+Z} W_i \;, \tag{4.2}$$

where the $W_i$'s are iid $\mathbb{Z}_+$-valued rv's with common pgf $B(z)$ of (1.9)–(1.10), and $Y$ and $Z$ are independent $\mathbb{Z}_+$-valued rv's (also independent of the $W_i$'s) and with respective pgf's

$$P_Y(z) = \Big(1 + c(1-z)^{\gamma/\delta_F}\Big)^{-1} \quad\text{and}\quad P_Z(z) = \Big(1 + c\,\eta^\gamma(1-z)^{\gamma/\delta_F}\Big)^{-1}. \tag{4.3}$$

This is shown as follows. Let $P(z)$ be the pgf of $\sum_{i=1}^{Y+Z} W_i$. By (1.10) and (4.3),

$$P(z) \,=\, P_{Y+Z}\big(B(z)\big) \,=\, P_Y\big(B(z)\big)\,P_Z\big(B(z)\big) \,=\, \big(1+c\,A(z)^\gamma\big)^{-1}\big(1+c\,\eta^\gamma A(z)^\gamma\big)^{-1},$$

or, $P(z) = P_\epsilon(z)\,P_\epsilon(F_t(z))$, $t = -\ln\eta$. The representation (4.2) follows then from (2.3).

The joint pgf of $(X_1, X_2, ..., X_k)$ is (by way of (2.4) and (1.3))

$$\begin{aligned}\Phi_k(z_1,...,z_k) \;&=\; \big(1 + c\,\eta^\gamma A(z_1)^\gamma\big)^{-1} \\ &\quad\times\left[\prod_{i=2}^{k}\Big(1 + c\,A\big(z_{i-1}F_t(z_i)\big)^\gamma\Big)^{-1}\right]\big(1 + c\,\eta^\gamma A(z_k)^\gamma\big)^{-1},\end{aligned} \tag{4.4}$$

where $t = -\ln\eta$.

Similarly to the discrete stable case of Section 3, an $F$-ML distribution with exponent $\gamma$ has a finite mean if and only if $\gamma = \delta_F$ and $B'(1) < \infty$ (or, equivalently, (2.14) holds). Therefore, a finite mean $F$-INMA(1) process with an $F$-ML innovation exists only if $\gamma = \delta_F$ and $B'(1) < \infty$. In this case $\mu_\epsilon = c\,B'(1)$. If we further assume that $B''(1) < \infty$, the variance of $\epsilon_n$ is $\sigma_\epsilon^2 = c\big(B''(1) + c\,B'(1)^2 + B'(1)\big)$. The mean and variance of $X_n$ as well as the correlation coefficient of $(X_n, X_{n+1})$ follow from Proposition 2.1, under the further assumption $\sum_{n=2}^\infty n\,(n-1)\,h_n < \infty$. We note that when $\gamma = \delta_F$, the distributions of the rv's $Y$ and $Z$ of (4.2) and (4.3) simplify respectively to a Geometric $\big(\frac{c}{1+c}\big)$ and a Geometric $\big(\frac{c\eta^\gamma}{1+c\eta^\gamma}\big)$.

The total number of elements, $T_k = \sum_{i=1}^k X_i$, that were present in the system during the time interval $[0,k]$ for an $F$-INMA(1) process with an $F$-ML marginal has pgf

$$P_{T_k}(z) = \Big[\big(1+c\,A(z)^\gamma\big)\big(1+c\,\eta^\gamma A(z)^\gamma\big)\Big]^{-1}\Big(1+c\,A\big(z\,F_t(z)\big)^\gamma\Big)^{1-k}, \quad t = -\ln\eta\,. \tag{4.5}$$

By (4.5), $T_k$ admits the representation $T_k \stackrel{d}{=} Y_2 + W_2 + Z_2$, where $Y_2$, $W_2$ and $Z_2$ are independent, $Y_2$ and $W_2$ have $F$-ML distributions with exponent $\gamma$, and $Z_2$ is a compounding of the distribution with pgf $zF_t(z)$ by the $(k-1)$-th convolution of the distribution of $\epsilon_n$.

Following McKenzie (1986), an $F$-INMA(1) process with an $F$-ML marginal distribution can be obtained by modifying (2.1) as follows:

$$X_n = \eta \odot \epsilon_n + B_n\,\epsilon_{n-1}\ , \tag{4.6}$$

where $0<\eta<1$, $(\epsilon_n,\, n\in\mathbb{Z})$ is a sequence of iid rv's with a common $F$-ML distribution with exponent $0<\gamma\le\delta_F$, $(B_n,\, n\in\mathbb{Z})$ is a sequence of iid Bernoulli$(1-\eta^\gamma)$ rv's, and $(\epsilon_n,\, n\in\mathbb{Z})$ and $(B_n,\, n\in\mathbb{Z})$ are independent. By (4.6), (4.1), and (1.7), the pgf $P(z)$ of $X_n$ is shown to be

$$P(z) = \bigl(1+c\,\eta^\gamma A(z)^\gamma\bigr)^{-1}\Bigl(\eta^\gamma+(1-\eta^\gamma)\bigl(1+c\,A(z)^\gamma\bigr)^{-1}\Bigr) = \bigl(1+c\,A(z)^\gamma\bigr)^{-1}\,.$$

The finite mean $F^{(0)}$-ML innovation sequence corresponding to the $F^{(0)}$-INMA(1) process of (2.1) (with $F^{(0)}$ as in (2.28)) reduces to a geometric innovation with probability of success $1/(1+c)$. Likewise, for the semigroup $F^{(\theta)}$, $0<\theta<1$, of (2.28), the finite mean $F^{(\theta)}$-ML innovation process for an $F^{(\theta)}$-INMA(1) process admits the representation $\epsilon_n \stackrel{d}{=} I_n\,\epsilon'_n$ where $(I_n,\, n\in\mathbb{Z})$ and $(\epsilon'_n,\, n\in\mathbb{Z})$ are independent sequences of $\mathbb{Z}_+$-valued iid rv's, $I_n$ is Bernoulli$(c/(1+c))$, and $\epsilon'_n$ has a (truncated at zero) geometric distribution with probability of success $\overline{\theta}/(1+c)$. Finally, the geometric INMA(1) process of McKenzie (1986) corresponds to the modified finite mean $(\gamma=1)$ $F$-INMA(1) process of (4.6) with an $F^{(0)}$-ML marginal distribution.

## 5. $F$-INMA(1) PROCESSES WITH A COMPOUND DISCRETE LINNIK INNOVATION SEQUENCE

A $\mathbb{Z}_+$-valued rv $X$ is said to have an $F$-compound discrete Linnik distribution if its pgf has the form

$$P(z) = \bigl(1+\lambda\,A(z)^\gamma\bigr)^{-r}\ , \tag{5.1}$$

for some $0<\gamma\le\delta_F$, $\lambda>0$, and $r>0$. van Harn and Steutel (1993) showed that $F$-compound discrete Linnik distributions are $F$-self-decomposable and arise as solutions to stability equations for $\mathbb{Z}_+$-valued processes with stationary independent increments. Aly and Bouzar (2005) constructed a $\mathbb{Z}_+$-valued stationary INAR(1) process with an $F$-compound discrete Linnik distribution. Note the case $r=1$ corresponds to the $F$-ML distribution of the previous section.

If $(X_n,\, n\in\mathbb{Z})$ is an $F$-INMA(1) process such that $\epsilon_n$ has the $F$-compound discrete Linnik distribution, then the distribution of $X_n$ has the following representation:

$$X_n \overset{d}{=} \sum_{i=1}^{Y+Z} W_i \ , \tag{5.2}$$

where the $W_i$'s are iid with common pgf $B(z)$, and $Y$ and $Z$ are $\mathbb{Z}_+$-valued independent rv's (also independent of the $W_i$'s) with respective pgf's

$$P_Y(z) = \Big(1 + c(1-z)^{-\gamma/\delta_F}\Big)^{-r} \quad \text{and} \quad P_Z(z) = \Big(1 + c\,\eta^\gamma(1-z)^{-\gamma/\delta_F}\Big)^{-r} . \tag{5.3}$$

The proof of (5.2)–(5.3) is identical to the one given in the case of the $F$-INMA(1) process with an $F$-ML innovation (see (4.2)–(4.3) of the previous section ). The details are omitted.

Formulas for the joint pgf of $(X_1, X_2, ..., X_k)$ as well as the pgf of $T_k = \sum_{i=1}^k X_i$ can be derived similarly to the $F$-ML case of the previous section.

Furthermore, a finite mean $F$-INMA(1) process with a compound discrete Linnik innovation sequence exists only if $\gamma = \delta_F$ and $B'(1) < \infty$. In this case $\mu_\epsilon = rc\,B'(1)$. If we further assume then $B''(1) < \infty$, the variance of $\epsilon_n$ is $\sigma_\epsilon^2 = rc\big(B''(1) + cB'(1)^2 + B'(1)\big)$. The mean and variance of $X_n$ as well as the correlation coefficient of $(X_n, X_{n+1})$ follow from Proposition 2.1, under the further assumption $\sum_{n=2}^\infty n\,(n-1)\,h_n < \infty$. We note that when $\gamma = \delta_F$, the distributions of the rv's $Y$ and $Z$ of (5.2) simplify respectively to a negative binomial $\big(\frac{c}{1+c},\, r\big)$ and a negative binomial $\big(\frac{c\eta^\gamma}{1+c\eta^\gamma},\, r\big)$.

## 6. $F$-INMA$(q)$ PROCESSES

**Definition 6.1.** A sequence $(X_n, n \in \mathbb{Z})$ of $\mathbb{Z}_+$-valued rv's is said to be an $F$-INMA$(q)$ process if for any $n \in \mathbb{Z}$,

$$X_n \;=\; \epsilon_n + \sum_{i=1}^{q} \eta_i \odot_F \epsilon_{n-i} \ , \tag{6.1}$$

where $(\epsilon_n, n \in \mathbb{Z})$ is a sequence of iid, $\mathbb{Z}_+$-valued rv's, and $0 < \eta_i < 1$, $i = 1, 2, ..., q$.

The generalized multiplications $\eta_i \odot_F \epsilon_{n-i}$, $i = 1, ..., q$, in (6.1) are performed independently. More precisely, we assume the existence of $q$ independent arrays $\big(Y_{j,n}^{(i)},\, j \geq 0,\, n \in \mathbb{Z}\big)$, $i = 1, 2, ..., q$, of iid $\mathbb{Z}_+$-valued rv's, independent of $(\epsilon_n, n \in \mathbb{Z})$, such that for each $i = 1, 2, ..., q$, the array's common pgf is $F_{t_i}(z)$, $t_i = -\log \eta_i$, and

$$\eta_i \odot_F \epsilon_{n-i} \overset{d}{=} \sum_{j=1}^{\epsilon_{n-i}} Y_{j,n-j}^{(i)} \ . \tag{6.2}$$

Equation (6.2) can be interpreted as follows. Subsequent to time $n$, each element of $\epsilon_n$ has $q$ nonoverlapping reproduction periods: $(n+i,\, n+i+1]$, $i=0,1,2,...,q-1$ with the distribution of offspring having pgf $F_{t_i}(z)$ over $(n+i,\, n+i+1]$. Each $\eta_i \odot_F \epsilon_n$ represents then the total number of offspring brought into the system by all $\epsilon_n$ elements. The offspring survive one unit of time and are replaced at time $n+1$ by the offspring from the next reproduction period. The offspring of $\epsilon_n$ are phased out of the system after $q$ units of time. It is important to note that for all $n \in \mathbb{Z}$ and all $i,j=1...,q$, $i \neq j$, $\eta_i \odot \epsilon_n$ and $\eta_j \odot \epsilon_n$ are independent, given $\epsilon_n$.

The process $(X_n,\, n\in\mathbb{Z})$ of (6.1) is necessarily stationary and its marginal distribution has pgf

$$P_X(z) = P_\epsilon(z) \prod_{i=1}^{q} P_\epsilon\big[F_{t_i}(z)\big] , \tag{6.3}$$

where $t_i = -\log \eta_i,\ i=1,...,q$.

Distributional properties of an $F$-INMA$(q)$ process are given in the following proposition. The proof is similar to the one given in the case $q=1$ in section 2 (Proposition 2.1). The details are omitted.

**Proposition 6.1.** *Let $(X_n,\, n\in\mathbb{Z})$ be an $F$-INMA$(q)$ process. Assume further that the mean $\mu_\epsilon$ and the variance $\sigma_\epsilon^2$ of $\epsilon_n$ are finite and that $\sum_{n=2}^{\infty} n(n-1)h_n < \infty$. Then (with $\eta_0=1$)*

**(1)** $E(X_n) = \mu_\epsilon \sum_{i=0}^{q} \eta_i^{\delta_F}$;

**(2)** $\mathrm{Var}(X_n) = \sigma_\epsilon^2\left(\sum_{i=0}^{q}\eta_i^{\delta_F}\right) + \mu_\epsilon\Big(1-U''(1)/U'(1)\Big)\left(\sum_{i=0}^{q}\eta_i^{\delta_F}\big(1-\eta_i^{\delta_F}\big)\right)$;

**(3)** *the ACRF of $(X_n,\, n\in\mathbb{Z})$ at lag $k$ is*

$$\rho(k) = \begin{cases} \left[\left(\sum_{i=0}^{q-k}\eta_i^{\delta_F}\eta_{i+k}^{\delta_F}\right)\sigma_\epsilon^2\right] \Big/ \left[\left(\sum_{i=0}^{q}\eta_i\,\delta_F\right)\sigma_\epsilon^2 \right. & \\ \qquad \left. + \,\mu_\epsilon\left(1-\dfrac{U''(1)}{U'(1)}\right)\left(\sum_{i=0}^{q}\eta_i^{\delta_F}\big(1-\eta_i^{\delta_F}\big)\right)\right], & 0\le k\le q\,, \\ 0, & k>q\,. \end{cases} \tag{6.4}$$

It is clear from (6.4) that an $F$-INMA$(q)$ process has the same correlation structure as the standard $MA(q)$ processes.

An $F$-INMA$(q)$ process with an $F$-stable innovation sequence has finite mean only if $\gamma=\delta_F$ and $B'(1)<\infty$. In addition, if $B''(1)<\infty$, then $\epsilon_n$ has finite mean and finite variance (recall, $\mu_\epsilon = \lambda B'(1)$ and $\sigma_\epsilon^2 = \lambda(B''(1) + B'(1))$). The mean and variance of $X_n$ as well as the correlation coefficient of $(X_n, X_{n+1})$, follow from Proposition 6.1, under the further assumption $\sum_{n=2}^{\infty} n\,(n-1)\,h_n < \infty$.

If $(X_n,\, n \in \mathbb{Z})$ is an $F$-INMA$\,(q)$ process such that $\epsilon_n$ has an $F$-stable distribution with pgf given by (2.13), then by (6.3) and (1.7) its marginal is also $F$-stable with pgf

$$P_X(z) = \exp\left\{-\lambda\left(\sum_{i=0}^{q}\eta_i^\gamma\right)\big(A(z)\big)^\gamma\right\}, \qquad \eta_0 = 1\ . \tag{6.5}$$

We note that the Poisson INMA$\,(q)$ process of McKenzie (1988) and the Poisson Geometric INMA$\,(q)$ process of Aly and Bouzar (1994) are special cases of $F$-INMA$\,(q)$ processes with a stable marginal for the semigroups $F^{(0)}$ and $F^{(\theta)}$ $(0 < \theta < 1)$ of (2.28), respectively.

If $(X_n,\, n \in \mathbb{Z})$ is an $F$-INMA$\,(q)$ process such that $\epsilon_n$ has the $F$-ML distribution of (4.1), then $X_n$ admits the following representation:

$$X_n \overset{d}{=} \sum_{i=1}^{Y+Z_1+\cdots+Z_{q-1}} W_i\ , \tag{6.6}$$

where the $W_i$'s are iid $\mathbb{Z}_+$-valued rv's with common pgf $B(z)$ of (1.9)–(1.10), and $Y$ and $Z_i$, $i = 1, ..., q-1$, are independent $\mathbb{Z}_+$-valued rv's (also independent of the $W_i$'s) and with respective pgf's

$$\begin{aligned} P_Y(z) &= \Big(1 + c(1-z)^{\gamma/\delta_F}\Big)^{-1}\ , \\ P_{Z_i}(z) &= \Big(1 + c\,\eta_i^\gamma(1-z)^{\gamma/\delta_F}\Big)^{-1}, \qquad i = 1, ..., q-1\ . \end{aligned} \tag{6.7}$$

$F$-INMA$\,(q)$ processes with compound discrete Linnik innovation sequences can be constructed in similar fashion. The details are omitted.

We note next the existence of an $F$-INMA process of infinite order ($F$-INMA$(\infty)$). Let $X_n$, $n \in \mathbb{Z}$ be a stationary $F$-INAR$\,(1)$ process, i.e.,

$$X_n = \eta \odot_F X_{n-1} + \epsilon_n\ , \qquad n \in \mathbb{Z}\ , \tag{6.8}$$

for some innovation sequence $(\epsilon_n,\, n \in \mathbb{Z})$ and some $0 < \eta < 1$. Then (see Aly and Bouzar, 2005) $X_n$, $n \in \mathbb{Z}$ admits the following $F$-INMA$(\infty)$ representation:

$$X_n = \sum_{i=0}^{\infty} \eta^i \odot_F \epsilon_{n-i}\ , \qquad n \in \mathbb{Z}\ . \tag{6.9}$$

We conclude this section by mentioning that classes of $F$-INMA$\,(q)$ processes with an autocorrelation structure different from (6.4) may result by assuming some form of dependence between the generalized multiplications in equation (6.1). Al-Osh and Alzaid (1988) and Brännäs and Hall (2001) proposed several INMA$(q)$ processes where dependence between the binomial thinnings in the governing equation was allowed.

## 7. $F$-INARMA$(1,q)$ PROCESSES

In this section the $F$-INAR(1) process of Aly and Bouzar (2005) is combined with the $F$-INMA$(q)$ process of the previous section to obtain a mixed process. Let $(\epsilon_n, n\in\mathbb{Z})$ be a sequence of iid rv's and define the $F$-INAR(1) process $(Y_n, n\in\mathbb{Z})$ by

$$Y_n = \eta \odot_F Y_{n-1} + \epsilon_n \ , \tag{7.1}$$

The $F$-INARMA$(1,q)$ process is defined as

$$X_n = Y_{n-q} + \sum_{i=1}^{q} \eta_i \odot_F \epsilon_{n+1-i} \ . \tag{7.2}$$

Note that both the $AR(1)$ and the $MA(q)$ components in (7.1)–(7.2) share the same innovation sequence $(\epsilon_n, n\in\mathbb{Z})$. Moreover, the generalized multiplications $\eta_i \odot_F \epsilon_{n+1-i}$, $i=1,...,q$, in (7.2) are performed independently. A representation of $(X_n, n\in\mathbb{Z})$ of (7.1)–(7.2) in terms of sequences of iid rv's can be easily obtained from the representations of its $AR(1)$ and $MA(q)$ components. The details are left out. If $(Y_n, n\in\mathbb{Z})$ is stationary (see Aly and Bouzar (2005) for sufficient conditions), then $(X_n, n\in\mathbb{Z})$ is also stationary. We will assume throughout the section that $(Y_n, n\in\mathbb{Z})$ is stationary. The joint pgf of higher order distributions of $(X_n, n\in\mathbb{Z})$ can be expressed in terms of the pgf's $P_Y(z)$ of $Y_n$, $P_\epsilon(z)$ of $\epsilon_n$, $F_t(z)$ $(t=-\ln\eta)$, and $F_{t_i}(z)$ $(t_i=-\ln\eta_i,\ i=1,...,q$. For example, the joint pgf of $(X_{n-1}, X_n)$ is shown to be

$$\phi_2(z_1,z_2) \;=\; P_Y\big[z_1 F_t(z)\big]\, P_\epsilon\big[F_{t_1}(z_2)\big] \prod_{i=1}^{q} P_\epsilon\big[F_{t_i}(z_1)\, F_{t_{i+1}}(z_2)\big] \ , \tag{7.3}$$

where $F_{t_{q+1}}(z_2) = z_2$.

Assume that $Y_n$ and $\epsilon_n$ have finite means ($\mu_\epsilon$ and $\mu$, respectively) and finite variances ($\sigma_\epsilon^2$ and $\sigma^2$ respectively). Assume further that $\sum_{n=2}^\infty n(n-1)h_n < \infty$. It can be shown that

$$\operatorname{Cov}\big(\eta \odot_F \epsilon_n,\, \eta' \odot_F \epsilon_n\big) = (\eta\,\eta')^{\delta_F}\,\sigma_\epsilon^2 \ , \qquad \eta,\eta' \in (0,1) \ , \tag{7.4}$$

$$\operatorname{Cov}(Y_{n-k}, Y_n) = \eta^{k\delta_F}\,\sigma^2 \ , \tag{7.5}$$

and

$$Y_{n-q} \stackrel{d}{=} \eta^k \odot_F Y_{n-q-k} + \sum_{i=0}^{k-1} \eta^i \odot_F \epsilon_{n-q-i} \ . \tag{7.6}$$

By (7.4)–(7.6) and the independence assumptions we obtain the *ACRF* at lag $k$ of $(X_n, n \in \mathbb{Z})$:

$$\rho(k) = \begin{cases} \dfrac{\eta^{k\delta_F}\, \sigma^2_{Y_0} + \eta^{(k-q)\delta_F}\Big(\sum_{l=1}^q \big(\eta^{l-1}\,\eta_l\big)^{\delta_F}\Big)\,\sigma^2_\epsilon}{\sigma^2_{Y_0} + \Big(\sum_{k=1}^q \eta_k^{2\delta_F}\Big)\,\sigma^2_\epsilon + \mu_\epsilon\Big(\sum_{k=1}^q \eta_k^{\delta_F}\big(1-\eta_k^{\delta_F}\big)\Big)\big(1-U''(1)/U'(1)\big)}\,, & k > q \\[2ex] \dfrac{\eta^{k\delta_F}\, \sigma^2_{Y_0} + \Big(\sum_{l=1}^{q-k} \big(\eta^l\,\eta_l\,\eta_k\big)^{\delta_F} + \eta^{(k-q)\delta_F} \sum_{l=q-k+1}^q \big(\eta^{l-1}\,\eta_l\big)^{\delta_F}\Big)\,\sigma^2_\epsilon}{\sigma^2_{Y_0} + \Big(\sum_{k=1}^q \eta_k^{2\delta_F}\Big)\,\sigma^2_\epsilon + \mu_\epsilon\Big(\sum_{k=1}^q \eta_k^{\delta_F}\big(1-\eta_k^{\delta_F}\big)\Big)\big(1-U''(1)/U'(1)\big)}\,, & k \le q \end{cases} \tag{7.7}$$

where $t = \ln\eta$ and $t_i = \ln\eta_i$, $i = 1, 2, ..., q$.

Let $(X_n, n \in \mathbb{Z})$ be a $F$-INARMA$\,(1,q)$ process. Assume that its $F$-INAR$\,(1)$ component $(Y_n, n \in \mathbb{Z})$ of (7.1) is stationary with an $F$-stable marginal distribution with pgf (2.13). Then the innovation sequence $(\epsilon_n, n \in \mathbb{Z})$ has also an $F$-stable marginal distribution with pgf (see Aly and Bouzar, 2005)

$$P_\epsilon(z) = \exp\Big[-\lambda\,(1-\eta^\gamma)\,A(z)^\gamma\Big] \ .$$

It follows that the associated $F$-INMA$\,(q)$ component in (7.2) has also an $F$-stable marginal with pgf

$$P_1(z) = \exp\left\{-\lambda\,(1-\eta^\gamma)\left(\sum_{i=1}^q \eta_i^\gamma\right)\big(A(z)\big)^\gamma\right\} .$$

Therefore, $(X_n, n \in \mathbb{Z})$ is stationary with an $F$-stable marginal distribution with pgf

$$P_X(z) = \exp\left\{-\lambda\left[1 + (1-\eta^\gamma)\left(\sum_{i=1}^q \eta_i^\gamma\right)\right]\big(A(z)\big)^\gamma\right\} . \tag{7.8}$$

If $(X_n, n \in \mathbb{Z})$ is an $F$-INARMA$\,(1,q)$ process such that its $F$-INAR$\,(1)$ component $(Y_n, n \in \mathbb{Z})$ has an $F$-ML marginal (with pgf (4.1)), then the innovation sequence $(\epsilon_n, n \in \mathbb{Z})$ admits the representation (see Aly and Bouzar, 2005)

$$\epsilon_n = I_n\, E_n \ , \tag{7.9}$$

where $(I_n, n \in \mathbb{Z})$ and $(E_n, n \in \mathbb{Z})$ are independent sequences of iid rv's such that $I_n$ is Bernoulli$(1-\eta^\gamma)$ and $E_n$ has the same distribution as $Y_n$. It follows from (4.1), (7.2) and (7.9) that $(X_n, n \in \mathbb{Z})$ is stationary with marginal pgf

$$P_X(z) = \frac{1}{1 + dA(z)^\gamma} \prod_{i=1}^q \left(\eta^\gamma + \frac{1-\eta^\gamma}{1 + d\eta_i^\gamma A(z)^\gamma}\right) . \tag{7.10}$$

## 8. CONCLUSION

We have presented a class of integer-valued time series that can be used to model count data. The models introduced in this paper may be seen as extensions of the classical branching processes of Galton–Watson–Bienaymé. Various distributional and regression properties were shown to be similar to those of the standard real-valued ARMA processes. Models with specific marginals such as stable distributions and Mittag–Leffler distributions were discussed in some detail and some examples were developed.

## ACKNOWLEDGMENTS

The authors are grateful to a referee for helpful comments and suggestions. The research of E.-E.A.A. Aly was done while the author was on sabbatical leave from Kuwait University.

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