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A bivariate Weibull distribution and its applications in reliability

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Abstract:

- In this article, the bivariate exponential distribution proposed by Downton ("Bivariate exponential distributions in reliability theory", *Journal of the Royal Statistical Society Series B*, 1970) is extended to a bivariate Weibull distribution, and it is called the Downton's bivariate Weibull (DBW) distribution. Statistical properties of the DBW distribution are explored and likelihood inference developed based on complete as well as right-censored bivariate data are discussed. Through extensive Monte Carlo simulations, performance of the point and interval estimates are evaluated. Two real datasets are analyzed for illustrative purposes. It is concluded that the DBW distribution is very useful to model bivariate data.

Keywords:

- *Bivariate lifetime data; bivariate Weibull distribution; maximum likelihood estimation; asymptotic confidence intervals; right censoring; moment-based estimates*

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1. INTRODUCTION

Lifetime data can often be two dimensional. For example, Jung and Bai [9] analysed field data under two-dimensional warranty, using bivariate data on age and mileage of cars. Bivariate life data pertaining to trucks working in the mines were analysed by Fuqing et al. [6]. In such applications, one of the main goals of analyses is to capture the dependence between components of the two-dimensional lifetimes. In some cases, the dependence is captured by using an approach based on copulas; see, for example Fuqing et al. [6]. In some other cases, particular bivariate distributions are used; for example, see Yuan [16], among others.

In this article, we present and discuss a bivariate Weibull distribution that can be conveniently used for modelling two-dimensional lifetime data. The distribution presented here is flexible, computationally convenient, and has desirable properties that can be exploited for modelling purposes.

A bivariate exponential distribution, known as the Downton's bivariate exponential (DBE) distribution [5], is as follows. If a random vector (Y_1, Y_2) follows the DBE distribution, the corresponding joint probability density function (PDF) is given by

(1.1)

$$f_{Y_1,Y_2}(y_1, y_2) = \frac{1}{\theta_1\theta_2(1-\rho)} \exp\left\{-\frac{1}{1-\rho}\left(\frac{y_1}{\theta_1}+\frac{y_2}{\theta_2}\right)\right\} I_0\left(\frac{2(\rho y_1 y_2)^{1/2}}{(1-\rho)\sqrt{\theta_1\theta_2}}\right), \quad y_1, y_2 > 0,$$

with $\theta_1, \theta_2 > 0$ and $0 \leq \rho < 1$. We express this as $(Y_1, Y_2) \sim DBE(\theta_1, \theta_2, \rho)$. Here, $I_0(\cdot)$ is the modified Bessel function of the first kind of order zero. Downton [5] explained this model from the perspective of a failure model in the reliability context; see also Kotz et al. [10] for a comprehensive description.

The DBE distribution has some excellent mathematical properties. For example, the DBE distribution originates from a shock model where two components of a system are subject to independent streams of shocks with exponentially distributed interarrival times. Also, the number of such shocks to cause failure is a random variable distributed according to the geometric distribution. For the DBE distribution, the marginal distributions of Y_1 and Y_2 are exponential, with scale parameters θ_1 and θ_2, respectively. The dependence between Y_1 and Y_2 is captured through the correlation parameter ρ; when $\rho = 0$, Y_1 and Y_2 are independent. The model is quite convenient to be used in reliability modelling due to its tractable mathematical nature. Especially, as the corresponding marginal distributions of Y_1 and Y_2 are univariate exponential distributions, the DBE distribution is particularly useful in situations where the hazard rates of the marginal distributions can be assumed (or verified) to be constants.

However, for bivariate lifetime data with non-constant marginal hazard rates, the DBE distribution is clearly not suitable. In such scenarios, it is more appropriate to model the bivariate life data by using a bivariate distribution that has univariate Weibull marginals, which can accommodate the non-constant marginal hazard rates. The motivation of this work is to develop a model that can accommodate non-constant hazard rates of the marginal lifetimes corresponding to the two components of a system, within the same context of arrival of independent streams of shocks to the two components. To achieve this, we have consider a non-linear transformation of the marginal lifetimes as described below. The resulting model is more appropriate for real datasets having non-constant marginal hazard rates.

Therefore, in this article, we discuss a generalization of the DBE distribution to a bivariate Weibull distribution which we call the Downton's bivariate Weibull (DBW) distribution. The DBW distribution partially retains the mathematical advantages of the DBE distribution with respect to its analytical tractability. In fact, the DBW distribution loses some of the interpretations of the DBE distribution, such as exponential interarrival times of the independent shock. However, as the DBW distribution has Weibull marginals, it is very flexible for modelling purposes; for example, it can accommodate bivariate lifetime data with non-constant marginal hazard rates.

In this article, we discuss the DBW distribution, its statistical properties, and its applications in detail. First, after constructing the DBW distribution, we explore some of its important properties which will be helpful in applying the distribution to real data. Then, we discuss likelihood inference for the DBW distribution, based on complete as well as right-censored lifetime data. Moment-based estimators for the distribution based on complete data are also discussed. The inferential methods are examined through extensive Monte Carlo simulation studies. Then, as an application, a two-dimensional warranty model is discussed. Analyses of two real datasets for illustrative purposes are presented. Based on our exploration, we conclude that the DBW distribution has desirable statistical properties, is computationally convenient, and is quite flexible for modelling purposes. We recommend its extensive use in modelling bivariate data, especially arising from lifetime experiments.

The paper is organized as follows. In Section 2, the DBW distribution with its statistical properties, and data generation algorithm are presented. Likelihood inference based on complete data from the DBW distribution, and a Monte Carlo simulation study to evaluate the point and interval estimates are presented in Section 3. This section also contains discussion of a procedure for obtaining moment-based estimators. In Section 4, inferences for the DBW distribution based on bivariate right-censored data are discussed, and a simulation study examining the performance of the estimates are presented. An application in the form of a two-dimensional warranty model is presented in Section 5. Two case studies based on real datasets are provided in Section 6 for illustrative purposes. Finally, the paper is concluded with some remarks in Section 7.

2. DOWNTON'S BIVARIATE WEIBULL DISTRIBUTION

Suppose $(Y_1, Y_2) \sim DBE(\theta_1, \theta_2, \rho)$. Consider the transformations $X_1 = Y_1^{\alpha_1}$ and $X_2 = Y_2^{\alpha_2}$, with $\alpha_1, \alpha_2 > 0$. The joint density of the transformed random variables X_1 and X_2 is then given by

$$f_{X_1,X_2}(x_1,x_2) = \frac{\eta_1\eta_2 x_1^{\eta_1-1} x_2^{\eta_2-1}}{\theta_1\theta_2(1-\rho)} \exp\left\{-\frac{1}{1-\rho}\left(\frac{x_1^{\eta_1}}{\theta_1}+\frac{x_2^{\eta_2}}{\theta_2}\right)\right\} I_0\left(\frac{2(\rho x_1^{\eta_1} x_2^{\eta_2})^{1/2}}{(1-\rho)\sqrt{\theta_1\theta_2}}\right), \quad x_1, x_2 > 0, \tag{2.1}$$

with $\theta_1, \theta_2, \eta_1, \eta_2 > 0$ and $0 \leq \rho < 1$, where $\eta_1 = 1/\alpha_1$ and $\eta_2 = 1/\alpha_2$. Thus, through this power transformation of the variables Y_1 and Y_2, we obtain a bivariate distribution which we call the Downton's bivariate Weibull (DBW) distribution. We write this as $(X_1, X_2) \sim DBW(\eta_1, \theta_1, \eta_2, \theta_2, \rho)$. A similar distribution, called the Nakagami-m distribution, was discussed by Sagias and Karagiannidis [15] in the context of modelling fading channels relating to digital communication systems. However, Sagias and Karagiannidis [15] derived this distribution starting from a bivariate Rayleigh distribution.

Using an infinite series representation of the Bessel function provided in Gradshteyn and Ryzhik [7] as

$$I_0(z) = \sum_{k=0}^{\infty} \frac{1}{(k!)^2}\left(\frac{z}{2}\right)^{2k},$$

we can obtain another representation of the PDF of the DBW distribution as

$$\begin{aligned} f_{X_1,X_2}(x_1,x_2) &= \eta_1\eta_2 \exp\left\{-\frac{1}{1-\rho}\left(\frac{x_1^{\eta_1}}{\theta_1}+\frac{x_2^{\eta_2}}{\theta_2}\right)\right\} \\ &\quad \times \sum_{k=0}^{\infty} \frac{\rho^k}{(k!)^2(1-\rho)^{2k+1}} \frac{x_1^{-1+(k+1)\eta_1} x_2^{-1+(k+1)\eta_2}}{(\theta_1\theta_2)^{k+1}}, \quad x_1, x_2 > 0. \end{aligned} \tag{2.2}$$

The corresponding joint cumulative density function (CDF) of X_1 and X_2 can be obtained as

$$\begin{aligned} F_{X_1,X_2}(x_1,x_2) = &\ 1 - \exp\left(-\frac{x_1^{\eta_1}}{\theta_1}\right) Q_1\left(\sqrt{\frac{2}{1-\rho}}\frac{x_2^{\eta_2/2}}{\sqrt{\theta_2}}, \sqrt{\frac{2\rho}{1-\rho}}\frac{x_1^{\eta_1/2}}{\sqrt{\theta_1}}\right) \\ & - \exp\left(-\frac{x_2^{\eta_2}}{\theta_2}\right)\left[1 - Q_1\left(\sqrt{\frac{2\rho}{1-\rho}}\frac{x_2^{\eta_2/2}}{\sqrt{\theta_2}}, \sqrt{\frac{2}{1-\rho}}\frac{x_1^{\eta_1/2}}{\sqrt{\theta_1}}\right)\right], \end{aligned} \tag{2.3}$$

where $Q_1(\cdot,\cdot)$ is the first order Marcum's Q-function [11]. The bivariate survival function of X_1 and X_2 is given by

$$\begin{aligned} S_{X_1,X_2}(x_1,x_2) &= 1 - F_{X_1}(x_1) - F_{X_2}(x_2) + F_{X_1,X_2}(x_1,x_2) \\ &= \exp\left(-\frac{x_1^{\eta_1}}{\theta_1}\right)\left[1 - Q_1\left(\sqrt{\frac{2}{1-\rho}}\frac{x_2^{\eta_2/2}}{\sqrt{\theta_2}}, \sqrt{\frac{2\rho}{1-\rho}}\frac{x_1^{\eta_1/2}}{\sqrt{\theta_1}}\right)\right] \\ &\quad + \exp\left(-\frac{x_2^{\eta_2}}{\theta_2}\right) Q_1\left(\sqrt{\frac{2\rho}{1-\rho}}\frac{x_2^{\eta_2/2}}{\sqrt{\theta_2}}, \sqrt{\frac{2}{1-\rho}}\frac{x_1^{\eta_1/2}}{\sqrt{\theta_1}}\right), \end{aligned} \tag{2.4}$$

where F_{X_1} and F_{X_2} are the marginal CDFs of X_1 and X_2, respectively. In Figures 1 and 2, the joint density function is plotted for different values of the parameters.

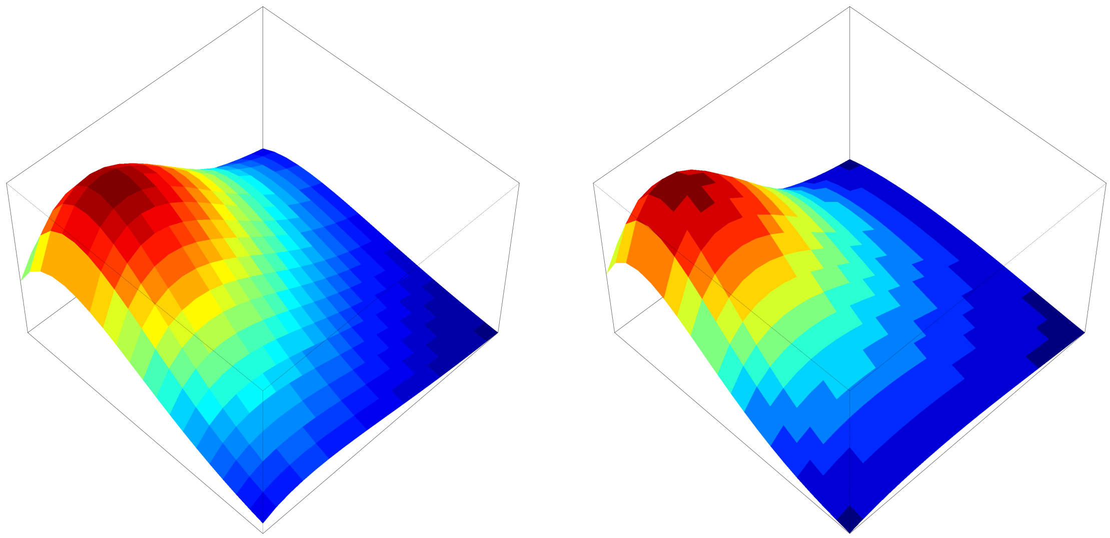

Figure 1: Plot of joint density: left: $\eta_1 = \eta_2 = 1.5$, $\theta_1 = \theta_2 = 1$, $\rho = 0$, right: $\eta_1 = \eta_2 = 1.5$, $\theta_1 = \theta_2 = 1$, $\rho = 0.25$

2.1. Some properties

If $(X_1, X_2) \sim \text{DBW}(\eta_1, \theta_1, \eta_2, \theta_2, \rho)$, the marginal distributions of X_1 and X_2 are univariate Weibull with parameters (η_1, θ_1) and (η_2, θ_2), respectively, with PDF

$$f_{X_i}(x_i) = \frac{\eta_i}{\theta_i} x_i^{\eta_i - 1} \exp\left\{ -\frac{x_i^{\eta_i}}{\theta_i} \right\}, \quad x_i > 0, \tag{2.5}$$

and CDF

$$F_{X_i}(x_i) = 1 - \exp\left\{ -\frac{x_i^{\eta_i}}{\theta_i} \right\}, \quad x_i > 0 \tag{2.6}$$

for $i = 1, 2$.

The conditional distributions can be obtained by using the joint density and the marginal densities. If $(X_1, X_2) \sim \text{DBW}(\eta_1, \theta_1, \eta_2, \theta_2, \rho)$, then the PDFs of the conditional distributions of X_1 given $X_2 = x_2$, and X_2 given $X_1 = x_1$ are

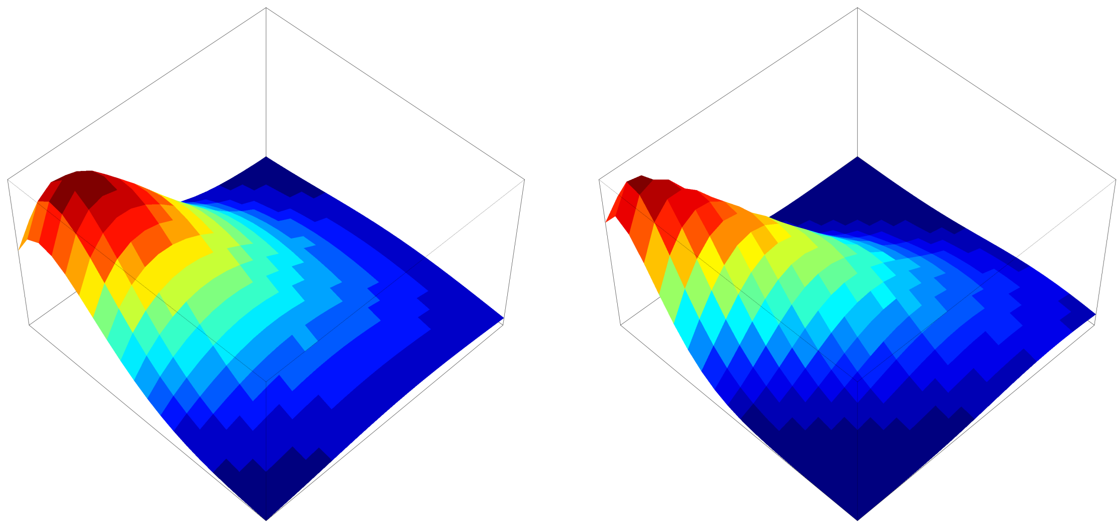

Figure 2: Plot of joint density: left: $\eta_1 = \eta_2 = 1.5$, $\theta_1 = \theta_2 = 1$, $\rho = 0.5$, right: $\eta_1 = \eta_2 = 1.5$, $\theta_1 = \theta_2 = 1$, $\rho = 0.75$

given by

(2.7)
$$f_{X_1|X_2=x_2}(x_1) = \frac{\eta_1 x_1^{\eta_1-1}}{\theta_1(1-\rho)} \exp\{-(A_1+A_2)\} \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \frac{\rho^k}{(1-\rho)^{2k}} \left(\frac{x_1^{\eta_1} x_2^{\eta_2}}{\theta_1\theta_2}\right)^k, \quad x_1 > 0$$

and

(2.8)
$$f_{X_2|X_1=x_1}(x_2) = \frac{\eta_2 x_2^{\eta_2-1}}{\theta_2(1-\rho)} \exp\{-(B_1+B_2)\} \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \frac{\rho^k}{(1-\rho)^{2k}} \left(\frac{x_1^{\eta_1} x_2^{\eta_2}}{\theta_1\theta_2}\right)^k, \quad x_2 > 0,$$

respectively. The CDFs of the conditional distributions of X_1 given $X_2 = x_2$, and X_2 given $X_1 = x_1$ are given by

$$F_{X_1|X_2=x_2}(x_1) = \exp\{-A_2\} \times \sum_{k=0}^{\infty} \frac{1}{(k!)^2} A_2^k \gamma(k+1, A_1), \quad x_1 > 0, \tag{2.9}$$

and

$$F_{X_2|X_1=x_1}(x_2) = \exp\{-B_2\} \times \sum_{k=0}^{\infty} \frac{1}{(k!)^2} B_2^k \gamma(k+1, B_1), \quad x_2 > 0, \tag{2.10}$$

respectively, where $\gamma(s,t) = \int_0^t u^{s-1} e^{-u} du$ is the lower incomplete gamma function, and

$$A_1 = \frac{1}{1-\rho} \frac{x_1^{\eta_1}}{\theta_1},\ A_2 = \frac{\rho}{1-\rho} \frac{x_2^{\eta_2}}{\theta_2},\ B_1 = \frac{1}{1-\rho} \frac{x_2^{\eta_2}}{\theta_2},\ B_2 = \frac{\rho}{1-\rho} \frac{x_1^{\eta_1}}{\theta_1}.$$

Alternative expressions involving the modified Bessel function for the conditional PDFs and CDFs can be obtained. If $(X_1, X_2) \sim \text{DBW}(\eta_1, \theta_1, \eta_2, \theta_2, \rho)$, then alternative expressions for the PDFs of the conditional distributions of X_1 given $X_2 = x_2$, and X_2 given $X_1 = x_1$, respectively, are given by

$$f_{X_1|X_2=x_2}(x_1) = \frac{\eta_1 x_1^{\eta_1-1}}{\theta_1(1-\rho)} \exp\{-(A_1+A_2)\} I_0\left(\frac{2(\rho x_1^{\eta_1} x_2^{\eta_2})^{1/2}}{(1-\rho)\sqrt{\theta_1\theta_2}}\right), \quad x_1 > 0 \tag{2.11}$$

and

$$f_{X_2|X_1=x_1}(x_2) = \frac{\eta_2 x_2^{\eta_2-1}}{\theta_2(1-\rho)} \exp\{-(B_1+B_2)\} I_0\left(\frac{2(\rho x_1^{\eta_1} x_2^{\eta_2})^{1/2}}{(1-\rho)\sqrt{\theta_1\theta_2}}\right), \quad x_2 > 0. \tag{2.12}$$

Similarly, alternative expressions for the CDFs of the conditional distributions of X_1 given $X_2 = x_2$, and X_2 given $X_1 = x_1$ are given by

$$\begin{aligned} F_{X_1|X_2=x_2}(x_1) &= A_1 \exp(-A_2) \int_0^1 e^{-A_1 t} I_0(2\sqrt{A_1 A_2 t})dt, \\ F_{X_2|X_1=x_1}(x_2) &= B_1 \exp(-B_2) \int_0^1 e^{-B_1 t} I_0(2\sqrt{B_1 B_2 t})dt, \end{aligned}$$

respectively, which can be shown as follows. Note that we can write

$$\begin{aligned} F_{X_1|X_2=x_2}(x_1) &= \exp\{-A_2\} \sum_{k=0}^{\infty} \frac{1}{(k!)^2} A_2^k \gamma(k+1, A_1) \\ &= \exp\{-A_2\} \sum_{k=0}^{\infty} \frac{1}{(k!)^2} A_2^k \left[k! \exp\{-A_1\} A_1^{k+1} \sum_{l=0}^{\infty} \frac{A_1^l}{(l+k+1)!}\right] \\ &= A_1 \exp\{-(A_1+A_2)\} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(1)_l}{(2)_{l+k}} \frac{(A_1 A_2)^k}{k!} \frac{A_1^l}{l!} \\ &= A_1 \exp\{-(A_1+A_2)\} \Phi_3(b=1, c=2, w=A_1, z=A_1 A_2) \\ &= A_1 \exp\{-(A_1+A_2)\} \left[\exp\{A_1\} \int_0^1 e^{-A_1 t} I_0(2\sqrt{A_1 A_2 t})dt\right] \\ &= A_1 \exp\{-A_2\} \int_0^1 e^{-A_1 t} I_0(2\sqrt{A_1 A_2 t})dt \end{aligned}$$

where

$$\Phi_3(b, c, w, z) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(b)_k}{(c)_{k+l}} \frac{w^k z^l}{k! l!}$$

is the Humbert function or the confluent Appell function [4].

The correlation parameter ρ captures the dependence between X_1 and X_2. If $(X_1, X_2) \sim \text{DBW}(\eta_1, \theta_1, \eta_2, \theta_2, \rho)$, X_1 and X_2 are independent if and only if $\rho = 0$. When $0 < \rho < 1$, X_1 and X_2 are correlated. However, note that ρ is not the Pearson's correlation coefficient between X_1 and X_2 for the DBW distribution.

Following [15], an expression for the product moments of the DBW distribution, in terms of the generalized hypergeometric function, can be obtained to

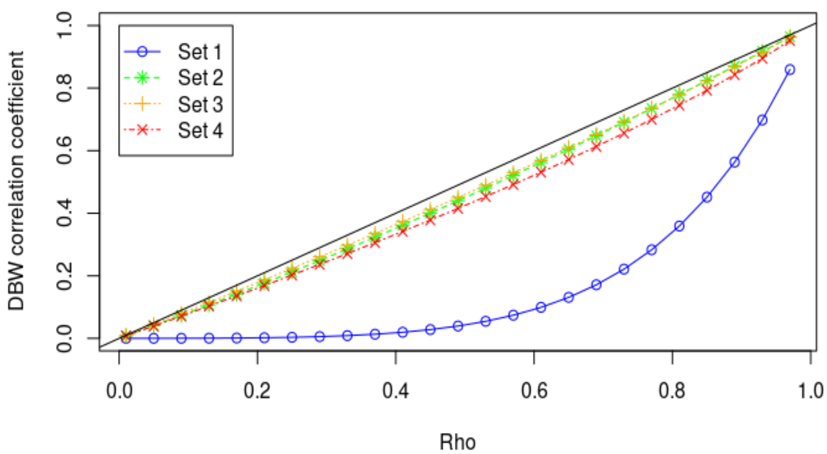

Figure 3: Plot of ρ and ρ_{PMC}; the values of the shape parameters for different sets are the following - Set 1: $\eta_1 = \eta_2 = 0.1$, Set 2: $\eta_1 = \eta_2 = 0.5$, Set 3: $\eta_1 = \eta_2 = 2.4$, Set 4: $\eta_1 = \eta_2 = 4.5$. The $y = x$ line is in black.

be

$$E(X_1^{m_1} X_2^{m_2}) = (1-\rho)^{1+\frac{m_1}{\eta_1}+\frac{m_2}{\eta_2}} \theta_1^{\frac{m_1}{\eta_1}} \theta_2^{\frac{m_2}{\eta_2}} \Gamma(1+\frac{m_1}{\eta_1})\Gamma(1+\frac{m_2}{\eta_2}) \times {}_2F_1\left(1+\frac{m_1}{\eta_1}, 1+\frac{m_2}{\eta_2}; 1; \rho\right),$$

where ${}_2F_1\,(\cdot;\cdot;\cdot)$ is the generalized hypergeometric function. Therefore, the product moment correlation coefficient of X_1 and X_2 is given by

$$\rho_{PMC} = \frac{\Gamma(1+\frac{1}{\eta_1})\Gamma(1+\frac{1}{\eta_2})[(1-\rho)^{1+\frac{1}{\eta_1}+\frac{1}{\eta_2}} \times {}_2F_1\left(1+\frac{1}{\eta_1}, 1+\frac{1}{\eta_2}; 1; \rho\right) - 1]}{\sqrt{\Gamma(1+\frac{2}{\eta_1}) - \{\Gamma(1+\frac{1}{\eta_1})\}^2}\sqrt{\Gamma(1+\frac{2}{\eta_2}) - \{\Gamma(1+\frac{1}{\eta_2})\}^2}}.$$

The expression for the product moment correlation ρ_{PMC} clearly shows that ρ is not the correlation coefficient between X_1 and X_2 for the DBW distribution and in fact only takes values in [0,1). Note that ρ_{PMC} depends only on η_1, η_2, and ρ; the scale parameters θ_1 and θ_2 have no impact on the correlation coefficient. Figure 3 gives a plot of different values of ρ and ρ_{PMC}, for different values of η_1 and η_2.

2.2. Data generation algorithm

Realisations from the DBW distribution can be generated using the marginal and conditional distributions. To generate an observation (x_1, x_2) from the

$DBW(\eta_1,\ \theta_1,\ \eta_2,\ \theta_2,\ \rho)$ distribution, the simplest route would be to generate x_1 from the marginal Weibull distribution, and then for the given value x_1, to generate x_2 from its conditional distribution. The conditional distribution in Eq.(2.10) is computationally efficient in this regard. The process of generating random variables from the DBW distribution is as follows:

ALGORITHM:

STEP 1: Generate x_1 from Weibull(η_1, θ_1).

STEP 2: Transform x_1 into z_1, where $z_1 = \frac{\rho}{1-\rho}\frac{x_1^{\eta_1}}{\theta_1}$.

STEP 3: Generate z_2 from the distribution with CDF

$$F_{Z_2|Z_1=z_1}(z_2) = z_2 \exp(-z_1) \int_0^1 e^{-z_2 t} I_0(2\sqrt{z_1 z_2 t})dt.$$

STEP 4: Obtain x_2, using the transformation $x_2 = (\theta_2(1-\rho)z_2)^{1/\eta_2}$.

Note, however, that the above algorithm should only be used when $0 < \rho < 1$. For $\rho = 0$, x_1 and x_2 can be generated directly from their respective marginal distributions.

3. INFERENCE BASED ON COMPLETE BIVARIATE DATA

3.1. Likelihood inference

Based on observed bivariate data, maximum likelihood estimates (MLEs) for the parameters of the DBW distribution may be obtained. Let $(x_{1i}, x_{2i}), i = 1, ..., n$, denote the observed bivariate data. The likelihood function is given by

$$L(\boldsymbol{\omega}) = \prod_{i=1}^{n} f_{X_1,X_2}(x_{1i}, x_{2i}),$$

where $\boldsymbol{\omega} = (\eta_1, \theta_1, \eta_2, \theta_2, \rho)$ is the vector of model parameters. Using the joint PDF of X_1 and X_2 given in Eq.(2.1), the log-likelihood function is

$$\begin{aligned} \log L(\boldsymbol{\omega}) &= n(\log\eta_1 + \log\eta_2 - \log\theta_1 - \log\theta_2 - \log(1-\rho)) \\ &\quad +(\eta_1 - 1)\sum_{i=1}^{n}\log x_{1i} + (\eta_2 - 1)\sum_{i=1}^{n}\log x_{2i} - \frac{1}{1-\rho}\sum_{i=1}^{n}\left(\frac{x_{1i}^{\eta_1}}{\theta_1} + \frac{x_{2i}^{\eta_2}}{\theta_2}\right) \\ &\quad + \sum_{i=1}^{n}\log I_0\left(\frac{2(\rho x_{1i}^{\eta_1} x_{2i}^{\eta_2})^{1/2}}{(1-\rho)\sqrt{\theta_1\theta_2}}\right). \end{aligned} \tag{3.1}$$

The log-likelihood equations with respect to $\eta_1,\ \theta_1,\ \eta_2,\ \theta_2$, and ρ are as follows:

(3.2)

$$\frac{\partial \log L}{\partial \eta_1} = \frac{n}{\eta_1} + \sum_{i=1}^{n} \log x_{1i} - \frac{1}{1-\rho} \sum_{i=1}^{n} \frac{x_{1i} \log x_{1i}}{\theta_1} + \sum_{i=1}^{n} \frac{I_0'\left(\frac{2(\rho x_{1i}^{\eta_1} x_{2i}^{\eta_2})^{1/2}}{(1-\rho)\sqrt{\theta_1\theta_2}}\right)\left(\frac{(\rho x_{1i}^{\eta_1} x_{2i}^{\eta_2})^{1/2}}{(1-\rho)\sqrt{\theta_1\theta_2}}\right) \log x_{1i}}{I_0\left(\frac{2(\rho x_{1i}^{\eta_1} x_{2i}^{\eta_2})^{1/2}}{(1-\rho)\sqrt{\theta_1\theta_2}}\right)},$$

$$\frac{\partial \log L}{\partial \theta_1} = -\frac{n}{\theta_1} + \frac{1}{1-\rho} \sum_{i=1}^{n} \frac{x_{1i}^{\eta_1}}{\theta_1^2} - \sum_{i=1}^{n} \frac{I_0'\left(\frac{2(\rho x_{1i}^{\eta_1} x_{2i}^{\eta_2})^{1/2}}{(1-\rho)\sqrt{\theta_1\theta_2}}\right)\left(\frac{(\rho x_{1i}^{\eta_1} x_{2i}^{\eta_2})^{1/2}}{(1-\rho)\sqrt{\theta_1^3\theta_2}}\right)}{I_0\left(\frac{2(\rho x_{1i}^{\eta_1} x_{2i}^{\eta_2})^{1/2}}{(1-\rho)\sqrt{\theta_1\theta_2}}\right)}, \tag{3.3}$$

(3.4)

$$\frac{\partial \log L}{\partial \eta_2} = \frac{n}{\eta_2} + \sum_{i=1}^{n} \log x_{2i} - \frac{1}{1-\rho} \sum_{i=1}^{n} \frac{x_{2i} \log x_{2i}}{\theta_2} + \sum_{i=1}^{n} \frac{I_0'\left(\frac{2(\rho x_{1i}^{\eta_1} x_{2i}^{\eta_2})^{1/2}}{(1-\rho)\sqrt{\theta_1\theta_2}}\right)\left(\frac{(\rho x_{1i}^{\eta_1} x_{2i}^{\eta_2})^{1/2}}{(1-\rho)\sqrt{\theta_1\theta_2}}\right) \log x_{2i}}{I_0\left(\frac{2(\rho x_{1i}^{\eta_1} x_{2i}^{\eta_2})^{1/2}}{(1-\rho)\sqrt{\theta_1\theta_2}}\right)},$$

$$\frac{\partial \log L}{\partial \theta_2} = -\frac{n}{\theta_2} + \frac{1}{1-\rho} \sum_{i=1}^{n} \frac{x_{2i}^{\eta_2}}{\theta_2^2} - \sum_{i=1}^{n} \frac{I_0'\left(\frac{2(\rho x_{1i}^{\eta_1} x_{2i}^{\eta_2})^{1/2}}{(1-\rho)\sqrt{\theta_1\theta_2}}\right)\left(\frac{(\rho x_{1i}^{\eta_1} x_{2i}^{\eta_2})^{1/2}}{(1-\rho)\sqrt{\theta_1\theta_2^3}}\right)}{I_0\left(\frac{2(\rho x_{1i}^{\eta_1} x_{2i}^{\eta_2})^{1/2}}{(1-\rho)\sqrt{\theta_1\theta_2}}\right)}, \tag{3.5}$$

(3.6)

$$\frac{\partial \log L}{\partial \rho} = \frac{n}{1-\rho} - \frac{1}{(1-\rho)^2} \sum_{i=1}^{n} \left(\frac{x_{1i}^{\eta_1}}{\theta_1} + \frac{x_{2i}^{\eta_2}}{\theta_2}\right) - \sum_{i=1}^{n} \frac{I_0'\left(\frac{2(\rho x_{1i}^{\eta_1} x_{2i}^{\eta_2})^{1/2}}{(1-\rho)\sqrt{\theta_1\theta_2}}\right)\left(\frac{(x_{1i}^{\eta_1} x_{2i}^{\eta_2})^{1/2}}{\sqrt{\rho\theta_1\theta_2}}\right)\frac{1+\rho}{(1-\rho)^2}}{I_0\left(\frac{2(\rho x_{1i}^{\eta_1} x_{2i}^{\eta_2})^{1/2}}{(1-\rho)\sqrt{\theta_1\theta_2}}\right)},$$

where $I_0'(z) = \frac{\partial}{\partial z} I_0(z)$. Using the relation

$$\frac{\partial}{\partial z} I_\nu(z) = \frac{\nu}{z} I_\nu(z) + I_{\nu+1}(z),$$

we readily obtain $I_0'(z) = I_1(z)$, where $I_1(z)$, the modified Bessel function of the first kind of order one, is given by the power series expansion

$$I_1(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{1+2k}}{k!\Gamma(k+2)}.$$

It may be observed that it is not possible to obtain explicit solutions to these log-likelihood equations, and one has to rely on numerical methods for obtaining the MLE $\widehat{\boldsymbol{\omega}} = (\widehat{\eta_1}, \widehat{\theta_1}, \widehat{\eta_2}, \widehat{\theta_2}, \widehat{\rho})$. Numerical methods such as the Newton-Raphson technique, or its modified versions can be employed to obtain $\widehat{\boldsymbol{\omega}}$.

Asymptotic confidence intervals for the model parameters may be obtained by using the observed Fisher information matrix, and asymptotic normality of the MLEs. The observed Fisher information matrix, denoted by $\boldsymbol{I}(\boldsymbol{\omega})$, is defined as the negative of the hessian of log-likelihood function in Eq.(3.1); that is,

$$\boldsymbol{I}(\boldsymbol{\omega}) = -\nabla^2(\log L(\boldsymbol{\omega})).$$

Now, under general regularity conditions, we know that the asymptotic distribution of the MLE $\widehat{\boldsymbol{\omega}}$ is a multivariate normal distribution, i.e.,

$$\sqrt{n}(\widehat{\boldsymbol{\omega}} - \boldsymbol{\omega}) \rightarrow \boldsymbol{N}_5(\boldsymbol{0}, \boldsymbol{I}^{-1}(\boldsymbol{\omega})|_{\boldsymbol{\omega}=\widehat{\boldsymbol{\omega}}}).$$

Using the above, asymptotic confidence intervals for the model parameters can be constructed. For example, for η_1, an asymptotic 100(1-β)% confidence interval is given by

$$(\widehat{\eta_1} - \tau_{\beta/2}\sqrt{\boldsymbol{I}_{1,1}^{-1}} \quad , \quad \widehat{\eta_1} + \tau_{\beta/2}\sqrt{\boldsymbol{I}_{1,1}^{-1}})$$

where $\boldsymbol{I}_{i,j}^{-1}$ is the (i,j)-th element of $\boldsymbol{I}^{-1}(\boldsymbol{\omega})|_{\boldsymbol{\omega}=\widehat{\boldsymbol{\omega}}}$, and τ_δ is the upper δ-percentile point of the standard normal distribution. Asymptotic confidence intervals for the parameters θ_1, η_2, θ_2, and ρ can be constructed in a similar fashion.

3.1.1. Choice of initial values

The numerical method for obtaining MLEs requires initial choices for the values of the parameters. Here, we develop a method for choosing initial parameter values. Note that the marginal distribution of X_1 (or X_2) does not depend on η_2 and θ_2 (or η_1 and θ_1). Based on this observation, we propose a method for computing closed form initial values.

When X follows a Weibull distribution with the PDF as given in Eq.(2.5), then $Z = \log X$ has an extreme-value distribution with PDF

$$f_Z(z) = \frac{1}{\sigma}\exp\left[\left(\frac{z-\mu}{\sigma}\right) - \exp\left(\frac{z-\mu}{\sigma}\right)\right], \quad -\infty < z < \infty, -\infty < \mu < \infty, \sigma > 0,$$

where μ and σ are location and scale parameters, respectively, with

$$\mu = \frac{1}{\eta}\log\theta, \quad \sigma = \frac{1}{\eta}.$$

It can be shown that

$$E(Z) = \mu - \gamma\sigma, \quad Var(Z) = \frac{\pi^2}{6}\sigma^2,$$

where $\gamma = 0.5722$ (approximately) is the Euler's constant. Using these results and relations, by equating the population moments with corresponding sample quantites and then by transforming back, we can easily work out approximate estimates for θ_1, θ_2, η_1, and η_2, as follows:

$$\theta_i = \exp\left(\bar{z}_i\left(\frac{\sqrt{6}}{\pi}s_i\right)^{-1} + \gamma\right), \quad \eta_i = \left(\frac{\sqrt{6}}{\pi}s_i\right)^{-1}, \quad i = 1, 2,$$

where $z_i = \log(x_i)$, $\bar{z}_i = \frac{1}{n}\sum_{i=1}^{n} z_i$, and $s_i = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(z_i - \bar{z})^2}$.

For the dependence parameter ρ, we choose the initial value by equating it to the sample Pearson's correlation coefficient estimated from the data, though it should be recalled here that ρ is not the Pearson's correlation coefficient for the DBW distribution. We have observed in our Monte Carlo simulation studies that the initial choices of parameter values obtained in this manner work quite efficiently for the numerical optimization technique to calculate the MLEs.

3.2. Estimators based on moments

Moment-based estimates of the parameters of the DBW distribution can be obtained by using results from the marginal and joint distributions. Note that, from the marginal distributions of X_1 and X_2, we have

$$E(X_i) = \theta_i^{1/\eta_i}\Gamma\left(1 + \frac{1}{\eta_i}\right), \quad E(X_i^2) = \theta_i^{2/\eta_i}\Gamma\left(1 + \frac{2}{\eta_i}\right), \quad i = 1, 2. \tag{3.7}$$

Suppose we have observed data as n pairs (x_{1j}, x_{2j}), $j = 1, ..., n$. Using Eq.(3.7), based on the observed data, we form the following equations, for $i = 1, 2$:

$$\tfrac{1}{n}\sum_{j=1}^{n} x_{ij} = \theta_i^{1/\eta_i}\Gamma\left(1 + \frac{1}{\eta_i}\right) \tag{3.8}$$

$$\tfrac{1}{n}\sum_{j=1}^{n} x_{ij}^2 = \theta_i^{2/\eta_i}\Gamma\left(1 + \frac{2}{\eta_i}\right) \tag{3.9}$$

which readily give

$$\frac{\Gamma(1 + \frac{2}{\eta_i})}{\{\Gamma(1 + \frac{1}{\eta_i})\}^2} = \frac{\frac{1}{n}\sum_{j=1}^{n} x_{ij}^2}{\overline{x}_i^2}, \tag{3.10}$$

where $\overline{x}_i = \frac{1}{n}\sum_{j=1}^{n} x_{ij}$. A moment-based estimator of η_i, say $\widetilde{\eta}_i$, can be obtained by numerically solving (3.10). Then, from (3.8), a moment-based estimator of θ_i, say $\widetilde{\theta}_i$, can be obtained as

$$\widetilde{\theta}_i = \left[\frac{\overline{x}_i}{\Gamma(1 + \frac{1}{\widetilde{\eta}_i})}\right]^{\widetilde{\eta}_i}, \quad i = 1, 2. \tag{3.11}$$

Now, note that when $(X_1, X_2) \sim DBW(\eta_1, \theta_1, \eta_2, \theta_2, \rho)$, we have $(Y_1, Y_2) \sim DBE(\theta_1, \theta_2, \rho)$, where $Y_i = X_i^{\eta_i}$, for $i = 1, 2$. A closed form expression for the moment $\mu'_{r_1,r_2} = E(X_1^{r_1} X_2^{r_2})$ of the (r_1, r_2)-th order of the $DBE(\theta_1, \theta_2, \rho)$ distribution can be obtained from Al-Saadi et al. [1] as

$$\mu'_{r_1,r_2} = r_1! r_2! \theta_1^{r_1} \theta_2^{r_2} \sum_{j=0}^{r_2} \binom{r_2}{j} \binom{r_1 + r_2 - j}{r_1} \rho^{r_2-j} (1-\rho)^j,$$

from which it follows that

$$\mu'_{1,1} = (1+\rho)\theta_1\theta_2. \tag{3.12}$$

Al-Saadi and Young [2] proposed moment-estimators for the parameters of the Downton's bivariate exponential (DBE) distribution based on this result. Balakrishnan and Ng [3] proposed methods for improving estimates of ρ for the DBE distribution through resampling schemes. For the $DBW(\eta_1, \theta_1, \eta_2, \theta_2, \rho)$ distribution, using the data (x_{1j}, x_{2j}), $j = 1, ..., n$, and using (3.12), we define

$$R = \frac{\sum_{j=1}^{n} y_{1j} y_{2j}}{n \widetilde{\theta}_1 \widetilde{\theta}_2} - 1, \tag{3.13}$$

where $Y_i = X_i^{\eta_i}$, $i = 1, 2$, and $\widetilde{\theta}_1$ and $\widetilde{\theta}_2$ are as given in (3.11). Finally, a moment-based estimator for ρ can be proposed as

$$\widetilde{\rho} = \begin{cases} 0, & \text{if} \quad R < 0 \\ R, & \text{if} \quad 0 \le R < 1 \\ 1, & \text{if} \quad R \ge 1. \end{cases} \tag{3.14}$$

The process described above can be organized in the form of an algorithm to compute the moment-based estimators $\widetilde{\theta}_1$, $\widetilde{\theta}_2$, $\widetilde{\eta}_1$, $\widetilde{\eta}_2$, and $\widetilde{\rho}$, using the data (x_{1j}, x_{2j}), $j = 1, ..., n$ as follows:
ALGORITHM 2:

STEP 1: Obtain $\widetilde{\eta}_i$, $i = 1, 2$, by numerically solving (3.10).

STEP 2: Obtain $\widetilde{\theta}_i$, $i = 1, 2$, from (3.11).

STEP 3: Transform the data into (y_{1j}, y_{2j}), where $y_{ij} = x_{ij}^{\widetilde{\eta}_i}$, $i = 1, 2$, $j = 1, ..., n$.

STEP 4: Based on the transformed data (y_{1j}, y_{2j}), obtain $\widetilde{\rho}$ from (3.14).

Note that the moment-based estimates of parameters may be used as initial values for numerical computation of the MLEs.

3.3. Numerical Experiments: Monte Carlo simulations

To examine performance of the MLEs $\widehat{\eta}_1, \widehat{\theta}_1, \widehat{\eta}_2, \widehat{\theta}_2$, and $\widehat{\rho}$, and the asymptotic confidence intervals, extensive Monte Carlo simulations are carried out by using the R software [14]. For generating observations from the DBW distribution, the Algorithm 1 in Section 2.2 is followed.

The scale parameters θ_1 and θ_2 are fixed at unity, without any loss of generality. Different values of the shape parameters η_1 and η_2, and the dependence parameter ρ are used. The values of η_1 and η_2 are set at 0.50, 0.75, 1.00, and 1.25; ρ is taken as 0.25 (small), 0.5 (moderate), and 0.75 (high). Finally, all these settings are repeated for three different sample sizes: 20, 40, and 100. The detailed results of this detailed Monte Carlo simulation study are presented in the online supplementary materials of this paper in which we report bias and mean squared error (MSE) of the MLEs. Coverage probability of the asymptotic 95% confidence intervals are also reported.

It may be noted that the bias and MSE of the MLEs of all the parameters are quite reasonable. Clearly, as one would expect, the bias and MSE decrease with increasing sample size. The coverage probability of the confidence intervals for all parameters are reasonably close to the nominal level of 95%, and as expected, they improve with increasing sample size. The coverage probability of the confidence interval for ρ when $\rho = 0.25$ in smaller samples ($n = 20, 40$) is somewhat less than the nominal level, but for larger samples ($n = 75, 100$) it improves, and gets closer to the nominal level. Overall, the simulation results reflect that the MLEs are reasonably close to the true parameter values for a very wide range of simulation settings.

For the accuracy of the asymptotic confidence intervals, an important factor of the asymptotic normality of the MLEs. Here, we present histograms of the MLEs corresponding to one of the simulation settings in Figure 4. As pointed out by one of the reviewers, the asymptotic normality of the MLE of the parameter ρ can be questionable, especially when the true value of ρ is close to 0 and 1. To address this issue, suitable transformations of the MLEs may be considered. We optimized the likelihood function on a different scale transforming the parameters, to avoid any such problem. Natural logarithm transformation was considered for θ_1, θ_2, η_1, and η_2, and logit transformation was considered for ρ. The histograms of the transformed estimates are given in Figure 5. Similar results are obtained for all simulation settings considered.

4. INFERENCE BASED ON RIGHT-CENSORED DATA

4.1. Maximum likelihood estimation

In reliability experiments, right-censoring, and by that we mean Type-I right censoring, is perhaps the most common among different censoring mechanisms. For the univariate case, it may be described as follows. In the right-censoring scheme, units under study are followed up to a pre-fixed time-point,

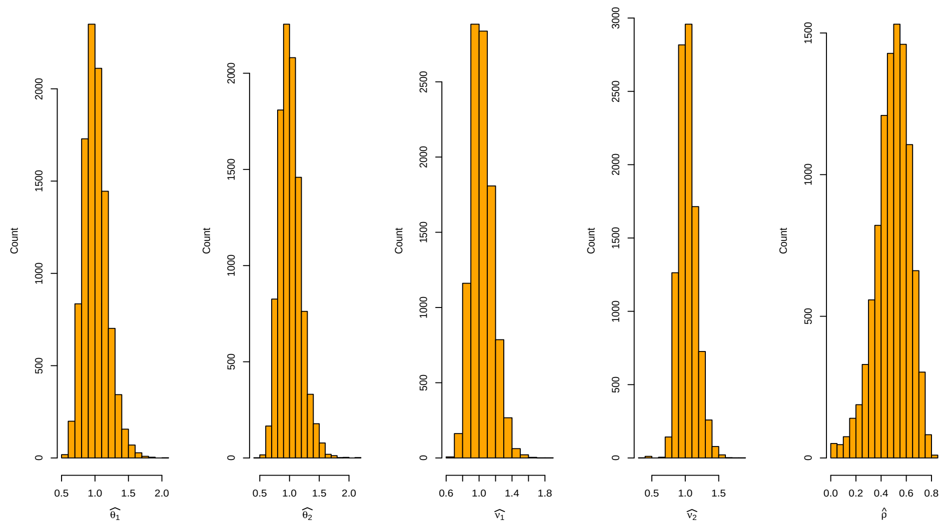

Figure 4: Histogram of the MLEs based on complete data when $\eta_1 = \eta_2 = 1$, $\theta_1 = \theta_2 = 1$, $\rho = 0.5$.

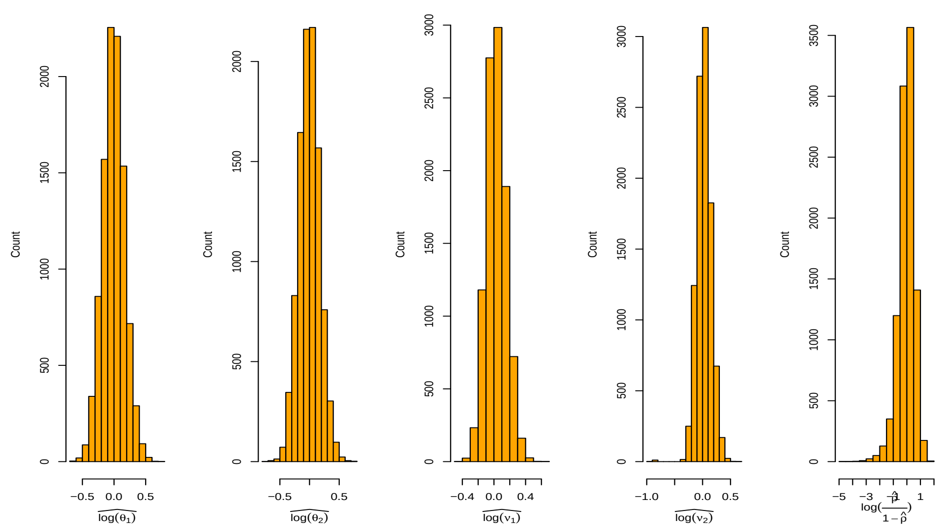

Figure 5: Histogram of the transformed MLEs based on complete data when $\eta_1 = \eta_2 = 1$, $\theta_1 = \theta_2 = 1$, $\rho = 0.5$.

often called the right-censoring point. The units which are still in working condition at this point, are called the right-censored units. The data on lifetimes obtained from such a study are then a mixture of observed failures and right censored lifetimes. Inferential procedures based on univariate data with right-censoring have been developed by several authors; see for example Meeker and

Escobar [13].

For bivariate lifetime data, naturally the right-censoring model needs to consider censoring on both components of the lifetime vector. We consider the following model for right-censoring in the bivariate case. First, assume that all the n units under study are followed starting from a common time point, say τ, and are followed until a pre-fixed time point, say T; the endpoint T is assumed to be common to both components of the lifetime vector (X_1, X_2) for each unit (though this assumption can be relaxed without complications). At the end of the study, there are four possibilities for each unit: (a) failure with respect to both components X_1 and X_2, (b) failure with respect to X_1, but X_2 is right censored at T, (c) failure with respect to X_2, but X_1 is right censored at T, (d) both components are right-censored at T. We assume non-informative censoring, i.e., the censoring mechanism is assumed to be independent of the lifetimes.

We can now construct the likelihood function for right-censored data. Let Δ_1 and Δ_2 denote indicators of censoring of a unit with corresponding to X_1 and X_2, respectively, i.e., for $i = 1, 2$,

$$\Delta_i = \begin{cases} 0, & \text{if} \quad X_i > T - \tau \\ 1, & \text{otherwise.} \end{cases}$$

Let S_{obs}, S_1, and S_2, and S_{cen} denote index sets such that

$$S_{obs} = \{j : \Delta_{1j} = 1, \Delta_{2j} = 1\}, \quad S_1 = \{j : \Delta_{1j} = 1, \Delta_{2j} = 0\}$$

$$S_2 = \{i : \Delta_{1j} = 0, \Delta_{2j} = 1\}, \quad S_{cen} = \{i : \Delta_{1j} = 0, \Delta_{2j} = 0\}.$$

Then the observed likelihood function for bivariate right-censored lifetime data is given by

$$\begin{aligned} L(\boldsymbol{\omega}) &= \prod_{j \in S_{obs}} f_{X_1,X_2}(x_{1j}, x_{2j}) \times \prod_{j \in S_1} f_{X_1}(x_{1j})[1 - F_{X_2|X_1=x_{1j}}(T-\tau)] \\ &\times \prod_{j \in S_2} f_{X_2}(x_{2j})[1 - F_{X_1|X_2=x_{2j}}(T-\tau)] \times \prod_{j \in S_{cen}} S_{X_1,X_2}(T-\tau, T-\tau). \end{aligned} \tag{4.1}$$

To further generalize the right-censoring model, consider the case where different units may start from different time points. For the j-th unit that starts from τ_j, the right-censoring time C_j is defined as the length of its life up to the fixed right censoring point T, and in that case, its observed lifetime corresponding to the two components X_1 and X_2 may be defined as

$$Z_{1j} = \text{Min}(X_{1j}, C_j), \quad Z_{2j} = \text{Min}(X_{2j}, C_j).$$

The likelihood function can then be generalized as

$$\begin{aligned} L(\boldsymbol{\omega}) &= \prod_{j=1}^{n} \{f_{X_1,X_2}(z_{1j}, z_{2j})\}^{\Delta_{1j}\Delta_{2j}} \{f_{X_1}(z_{1j})[1 - F_{X_2|X_1=z_{1j}}(C_j)]\}^{\Delta_{1j}(1-\Delta_{2j})} \\ &\times \{f_{X_2}(z_{2j})[1 - F_{X_1|X_2=z_{2j}}(C_j)]\}^{(1-\Delta_{1j})\Delta_{2j}} \{S_{X_1,X_2}(C_j, C_j)\}^{(1-\Delta_{1j})(1-\Delta_{2j})}. \end{aligned} \tag{4.2}$$

It is quite clear that explicit, closed-form MLE of the parameters $\widehat{\boldsymbol{\omega}} = (\widehat{\eta}_1, \widehat{\theta}_1, \widehat{\eta}_2, \widehat{\theta}_2, \widehat{\rho})$ cannot be obtained by optimizing Eq.(4.1) or Eq.(4.2); numerical techniques must be employed to compute the MLEs. Asymptotic confidence intervals for the parameters may be constructed by using asymptotic variance-covariance matrix which is possible to obtain through the observed Fisher information matrix, in a similar fashon as discussed in Section 3. It is of importance to mention here that when all the observations are right-censored, i.e., $\Delta_{1i} = 0$ and $\Delta_{2i} = 0$ for all i, the likelihood estimates do not converge. In statistical analysis of reliability data, this is a quite common issue that occurs due to insufficiency of the available information regarding the underlying lifetime distribution in case of no observed failures.

4.2. Numerical experiments: Monte Carlo simulations

Detailed Monte Carlo simulations by using the R software are performed to assess the performance of the MLEs of the DBW disribution based on right-censored data. The bias and mean squared error (MSE) of the point estimates, and coverage probability of the asymptotic confidence intervals are estimated through simulations.

Three different sample sizes are used: $n = 20$ (small), 40 (moderate), and 100 (large). The scale parameters θ_1 and θ_2 are taken to be unity, without loss of generality. Three different values, namely, 0.75, 1.00, and 1.25 are used for the shape parameters η_1 and η_2. The dependence parameter ρ is fixed at 0.25 (moderate) and 0.75 (high). These settings are used along with different right-censoring rates. The detailed simulation results are provided in the online supplementary materials of this paper.

It is observed that bias and MSE of the MLEs in case of right-censored data are quite reasonable throughout, demonstrating expected trend of reducing with increasing sample size. The coverage probability of the confidence intervals for all parameters η_1, θ_1, η_2, and θ_2 are very close to the nominal level of 95%. The coverage probability of the confidence intervals for ρ is occasionally (e.g., when true value of ρ is 0.25) away from the nominal level of 95% for smaller sample sizes ($n = 20$), but it gradually improves with increase in sample size, and gets very close to the nominal level of 95% for larger sample size ($n = 100$). In summary, it may be concluded that the MLEs of parameters of the DBW distribution based on right-censored data are quite efficient in estimating the true parameters for a wide range of simulation settings.

In this case of time-censored data also, like the case with complete data, we optimized the likelihood function on transformed scale of the parameters. We provide histograms of the MLEs based on time-censored data, on the original scale as well as transformed scale, for one of the simulation settings in Figures 6 and 7. This demonstrates asymptotic normality of the MLEs.

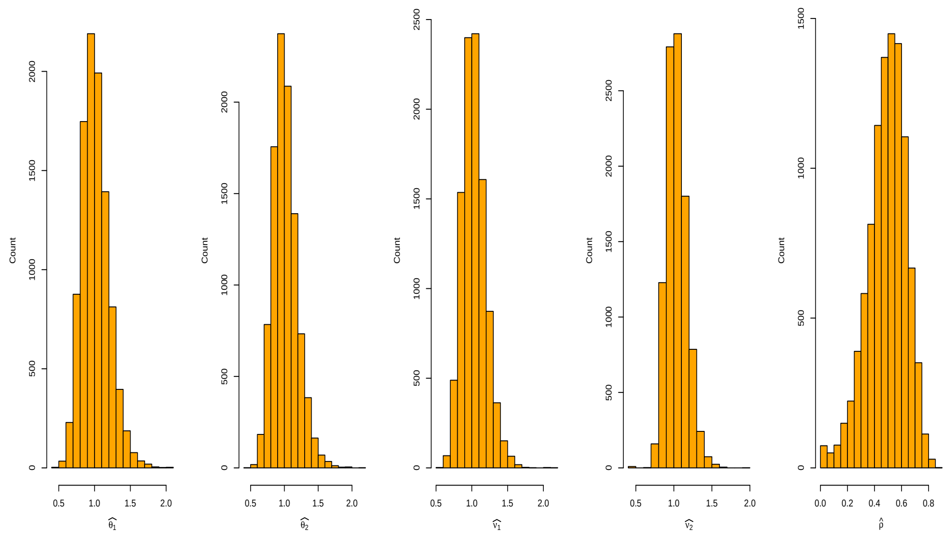

Figure 6: Histogram of the MLEs based on censored data when $\eta_1 = \eta_2 = 1$, $\theta_1 = \theta_2 = 1$, $\rho = 0.5$.

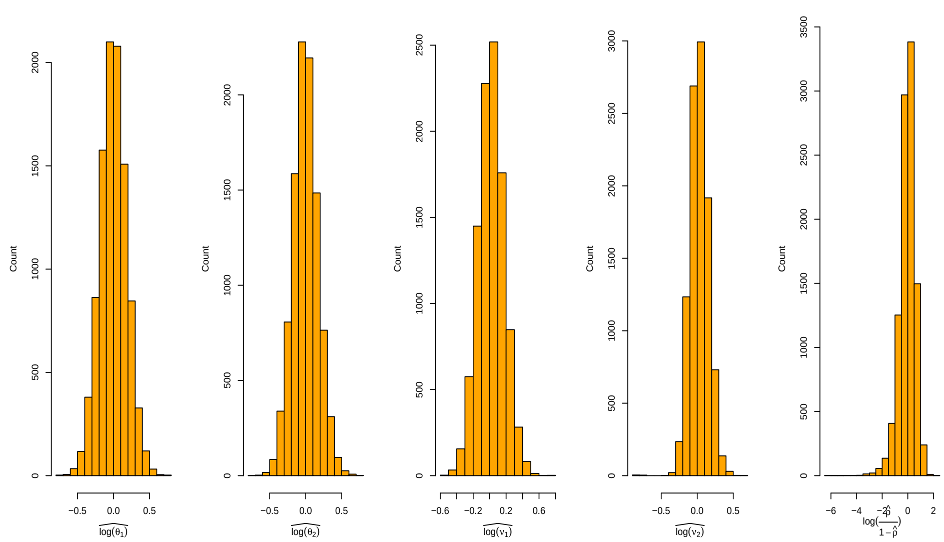

Figure 7: Histogram of the transformed MLEs based on censored data when $\eta_1 = \eta_2 = 1$, $\theta_1 = \theta_2 = 1$, $\rho = 0.5$.

5. APPLICATION TO A TWO-DIMENSIONAL WARRANTY MODEL

Several products such as automobiles, industry equipments, and heavy duty machines are sold under a two-dimensional warranty. For such products, their age as well as usage information are used to evaluate a warranty claim [9]. For example, a manufacturer may sell a car under a warranty of 5 years or 50,000 kilometers.

Naturally, a bivariate distributional model will be suitable for analysing reliability data with a two-dimensional warranty. Jung and Bai [9] proposed a model for analysing two-dimensional warranty data. Motivated by the work of Jung and Bai [9], here we propose a methodology to model two-dimensional warranty data by using a bivariate distribution. Our methodology generalizes the method proposed by Jung and Bai [9] by using more information while estimating the underlying bivariate distribution in terms of different status of failures of units sold under two-dimensional warranty.

Consider a product with two measurable quantities with respect to a two-dimensional warranty; let the random variables U and V denote these measurable quantities (for example, age and mileage of a car, respectively). Suppose the product is sold under a two-dimensional warranty (ψ, ξ); that is, a failure is considered to be within warranty when $(U, V) \in \psi \circledast \xi$, where

$$\psi \circledast \xi = \{(u, v) : 0 \leq u \leq \psi, 0 \leq v \leq \xi\}.$$

Suppose U and V follow a bivariate distribution with CDF $F_{U,V}(u, v; \boldsymbol{\omega})$ with corresponding PDF $f_{U,V}(u, v; \boldsymbol{\omega})$, where $\boldsymbol{\omega}$ is the vector of relevant model parameters. Our aim is to estimate the distribution $F_{U,V}(u, v; \boldsymbol{\omega})$ based on field-failure data of the product, so that relevant probabilities for the two-dimensional warranty can be estimated from $\widehat{F}_{U,V}(u, v; \widehat{\boldsymbol{\omega}})$.

Consider n units of the concerned product that are sold at a common time $(U_{start}, V_{start}) = (0, 0)$, under a two-dimensional warranty (ψ, ξ). Further assume that the manufacturer observes the status of the units at $(U_{end}, V_{end}) = (\psi, \xi)$. For each unit, there are four possibilities at this stage:
(a) failure within the warranty region, i.e., $(u, v) \in \psi \circledast \xi$; in this case its contribution to the likelihood function would be $f_{U,V}(u, v)$,
(b) warranty expired with respect to U, but not with respect to V, i.e., $U > \psi$ and $0 \leq V \leq \xi$; in this case, its contribution to the likelihood function would be $f_U(u)[1 - F_{V|U=u}(\xi)]$,
(c) warranty expired with respect to V, but not with respect to U, i.e., $0 \leq U \leq \psi$ and $V > \xi$; in this case its contribution to the likelihood function would be $f_V(v)[1 - F_{U|V=v}(\psi)]$
(d) warranty expired with respect to both U and V, i.e., $U > \psi$ and $V > \xi$; in this case its contribution to the likelihood function would be $S_{U,V}(\psi, \xi)$, where $S_{(U,V)}(\cdot, \cdot)$ is the joint survival function of (U, V).

Clearly, in cases (b), (c), and (d) above, at least one of the random variables is right-censored. Analysing product field-failure data under two-dimensional warranty is particularly challenging due to the right censored observations. Considering contributions of the n units under study according to their status with respect to (a), (b), (c), and (d), the likelihood function for field-failure data under two-dimensional warranty is given by

$$\begin{aligned} L(\boldsymbol{\omega}) &= \prod_{i=1}^{n} \{f_{U,V}(u_i, v_i)\}^{\Delta_{1i}\Delta_{2i}} \{f_U(u_i)[1 - F_{V|U=u_i}(\xi)]\}^{\Delta_{1i}(1-\Delta_{2i})} \\ &\times \{f_V(v_i)[1 - F_{U|V=v_i}(\psi)]\}^{(1-\Delta_{1i})\Delta_{2i}} \{S_{U,V}(\psi, \xi)\}^{(1-\Delta_{1i})(1-\Delta_{2i})}, \end{aligned} \tag{5.1}$$

where

$$\Delta_1 = \begin{cases} 0, & \text{if} \quad U > \psi \\ 1, & \text{otherwise,} \end{cases}$$

and

$$\Delta_2 = \begin{cases} 0, & \text{if} \quad V > \xi \\ 1, & \text{otherwise.} \end{cases}$$

It can be seen immediately that the likelihood in Eq.(5.1) follows from the general right-censored likelihood in Eq.(4.2). The likelihood in Eq.(5.1) can be maximized using numerical approach to obtain the MLE $\widehat{\boldsymbol{\omega}}$. Then, using $\widehat{\boldsymbol{\omega}}$, it is straightforward to estimate probabilities such as $\widehat{F}_{U,V}(\psi, \xi; \widehat{\boldsymbol{\omega}})$, i.e., the probability of a unit failing within the warranty region.

As the DBW distribution is a flexible model with Weibull marginals, it can accommodate a wide range of lifetime data, successfully capturing non-constant marginal hazard rates as well the dependence between the components. Therefore, we propose to model the joint distribution of U and V by the DBW distribution. When

$$(U, V) \sim DBW(\eta_1, \theta_1, \eta_2, \theta_2, \rho),$$

plugging-in the joint PDF, conditional CDFs, and joint survival function of the distribution in Eq.(5.1), we can estimate the parameters by the MLE $\widehat{\boldsymbol{\omega}} = (\widehat{\eta}_1, \widehat{\theta}_1, \widehat{\eta}_2, \widehat{\theta}_2, \widehat{\rho})$. Then, the probability of a unit failing within the warranty region is given by

$$\begin{aligned} \widehat{F}_{U,V}(\psi, \xi; \widehat{\boldsymbol{\omega}}) =& \ 1 - \exp\left(-\frac{\psi^{\widehat{\eta}_1}}{\widehat{\theta}_1}\right) Q_1\left(\sqrt{\frac{2}{1-\widehat{\rho}}}\frac{\xi^{\widehat{\eta}_2/2}}{\sqrt{\widehat{\theta}_2}}, \sqrt{\frac{2\widehat{\rho}}{1-\widehat{\rho}}}\frac{\psi^{\widehat{\eta}_1/2}}{\sqrt{\widehat{\theta}_1}}\right) \\ & - \exp\left(-\frac{\xi^{\widehat{\eta}_2}}{\widehat{\theta}_2}\right)\left[1 - Q_1\left(\sqrt{\frac{2\widehat{\rho}}{1-\widehat{\rho}}}\frac{\xi^{\widehat{\eta}_2/2}}{\sqrt{\widehat{\theta}_2}}, \sqrt{\frac{2}{1-\widehat{\rho}}}\frac{\psi^{\widehat{\eta}_1/2}}{\sqrt{\widehat{\theta}_1}}\right)\right]. \end{aligned} \tag{5.2}$$

An illustrative example of this warranty model is given in Section 6.1.

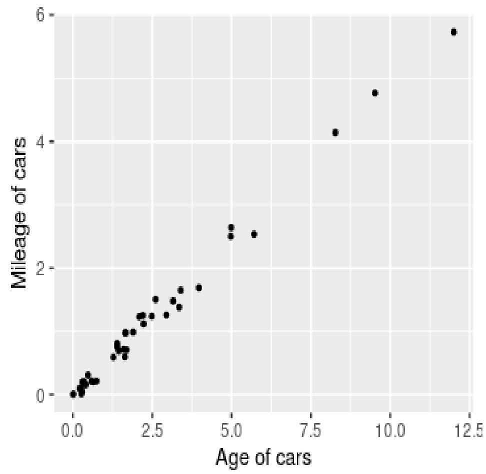

Figure 8: Scatterplot of Age (X_1) and Mileage (X_2) of cars for the car warranty data.

6. Case studies

In this section, we provide analyses of two real datasets. The first example is based on the car warranty data of Jung and Bai [9]. The second example is based on a general bivariate data on bone mineral density, originally from Johnson and Wichern [8]. The car warranty data demonstrates the use of the DBW distribution for reliability data, including its application in the two-dimensional warranty model discussed above. The the bone mineral density data is not a lifetime data as such, we present analysis of this data to demonstrate the versatility of the DBW distribution in modelling different types of data.

6.1. Real Data 1: Car warranty data

The car warranty dataset, analysed by Jung and Bai [9], and Yuan [16], contains data on age and mileage of 40 cars. Age and mileage of a car are two measurable quantities to assess its lifetime. Yuan [16] analysed this data to obtain generalized moment estimates of a bivariate Weibull distribution, while Jung and Bai [9] considered a two-dimensional warranty model.

Suppose U and V are the age and mileage of a car, respectively. In Figure 8, a scatterplot of U and V is given. It is clear from this scatterplot that age and mileage are highly correlated, which suggests that a bivariate distribution that captures the dependence between them will be suitable to model the car warranty data.

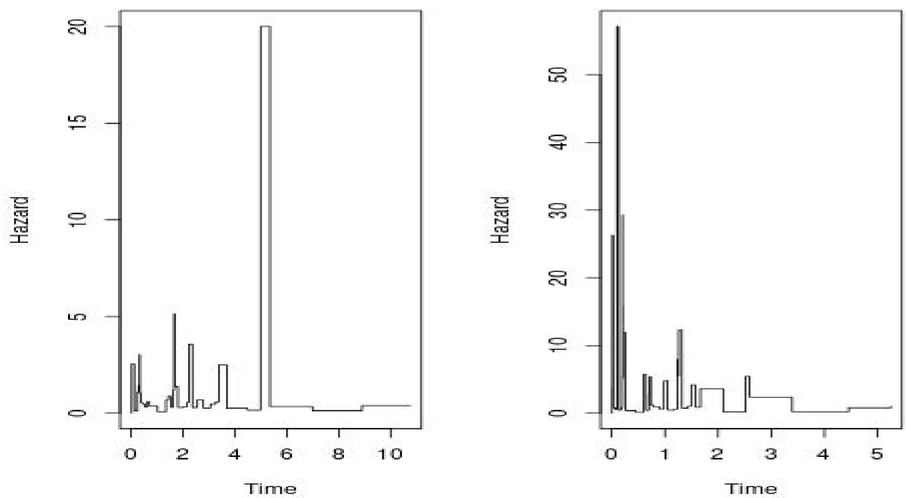

Figure 9: Plot of hazard functions for the marginal distributions for the car warranty data: car age (left), and car mileage (right).

Right-censoring Point	Parameter	MLE	Asymptotic 95% CI	$\widehat{F}_{U,V}(\psi,\xi)$
(0.5,0.5)	θ_1	3.909	(2.218, 6.891)	$\widehat{F}_{U,V}(0.5, 0.5) = 0.250$
	θ_2	2.679	(1.573, 4.562)	
	η_1	0.428	(0.258, 0.710)	
	η_2	0.379	(0.232, 0.620)	
	ρ	0.993	(0.983, 0.998)	
(1,1)	θ_1	2.537	(1.597, 4.032)	$\widehat{F}_{U,V}(1, 1) = 0.469$
	θ_2	1.579	(1.049, 2.376)	
	η_1	0.404	(0.254, 0.642)	
	η_2	0.414	(0.271, 0.632)	
	ρ	0.991	(0.977, 0.996)	
(2,2)	θ_1	1.778	(1.213, 2.609)	$\widehat{F}_{U,V}(2, 2; \widehat{\boldsymbol{\omega}}) = 0.763$
	θ_2	0.952	(0.695, 1.304)	
	η_1	0.719	(0.550, 0.941)	
	η_2	0.716	(0.546, 0.940)	
	ρ	0.985	(0.973, 0.992)	

Table 1: MLEs and asymptotic confidence intervals of parameters of the DBW model based on Warranty Data for different right-censoring points

An option is to use the DBE distribution for this data. However, to observe that the marginal distibutions of age and mileage have non-constant hazard rates, see Figure 9. From this plot, it is clear that the DBE distribution will not be suitable for the car warranty data, as the marginal distributions of U and V cannot be modelled by exponential distributions. Therefore, to accommodate the non-constant marginal hazards rates, we model the car warranty data by the DBW distribution, where the marginals are Weibull distributions. It may be noted here that Jung and Bai [9] also provided justification for using the Weibull distribution as the marginal models for age and mileage in the car warranty data; though their bivariate model for age and mileage was not the DBW distribution.

The warranty data contains complete observations on U and V. However, for illustrative purposes, here we artificially impose right censoring to analyse the data; see the results presented in Table 1. Note that for greater amount of right-censored units, the confidence intervals of the parameters are wider, implying larger standard errors. Naturally, this implies that the estimates are relatively more reliable for smaller right censoring percentages, as expected. The MLEs can be used further to compute suitable porbabilities from the estimated model. Table 1 also gives the estimated probabilities for units to fail within the warranty region, for different two-dimensional warranty schemes: $(\psi, \xi) = (0.5, 0.5)$, (1,1), (2,2). It is, of course, possible to use different values for ψ and ξ, for example, $(\psi, \xi) = (1,2)$ etc. for real life applications.

6.2. Real Data 2: Bone mineral density data

The bone mineral density data from Johnson and Wichern [8] consists of measurements on bone mineral density (BMD) in the unit g/cm^2 for 24 adults. Within each observation, the first value (X_1) represents the BMD of the bone dominant radius before the start of the study, and the second value (X_2) represents the BMD of the bone after the study. A scatterplot of the observed data indicates that the two variables are highly correlated.

Assuming DBW distribution for (X_1, X_2), we obtain the MLEs of the model parameters based on the given BMD data; Table 2 gives the results. We also calculate asymptotic 95% confidence intervals for the parameters.

Parameter	Estimate	95% CI
θ_1	0.407	(0.265, 0.623)
θ_2	0.372	(0.236, 0.586)
η_1	7.026	(5.427, 9.096)
η_2	8.100	(5.846, 11.222)
ρ	0.935	(0.860, 0.972)

Table 2: Estimates of parameters of the DBW model based on the BMD data

To check if the model fit is reasonable, in Figure 11 we plot the estimated

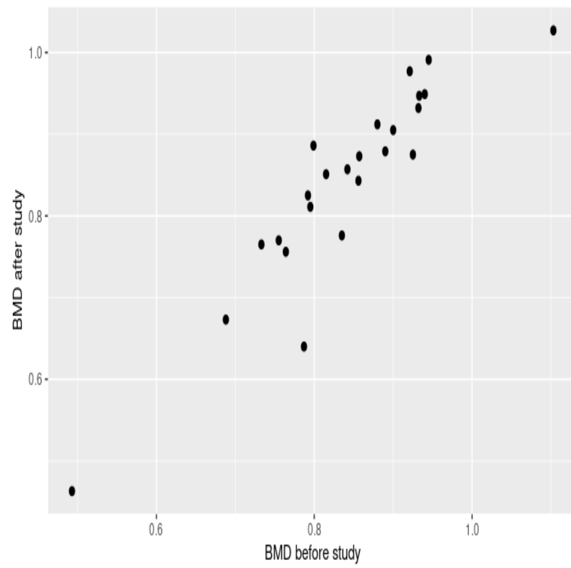

Figure 10: Plot of BMD before and after study

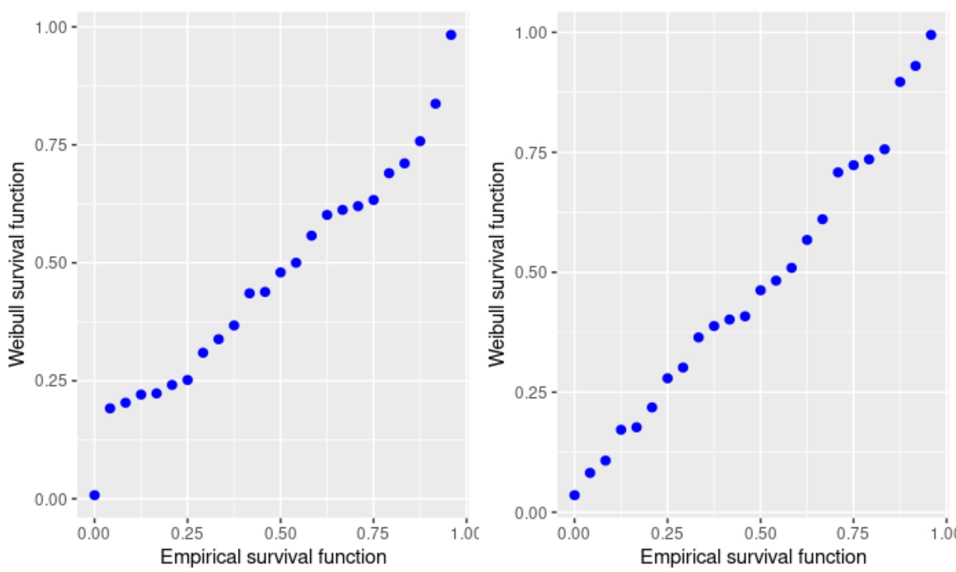

Figure 11: Plot of empirical survival function and Weibull survival function for the marginal distributions at the ordered observations with estimates parameters for X_1 (left) and X_2 (right)

Weibull survival function $\widehat{S}(t) = \exp\left\{-\left(\frac{t^{\hat{\eta}}}{\hat{\theta}}\right)\right\}$ against the empirical survival function, evaluated at the ordered observed values, separately for X_1 and X_2. Noting that the plotted points roughly form a straight line which indicates that the estimated values of the survival function are in agreement, we conclude that the model fit is reasonable.

Finally, we can compare the suitability of the DBW model with other bivariate models available in the literature. For this purpose, here we consider the Marshall-Olkin bivariate Weibull (MOBW) distribution which was proposed by Marshall and Olkin as an extension of a bivariate exponential distribution that is now known as Marshall-Olkin bivariate exponential (MOBE) distribution [12]. For the DBW and MOBW models, we calculate the Akaike's Information Criterion (AIC), and compare the values; the model with the lower AIC value would be more suitable for a given data. For the bone mineral density data, we obtain AIC_{DBW} = -103.297, and AIC_{MOBW} = -55.457. This suggests that the DBW distribution is a more suitable model for the bone mineral density data.

7. Conclusion

In this article, a bivariate Weibull distribution, which we call the Downton's bivariate Weibull (DBW) distribution, is considered. Important statistical properties of the DBW distribution are studied. Then, inferences based on complete, and right-censored bivariate data are discussed for this distribution. Through extensive Monte Carlo simulation studies, it is observed that the point and interval estimates of the parameters of this distribution perform quite well - for both complete as well as right-censored data. A two-dimensional warranty model is discussed, and the application of the DBW distribution in the warranty model is considered. For illustrative purposes, two case studies based on real datasets are provided.

The DBW distribution has desirable statistical properties. It is quite flexible for modelling purposes, and it computationally convenient. It successfully captures the dependence between the components of a lifetime vector. In particular, the DBW distribution is very useful in modelling bivariate lifetime data when the marginal distributions indicate non-constant hazard rates. Based on our explorations presented in this paper, we strongly recommend its use in real life, particularly to model bivariate reliability data.

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