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Survival Copula Entropy and Dependence in Bivariate Distributions

Authors: N. Unnikrishnan Nair
– Department of Statistics, Cochin University of Science and Technology, Cochin 682 022, Kerala, India
unnikrishnannair4@gmail.com

S. M. Sunoj ✉
– Department of Statistics, Cochin University of Science and Technology, Cochin 682 022, Kerala, India
smsunoj@cusat.ac.in

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Abstract:

- In the present work we propose survival copula entropy as an alternative to Shannon entropy, cumulative residual entropy and copula entropy measures in computing the uncertainty in bivariate populations. We examine the relationships between the various measures. The properties of survival copula entropy are discussed, especially its applications to ascertain the nature and extent of uncertainty among copulas.

Keywords:

- *bivariate Shannon entropy; copula entropy; cumulative residual entropy; dependence measures.*

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- 94A17, 62H05.

✉ Corresponding author

1. INTRODUCTION

Knowledge on the nature and extent of the joint behaviour of random quantities is a topic of considerable interest in all fields of scientific activity, In this context the joint distribution of random variables is an indispensable tool in analysing various aspects of interrelationship among the constituent variables, Among various measures developed for understanding the amount of uncertainty prevailing in the outcomes generated by the distribution, entropy has established itself as an efficient mechanism in a variety of fields. The basic measure of uncertainty employed in the bivariate case is the Shannon entropy defined as

$$h(X,Y) = h_{X,Y} = -\int_S\int f(x,y)\log f(x,y)dxdy, \tag{1.1}$$

where $f(x,y)$ is the probability density function of the random vector (X,Y) with support S. Since the introduction of (1.1), several modifications were introduced by way of additional parameters to impart more flexibility, measures with structural changes, replacing joint density by conditional ones etc., to provide a wide range of new measures. The structural modification to obtain a new measure by replacing the density by the survival function is due to Rao *et al.* (2004) in the univariate case, motivated researchers to apply the same logic in the bivariate case resulting in the bivariate version of cumulative residual entropy

$$H(X,Y) = H_{X,Y} = -\int_S\int \bar{F}(x,y)\log \bar{F}(x,y)dxdy, \tag{1.2}$$

where $\bar{F}(x,y) = P(X > x, Y > y)$ is the survival function of (X,Y). The expression (1.2) is implied as a particular case in the definition of bivariate cumulative residual entropy in Rajesh *et al.* (2014). A critical aspect to be considered in using bivariate distribution for modelling is the dependence relation existing between X and Y. In this respect copulas are found to be more general and flexible as they provide means of obtaining the joint distribution through the marginals connected by known dependence relationships. There are three approaches to study the nature and extent of dependence in copulas. The first is through global measures that specify the association like the Pearson's correlation coefficient, Spearman's rho, Kendall's tau, Blomquist's beta, etc. A second alternative is to use dependence concepts like total positivity, quadratic dependence and stochastic increase. Finally, we have time dependent measures of association which are used when analysing data on duration variables where the time elapsed since the commencement of observation is vital. See Nair and Sankaran (2010) for a review of materials in this connection. Any one of these methods, depending on the appropriateness of the techniques chosen in the problem at hand, will enable us to know whether dependence is positive or negative and also to compare the degree of association. In view of the flexibility of the copula over distribution functions, Ma and Sun (2008) proposed the copula entropy

$$i(X,Y) = i_{X,Y} = -\int_0^1\int_0^1 c(u,v)\log c(u,v)dudv, \tag{1.3}$$

as a measure of uncertainty in the copula density $c(u,v)$ associated with (X,Y). It provides a tool to connect copulas and information theory. Since its inception the measure (1.3) was used to ascertain structural learning, dependence aspects, variable selection, casual discovery, etc in various disciplines such as hydrology (Chen *et al.*, 2013), biology (Charzyńska and Gambin, 2015), neuroscience (Kayser *et al.*, 2015) and medicine (Ma, 2019a,b; Mesiar and

Sheikhi, 2021) etc. The generality and range of application the copula entropy motivate the investigation of more refined measures of uncertainty for copulas. Further as an index of information, the negative values of (1.3) are difficult to interpret and it is more preferable to have a measure that assumes positive values only, and that too in a finite interval. The form of copula density in many standard cases is analytically and computationally more complicated to work with, when compared to the usual copula or the survival copula. To study various aspects of dependence, most concepts in that area are expressed in terms of the copula than its density. Moreover inference procedures available in literature for copulas can be made use of in estimating and testing copula-based entropies. These facts suggest proposing an uncertainty measure based on copulas and investigating its properties. The objective of the present work is thus to make a preliminary study of a new measure of uncertainty in terms of the survival copulas, in the same manner as the development of (1.2) from (1.1).

A summary of the present work is as follows. In Section 2, we define the survival copula entropy and obtain some relationships between cumulative residual entropy, copula entropy and survival copula entropy. Following this, in Section 3 the properties of the new entropy especially its role as a measure of dependence is discussed. In Section 4, application of survival copula entropy to some real situations is demonstrated. The paper ends with a brief conclusion in Section 5.

2. SURVIVAL COPULA ENTROPY

As mentioned in the introduction, let (X, Y) be a random vector with distribution function $F(x, y)$ and survival function $\bar{F}(x, y)$. Recall that a copula is a function $C : I^2 \to I$, where I is the unit interval [0, 1], such that

$$C(0, v) = C(u, 0) = 0; \;\; C(u, 1) = C(1, u) = u,$$

for all u, v in I and C is 2-increasing so that the C-volume of the rectangle $[a, b] \times [k, d]$, $V_C([a, b] \times [k, d]) \geq 0$ for all rectangles in I^2. The function C induces a probability measure on I^2 via $V_C([a, b] \times [k, d]) = C(u, v)$. When C is absolutely continuous we have the copula density $c(u, v) = \frac{\partial^2 C}{\partial u \partial v}$. A survival copula $\bar{C} : I^2 \to I$ satisfies

$$\bar{C}(u, 1) = 0 = \bar{C}(1, u) \;\text{ and }\; \bar{C}(u, 0) = u = \bar{C}(0, u),$$

for all u in I and volume $V_{\bar{C}}([a, b] \times [k, d]) \geq 0$. Further,

$$\bar{C}(u, v) = u + v - 1 + C(1 - u, 1 - v).$$

With these basic notions we define the measures of uncertainty with reference to C and $\bar{C}$.

Definition 2.1. The survival copula entropy (SCE) associated with the survival copula $\bar{C}$ of (X, Y) is defined as

$$I_{\bar{C}}(X, Y) = -\int_0^1 \int_0^1 \bar{C}(u, v) \log \bar{C}(u, v) du dv. \tag{2.1}$$

Example 2.1. Consider the Gumbel-Barnett family

$$\bar{C}(u,v) = u\ v\ \exp[-\theta \log u \log v],\ 0 \le \theta \le 1.$$

$$\begin{aligned} I_{\bar{C}}(X,Y) &= -\int_0^1 \int_0^1 uve^{-\theta \log u \log v}[\log u + \log v - \theta \log u \log v]dudv \\ &= -e^{\frac{4}{\theta}} EI\Big(-\frac{4}{\theta}\Big), \end{aligned}$$

where $EI(z) = -\int_{-z}^{\infty} \frac{e^{-t}}{t}dt$. The value of $I_{\bar{C}}$ for selected values of θ are given in Table 1.

Table 1: Survival copula entropy of the Gumbel-Barnett copula.

θ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$I_{\bar{C}}$	0.24404	0.23859	0.23357	0.22570	0.22455	0.22048	0.21664	0.21302	0.20960

The quantity in (2.1) is obviously a measure of uncertainty since $\bar{C}(u,v)$ is a bivariate survival function with uniform marginals and (2.1) is thus a cumulative entropy, by definition (1.2). As a measure of uncertainty in bivariate distributions, it is of interest to examine its structure in relation to the existing similar measures like cumulative entropy (1.2) and the copula entropy (1.3). For this purpose we assume that the marginal survival functions $\bar{F}_X$ and $\bar{F}_Y$ of X and Y are continuous, strictly decreasing over the half-line $[0,\infty)$ with $\bar{F}_X(0) = 1 = \bar{F}_Y(0)$. Then the bivariate cumulative entropy of (X,Y) can be written as

$$\begin{aligned} H(X,Y) &= -\int_0^\infty \int_0^\infty \bar{F}(x,y) \log \bar{F}(x,y) dxdy \\ &= -\int_0^1 \int_0^1 \bar{F}\big(\bar{F}_X^{-1}(u), \bar{F}_Y^{-1}(v)\big) \log \bar{F}\big(\bar{F}_X^{-1}(u), \bar{F}_Y^{-1}(v)\big) d\bar{F}_X^{-1}(u) d\bar{F}_Y^{-1}(v) \\ &= -\int_0^1 \int_0^1 \bar{C}(u,v) \log \bar{C}(u,v) d\bar{F}_X^{-1}(u) d\bar{F}_Y^{-1}(v), \end{aligned} \tag{2.2}$$

by Sklar's theorem. It is evident that the copula version of (1.2) is not in general the same as SCE. However for bivariate uniform distributions like

$$\bar{F}(x,y) = (1-x)(1-y)[1+\theta xy],\ 0 \le x, y \le 1,$$

the quantities $H_{XY}(x,y)$ and $I_{\bar{C}}(X,Y)$ remain the same. Some similar calculations reveal that

$$I_{\bar{C}}(X,Y) = -\int_0^\infty \int_0^\infty \big[\bar{F}(x,y) \log \bar{F}(x,y)\big] d\bar{F}_X d\bar{F}_Y, \tag{2.3}$$

showing that the distribution function counterpart of SCE is not identical with (1.2). Thus survival copula entropy is a different measure than the other comparable measures of uncertainty of bivariate distributions as can be seen from this and the following discussions.

We now examine the relationships between the two copula-based entropies $i(X,Y)$ and $I(X,Y)$. Although there exists nice relationship $c(u,v) = \frac{\partial^2 C(u,v)}{\partial u \partial v}$ between c and C, a simple equation connecting their entropies appears to be elusive and the same is true even for specific copulas. However, if we consider the random vector (X_E, Y_E) associated with (X, Y) through

$$f_{X_E,Y_E}(x,y) = \frac{\bar{F}(x,y)}{E(XY)}, \tag{2.4}$$

where f_{X_E,Y_E} stands for the density function of (X_E, Y_E), some relationships useful in different contexts can be derived. Note that (2.4) is often called the equilibrium distribution of (X, Y). For a detailed discussion of the properties and applications of such distributions, we refer to Nair and Preeth (2009) and Navarro and Sarabia (2010). Using (2.4) and (1.1) we can write the Shannon entropy of (X_E, Y_E) as

$$\begin{aligned} h_{X_E,Y_E} &= -\int_0^\infty \int_0^\infty \frac{\bar{F}(x,y)}{E(XY)} \left[\log \bar{F}(x,y) - E(XY)\right] dxdy \\ &= \frac{H_{X,Y}}{E(XY)} + \log E(XY), \end{aligned} \tag{2.5}$$

showing $H_{X,Y}$ as a change of origin and scale in h_{X_E,Y_E}.

From Ma and Sun (2011), the mutual information among (X, Y)

$$\begin{aligned} M(X,Y) &= -\int_0^\infty \int_0^\infty f(x,y) \log \frac{f(x,y)}{f_X(x) f_Y(y)} dxdy \\ &= h_X + h_Y - h_{X,Y}, \end{aligned}$$

is the negative of the copula entropy $i_{X,Y}$ where h_X and h_Y respectively denote the Shannon entropy of X and Y and so,

$$-i_{X,Y} = h_X + h_Y - h_{X,Y}$$

giving

$$i_{X_E,Y_E} = h_{X_E,Y_E} - h_{X_E} - h_{Y_E}. \tag{2.6}$$

From the definition of the univariate equilibrium distribution $f_{X_E} = \frac{\bar{F}_X(x)}{E(X)}$, we get

$$h_{X_E} = \frac{H_X}{E(X)} + \log E(X), \tag{2.7}$$

and similarly,

$$h_{Y_E} = \frac{H_Y}{E(Y)} + \log E(Y), \tag{2.8}$$

where $H_X = -\int_0^\infty \bar{F}_X(x) \log \bar{F}_X(x) dx$ is the cumulative residual entropy of X and H_Y is similarly defined (see Rao *et al.*, 2004). From equations (2.5) through (2.8) the expression for copula entropy is obtained in terms of cumulative entropies as

$$i_{X_E,Y_E} = \frac{H_{X,Y}}{E(XY)} - \frac{H_X}{E(X)} - \frac{H_Y}{E(Y)} + \log \frac{E(XY)}{E(X)E(Y)}. \tag{2.9}$$

Finally from (2.3),

$$\begin{aligned} I_{\bar{C}}(X,Y) &= -\int_0^\infty \int_0^\infty E(XY) f_E(x,y) [\log E(XY) f_E(x,y)] f_X(x) f_Y(y) dxdy \\ &= -\int_0^1 \int_0^1 E(XY) f_E\left(\bar{F}_X^{-1}, \bar{F}_Y^{-1}\right) \left[\log E(XY) f_E\left(\bar{F}_X^{-1}, \bar{F}_Y^{-1}\right)\right] dudv. \end{aligned} \tag{2.10}$$

From

$$f(x,y)=\frac{\partial^2\bar{F}(x,y)}{\partial x\partial y},$$

we have

$$\begin{aligned} f\big(\bar{F}_X^{-1},\bar{F}_Y^{-1}\big) &= \frac{\partial^2\bar{F}\big(\bar{F}_X^{-1},\bar{F}_Y^{-1}\big)}{\partial\bar{F}_X^{-1}\partial\bar{F}_Y^{-1}} \\ &= \frac{\partial^2 C(u,v)}{\partial u\partial v}\frac{\partial u\partial v}{\partial\bar{F}_X^{-1}\partial\bar{F}_Y^{-1}} \\ &= c(u,v)\frac{\partial u\partial v}{\partial\bar{F}_X^{-1}\partial\bar{F}_Y^{-1}}. \end{aligned}$$

Thus from (2.10), the survival copula entropy is related to the copula entropy as

$$\begin{aligned} I_{\bar{C}}(X,Y) = -\int_0^1\int_0^1 & E(XY)f_X\big(\bar{F}_X^{-1}\big)f_Y\big(\bar{F}_Y^{-1}\big)c(u,v) \\ & \log\big[E(XY)f_X\big(\bar{F}_X^{-1}\big)f_Y\big(\bar{F}_Y^{-1}\big)c(u,v)\big]dudv. \end{aligned} \tag{2.11}$$

In the next example we demonstrate how the above results work out in a specific distribution.

Example 2.2. Let (X,Y) follows bivariate Pareto distribution

$$\bar{F}(x,y)=(1+x+y)^{-\theta},\ \theta>0;\ x,y>0.$$

Then we have

$$\bar{F}_X(x)=(1+x)^{-\theta};\ \ \bar{F}_Y(y)=(1+y)^{-\theta},$$

$$E(XY)=\int_0^\infty\int_0^\infty(1+x+y)^{-\theta}dxdy=[(\theta-1)(\theta-2)]^{-1},\quad \theta>2 \tag{2.12}$$

and

$$f_{X_E,Y_E}(x,y)=(\theta-1)(\theta-2)(1+x+y)^{-\theta},$$

$$\begin{aligned} H_{XY} &= -\int_0^\infty\int_0^\infty(1+x+y)^{-\theta}(-\theta\log(1+x+y))dxdy \\ &= \frac{(2\theta-3)\theta}{(\theta-1)(\theta-2)} \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} h_{X_E,Y_E} &= -\int_0^\infty\int_0^\infty(\theta-1)(\theta-2)(1+x+y)^{-\theta}\Big[\log(\theta-1)(\theta-2)+\log(1+x+y)^{-\theta}\Big]dxdy \\ &= (2\theta-3)\theta+\log(\theta-1)(\theta-2). \end{aligned} \tag{2.14}$$

The formula (2.5) is verified from (2.12), (2.13) and (2.14). Also

$$H_X=-\int_0^\infty(1+x)^{-\theta}(-\theta\log(1+x))dx=\frac{\theta}{(\theta-1)^2}.$$

Similarly $h_Y=\frac{\theta}{(\theta-1)^2}$ and $E(X)=E(Y)=\frac{1}{(\theta-1)}$. Hence from (2.9)

$$\begin{aligned} i_{X_E,Y_E} &= (2\theta-3)\theta-\frac{\theta}{\theta-1}-\frac{\theta}{\theta-1}+\log\frac{(\theta-1)^2}{(\theta-1)(\theta-2)} \\ &= \frac{2\theta^2-7\theta+1}{\theta-1}+\log\frac{\theta-1}{\theta-2},\qquad \theta>2. \end{aligned}$$

The survival copula is

$$\bar{C}_{X,Y}(u,v) = \left(u^{-\frac{1}{\theta}} + v^{-\frac{1}{\theta}} - 1\right)^{-\theta}, \tag{2.15}$$

which is the Clayton family. Also

$$c(u,v) = \frac{\theta+1}{\theta} \frac{u^{-\frac{1}{\theta}-1} v^{-\frac{1}{\theta}-1}}{\left(u^{-\frac{1}{\theta}} + v^{-\frac{1}{\theta}-1}\right)^{\theta+2}} (\theta+2). \tag{2.16}$$

One can directly calculate both $I_{\bar{C}}$ and $i(X,Y)$ from (2.15) and (2.16). We may also use the fact that

$$\bar{C}(u,v) = \frac{\theta+1}{\theta} \frac{u^{-\frac{1}{\theta}-1} v^{-\frac{1}{\theta}-1}}{\left(u^{-\frac{1}{\theta}} + v^{-\frac{1}{\theta}-1}\right)^{\theta+2}} c(u,v)$$

or formula (2.11). Note that when applying (2.11), $f_X(x) = \theta(1+x)^{-\theta-1}$ so that $f_X\left(\bar{F}_X^{-1}\right) = \theta u^{-\frac{\theta+1}{\theta}}$ and $f_Y\left(\bar{F}_Y^{-1}\right) = \theta v^{-\frac{\theta+1}{\theta}}$

We have so far discussed the entropy function derived from the survival copula. One can also define the entropy based on the usual copula C.

Definition 2.2. The cumulative copula entropy (CCE) of (X,Y) is defined as

$$I_C(X,Y) = -\int_0^1 \int_0^1 C(u,v) \log C(u,v) du dv.$$

Example 2.3. The Farlie-Gumbel-Morgestern copula

$$C(u,v) = uv[1 + \theta(1-u)(1-v)], \quad -1 \le u \le 1$$

has CCE given by

$$I_C = -\int_0^1 \int_0^1 uv(1+\theta(1-u)(1-v))[\log u + \log v + \log(1+\theta(1-u)(1-v))] du dv.$$

The integral does not converge for $\theta > 0$. For $\theta < 0$,

$$\begin{aligned} I_C = \left(168\theta^2\right)^{-1} \Big[& -690 + 84\theta^2 + 10\theta^3 - 3\pi^2 + 9\pi^2\theta + 3(17 + 9\theta + 9\theta^2 + \theta^3) \\ & - 6(-1+3\theta)\log(-\theta)\log(1+\theta) + 18(1-3\theta)\Big] \, PolyLog[2, 1+\theta], \end{aligned}$$

where $PolyLog(n,z) = \sum_{k=1}^{\infty} z^k/k^n$.

Values of the entropy function for some values of θ are given in Table 2.

Table 2: CCE for FGM copula.

θ	-0.1	-0.2	-0.3	-0.4	-0.5	-0.6	-0.7	-0.8	-0.9
$I_{\bar{C}}$	0.24811	0.24615	0.24412	0.24200	0.23981	0.23753	0.23516	0.23268	0.23013

Remark 2.1. In general I_C and $I_{\bar{C}}$ are different for the random vector (X, Y). When (X, Y) is radially symmetric, that is for any (u, v) in I^2, the rectangles $[0, u] \times [0, v]$ and $[1-u, 1] \times [1-v, 1]$ have equal C-volume, then $C = \bar{C}$ and the entropy satisfies $I_C = I_{\bar{C}}$. Since the algebra involved in deriving various results in I_C is similar to those in $I_{\bar{C}}$, we restrict our subsequent discussions to the latter case.

3. PROPERTIES OF SURVIVAL COPULA ENTROPY

An important aspect in favour of SCE among other measures is that its values lies in a finite interval which makes it easier for comparison and interpretation.

Proposition 3.1. *The SCE satisfies*

$$\frac{1}{18} \leq I_{\bar{C}}(X, Y) \leq \frac{1}{3}. \tag{3.1}$$

Proof: It is well known that for every copula C and for all (u, v) in I^2,

$$W(u, v) \leq C(u, v) \leq M(u, v),$$

where $M(u, v) = \min(u, v)$ and $W = \max(u + v - 1, 0)$ are copula versions of the Fréchet-Hoeffding bounds of a bivariate distribution in $\mathbb{R}^2$. Let $\bar{M}$ and $\bar{W}$ be the survival copulas corresponding to M and W respectively. Then

$$\begin{aligned}
\bar{M}(u, v) &= u + v - 1 + \min(1-u, 1-v) \\
&= u + v - \max(u, v) \\
&= \min(u, v) = M.
\end{aligned}$$

Similarly $\bar{W} = W$. Thus $W(u, v) \leq \bar{C}(u, v) \leq M(u, v)$ so that

$$\begin{aligned}
I_{\bar{C}}(X, Y) &= -\int_0^1 \int_0^1 \bar{C}(u, v) \log \bar{C}(u, v) du dv \\
&\geq -\int_0^1 \int_0^1 \bar{M}(u, v) \log \bar{M}(u, v) du dv
\end{aligned}$$

or

$$\begin{aligned}
I_{\bar{C}}(X, Y) &\geq -\int_0^1 \int_0^1 \min(u, v) \log \min(u, v) du dv \\
&= -\int_0^1 \int_u^1 v \ \log u \ du dv - \int_0^1 \int_0^u v \ \log v \ du dv \\
&= \frac{1}{18}, \quad \text{using } \ 0 \log 0 = 0.
\end{aligned}$$

Also

$$
\begin{aligned}
I_{\bar{C}}(X,Y) &\leq -\int_0^1\int_0^1 \bar{W}(u,v)\log\bar{W}(u,v)dudv \\
&= \int_0^1\int_0^1 \max(u+v-1,0)\log\max(u+v-1,0)dudv \\
&= \int_0^1\left(u-\frac{1}{2}\right)\log udu - \int_0^1\int_{1-u}^1\left(\frac{v\left(u+\frac{v}{2}-1\right)}{u+v-1}dv\right)du \\
&= \int_0^1\left(u-\frac{1}{2}\right)\log udu - \int_0^1\int_{1-u}^1\left[\frac{v}{2}+\frac{3}{2}(1-u)+\frac{3}{2}\left(\frac{(1-u)^2}{u+v-1}\right)\right]dudv \\
&= \frac{1}{3}.
\end{aligned}
$$

□

Remark 3.1. It is not necessary that the SCE for all copulas attain the above bounds. For example, the Clayton copula contains M and W and hence their entropies lie in $\left[\frac{1}{18},\frac{1}{3}\right]$. At the same time the Gumbel-Barnett family does not include M and W and the bounds prescribed by M and W are not attained for this family. On the other hand $C_2 = \max(\theta uv + (1-\theta)(u+v-1), 0)$ contains W but not M, while $C_3 = \theta/\log\left(e^{\theta/u}+e^{\theta/v}-e^{-v}\right)$ has M as a member, but not W. Further C and $\bar{C}$ have the same entropy if and only if C is radially symmetric.

3.1. Ordering copulas via entropy

There are many situations where the data on the same random variable comes from different sources and the problem is to choose the more informative one for analysis. In such circumstances the entropies in each case has to be compared. The ordering of copulas comes handy in comparing the entropies. If C_1 and C_2 are two copulas, we say that C_1 is smaller (larger) than C_2 in concordance order, if $C_1(u,v) \leq (\geq) C_2(u,v)$ for all u,v in I, and is denoted by $C_1 \prec (\succ) C_2$. Note that $C_1 \prec (\succ) C_2 \Leftrightarrow \bar{C}_1 \prec (\succ) \bar{C}_2$. The following proposition is immediate.

Proposition 3.2.

$$\bar{C}_1 \prec (\succ) \bar{C}_2 \implies I_{\bar{C}_1} \leq (\geq) I_{\bar{C}_2}. \tag{3.2}$$

With some additional assumptions the converse of (3.2) is also true.

Proposition 3.3. *Let $\mathcal{A}$ denote the class of copulas that are concordance ordered, that is, for elements $\bar{C}_1$ and $\bar{C}_2 \in \mathcal{A}$, we have either $\bar{C}_1(u,v) \leq \bar{C}_2(u,v)$ or $\bar{C}_1(u,v) \geq \bar{C}_2(u,v)$. Then*

$$I_{\bar{C}_1} \leq (\geq) I_{\bar{C}_2} \implies \bar{C}_1(u,v) \leq (\geq) \bar{C}_2(u,v) \text{ for all } u,v \text{ in } I. \tag{3.3}$$

Proof: To prove the above implication assume that $I_{\bar{C}_1} \geq I_{\bar{C}_2}$ which is equivalent to

$$\int_0^1 \int_0^1 \bar{C}_1(u,v) \log \bar{C}_1(u,v) dudv \geq \int_0^1 \int_0^1 \bar{C}_2(u,v) \log \bar{C}_2(u,v) dudv. \tag{3.4}$$

Since C_1 and C_2 are ordered, if $C_1 \leq C_2$ then $ref3.3$ is violated and hence $C_1 \geq C_2$. □

As an example, from the values of $I_{\bar{C}}$ given in Tables 1 and 2, it is seen that entropies are decreasing. It is well known that the corresponding copulas are also decreasing functions of θ in their assumed ranges.

3.2. Survival copula entropy and dependence

An important use of SCE is assessing the nature of dependence between X and Y, thus making a connection between entropy and dependence. The above discussion on concordances ordering and entropy have significant implications in ascertaining the mode of dependence and the SCE. An important and perhaps mostly used dependence concept is positive (negative) quadrant dependence PQD (NQD). Recall that a copula C is PQD (NQD) if $\bar{C}(u,v) \geq (\leq) uv$, for u, v in I.

Proposition 3.4. *The vector (X, Y) is PQD (NQD), then*

$$I_{\bar{C}} \geq (\leq) \frac{1}{4}. \tag{3.5}$$

Proof:

$$(X,Y) \text{ is PQD (NQD)} \implies C(u,v) \geq (\leq) \prod(u,v) = uv.$$

From

$$\int_0^1 \int_0^1 C(u,v) \log C(u,v) dudv \geq (\leq) \int_0^1 \int_0^1 uv(\log u + \log v) dudv = \frac{1}{4},$$

the result follows. □

Remark 3.2. In view of (3.1), PQD (NQD) random vectors are sought the interval $[\frac{1}{4}, \frac{1}{3}]$ and $[\frac{1}{18}, \frac{1}{4}]$ respectively. Further, if the random variables X and Y are independent then $I_{\bar{C}} = \frac{1}{4}$. The next proposition gives a criterion to check whether which of two random variables are more positively dependent.

Proposition 3.5. *For copulas $\bar{C}_1, \bar{C}_2 \in \mathcal{A}$, $\bar{C}_1$ is more PQD than $\bar{C}_2$ if and only if $I_{\bar{C}_1} \geq I_{\bar{C}_2}$.*

Proof: We say that $\bar{C}_1$ is more PQD than $\bar{C}_2$ if $\bar{C}_1 \succ \bar{C}_2$. Thus

$$\bar{C}_1 \text{ is more PQD than } \bar{C}_2 \Leftrightarrow C_1(u,v) \geq C_2(u,v) \text{ for all } u, v$$

$$\Leftrightarrow I_{\bar{C}_1} \geq I_{\bar{C}_2},$$

by (3.2) and Remark 3.2. □

We can also compare members of a specified family of copulas $\bar{C}_\theta(u,v)$ indexed by a parameter $\theta \in \Theta$. The family $\{\bar{C}_\theta\}, \theta \in \Theta$ is positively (negatively) ordered whenever $\bar{C}_{\theta_1} \prec (\succ)\ \bar{C}_{\theta_2}$ for $\theta_1, \theta_2 \in \Theta, \theta_1 \leq (\geq)\theta_2$. In this case we have the next proposition that gives a criterion to distinguish between more positive dependent among families of copulas.

Proposition 3.6. *Let $\{\bar{C}_\theta\}$ be positively (negatively) ordered. Then $I_{\bar{C}_{\theta_1}} \leq (\geq) I_{\bar{C}_{\theta_2}}$, for all $\theta_1, \theta_2 \in \Theta, \theta_1 \leq (\geq)\theta_2$.*

Example 3.1. The Gumbel-Barnett family in Example 2.1 is negatively ordered as can be verified from Table 1 and FGM copula in Table 2 is positively ordered.

Remark 3.3. For many standard copula families, it is algebraically difficult to establish whether it is positively or negatively ordered. Proposition 3.4 gives a relative simple alternative tool to resolve this problem.

Remark 3.4. The relationship CCE has with well known measures of dependence is also worth examination. The measures in common use are the

Kendall's tau, $\tau = 4\int\int_{I^2} C(u,v)\frac{\partial^2 C}{\partial u \partial v} dudv - 1,$

Spearman's rho, $\rho = 12\int\int_{I^2} C(u,v)dudv - 3,$

Blomqvist's beta, $\beta = 4C\left(\frac{1}{2}, \frac{1}{2}\right) - 1,$

Gini coefficient, $\xi = 4\left[\int\int_{I^2} C(u, 1-u)du - \int\int_{I^2} (u - C(u,u))du\right],$ and

the product moment correlation coefficient

$$r = [D(X)D(Y)]^{-1}\int\int_{I^2}[C(u,v) - uv]dF^{-1}(u)dG^{-1}(u)$$

where $D(X), D(Y)$ are standard deviations of X and Y, and F and G are their marginal distribution functions. It is known that (Nelson, 2006) when X and Y are PQD:

(i) $3\tau \geq \rho \geq 0, \quad \beta \geq 0$ and $\xi \geq 0$.

(ii) For non-decreasing function $p(x)$ and $q(y)$ whose expectations are finite and

$$E(p(X)q(Y)) < \infty, \qquad \mathrm{Cov}(p(X), q(Y)) \geq 0$$

and conversely, implying that $\mathrm{Cov}(X,Y)$ and hence $r \geq 0$. Combining these with our earlier propositions we find that when the cumulative copula $I_{\bar{C}}$ lies in the interval $[\frac{1}{4}, \frac{1}{3}]$, we have positive dependence in the sense of PQD as well as τ, ρ, β, ξ and r. In general it is difficult to find expressions that connects $\bar{C}$ with the various coefficients τ, ρ, etc, but one can obtain formulas in respect of specific copulas. For example, in the Gumbel-Barnett copula, the Spearman's coefficient is $\rho = 12\Big[-\frac{e^{-\frac{4}{\theta}}}{\theta}\ EI\big(-\frac{4}{\theta}\big)\Big] - 3$ so that from Example 2.1,

$$\rho = 12\left(\frac{I_{\bar{C}}}{\theta}\right) - 3.$$

In this copula, $I_{\bar{C}}$ is a decreasing function of θ and the maximum of $I_{\bar{C}}$ occurs at $\theta = 0$, the case of independence in which case $I_{\bar{C}} = 0.25$. Hence for this copula $\rho \leq 0$ so that there is negative dependence for all $0 \leq \theta \leq 1$. By virtue of Propositions 3.2 and 3.4, we also conclude that as the $I_{\bar{C}}$ value decreases from 0.25, so does the extent of negative dependence between X and Y.

3.3. Effect of transformations

There are occasions where transformations have to be applied to the baseline random variables to facilitate easier analysis. It is of interest to know how the SCE is affected by such transformations.

Proposition 3.7. *Let $T(X)$ and $W(Y)$ be strictly monotone transformations of X and Y. Let the corresponding SCE's be $I(T(X), W(Y))$ and $I(X, Y)$ respectively. Then*

(i) *$I(T(X), W(Y)) = I(X, Y)$ when $T(X)$ and $W(Y)$ are both strictly increasing,*

(ii) *$I(T(X), W(Y)) = I(X, Y)$ when $T(X)$ and $W(Y)$ are both strictly decreasing and I is the cumulative copula entropy of (X, Y) and*

(iii) $$I(T(X), W(Y)) = \begin{cases} \int_0^1 \int_0^1 \big[v - \bar{C}(u,v)\big] \log\big[v - \bar{C}(u,v)\big] dudv, \\ \int_0^1 \int_0^1 \big[u - \bar{C}(u,v)\big] \log\big[u - \bar{C}(u,v)\big] dudv, \end{cases}$$

where $T(X)$ is strictly increasing (decreasing) and $W(Y)$ is strictly decreasing (increasing).

Proof: Proceeding as in Theorems 2.4.3 and 2.4.4 in (Nelson, 2006, p. 25, 26) we find that

$$\bar{C}_{T(X),W(Y)}(u,v) = \bar{C}_{X,Y}(u,v)$$

when both $T(X)$ and $T(Y)$ are increasing,

$$\bar{C}_{T(X),W(Y)}(u,v) = v - \bar{C}_{X,Y}(1-u,v)$$

when $T(X)$ is increasing and $W(Y)$ is decreasing,

$$\bar{C}_{T(X),W(Y)}(u,v) = u - \bar{C}_{X,Y}(u,1-v)$$

when $T(X)$ is decreasing and $W(Y)$ is increasing, and

$$\bar{C}_{T(X),W(Y)}(u,v) = C_{X,Y}(u,v)$$

when $T(X)$ and $W(Y)$ are strictly decreasing. Then first and last results directly establish (i) and (ii) of the Proposition and (iii) is obtained from the 2nd and 3rd results with a transformation of $(1-u)$ $(1-v)$ to u (v). It may be noted that entropy of $(T(X), W(Y))$ is independent of the form of the two functions. □

Example 3.2. The linear transformations $T(x) = \alpha x + \beta$ and $W(y) = \gamma y + \phi$ are common in data analysis. When $\alpha > 0$, $\gamma > 0$, in this case

$$\bar{C}(u,v) = u\ v\ \exp[-\theta \log u \log v], \qquad 0 \le \theta \le 1,$$

and

$$I_{\bar{C}}(\alpha X + \beta, \gamma Y + \phi) = -e^{\frac{4}{\theta}} EI\left(-\frac{4}{\theta}\right) = I_{\bar{C}}(X,Y), \qquad \text{for} \quad \alpha, \beta, \gamma, \phi > 0.$$

4. APPLICATIONS

In this section we demonstrate how the results obtained in the previous sections can be implemented in a practical problem. The example considered relates to an investigation on 20 individuals for isolated aortic regurgitations before and after surgery and 20 persons for isolated mitral regurgitation analysed in Kumar and Shoukri (2007). Data on pre-operative ejection fraction (X) and post-operative ejection fraction (Y) arranged in order of magnitude are

$$X: \ 0.29, 0.36, 0.39, 0.41, 0.50, 0.53, 0.54, 0.55, 0.56, 0.56, 0.56, 0.58, 0.60, 0.60, 0.62, 0.64, 0.64, 0.67, 0.80, 0.87,$$

$$Y: \ 0.17, 0.24, 0.26, 0.26, 0.27, 0.29, 0.30, 0.32, 0.33, 0.33, 0.34, 0.38, 0.47, 0.47, 0.50, 0.56, 0.58, 0.59, 0.62, 0.63.$$

The first step in the analysis is the estimation of the SCE. We consider the empirical survival copula for a random sample $(x_1, y_1), (x_2, y_2), ..., (x_n, y_n)$ from a continuous bivariate distribution given by

$$\bar{C}\left(\frac{i}{n}, \frac{j}{n}\right) = \frac{\left(\text{number of pairs in the sample with } x > x_{(i)}, y > y_{(j)}\right)}{n},$$

where $x_{(i)}\left(y_{(j)}\right)$ is the i-th (j-th) order statistic of the observations on $X(Y)$. Using $\bar{C}\left(\frac{i}{n}, \frac{j}{n}\right)$ as the estimator of $\bar{C}(u,v)$, the resubstitution estimator of $I_{\bar{C}}$ is

$$\hat{I}_{\bar{C}} = \frac{2}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \bar{C}\left(\frac{i}{n}, \frac{j}{n}\right) \log \bar{C}\left(\frac{i}{n}, \frac{j}{n}\right),$$

at those points for which $\frac{i}{n} + \frac{j}{n} - 1 > 0$. The estimated value for the given sample is $\hat{I}_{\bar{C}} = 0.3049$. Kumar and Shoukri (2007) in their analysis, found that the Pearson correlation coefficient $r = 0.6870$, Kendall's rank correlation $\tau = 0.5050$ and Spearman's rank correlation $\rho = 0.6970$. Thus all the measures indicate positive dependence between the two sets of observations. Using Proposition 3.4, our nonparametric estimate also support this conclusion irrespective of the copula model, since $\hat{I}_{\bar{C}} > \frac{1}{4}$. Further, the data gives satisfactory evidence for the PQD property of the underlying copula.

5. CONCLUSION

In this work we have proposed a measure of uncertainty based on survival copula and examined some of its properties. Apart from being useful to evaluate uncertainty it can be of application in assessing copula properties like independence and their ordering. The proposed concept is more general than distribution-based counterparts and has some advantages over them and the existing copula entropy.

DATA AVAILABILITY STATEMENTS

Authors can confirm that all relevant data are included in the article.

CONFLICT OF INTEREST STATEMENT

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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