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Statistical Inference for the Inverse Lindley Distribution Based on Lower Record Values

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Abstract:

- In this paper, we discuss the problem of classical and Bayesian estimation of the parameter of the inverse Lindley distribution based on lower records, as well as the prediction of a future record value. We obtain the maximum likelihood estimator, the approximate confidence interval, as well as two bootstrap-type confidence intervals for the parameter based on the inverse Lindley distribution records. In the context of Bayesian estimation, we use the Tierney and Kadane's method and two Markov chain Monte Carlo approaches. The future record values are also explored using the maximum likelihood and Bayesian approaches. The highest conditional density, as well as Bayesian intervals, are also constructed for a future lower record. A simulation study and a real data example are also given for the sake of comparison and illustration.

Keywords:

- *Bayesian estimation and prediction; general entropy loss function; inverse Lindley distribution; lower record values; maximum likelihood estimation.*

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- 62F10, 62C10.

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1. INTRODUCTION

In recent years, the inverse Lindley distribution (ILD) has attracted the attention of several authors. It was first introduced by [28] and its stress-strength reliability was explored under classical and Bayesian models. In addition, its application to head and neck cancer data was demonstrated. Let Y have a Lindley distribution with parameter θ, and define $X = \frac{1}{Y}$, then X has an inverse Lindley distribution with parameter θ $\big($notationally $X \sim ILD(\theta)\big)$ and the probability density function (PDF) of X is obtained to be

$$f(x;\theta) = \frac{\theta^2}{1+\theta}\left(\frac{x+1}{x^3}\right)\mathrm{e}^{-\frac{\theta}{x}}, \quad x>0, \quad \theta>0. \tag{1.1}$$

The corresponding cumulative distribution function (CDF) is given by

$$F(x;\theta) = \left(1+\frac{\theta}{(1+\theta)x}\right)\mathrm{e}^{-\frac{\theta}{x}}, \quad x>0, \quad \theta>0. \tag{1.2}$$

The PDF (1.1), is a mixture of the inverse exponential distribution with parameter θ and the inverse gamma distribution with shape parameter 2 and scale parameter θ, namely (1.1) can be written as

$$f(x;\theta) = pf_1(x) + (1-p)f_2(x),$$

where

$$f_1(x) = \theta x^{-2}\mathrm{e}^{-\frac{\theta}{x}}, \quad f_2(x) = \theta^2 x^{-3}\mathrm{e}^{-\frac{\theta}{x}}, \quad \text{and} \quad p = \frac{\theta}{1+\theta}.$$

[7] and [8] discussed the problem of estimation of the parameter of the ILD under Type-I censored data and hybrid censored data, respectively. Many generalizations of the ILD have been introduced. For example, [5] studied the generalized inverse Lindley distribution and presented an application to Danish fire insurance data.

Record values are of great significance in many real-life situations such as in industry, weather, and life-testing events. Record values and their basic properties have been discussed by [12], [27], [25], and [3] among others. Recently, [4] and [17] worked on the inferential problems for the Lindley distribution, and [29] focused on the inference for the generalized Lindley distribution based on record data. Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed (iid) random variables with CDF $F(x;\theta)$ and PDF $f(x;\theta)$. Then the observation X_j is a lower record value if it is smaller than all its preceding observations, namely $X_j < X_i$, $\forall i < j$. In other words, let $L(1) = 1$ and $L(m) = \min\{j | j > L(m-1),\ X_j < X_{L(m-1)}\}$ for $m > 1$. Then $X_{L(m)}$ is the m-th lower record value, and the sequence $\{L(m), m \geq 1\}$ represents the record times. The PDF of $X_{L(m)}$ for $m \geq 1$ is given by (see e.g. [3])

$$f_{X_{L(m)}}(x) = \frac{1}{(m-1)!}\big[-\ln\big(F(x;\theta)\big)\big]^{m-1} f(x;\theta), \quad x>0, \quad m\geq 1.$$

The joint PDF of $X_{L(m)}$ and $X_{L(n)}$, for $1 \leq m < n$ and $x < y$, is

$$f_{X_{L(m)},X_{L(n)}}(x,y;\theta) = \frac{1}{(m-1)!(n-m-1)!}\big[-\ln\big(F(x;\theta)\big)\big]^{m-1} \times\big[\ln\big(F(x;\theta)\big)-\ln\big(F(y;\theta)\big)\big]^{n-m-1}\frac{f(x;\theta)}{F(x;\theta)}f(y;\theta).$$

In addition, suppose that $\mathbf{x}=(x_1,\dots,x_m)$ is the observed vector of $\big(X_{L(1)},\dots,X_{L(m)}\big)$, then the likelihood function of θ given the m lower records can be expressed as

$$L(\theta|\boldsymbol{x}) = f(x_m;\theta)\prod_{i=1}^{m-1}\frac{f(x_i;\theta)}{F(x_i;\theta)}, \quad x>0, \quad m\geq 1. \tag{1.3}$$

So, for the ILD, the PDF of m-th lower record is given by

$$f_{X_{L(m)}}(x) = \frac{1}{\Gamma(m)}\left[-\ln\left(1+\frac{\theta}{(1+\theta)x}\right)+\frac{\theta}{x}\right]^{m-1}\frac{\theta^2}{1+\theta}\left(\frac{1+x}{x^3}\right)\mathrm{e}^{-\frac{\theta}{x}}, \quad x>0, \quad \theta>0. \tag{1.4}$$

The main aim of this article is to present both frequentist and Bayesian methodology to estimate the parameter of the ILD based on lower records and to predict a future record based on past observed record values. The rest of the paper is organized as follows: In Section 2, we use the maximum likelihood (ML) method as a frequentist methodology to obtain a point estimator of the parameter. Besides, the asymptotic confidence interval (CI) as well as two different bootstrap-type CIs are obtained. We also consider the problem of Bayesian estimation of the unknown parameter in this section. In Section 3, the problem of predicting a future record value is discussed based on using both classical and Bayesian procedures. In Section 4, a Monte Carlo simulation study is conducted to evaluate the performances of the proposed estimators in the sense of estimated bias and their associated estimated risks. In Section 5, the applicability of the paper results, is shown using an application to real data. Finally, the paper ends with some conclusions in Section 6.

2. PARAMETER ESTIMATION

In this section, we use both classical and Bayesian methods of estimation to evaluate the parameter of the inverse Lindley distribution based on lower records.

2.1. Maximum likelihood estimation

In this subsection, we discuss the process of obtaining the ML estimator of parameter θ based on lower record values for $ILD(\theta)$. Suppose that

$X_{L(1)}, \ldots, X_{L(m)}$ are the first m record statistics arising from a sequence of iid random variables from $ILD(\theta)$ with PDF (1.1) and $\boldsymbol{x} = (x_1, x_2, \ldots, x_m)$ is the observed vector of $(X_{L(1)}, \ldots, X_{L(m)})$. The likelihood function of the parameter given $\boldsymbol{x}$ is as follows

$$L(\theta|\boldsymbol{x}) = \frac{\theta^{2m} e^{-\frac{\theta}{x_m}}}{x_m(1+\theta)} \prod_{i=1}^{m} \frac{1+x_i}{x_i^2} \prod_{i=1}^{m-1} \frac{1}{\theta(1+x_i)+x_i}.$$

Hence, the log-likelihood function is

$$l(\theta) = \ln L(\theta|\boldsymbol{x}) = 2m \ln\theta - \ln(1+\theta) - \frac{\theta}{x_m} - \sum_{i=1}^{m-1} \ln\big(\theta(1+x_i)+x_i\big) + A(\boldsymbol{x}), \tag{2.1}$$

where $A(\boldsymbol{x}) = \sum_{i=1}^{m} \ln(1+x_i) - \ln x_m - 2\sum_{i=1}^{m} \ln x_i$.

The ML estimate of θ can be obtained by maximizing (2.1) with respect to θ. Upon differentiating (2.1) with respect to θ and equating it with zero, we have

$$\frac{\partial l(\theta)}{\partial\theta} = \frac{2m}{\theta} - \frac{1}{1+\theta} - \frac{1}{x_m} - \sum_{i=1}^{m-1} \frac{1+x_i}{\theta(1+x_i)+x_i} = 0.$$

It can be shown that the solution of (2.1) can be obtained as a fixed point solution of $h(\theta) = \theta$ where

$$h(\theta) = 2m\left(\frac{1}{1+\theta} + \frac{1}{x_m} + \sum_{i=1}^{m-1} \frac{1+x_i}{\theta(1+x_i)+x_i}\right)^{-1}.$$

Next, we show the uniqueness and existence of the ML estimate of θ. To this end, let $v_1(\theta) = h(\theta)$ and $v_2(\theta) = \theta$. It can be easily verified that $v_1(\theta)$ is an increasing function with

$$v_1(0) = 2m\Big(\sum_{i=1}^{m} \frac{1}{x_i} + m\Big)^{-1}, \quad v_1(\infty) = 2m\, x_m.$$

So $v_1(\theta)$ starts from a positive real value at 0 and increases to $2m\, x_m$, which is a finite value. For large θ, $v_1(\theta)$ is a finite value whereas $v_2(\theta) \to \infty$ as θ goes to ∞. This implies that there exists one real positive root, say $\hat{\theta}$, such that $h(\hat{\theta}) = \hat{\theta}$.

2.2. Asymptotic confidence interval

It seems that the ML estimate of θ does not possess an explicit form, and therefore it is not easy to obtain the variance of $\widehat{\theta}$, where $\widehat{\theta}$ denotes the ML estimator (MLE) of θ. Consequently, we cannot get the exact distribution of the MLE and the exact bounds for the parameter. The intent is to use the large-sample approximation. The asymptotic distribution of $\widehat{\theta}$ is ([21])

$$(\widehat{\theta} - \theta) \xrightarrow{D} N(0, I^{-1}_{X_{L(1)},\ldots,X_{L(m)}}(\theta)),$$

where $I^{-1}_{X_{L(1)},\ldots,X_{L(m)}}(\theta)$ is the inverse of the Fisher information of the first m lower records about the unknown parameter θ and $\xrightarrow{D}$ stands for convergence in distribution. Since θ is unknown, we estimate the asymptotic variance of $\widehat{\theta}$ based on the inverse of the observed Fisher information of the first m lower records, in other words, we have

$$\widehat{\mathrm{Var}}(\widehat{\theta}) = \left(\tilde{I}_{X_{L(1)},\cdots,X_{L(m)}}(\widehat{\theta})\right)^{-1},$$

where

$$\tilde{I}_{X_{L(1)},\cdots,X_{L(m)}}(\widehat{\theta}) = \frac{2m}{\widehat{\theta}^2} - \frac{1}{(1+\widehat{\theta})^2} - \sum_{i=1}^{m-1}\left(\frac{1+X_{L(i)}}{\widehat{\theta}(1+X_{L(i)})+X_{L(i)}}\right)^2.$$

Using the above element, one can derive the approximate $100(1-\alpha)\%$ CI of the parameter θ as follows

$$\widehat{\theta} \pm z_{\frac{\alpha}{2}}\sqrt{\widehat{\mathrm{Var}}(\widehat{\theta})},$$

where $z_{\frac{\alpha}{2}}$ is the upper $\frac{\alpha}{2}$ quantile of the standard normal distribution.

2.3. Bootstrap confidence interval

In this subsection, two different bootstrap confidence intervals are proposed. The first one is the bootstrap percentile ($Boot-P$) CI and the second one is the basic bootstrap ($Boot-B$)CI [13, 16]. The following algorithm is used to generate parametric bootstrap samples.

Algorithm 1

Step 1: Compute the ML estimate of θ, denoted by $\hat{\theta}$, based on the observed lower records.

Step 2: Generate the bootstrap lower record sample $X^*_{L(1)},\ldots,X^*_{L(m)}$, from $ILD(\hat{\theta})$.

Step 3: Compute the ML estimate of θ based on the generated bootstrap sample in Step 2, denoted by $\hat{\theta}^*_1$.

Step 4: Repeat Steps 2 and 3, B times, and store $\hat{\theta}^*_i$ for $i=1,\ldots,B$, say $\{\hat{\theta}^*_1,\cdots,\hat{\theta}^*_B\}$.

i) $Boot-P$ method

Arrange $\hat{\theta}^*_i$'s in an ascending order and let θ^*_i be the i-th ordered member of $\{\hat{\theta}^*_1,\cdots,\hat{\theta}^*_B\}$, then the $100(1-\gamma)\%$ bootstrap percentile CI for θ is given by

$$\left(\theta^*_{(B+1)\frac{\gamma}{2}}, \theta^*_{(B+1)(1-\frac{\gamma}{2})}\right).$$

ii) *Boot* $-$ *B* method
The $100(1-\gamma)\%$ basic bootstrap CI for θ is given by

$$\left(2\hat{\theta}-\theta^*_{(B+1)(1-\frac{\gamma}{2})}, 2\hat{\theta}-\theta^*_{(B+1)(\frac{\gamma}{2})}\right).$$

2.4. Bayesian estimation

In this subsection, we work on Bayesian estimation of the unknown parameter θ in the ILD, based on lower record values. It should be noted that all the relations given in this subsection hold for the general case of one-dimensional parameter θ. In the context of Bayes estimation, the parameter is assumed to be a random variable with a prior distribution, $\pi(\theta)$. Let $\boldsymbol{X}$ denote the informative sample and $L\big(\theta,\delta(\boldsymbol{X})\big)$ denote the loss function, where $\delta(\boldsymbol{X})$ is an estimator of θ. The Bayes estimator of θ is derived through minimizing the posterior risk $E[L(\theta,\delta(\boldsymbol{X}))|\boldsymbol{X}]$ with respect to δ. In the literature, the squared error (SE) loss function is one of the common loss functions that has been frequently used for estimation problems, which is defined as $L\big(\theta,\delta(\boldsymbol{X})\big)=\big(\delta(\boldsymbol{X})-\theta\big)^2$. The Bayes estimator of θ is given by $\delta_{SE}(\boldsymbol{X})=E(\theta|\boldsymbol{X})$ under the SE loss function, provided that the mentioned expectation exists and is finite. The SE loss function, as a symmetric function, allocates equivalent losses to the overestimation and underestimation. However, in some practical situations, overestimation and underestimation are not of the same importance, and the use of symmetric loss functions seems inappropriate. [32] proposed an asymmetric loss function, called the linear-exponential (LE or linex) loss function, which is defined as

$$L\big(\theta,\delta(\boldsymbol{X})\big)=b^*\left[\mathrm{e}^{c(\delta(\boldsymbol{X})-\theta)}-c(\delta(\boldsymbol{X})-\theta)-1\right],\quad c\neq 0,\quad b^*>0,$$

where b^* and c are the parameters of the function. Without loss of generality, we can assume $b^*=1$ whereas c has to be determined carefully. Positive values of c are considered when the overestimation is more serious than underestimation, while the negative values are considered when the underestimation is more serious than overestimation (see e.g. [34]). The Bayes estimator of θ under the LE loss function is given by

$$\delta_{LE}(\boldsymbol{X})=\frac{-1}{c}\ln E(\mathrm{e}^{-c\theta}|\boldsymbol{X}),\quad c\neq 0.$$

provided that the above expectation exists and is finite.

Another asymmetric loss function, proposed by [10], is the general entropy (GE) loss function, which is defined as

$$L\big(\theta,\delta(\boldsymbol{X})\big)=w\left[\left(\frac{\delta(\boldsymbol{X})}{\theta}\right)^p-p\ln\left(\frac{\delta(\boldsymbol{X})}{\theta}\right)-1\right],\quad p\neq 0,\quad w>0.$$

Without loss of generality, we assume $w=1$. The Bayes estimator of θ under the GE loss function is given by

$$\delta_{GE}(\boldsymbol{X})=[E(\theta^{-p}|\boldsymbol{X})]^{-\frac{1}{p}},\quad p\neq 0,$$

provided that the above expectation exists and is finite. Now, assume that θ has a gamma prior distribution with the following PDF

$$\pi(\theta) = \frac{b^a}{\Gamma(a)}\theta^{a-1}\mathrm{e}^{-b\theta}; \qquad a > 0, b > 0, \theta > 0. \tag{2.2}$$

From the likelihood function (2.1) and the prior distribution (2.2), the posterior density function can be obtained to be

$$\begin{aligned} \pi(\theta|\boldsymbol{x}) &= \frac{L(\theta|\boldsymbol{x})\pi(\theta)}{\int_0^\infty L(\theta|\boldsymbol{x})\pi(\theta)\mathrm{d}\theta} \\ &= m(\boldsymbol{x})\theta^{2m+a-1}\mathrm{e}^{-\theta(\frac{1}{x_m}+b)}\Big\{(1+\theta)\prod_{i=1}^{m-1}\big(\theta(1+x_i)+x_i\big)\Big\}^{-1}, \end{aligned} \tag{2.3}$$

where

$$m(\boldsymbol{x}) = \frac{1}{\int_0^\infty \theta^{2m+a-1}\mathrm{e}^{-\theta(\frac{1}{x_m}+b)}\Big\{(1+\theta)\prod_{i=1}^{m-1}\big(\theta(1+x_i)+x_i\big)\Big\}^{-1}\mathrm{d}\theta}.$$

2.5. Tierney and Kadane's approximation

This subsection presents the approximate Bayes estimates of θ under the SE, LE, and GE loss functions using the Tierney and Kadane's (TK) approximation method. [31] used Laplace's formula to approximate posterior moments. To apply the TK approximation method, suppose that $F(\theta) = \frac{1}{m}\ln\pi(\theta) + \frac{1}{m}l(\theta)$ and $F^*(\theta) = F(\theta) + \frac{1}{m}\ln g(\theta)$ where $l(\theta)$ is the log-likelihood function of θ, $\pi(\theta)$ is the prior density, and $g(\theta)$ should be a smooth positive function on the parameter space. We know that posterior moment of $g(\theta)$ is

$$E(g(\theta)|\boldsymbol{x}) = \int_0^\infty g(\theta)\cdot\pi(\theta|\boldsymbol{x})\mathrm{d}\theta. \tag{2.4}$$

The expression (2.4) can be rewritten as

$$E(g(\theta)|\boldsymbol{x}) = \frac{\int_0^\infty \mathrm{e}^{mF^*(\theta)}\mathrm{d}\theta}{\int_0^\infty \mathrm{e}^{mF(\theta)}\mathrm{d}\theta}. \tag{2.5}$$

Using the TK method, the approximate form of (2.5) becomes

$$\widehat{E}(g(\theta)|\boldsymbol{x}) = \big(\frac{\sigma^*}{\sigma}\big)\exp\Big(m\big(F^*(\tilde{\theta}^*) - F(\tilde{\theta})\big)\Big),$$

where $\tilde{\theta}$ and $\tilde{\theta}^*$ are the modes of $F(\theta)$ and $F^*(\theta)$, respectively and

$$\sigma^2 = -\frac{1}{F''(\theta)|_{\theta=\tilde{\theta}}} \quad \text{and} \quad {\sigma^*}^2 = -\frac{1}{{F^*}''(\theta)|_{\theta=\tilde{\theta}^*}},$$

where $F''(\cdot)$ and $F^{*''}(\cdot)$ denote the second order derivatives of $F(\theta)$ and $F^*(\theta)$, respectively.

Now, let

$$G(\theta,k_1,k_2)=\frac{1}{m}\Big[(2m+a-1+k_1)\ln(\theta)-\ln(1+\theta)-\theta\big(b+k_2+\frac{1}{x_m}\big)-\sum_{i=1}^{m-1}\ln\big(\theta(1+x_i)+x_i\big)+B(\boldsymbol{x})\Big],$$

where $B(\boldsymbol{x})=\sum_{i=1}^{m}\ln(1+x_i)-\ln x_m-2\sum_{i=1}^{m}\ln x_i+a\ln b-\ln\Gamma(a)$ and k_1 and k_2 are real numbers. Then, $F(\theta)=G(\theta,0,0)$ and $F^*(\theta)=G(\theta,k_1^*,k_2^*)$, where

$$(2.6)\qquad k_1^*=\begin{cases}1 & \text{under } SE,\\ 0 & \text{under } LE,\\ -p & \text{under } GE,\end{cases}\qquad \text{and}\qquad k_2^*=\begin{cases}0 & \text{under } SE \text{ and } GE,\\ c & \text{under } LE.\end{cases}$$

Let $G^*(\theta,k_1,k_2)=\dfrac{\partial G(\theta,k_1,k_2)}{\partial\theta}$. Then, we have

$$G^*(\theta,k_1,k_2)=\frac{1}{m}\Big[\frac{2m+a-1+k_1}{\theta}-\frac{1}{1+\theta}-\big(b+k_2+\frac{1}{x_m}\big)-\sum_{i=1}^{m-1}\frac{1+x_i}{\theta(1+x_i)+x_i}\Big].$$

Note that $\dfrac{\partial^2G(\theta,k_1,k_2)}{\partial\theta^2}$ is free of k_2, so we let $G^{**}(\theta,k_1)=\dfrac{\partial^2G(\theta,k_1,k_2)}{\partial\theta^2}$ and we have

$$G^{**}(\theta,k_1)=\frac{1}{m}\Big[-\frac{2m+a-1+k_1}{\theta^2}+\frac{1}{(1+\theta)^2}+\sum_{i=1}^{m-1}\Big(\frac{1+x_i}{\theta(1+x_i)+x_i}\Big)^2\Big].$$

Let $F'(\cdot)$ and $F^{*'}(\cdot)$ denote the first order derivatives of $F(\theta)$ and $F^*(\theta)$, respectively. Then, $F'(\theta)=G^*(\theta,0,0)$ and $F^{*'}(\theta)=G^*(\theta,k_1^*,k_2^*)$, where k_1^* and k_2^* are given in (2.6).

Moreover, $F''(\theta)=G^{**}(\theta,0)$ and $F^{*''}(\theta)=G^{**}(\theta,k_1^*)$. Consequently, we get

$$\widehat{E}(g(\theta)|\boldsymbol{x})=\sqrt{\frac{F''(\theta)|_{\theta=\tilde\theta}}{F^{*''}(\theta)|_{\theta=\tilde\theta^*}}}\exp\Big(m\big(F^*(\tilde\theta^*)-F(\tilde\theta)\big)\Big),$$

where $\tilde\theta$ and $\tilde\theta^*$ can be derived from $F^{'}(\theta)=0$ and $F^{*'}(\theta)=0$, respectively, and

$$g(\theta)=\begin{cases}\theta & \text{under } SE,\\ \exp(-c\theta) & \text{under } LE,\\ \theta^{-p} & \text{under } GE.\end{cases}$$

Therefore, the approximate Bayes estimates of θ under the SE, LE and GE loss functions are given by

$$\begin{aligned}\widetilde\theta_{SE}&=\widehat{E}\big(g_{SE}(\theta)|\boldsymbol{x}\big)\\ \widetilde\theta_{LE}&=-\frac{1}{c}\ln\big[\widehat{E}\big(g_{LE}(\theta)|\boldsymbol{x}\big)\big],\quad c\neq0\\ \widetilde\theta_{GE}&=\big[\widehat{E}\big(g_{GE}(\theta)|\boldsymbol{x}\big)\big]^{-\frac{1}{p}},\quad p\neq0,\end{aligned}$$

respectively, where $g_{SE}(\theta)=\theta$, $g_{LE}(\theta)=\mathrm{e}^{-c\theta}$, and $g_{GE}(\theta)=\theta^{-p}$.

2.6. *MCMC* methods

In this subsection, we consider two Markov chain Monte Carlo ($MCMC$) methods to generate samples from the posterior distribution and then compute the approximate Bayes estimates of the parameter θ under the SE, LE, and GE loss functions. Two important subclasses of $MCMC$ methods, which are considered here, are importance sampling (IS) and Metropolis-Hastings (MH) methods, see [24] and [20] for the details of the MH algorithm.

To implement the IS procedure, we rewrite the posterior density function (2.3) as follows

$$\pi(\theta|\boldsymbol{x}) = C(\boldsymbol{x})\,\text{gamma}\big(\theta; 2m+a, \frac{1}{x_m}+b\big)h(\theta),$$

where $\text{gamma}(\theta; 2m+a, \frac{1}{x_m}+b)$ is the density of the gamma distribution with shape and rate parameters $2m+a$ and $\frac{1}{x_m}+b$, respectively, $C(\boldsymbol{x}) = \frac{m(\boldsymbol{x})\Gamma(2m+a)}{\left(x_m^{-1}+b\right)^{2m+a}}$ and

$$h(\theta) = \left\{(1+\theta)\prod_{i=1}^{m-1}\big(\theta(1+x_i)+x_i\big)\right\}^{-1}.$$

Now, let $G(\theta|\boldsymbol{x}) = \text{gamma}(\theta; 2m+a, \frac{1}{x_m}+b)h(\theta)$. Then the Bayes estimate of θ under the SE loss function is given by

$$\widehat{\theta}_{SE} = \frac{\int_0^\infty \theta G(\theta|\boldsymbol{x})\mathrm{d}\theta}{\int_0^\infty G(\theta|\boldsymbol{x})\mathrm{d}\theta}. \tag{2.7}$$

Consider the following algorithm.

Algorithm 2

Step 1: Generate θ from the gamma distribution with shape and rate parameters respectively as $2m+a$ and $\frac{1}{x_m}+b$.
Step 2: Repeat Step 1, N times to obtain the importance sample $\theta_1, \theta_2, \ldots, \theta_N$.

The approximate value of (2.7), which is the approximate Bayes estimate of θ under the SE loss function, can be obtained as

$$\hat{\theta}_{SE} = \frac{\sum\limits_{i=1}^{N}\theta_i h(\theta_i)}{\sum\limits_{i=1}^{N}h(\theta_i)} = \sum_{i=1}^{N}\theta_i w_i,$$

where $w_i = \frac{h(\theta_i)}{\sum_{i=1}^{N}h(\theta_i)}$. Besides, the approximate Bayes estimates of θ under the LE and GE loss functions are given by

$$\hat{\theta}_{LE} = -\frac{1}{c}\ln\Big(\sum_{i=1}^{N}\mathrm{e}_i^{-c\theta}w_i\Big), \quad \text{and} \quad \hat{\theta}_{GE} = \Big(\sum_{i=1}^{N}\theta_i^{-p}w_i\Big)^{-\frac{1}{p}},$$

respectively.

In the sequel, we use the MH algorithm to approximate the Bayes estimates of the parameter of the ILD. Here, we consider the normal distribution as a symmetric proposal distribution. According to [14], we write the MH algorithm steps as follows:

Algorithm 3

Step 1: Set an initial value $\theta^{(0)}$, we propose to consider the ML estimate of θ as the initial value.

Step 2: For $j = 1, \dots, N'$, repeat the following steps.

- Set $\theta = \theta^{(j-1)}$.
- Following [14], generate a new candidate parameter value δ from $N\big(\ln(\theta), \frac{S_{\theta_0}}{[\theta^{(0)}]^2}\big)$, where S_{θ_0} can be obtained using the inverse of the observed Fisher information as follows

$$S_{\theta_0} = \left\{ \frac{2m}{\theta^2} - \frac{1}{(1+\theta)^2} - \sum_{i=1}^{m-1} \left(\frac{1+x_i}{\theta(1+x_i)+x_i} \right)^2 \right\}^{-1} \Bigg|_{\theta=\theta^{(0)}}.$$

- Set $\theta' = \exp(\delta)$.
- Calculate $P = \min\big\{1, \frac{\pi(\theta'|\boldsymbol{x})q(\theta|\theta')}{\pi(\theta|\boldsymbol{x})q(\theta'|\theta)}\big\}$, where $q(x|b)$ is the density of the log-normal distribution with parameters $\ln(b)$ and $\frac{S_{\theta_0}}{[\theta^{(0)}]^2}$.
- Update $\theta^{(j)} = \theta'$ with probability P, otherwise set $\theta^{(j)} = \theta$.

We may discard the first k generated data, where k is the burn-in period. Suppose $\{\theta_l,\ l = 1, \dots, M\}$ is a sample produced according to Algorithm 3 with $M = N' - k$. Therefore, the approximate Bayes estimates of θ under the SE, LE and GE loss functions are given by

$$\tilde{\theta}^*_{SE} = \frac{1}{M} \sum_{l=1}^{M} \theta_l, \quad \text{and} \quad \tilde{\theta}^*_{LE} = -\frac{1}{c} \ln\big(\frac{1}{M} \sum_{l=1}^{M} \mathrm{e}^{-c\theta_l}\big),$$

and

$$\tilde{\theta}^*_{GE} = \big(\frac{1}{M} \sum_{l=1}^{M} \theta_l^{-p}\big)^{-\frac{1}{p}},$$

respectively.

3. PREDICTION of a FUTURE RECORD VALUE

Prediction of future records has been studied by many authors, see for example [15], [9], and [2]. In this section, we study the problem of predicting a future record value, given a sample of observed past record values.

3.1. Maximum likelihood prediction

Suppose that the first m lower record values $X_{L(1)}, \dots, X_{L(m)}$ are available from a population with PDF $f(x;\theta)$ and CDF $F(x;\theta)$. Let $Z = X_{L(n)}$, $n > m$, is an unobserved future record value. Then, the joint PDF of Z and $X_{L(1)}, \dots, X_{L(m)}$ is given by [6], which can also be obtained from the Markovian property of records, see e.g. [3]. Here, using the result given by [6] and from (1.1) and (1.2), the logarithm of the predictive likelihood function of the parameter and Z for the ILD is given by

$$\begin{aligned}\ln L(z,\theta;\boldsymbol{x}) = {} & (2m+2)\ln(\theta) - \ln(1+\theta) - \ln\Gamma(n-m) + \ln(z+1) - 3\ln(z) \\ & -\frac{\theta}{z} + (n-m-1)\ln\left(\frac{\theta}{z} - \frac{\theta}{x_m} + \ln\left(\frac{z[(1+\theta)x_m+\theta]}{x_m[(1+\theta)z+\theta]}\right)\right) \\ & -\sum_{i=1}^{m}\ln\left(\theta(1+x_i)+x_i\right) + \sum_{i=1}^{m}\ln(1+x_i) - 2\sum_{i=1}^{m}\ln(x_i), \quad z < x_m. \end{aligned} \tag{3.1}$$

Maximizing (3.1) with respect to θ and z, we could find the ML prediction of Z and the predictive maximum likelihood estimate of θ. Upon differentiating (3.1) partially with respect to θ and z and equating the results with zero, we have the following equations

$$\begin{aligned}\frac{\partial \ln L(z,\theta;\boldsymbol{x})}{\partial\theta} = {} & \frac{2m+2}{\theta} - \frac{1}{1+\theta} - \frac{1}{z} - \sum_{i=1}^{m}\frac{1+x_i}{\theta(1+x_i)+x_i} \\ & + \frac{(n-m-1)\left(\frac{1}{z} - \frac{1}{x_m} + \frac{x_m+1}{(1+\theta)x_m+\theta} - \frac{z+1}{(1+\theta)z+\theta}\right)}{\frac{\theta}{z} - \frac{\theta}{x_m} + \ln\left(\frac{z[(1+\theta)x_m+\theta]}{x_m[(1+\theta)z+\theta]}\right)} = 0, \\ \frac{\partial \ln L(z,\theta;\boldsymbol{x})}{\partial z} = {} & \frac{1}{1+z} - \frac{3}{z} + \frac{\theta}{z^2} \\ & + \frac{(n-m-1)\left(\frac{1}{z} - \frac{\theta}{z^2} - \frac{\theta+1}{(1+\theta)z+\theta}\right)}{\frac{\theta}{z} - \frac{\theta}{x_m} + \ln\left(\frac{z[(1+\theta)x_m+\theta]}{x_m[(1+\theta)z+\theta]}\right)} = 0. \end{aligned} \tag{3.2}$$

A numerical procedure can help us to find the solutions of the above equations.

One may also find the approximate ML (AML) prediction of Z by means of solving (3.2) after replacing θ with its ML estimate.

3.2. Interval prediction

In this subsection, we study the problem of interval prediction of a future record based on observed past lower record values coming from the ILD. Shortest and equal tails intervals have been nicely discussed in [18]. As mentioned earlier, record values satisfy the Markovian property (see e.g. [3]), in the sense that the conditional density of $Z = X_{L(n)}\,(n > m \geq 1)$ given the set of the first m lower records $(X_{L(1)}, \ldots, X_{L(m)}) = (x_1, \ldots, x_m)$ is the same as the conditional density of Z given $X_{L(m)} = x_m$. From (1.1) and (1.2), the conditional PDF of Z given x_m for the ILD becomes

(3.3)
$$f_Z(z|x_m;\theta) = \frac{\theta^2(1+z)x_m\Big(\frac{\theta}{z} - \frac{\theta}{x_m} + \ln\big(\frac{z[(1+\theta)x_m+\theta]}{x_m[(1+\theta)z+\theta]}\big)\Big)^{n-m-1}}{(n-m-1)![\theta(1+x_m)+x_m]z^3}\,\mathrm{e}^{-\theta(\frac{1}{z}-\frac{1}{x_m})},$$

where $z < x_m$.

As a consequence of (3.3), it can be proved that (see Appendix)

$$U = \frac{\theta}{Z} - \frac{\theta}{x_m} + \ln\Big(\frac{Z[(1+\theta)x_m+\theta]}{x_m[(1+\theta)Z+\theta]}\Big)\Big| X_{L(m)} = x_m \sim Gamma(n-m,1), \tag{3.4}$$

with the following density

$$g_U(z) = \frac{1}{\Gamma(n-m)} z^{n-m-1}\mathrm{e}^{-z}, \quad z > 0.$$

Then, the highest conditional density (HCD) interval for U at the level of $(1-\alpha)$ is in the form of $[c_1, c_2]$ if

$$[c_1, c_2] = \{c : c \geq 0, g_U(c) \geq k\},$$

for some $k > 0$, where $\int_{c_1}^{c_2} g_U(c)\mathrm{d}c = 1-\alpha$.

If $n > m+1$, then $g_U(c)$ is a unimodal PDF whose maximum value is achieved at $v = n-m-1 > 0$. In this case, c_1 and c_2 are the solutions of the following non-linear equations (see e.g. [11])

$$\int_{c_1}^{c_2} g_U(c)\mathrm{d}c = 1-\alpha, \qquad \text{and} \qquad g_U(c_1) = g_U(c_2).$$

The above equations can be reexpressed as follows

$$\gamma(c_2, n-m) - \gamma(c_1, n-m) = 1-\alpha, \qquad \text{and} \qquad \frac{c_1}{c_2} = \exp\big(-\frac{c_2-c_1}{n-m-1}\big),$$

where $\gamma(c,a) = \frac{1}{\Gamma(a)}\int_0^c x^{a-1}\mathrm{e}^{-x}\mathrm{d}x$ is the incomplete gamma function.

Thus, a $100(1-\alpha)\%$ prediction interval (PI) of Z based on the above (HCD) method is in the form of (L^*, U^*), where L^* and U^* satisfy the following non-linear equations

$$\frac{L^*\big(x_m(1+\theta)+\theta\big)}{x_m\big(L^*(1+\theta)+\theta\big)} = \exp\big(-\frac{\theta}{L^*}+\frac{\theta}{x_m}+c_2\big),$$

and

$$\frac{U^*\big(x_m(1+\theta)+\theta\big)}{x_m\big(U^*(1+\theta)+\theta\big)} = \exp\big(-\frac{\theta}{U^*}+\frac{\theta}{x_m}+c_1\big),$$

respectively. If θ is unknown, then it can be replaced by its MLE, which leads to a $100(1-\alpha)\%$ approximate PI (API) for Z.

Next, we consider the case when $n = m+1$, where $g_U(c)$ is a decreasing function with $g_U(0) = 1$ and $g_U(\infty) = 0$. So, we find the interval of the form $[0, c_1]$ where c_1 satisfies the following equation

$$\int_0^{c_1} g_U(c)\mathrm{d}c = 1-\alpha.$$

Therefore, $c_1 = -\ln\alpha$ and a $100(1-\alpha)\%$ PI for Z will be in the form of (L^*, x_m), where L^* satisfies the following equation

$$\frac{\alpha L^*\big(x_m(1+\theta)+\theta\big)}{x_m\big(L^*(1+\theta)+\theta\big)} = \exp\big(-\frac{\theta}{L^*}+\frac{\theta}{x_m}\big).$$

3.3. Bayesian prediction

In this subsection, we consider the prediction of a future record based on a Bayesian approach under the SE, LE, and GE loss functions. Suppose that the first m lower records $X_{L(1)}, \dots, X_{L(m)}$ are available from the ILD and we wish to predict the nth lower record $Z = X_{L(n)}$, $n > m$, based on the observed vector $\boldsymbol{x}$. From (2.3) and (3.3), the Bayes predictive density of Z given $\boldsymbol{x}$ is given by

$$\begin{aligned} f_Z(z|\boldsymbol{x}) &= \int_0^\infty f_Z(z|x_m;\theta)\pi(\theta|\boldsymbol{x})\,\mathrm{d}\theta \\ &= \frac{(1+z)x_m\, m(\boldsymbol{x})}{z^3\,\Gamma(n-m)}\int_0^\infty \Big(\frac{\theta}{z}-\frac{\theta}{x_m}+\ln\Big(\frac{z[(1+\theta)x_m+\theta]}{x_m[(1+\theta)z+\theta]}\Big)\Big)^{n-m-1} \\ &\quad\times\frac{\theta^{2m+a+1}}{x_m+\theta(1+x_m)}\mathrm{e}^{-\theta(\frac{1}{z}+b)}\Big[(1+\theta)\prod_{i=1}^{m-1}\big(x_i+\theta(1+x_i)\big)\Big]^{-1}\mathrm{d}\theta. \end{aligned}$$

In the particular case of $n = m+1$, the Bayes predictive density function of Z simplifies as

$$f_Z(z|\boldsymbol{x}) = \frac{(1+z)x_m\, m(\boldsymbol{x})}{z^3}\int_0^\infty \frac{\theta^{2m+a+1}\mathrm{e}^{-\theta(\frac{1}{z}+b)}}{x_m+\theta(1+x_m)}\Big[(1+\theta)\prod_{i=1}^{m-1}\big(x_i+\theta(1+x_i)\big)\Big]^{-1}\mathrm{d}\theta.$$

The Bayesian prediction of the nth lower record under the SE loss function is given by

$$\hat{Z}_{BS} = \hat{E}(Z|\boldsymbol{x}) = \int_0^{x_m} z f_Z(z|\boldsymbol{x})\mathrm{d}z,$$

and the Bayesian predictions of Z under the LE and GE loss functions are

$$\hat{Z}_{BL} = -\frac{1}{c}\ln \hat{E}(\mathrm{e}^{-cZ}|\boldsymbol{x}) = -\frac{1}{c}\ln\Big(\int_0^{x_m} \mathrm{e}^{-cz} f_Z(z|\boldsymbol{x})\mathrm{d}z\Big),$$

and

$$\hat{Z}_{BG} = [\hat{E}(Z^{-p}|\boldsymbol{x})]^{-\frac{1}{p}} = \Big[\int_0^{x_m} z^{-p} f_Z(z|\boldsymbol{x})\mathrm{d}z\Big]^{-\frac{1}{p}},$$

respectively, provided that the above integrals exist and are finite.

The predictive limits of a $100(1-\tau)\%$ two-sided PI for the future lower record $Z = X_{L(n)}$ can be obtained by solving the following two equations simultaneously with respect to L^{**} and U^{**}

$$\int_{L^{**}}^{\infty} f_Z(z|\boldsymbol{x})\mathrm{d}z = 1 - \frac{\tau}{2}, \qquad \text{and} \qquad \int_{U^{**}}^{\infty} f_Z(z|\boldsymbol{x})\mathrm{d}z = \frac{\tau}{2}.$$

4. A SIMULATION STUDY

In this section, we performed a simulation study to assess the performance of the point and interval estimators of θ and predictors of a future record value coming from the ILD. With this in mind, in each iteration of the simulation, we generate m lower records from the ILD with parameter θ and then we compute the ML estimate, the approximate Bayes estimates under the SE, LE, and GE loss functions using the TK, IS, and MH methods. The 95% asymptotic CIs, as well as the two bootstrap-type CIs are obtained. In the context of prediction, we compute AML prediction and 95% PI (based on the HCD method) for the $(m+1)$th lower record value. The following setting has been applied: We consider three different values for the number of lower records as $m = 3, 4, 5$ and three different values for the parameter as $\theta = 0.5, 1, 2$. The number of bootstrap repetitions is taken to be $B = 1000$. In the context of the Bayesian estimation, two gamma priors have been applied, Prior 1 with $(a_1, b_1) = (0.2, 1.5)$ and Prior 2 with $(a_2, b_2) = (3, 1)$. Besides, we take $c = -0.2, 0.2$ for the LE loss function and $p = -0.2, 0.2$ for the GE loss function. The results of the simulation study are based on $N = 1000$ iterations.

The assessment of the performances of the point estimators is based on estimated risks (ERs) under the SE, LE, and GE functions, and the evaluation of CIs is based on average length (AL) and coverage probability (CP). Let $\tilde{\theta}$ be an estimator of θ and $\tilde{\theta}_i$ be the corresponding estimate obtained in the ith iteration. Then the estimated bias (bias for short) and ERs of $\tilde{\theta}$ under the

SE, LE, and GE loss functions are given by

$$\text{Bias}(\tilde{\theta}) = \frac{1}{N}\sum_{i=1}^{N}\left(\tilde{\theta}_i - \theta\right), \tag{4.1}$$

$$ER_{SE}(\tilde{\theta}) = \frac{1}{N}\sum_{i=1}^{N}\left(\tilde{\theta}_i - \theta\right)^2, \tag{4.2}$$

$$ER_{LE}(\tilde{\theta}) = \frac{1}{N}\sum_{i=1}^{N}\left[\mathrm{e}^{c(\tilde{\theta}_i-\theta)} - c(\tilde{\theta}_i - \theta) - 1\right],$$

$$ER_{GE}(\tilde{\theta}) = \frac{1}{N}\sum_{i=1}^{N}\left[\left(\frac{\tilde{\theta}_i}{\theta}\right)^p - p\ln\left(\frac{\tilde{\theta}_i}{\theta}\right) - 1\right],$$

respectively.

Besides, we compute the empirical biases (biases for short) and mean squared prediction errors ($EMSPE$s) of the AML predictors (which can be formulated similarly as (4.1) and (4.2), respectively) and the ALs and CPs of the interval predictors.

The simulation results related to the point estimation are presented in Tables 1-6. The following abbreviations are used in Tables 1-6: BS (Bayes estimator under the SE loss function), BLc_1 (Bayes estimator under the SE loss function with $c_1 = 0.2$), BLc_2 (Bayes estimator under the SE loss function with $c_2 = -0.2$), BGp_1 (Bayes estimator under the GE loss function with $p_1 = 0.2$) and BGp_2 (Bayes estimator under the GE loss function with $p_2 = -0.2$). It is observed from Tables 1-6 that in all estimation methods, ERs are decreasing with respect to the number of records except for the case under Prior 2 when $\theta = 2$. We also observe that the ERs are close to each other for the TK, IS, and MH methods. Furthermore, for Prior 1, the ERs of the Bayes estimators are less than or equal to those of the ML estimators (a few exceptions exist), whereas for Prior 2 when $\theta = 0.5$, the ML estimators outperform the Bayes estimators in the sense of ER and bias. Prior 1 produces smaller ERs than Prior 2, when $\theta = 0.5$ and 1, which is also true for $\theta = 2$ in the most cases.

The performances of the asymptotic CIs and two different bootstrap CIs ($Boot{-}B$ and $Boot{-}P$ methods) are compared in terms of their ALs and CPs in Table 7. Table 7 shows that in all three methods, the AL of the CI decreases as the number of records increases. Besides, in all cases, the CPs of the asymptotic CIs are more than the corresponding CPs of the bootstrap CIs., and the ALs of the asymptotic CIs are less than those of the others. We also observe that the $Boot{-}B$ CIs perform better than the $Boot{-}P$ CIs in the sense of CP.

Finally, Table 8 presents the biases and $EMSPE$s of the AML predictors as well as the ALs and CPs of the APIs for the $(m+1)$th lower record value. From Table 8, we observe that for all values of θ, the AL, bias, and $EMSPE$ decrease as the number of records increases.

m	Method / ER	ML	TK					IS					MH				
			BS	BLc_1	BLc_2	BGp_1	BGp_2	BS	BLc_1	BLc_2	BGp_1	BGp_2	BS	BLc_1	BLc_2	BGp_1	BGp_2
3	Bias	0.175	0.064	0.052	0.067	0.001	0.022	0.060	0.053	0.068	−0.003	0.018	0.060	0.053	0.068	−0.003	0.018
	ER_{SE}	0.218	0.063	0.057	0.066	0.046	0.051	0.062	0.057	0.066	0.045	0.050	0.064	0.060	0.069	0.047	0.052
	ER_{LE} $(c = 0.2)$	0.005	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001
	ER_{LE} $(c = -0.2)$	0.004	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001
	ER_{GE} $(p = 0.2)$	0.006	0.003	0.003	0.003	0.003	0.003	0.003	0.003	0.003	0.003	0.003	0.003	0.003	0.003	0.003	0.003
	ER_{GE} $(p = -0.2)$	0.005	0.003	0.003	0.003	0.003	0.003	0.003	0.003	0.003	0.003	0.003	0.003	0.003	0.003	0.003	0.003
4	Bias	0.132	0.062	0.054	0.065	0.011	0.028	0.060	0.054	0.065	0.010	0.026	0.060	0.054	0.065	0.009	0.026
	ER_{SE}	0.097	0.044	0.041	0.046	0.033	0.036	0.044	0.041	0.046	0.033	0.036	0.044	0.042	0.047	0.033	0.036
	ER_{LE} $(c = 0.2)$	0.002	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001
	ER_{LE} $(c = -0.2)$	0.002	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001
	ER_{GE} $(p = 0.2)$	0.004	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002
	ER_{GE} $(p = -0.2)$	0.003	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002
5	Bias	0.124	0.070	0.063	0.072	0.027	0.041	0.068	0.064	0.073	0.026	0.040	0.069	0.064	0.073	0.026	0.040
	ER_{SE}	0.060	0.031	0.029	0.032	0.023	0.026	0.031	0.030	0.033	0.024	0.026	0.031	0.030	0.033	0.023	0.026
	ER_{LE} $(c = 0.2)$	0.001	0.001	0.001	0.001	0.000	0.001	0.001	0.001	0.001	0.000	0.001	0.001	0.001	0.001	0.000	0.001
	ER_{LE} $(c = -0.2)$	0.001	0.001	0.001	0.001	0.000	0.001	0.001	0.001	0.001	0.000	0.001	0.001	0.001	0.001	0.000	0.001
	ER_{GE} $(p = 0.2)$	0.002	0.002	0.002	0.002	0.001	0.001	0.002	0.002	0.002	0.001	0.001	0.002	0.002	0.002	0.001	0.001
	ER_{GE} $(p = -0.2)$	0.002	0.002	0.001	0.002	0.001	0.001	0.002	0.001	0.002	0.001	0.001	0.002	0.001	0.002	0.001	0.001

Table 1: estimated biases and ERs of point estimators of θ for Prior $1 = (0.2, 1.5)$ and $\theta = 0.5$.

m	ER \ Method	ML	TK BS	TK BLc_1	TK BLc_2	TK BGp_1	TK BGp_2	IS BS	IS BLc_1	IS BLc_2	IS BGp_1	IS BGp_2	MH BS	MH BLc_1	MH BLc_2	MH BGp_1	MH BGp_2
3	Bias	0.348	−0.066	−0.092	−0.053	−0.172	−0.136	−0.071	−0.090	−0.052	−0.178	−0.142	−0.072	−0.090	−0.052	−0.178	−0.142
	ER_{SE}	0.890	0.108	0.101	0.114	0.110	0.106	0.107	0.102	0.115	0.111	0.107	0.107	0.102	0.115	0.111	0.107
	ER_{LE} $(c = 0.2)$	0.023	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002
	ER_{LE} $(c = -0.2)$	0.015	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002
	ER_{GE} $(p = 0.2)$	0.006	0.002	0.002	0.002	0.003	0.003	0.002	0.002	0.002	0.003	0.003	0.002	0.002	0.002	0.003	0.003
	ER_{GE} $(p = -0.2)$	0.005	0.003	0.003	0.003	0.003	0.003	0.003	0.003	0.003	0.004	0.003	0.003	0.003	0.003	0.004	0.003
4	Bias	0.264	−0.026	−0.047	−0.015	−0.115	−0.085	−0.029	−0.045	−0.013	−0.117	−0.088	−0.030	−0.046	−0.014	−0.118	−0.089
	ER_{SE}	0.428	0.083	0.078	0.088	0.081	0.080	0.082	0.078	0.088	0.081	0.079	0.083	0.078	0.088	0.082	0.080
	ER_{LE} $(c = 0.2)$	0.010	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002
	ER_{LE} $(c = -0.2)$	0.007	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002
	ER_{GE} $(p = 0.2)$	0.003	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002
	ER_{GE} $(p = -0.2)$	0.003	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002
5	Bias	0.269	0.023	0.005	0.034	−0.055	−0.029	0.023	0.009	0.038	−0.053	−0.028	0.019	0.005	0.034	−0.058	−0.033
	ER_{SE}	0.320	0.075	0.069	0.080	0.066	0.067	0.075	0.070	0.081	0.067	0.068	0.076	0.071	0.082	0.068	0.069
	ER_{LE} $(c = 0.2)$	0.007	0.002	0.001	0.002	0.001	0.001	0.002	0.001	0.002	0.001	0.001	0.002	0.001	0.002	0.001	0.001
	ER_{LE} $(c = -0.2)$	0.006	0.001	0.001	0.002	0.001	0.001	0.001	0.001	0.002	0.001	0.001	0.001	0.001	0.002	0.001	0.001
	ER_{GE} $(p = 0.2)$	0.003	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001
	ER_{GE} $(p = -0.2)$	0.003	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001

Table 2: The estimated biases and ERs of point estimators of θ for Prior $1 = (0.2, 1.5)$ and $\theta = 1$.

m	Method / ER	ML	TK					IS					MH				
			BS	BLc_1	BLc_2	BGp_1	BGp_2	BS	BLc_1	BLc_2	BGp_1	BGp_2	BS	BLc_1	BLc_2	BGp_1	BGp_2
3	Bias	0.801	−0.588	−0.642	−0.556	−0.751	−0.697	−0.600	−0.640	−0.555	−0.763	−0.709	−0.599	−0.64	−0.555	−0.762	−0.708
	ER_{SE}	4.919	0.470	0.520	0.449	0.661	0.591	0.482	0.518	0.447	0.677	0.605	0.486	0.522	0.453	0.680	0.609
	ER_{LE} $(c = 0.2)$	0.341	0.009	0.010	0.008	0.012	0.011	0.009	0.010	0.008	0.013	0.011	0.009	0.010	0.009	0.013	0.011
	ER_{LE} $(c = -0.2)$	0.061	0.010	0.011	0.009	0.014	0.013	0.010	0.011	0.009	0.014	0.013	0.010	0.011	0.010	0.015	0.013
	ER_{GE} $(p = 0.2)$	0.006	0.004	0.004	0.004	0.006	0.005	0.004	0.004	0.004	0.006	0.005	0.004	0.004	0.004	0.006	0.005
	ER_{GE} $(p = -0.2)$	0.006	0.004	0.005	0.004	0.007	0.006	0.004	0.005	0.004	0.007	0.006	0.004	0.005	0.004	0.007	0.006
4	Bias	0.612	−0.449	−0.497	−0.415	−0.593	−0.545	−0.456	−0.495	−0.414	−0.599	−0.551	−0.457	−0.495	−0.415	−0.599	−0.552
	ER_{SE}	2.344	0.330	0.360	0.314	0.455	0.409	0.339	0.363	0.317	0.469	0.420	0.339	0.363	0.319	0.466	0.419
	ER_{LE} $(c = 0.2)$	0.082	0.006	0.007	0.006	0.009	0.008	0.006	0.007	0.006	0.009	0.008	0.006	0.007	0.006	0.009	0.008
	ER_{LE} $(c = -0.2)$	0.034	0.007	0.008	0.007	0.010	0.009	0.007	0.008	0.007	0.010	0.009	0.007	0.008	0.007	0.010	0.009
	ER_{GE} $(p = 0.2)$	0.004	0.002	0.003	0.002	0.004	0.003	0.003	0.003	0.002	0.004	0.003	0.003	0.003	0.002	0.004	0.003
	ER_{GE} $(p = -0.2)$	0.004	0.003	0.003	0.002	0.004	0.003	0.003	0.003	0.002	0.004	0.004	0.003	0.003	0.003	0.004	0.004
5	Bias	0.719	−0.280	−0.326	−0.243	−0.414	−0.369	−0.283	−0.323	−0.240	−0.414	−0.371	−0.288	−0.328	−0.246	−0.420	−0.376
	ER_{SE}	1.971	0.220	0.233	0.215	0.290	0.262	0.224	0.235	0.218	0.297	0.269	0.229	0.239	0.223	0.300	0.272
	ER_{LE} $(c = 0.2)$	0.057	0.004	0.005	0.004	0.006	0.005	0.004	0.005	0.004	0.006	0.005	0.004	0.005	0.004	0.006	0.005
	ER_{LE} $(c = -0.2)$	0.031	0.005	0.005	0.004	0.006	0.005	0.005	0.005	0.004	0.006	0.006	0.005	0.005	0.005	0.006	0.006
	ER_{GE} $(p = 0.2)$	0.004	0.001	0.002	0.001	0.002	0.002	0.001	0.002	0.001	0.002	0.002	0.002	0.002	0.001	0.002	0.002
	ER_{GE} $(p = -0.2)$	0.003	0.002	0.002	0.001	0.002	0.002	0.002	0.002	0.001	0.002	0.002	0.002	0.002	0.001	0.002	0.002

Table 3: The estimated biases and ERs of point estimators of θ for Prior $1 = (0.2, 1.5)$ and $\theta = 2$.

m	Method / ER	ML	TK BS	TK BLc_1	TK BLc_2	TK BGp_1	TK BGp_2	IS BS	IS BLc_1	IS BLc_2	IS BGp_1	IS BGp_2	MH BS	MH BLc_1	MH BLc_2	MH BGp_1	MH BGp_2
3	Bias	0.175	0.442	0.425	0.454	0.368	0.393	0.44	0.426	0.455	0.367	0.391	0.439	0.425	0.454	0.366	0.390
	ER_{SE}	0.218	0.404	0.373	0.432	0.312	0.341	0.401	0.373	0.432	0.308	0.338	0.403	0.376	0.434	0.312	0.341
	ER_{LE} $(c = 0.2)$	0.005	0.009	0.008	0.010	0.007	0.007	0.009	0.008	0.010	0.007	0.007	0.009	0.008	0.010	0.007	0.007
	ER_{LE} $(c = -0.2)$	0.004	0.007	0.007	0.008	0.006	0.006	0.007	0.007	0.008	0.006	0.006	0.007	0.007	0.008	0.006	0.006
	ER_{GE} $(p = 0.2)$	0.006	0.010	0.010	0.010	0.008	0.009	0.010	0.010	0.010	0.008	0.009	0.010	0.010	0.010	0.008	0.009
	ER_{GE} $(p = -0.2)$	0.005	0.009	0.008	0.009	0.007	0.008	0.009	0.008	0.009	0.007	0.008	0.009	0.008	0.009	0.007	0.008
4	Bias	0.132	0.351	0.341	0.359	0.295	0.314	0.351	0.342	0.361	0.295	0.314	0.350	0.340	0.359	0.293	0.312
	ER_{SE}	0.097	0.232	0.218	0.242	0.181	0.197	0.231	0.219	0.243	0.180	0.196	0.231	0.219	0.244	0.180	0.196
	ER_{LE} $(c = 0.2)$	0.002	0.005	0.005	0.005	0.004	0.004	0.005	0.005	0.005	0.004	0.004	0.005	0.005	0.005	0.004	0.004
	ER_{LE} $(c = -0.2)$	0.002	0.004	0.004	0.005	0.003	0.004	0.004	0.004	0.005	0.003	0.004	0.004	0.004	0.005	0.003	0.004
	ER_{GE} $(p = 0.2)$	0.004	0.007	0.007	0.007	0.006	0.006	0.007	0.007	0.007	0.006	0.006	0.007	0.007	0.007	0.006	0.006
	ER_{GE} $(p = -0.2)$	0.003	0.006	0.006	0.007	0.005	0.006	0.006	0.006	0.007	0.005	0.006	0.006	0.006	0.007	0.005	0.006
5	Bias	0.124	0.311	0.303	0.317	0.264	0.279	0.310	0.303	0.317	0.263	0.279	0.309	0.302	0.317	0.262	0.278
	ER_{SE}	0.060	0.157	0.150	0.164	0.123	0.134	0.158	0.151	0.165	0.124	0.135	0.158	0.151	0.165	0.124	0.134
	ER_{LE} $(c = 0.2)$	0.001	0.003	0.003	0.003	0.003	0.003	0.003	0.003	0.003	0.003	0.003	0.003	0.003	0.003	0.003	0.003
	ER_{LE} $(c = -0.2)$	0.001	0.003	0.003	0.003	0.002	0.003	0.003	0.003	0.003	0.002	0.003	0.003	0.003	0.003	0.002	0.003
	ER_{GE} $(p = 0.2)$	0.002	0.006	0.006	0.006	0.005	0.005	0.006	0.006	0.006	0.005	0.005	0.006	0.006	0.006	0.005	0.005
	ER_{GE} $(p = -0.2)$	0.002	0.005	0.005	0.005	0.004	0.005	0.005	0.005	0.005	0.004	0.005	0.005	0.005	0.005	0.004	0.005

Table 4: The estimated biases and ERs of point estimators of θ for Prior $2 = (3, 1)$ and $\theta = 0.5$.

m	ER \ Method	ML	TK					IS					MH				
			BS	BLc_1	BLc_2	BGp_1	BGp_2	BS	BLc_1	BLc_2	BGp_1	BGp_2	BS	BLc_1	BLc_2	BGp_1	BGp_2
3	Bias	0.348	0.646	0.601	0.687	0.516	0.559	0.645	0.604	0.689	0.515	0.558	0.639	0.598	0.684	0.509	0.552
	ER_{SE}	0.890	0.865	0.757	0.975	0.641	0.712	0.865	0.766	0.983	0.644	0.713	0.851	0.753	0.967	0.630	0.700
	ER_{LE} $(c = 0.2)$	0.023	0.020	0.017	0.022	0.014	0.016	0.020	0.017	0.023	0.014	0.016	0.019	0.017	0.022	0.014	0.016
	ER_{LE} $(c = -0.2)$	0.015	0.015	0.014	0.017	0.012	0.013	0.015	0.014	0.017	0.012	0.013	0.015	0.014	0.017	0.011	0.013
	ER_{GE} $(p = 0.2)$	0.006	0.007	0.006	0.007	0.005	0.006	0.007	0.006	0.007	0.005	0.006	0.007	0.006	0.007	0.005	0.006
	ER_{GE} $(p = -0.2)$	0.005	0.006	0.006	0.007	0.005	0.005	0.006	0.006	0.007	0.005	0.005	0.006	0.006	0.007	0.005	0.005
4	Bias	0.264	0.545	0.512	0.574	0.441	0.476	0.544	0.515	0.576	0.441	0.475	0.544	0.514	0.575	0.440	0.475
	ER_{SE}	0.428	0.570	0.511	0.629	0.430	0.474	0.574	0.519	0.636	0.437	0.480	0.573	0.517	0.636	0.433	0.477
	ER_{LE} $(c = 0.2)$	0.010	0.013	0.011	0.014	0.009	0.010	0.013	0.011	0.014	0.010	0.011	0.013	0.011	0.014	0.010	0.011
	ER_{LE} $(c = -0.2)$	0.007	0.010	0.009	0.011	0.008	0.009	0.010	0.010	0.012	0.008	0.009	0.010	0.009	0.012	0.008	0.009
	ER_{GE} $(p = 0.2)$	0.003	0.005	0.005	0.005	0.004	0.004	0.005	0.005	0.005	0.004	0.004	0.005	0.005	0.005	0.004	0.004
	ER_{GE} $(p = -0.2)$	0.003	0.005	0.004	0.005	0.004	0.004	0.005	0.004	0.005	0.004	0.004	0.005	0.004	0.005	0.004	0.004
5	Bias	0.269	0.518	0.491	0.541	0.428	0.458	0.518	0.494	0.544	0.430	0.459	0.516	0.491	0.541	0.427	0.457
	ER_{SE}	0.320	0.475	0.432	0.517	0.365	0.400	0.481	0.442	0.525	0.375	0.408	0.474	0.434	0.518	0.366	0.400
	ER_{LE} $(c = 0.2)$	0.007	0.010	0.009	0.011	0.008	0.009	0.011	0.010	0.012	0.008	0.009	0.010	0.009	0.011	0.008	0.009
	ER_{LE} $(c = -0.2)$	0.006	0.009	0.008	0.009	0.007	0.007	0.009	0.008	0.010	0.007	0.008	0.009	0.008	0.010	0.007	0.007
	ER_{GE} $(p = 0.2)$	0.003	0.005	0.004	0.005	0.004	0.004	0.005	0.004	0.005	0.004	0.004	0.005	0.004	0.005	0.004	0.004
	ER_{GE} $(p = -0.2)$	0.003	0.004	0.004	0.004	0.003	0.004	0.004	0.004	0.004	0.003	0.004	0.004	0.004	0.004	0.003	0.004

Table 5: The estimated biases and ERs of point estimators of θ for Prior $2 = (3, 1)$ and $\theta = 1$.

m	Method / ER	ML	TK					IS					MH				
			BS	BLc_1	BLc_2	BGp_1	BGp_2	BS	BLc_1	BLc_2	BGp_1	BGp_2	BS	BLc_1	BLc_2	BGp_1	BGp_2
3	Bias	0.801	0.668	0.558	0.777	0.452	0.524	0.664	0.562	0.780	0.449	0.520	0.662	0.56	0.778	0.446	0.518
	ER_{SE}	4.919	1.139	0.895	1.430	0.782	0.890	1.138	0.909	1.442	0.788	0.893	1.135	0.906	1.441	0.782	0.888
	ER_{LE} $(c = 0.2)$	0.341	0.026	0.020	0.033	0.017	0.020	0.026	0.020	0.034	0.018	0.020	0.026	0.020	0.034	0.017	0.020
	ER_{LE} $(c = -0.2)$	0.061	0.020	0.016	0.025	0.014	0.016	0.020	0.016	0.025	0.014	0.016	0.020	0.016	0.025	0.014	0.016
	ER_{GE} $(p = 0.2)$	0.006	0.003	0.003	0.004	0.002	0.003	0.003	0.003	0.004	0.002	0.003	0.003	0.003	0.004	0.002	0.003
	ER_{GE} $(p = -0.2)$	0.006	0.003	0.002	0.003	0.002	0.002	0.003	0.002	0.003	0.002	0.002	0.003	0.002	0.003	0.002	0.002
4	Bias	0.612	0.630	0.540	0.717	0.447	0.508	0.627	0.544	0.720	0.446	0.507	0.623	0.539	0.717	0.440	0.501
	ER_{SE}	2.344	0.941	0.762	1.149	0.668	0.751	0.944	0.776	1.159	0.678	0.758	0.939	0.768	1.156	0.665	0.749
	ER_{LE} $(c = 0.2)$	0.082	0.021	0.017	0.027	0.015	0.017	0.021	0.017	0.027	0.015	0.017	0.021	0.017	0.027	0.015	0.017
	ER_{LE} $(c = -0.2)$	0.034	0.017	0.014	0.020	0.012	0.014	0.017	0.014	0.020	0.012	0.014	0.017	0.014	0.020	0.012	0.013
	ER_{GE} $(p = 0.2)$	0.004	0.003	0.002	0.003	0.002	0.002	0.003	0.002	0.003	0.002	0.002	0.003	0.002	0.003	0.002	0.002
	ER_{GE} $(p = -0.2)$	0.004	0.002	0.002	0.003	0.002	0.002	0.002	0.002	0.003	0.002	0.002	0.002	0.002	0.003	0.002	0.002
5	Bias	0.719	0.725	0.642	0.807	0.559	0.615	0.723	0.645	0.808	0.559	0.613	0.722	0.643	0.808	0.556	0.611
	ER_{SE}	1.971	1.036	0.859	1.236	0.759	0.845	1.036	0.871	1.241	0.771	0.852	1.032	0.864	1.240	0.758	0.843
	ER_{LE} $(c = 0.2)$	0.057	0.024	0.019	0.029	0.017	0.019	0.024	0.020	0.029	0.017	0.019	0.024	0.019	0.029	0.017	0.019
	ER_{LE} $(c = -0.2)$	0.031	0.018	0.015	0.022	0.014	0.015	0.018	0.016	0.022	0.014	0.015	0.018	0.015	0.022	0.014	0.015
	ER_{GE} $(p = 0.2)$	0.004	0.003	0.002	0.003	0.002	0.002	0.003	0.002	0.003	0.002	0.002	0.003	0.002	0.003	0.002	0.002
	ER_{GE} $(p = -0.2)$	0.003	0.003	0.002	0.003	0.002	0.002	0.003	0.002	0.003	0.002	0.002	0.003	0.002	0.003	0.002	0.002

Table 6: The estimated biases and ERs of point estimators of θ for Prior 2 $= (3, 1)$ and $\theta = 2$.

			Method		
θ	m		Asymptotic	$Boot-B$	$Boot-P$
0.5	3	AL	1.222	1.938	1.938
		CP	0.985	0.868	0.840
	4	AL	1.005	1.362	1.362
		CP	0.997	0.902	0.788
	5	AL	0.900	1.113	1.113
		CP	0.999	0.927	0.668
1	3	AL	2.517	4.328	4.328
		CP	0.984	0.863	0.838
	4	AL	2.062	2.956	2.956
		CP	0.999	0.916	0.787
	5	AL	1.872	2.443	2.443
		CP	0.998	0.929	0.678
2	3	AL	5.481	10.002	10.002
		CP	0.982	0.879	0.812
	4	AL	4.431	6.771	6.771
		CP	0.995	0.918	0.772
	5	AL	4.163	5.750	5.750
		CP	0.998	0.942	0.633

Table 7: ALs and CPs of 95% CIs of θ.

θ	m	AML Predictor		API based on the HCD method	
0.5	3	Bias	0.041	AL	0.070
		$EMSPE$	0.004	CP	0.886
	4	Bias	0.026	AL	0.046
		$EMSPE$	0.002	CP	0.899
	5	Bias	0.016	AL	0.033
		$EMSPE$	0.000	CP	0.887
1	3	Bias	0.095	AL	0.156
		$EMSPE$	0.037	CP	0.887
	4	Bias	0.049	AL	0.098
		$EMSPE$	0.007	CP	0.917
	5	Bias	0.036	AL	0.071
		$EMSPE$	0.003	CP	0.887
2	3	Bias	0.225	AL	0.362
		$EMSPE$	0.226	CP	0.884
	4	Bias	0.130	AL	0.223
		$EMSPE$	0.055	CP	0.880
	5	Bias	0.090	AL	0.162
		$EMSPE$	0.026	CP	0.883

Table 8: The estimated biases and $EMSPE$s of the AML predictors and the ALs and CPs of the 95% APIs based on the HCD methods.

5. REAL DATA EXAMPLE

In this section, we use a real data set to illustrate the estimation and prediction procedures for the ILD. The data are the monthly rainfall during December recorded at Los Angeles civic center from 2001 to 2016 (see the website of Los Angeles Almanac: `www.laalmanac.com/weather/we08aa.htm`).

1.38	3.31	1.35	8.77	1.03	0.81	1.73	2.79
2.89	10.23	1.01	2.16	0.20	3.88	0.57	4.55

To assess the suitability of the inverse Lindley distribution for the provided dataset, various statistical tests and criteria were applied, including the Kolmogorov-Smirnov (K–S) test, Akaike information criterion (AIC), and Bayesian information criterion (BIC). The fitness results for the ILD were compared with those for the inverse xgamma distribution introduced by [33], with PDF $f(x) = \frac{\theta^2}{x^2(1+\theta)}(1 + \frac{\theta}{2x^2})\exp(\frac{-\theta}{x})$, the inverse Maxwell distribution introduced by [30], with PDF $f(x) = \frac{4\theta^{1.5}}{\sqrt{\pi}x^4}\exp(-\frac{\theta}{x^2})$, and the inverse Rayleigh distribution with PDF $f(x) = \frac{2\theta}{x^3}\exp(-\frac{\theta}{x^2})$. The results of the $K-S$ test, AIC, and BIC collectively support the appropriateness of the inverse Lindley distribution for the dataset. Specifically, the $K-S$ test yielded a p-value of 0.8047 for the ILD, as opposed to 0.6995 for the inverse xgamma, 0.000065 for the inverse Maxwell, and 0.001386 for the inverse Rayleigh distributions. This indicates that both the inverse Lindley and inverse xgamma distributions are suitable for these data. The AIC and BIC values for the ILD were obtained to be 71.9553 and 72.7279, respectively. In contrast, for the inverse xgamma distribution, the AIC and BIC values were computed as 72.7797 and 73.5523, suggesting that the inverse Lindley distribution is more appropriate for modeling this dataset.

From the original data set, we have extracted the first five lower records as follows: 1.38, 1.35, 1.03, 0.81, 0.20. Here, we use the same priors used in the simulation study, which are Prior 1 and Prior 2. We calculated the point and interval estimates for the unknown parameter θ based on the observed five lower records. Besides, we computed the AML prediction and the 95% API for the $6th$ lower record value. Table 9 represents our numerical findings.

Point Estimation					
		MLE	TK	IS	MH
Prior 1	SE	1.315	1.090	1.121	1.009
	$LE(c = 0.2)$		1.073	1.108	0.997
	$LE(c = -0.2)$		1.102	1.135	1.021
	$GE(p = -0.2)$		1.013	1.052	0.940
	$GE(p = -0.2)$		1.039	1.074	0.963
Prior 2	SE		1.578	1.588	1.625
	$LE(c = 0.2)$		1.553	1.566	1.604
	$LE(c = -0.2)$		1.600	1.610	1.648
	$GE(p = -0.2)$		1.489	1.508	1.546
	$GE(p = -0.2)$		1.519	1.535	1.572
Interval Estimation					
	95% Asymptotic CI	95% $Boot-B$ CI		95% $Boot-P$ CI	
	(0.382, 2.248)	(−0.814, 1.679)		(0.951, 3.443)	
Prediction					
	AML prediction		95% API		
	0.200		(0.133, 0.200)		

Table 9: The numerical results of the example.

6. CONCLUSIONS

The inverse Lindley distribution, introduced by [28], offers a versatile distribution with an inverted bathtub-shaped hazard rate function. [28] demonstrated its applicability to real-world data, specifically survival times of head and neck cancer patients. Since its inception, various authors have explored inferential aspects of the inverse Lindley distribution (ILD).

This paper focuses on the estimation of the unknown parameter of the ILD when the first m record values are available. The classical and Bayesian procedures were employed for parameter estimation, and attention was given to predicting a future record value. The article includes a simulation study and a real data application to illustrate the proposed procedures. A comparative analysis involved the maximum likelihood estimator and different Bayes estimators under squared error, linear-exponential, and general entropy loss functions, considering average empirical biases and associated estimated risks. The asymptotic and two bootstrap-type confidence intervals were assessed for their coverage probabilities and average lengths. Notably, the asymptotic confidence intervals demonstrated shorter lengths and larger coverage probabilities compared to bootstrap confidence intervals. Furthermore, Bayesian methods with small prior variance emerged as more preferable than classical methods.

The exploration extends to the estimation problem for $R = P(X < Y)$, utilizing two sequences of lower record values from two inverse Lindley populations with different parameters. Future work is suggested on inferential challenges for generalizations of the ILD based on record data. Additionally, the paper proposes investigating estimation and prediction problems for the ILD using alternative data types, such as progressively type I and type II censored data, hybrid censored data, progressively first failure censored data, and more. The authors anticipate reporting findings on some of these topics in future research endeavors. All computations were carried out using the statistical software R [26] and the packages AdequacyModel [22], LindleyR [23], lamW [1], and nleqslv [19] therein.

Appendix

Here, we want to prove (3.4). From (3.3), the conditional PDF of Z given the last observed record x_m for the ILD, $f_Z(z) \equiv f_Z(z|x_m; \theta)$, can be rewritten as

$$f_Z(z) = \frac{\left(\frac{\theta}{z} - \frac{\theta}{x_m} + \ln\left(\frac{z[(1+\theta)x_m+\theta]}{x_m[(1+\theta)z+\theta]}\right)\right)^{n-m-1}}{(n-m-1)!} \times \frac{\theta^2(1+z)}{z^2[(1+\theta)z+\theta]}$$
$$\times \frac{x_m[(1+\theta)z+\theta]}{z[(1+\theta)x_m+\theta]} \mathrm{e}^{-\theta(\frac{1}{z}-\frac{1}{x_m})}. \tag{6.1}$$

Let

$$u = g^*(z) = \frac{\theta}{z} - \frac{\theta}{x_m} + \ln\left(\frac{z[(1+\theta)x_m+\theta]}{x_m[(1+\theta)z+\theta]}\right). \tag{6.2}$$

Then, the jacobian is obtained to be

$$J = \frac{\partial g^*(z)}{\partial z} = -\frac{\theta^2(1+z)}{z^2[(1+\theta)z+\theta]}. \tag{6.3}$$

In addition, from (6.2), we get

$$\mathrm{e}^{-u} = \frac{x_m[(1+\theta)z+\theta]}{z[(1+\theta)x_m+\theta]} \mathrm{e}^{-\theta(\frac{1}{z}-\frac{1}{x_m})}. \tag{6.4}$$

Note that the PDF of U, given in (3.4), can be written as $g_U(u) = \frac{f_Z(g^{*^{-1}}(u))}{|J|}$, where $g^{*^{-1}}(\cdot)$ is the inverse function of $g^*(\cdot)$. So, the result follows from (6.1), (6.2), (6.3), and (6.4).

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