REVSTAT – Statistical Journal
Volume 0, Number 0, Month 0000, 000-000
https://doi.org/00.00000/revstat.v00i0.000

BAYESIAN AND FREQUENTIST ESTIMATION OF STRESS-STRENGTH RELIABILITY FROM A NEW EXTENDED BURR XII DISTRIBUTION

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Received: Month 0000 Revised: Month 0000 Accepted: Month 0000

Abstract:

- In this article, we propose and study a new three-parameter heavy-tailed distribution that unifies the Burr type XII and power inverted Topp-Leone distributions in an original manner. This unification is made through the use of a simple 'shift parameter'. Among its interesting functionalities, it exhibits possibly decreasing and unimodal probability density and hazard rate functions. We examine its quantile function, stochastic dominance, ordinary moments, weighted moments, incomplete moments, and stress-strength reliability coefficient. Then, the classical and Bayesian approaches are developed to estimate the model and stress-strength reliability parameters. Bayes estimates are obtained under the squared error and entropy loss functions. Simulated data are considered to point out the performance of the derived estimates based on the mean squared error. In the final part, the potential of the new model is exemplified by the analysis of two engineering data sets, showing that it is preferable to other reputable and comparable models.

Keywords:

- *Burr distribution; Bayesian inference; Maximum likelihood method; Stress-strength reliability; Data analysis.*

AMS Subject Classification:

• 62E15, 60E05, 62F10.

1. INTRODUCTION

In the analysis of survival data, the researcher attempts to make predictions about the lifetime of all elements / systems by fitting a statistical distribution / model. The underlying distribution of a dataset can then be used to estimate component life characteristics, such as reliability or probability of failure at any given time, average life, failure rate, etc. Reliability is used to assess the characteristics of strength and failure, compare several different models, predict product reliability, etc. In recent years, the Burr type XII (BXII) distribution created in [8] has gained great applicability in the field of reliability / survival analysis and has been discussed by many authors. It is widely recognized as one of the most straightforward and applicable heavy-tailed distributions. The fundamental properties and estimation methods based on the BXII distribution have been derived in [36], [38], [28], [29] and [37]. Due to its flexibility for data modeling, some extensions of the BXII distribution have been introduced in the literature. Among them are the beta BXII distribution (see [32]), Kumaraswamy BXII distribution (see [31]), beta exponentiated BXII distribution (see [25]), Marshall-Olkin BXII distribution (see [6]), McDonald BXII distribution (see [15]), Weibull BXII distribution (see [3]), Kumaraswamy exponentiated distribution (see [26]), generalized Burr-G distribution (see [30]), Topp-Leone BXII distribution (see [34]), transmuted BXII distribution (see [3]), generalized BXII power series distribution (see [13]) and modified BXII distribution (see [19]).

Along with these extended BXII distributions, other successful distributions for modeling survival phenomena have been established in recent years. This is the case for the 'power inverted Topp-Leone (PITL) distribution' invented in [2], which also belongs to the heavy-tailed family of distributions. The first thing to know about the PITL distribution is mathematical; the PITL distribution is the distribution of $(1 - X)^{1/c}X^{-1/c}$, where $c > 0$ and X is a random variable with the classical one-parameter Topp-Leone distribution. It is also the power version of the inverted Topp-Leone (ITL) distribution proposed in [16]. The PITL distribution is motivated in [2] by the following advantages: (i) it benefits from more flexibility compared with the ITL distribution on several aspects, including the shape possibilities of the associated probability density function (pdf) and hazard rate function (hrf), (ii) the inferences of the PITL model are quite manageable with the standard estimation methods, (iii) precise acceptance sampling plans can be developed without difficulty, and (iv) the PITL model is better than other competitive models, a claim illustrated with the analysis of the vinyl chloride data from [7] and the precipitation data from [17].

The purpose of this article is to create an original three-parameter heavy-tailed distribution that unifies the BXII and PITL distributions and to present its main statistical properties. A new tuning parameter that permits a shift between these two famous distributions largely controls this unification. It thus makes it possible to reach a wide range of intermediate distributions with equivalent interests and potentials. The proposed distribution is called the new extended BXII (NEB) distribution. In the first part of the article, we discuss the main characteristics of the NEB distribution, with an emphasis on the role of the shift parameter. Also, some of its functionalities and distributional measures are derived. Among others, we show that the pdf and hrf may be both decreasing and unimodal, which remains a rare feature for a three-parameter heavy-tailed distributions. Then, we examine the quantile function (qf), stochastic dominance, ordinary moments, weighted moments, incomplete moments, and an important measure of system performance: the stress-strength reliability coefficient, defined on the basis of two independent random variables with the NEB distribution. The historical motivations behind this coefficient in a general setting can be found in [12]. The second part of the article is devoted to the inferences of the NEB model. This includes properties, estimation of the model parameters, and estimation of the stress-strength reliability coefficient through classical and Bayesian methods. We now emphasize that the problem of estimating the stress-strength reliability is widely discussed in many articles and remains a common demand in mechanical reliability systems. For the consideration of various lifetime models, we may refer to [27], [22], [33], [21] and, more recently, [9] and [23], and the references cited therein. Following the spirit of these works, the estimation of the stress-strength reliability coefficient in the context of the NEB distribution opens some perspectives in reliability studies. In this regard, we analyze two sets of engineering data. Additionally, statistical comparisons with existing lifetime models that incorporate three or four parameters derived from the BXII model are carried out, and the results are satisfactory for the NEB model.

From the above consideration, we organize the paper as follows: Section 2 defines the NEB distribution along with a selection of its properties. Section 3 concerns the parameters and stress-strength reliability estimates via the maximum likelihood approach, with discussions on their asymptotic distributions. Then, in Section 4, the Bayes estimates are obtained under two different loss functions assuming uniform and gamma prior distributions for the parameters. Sections 5 and 6 provide the applicability of the new distribution and obtain the performance of the estimates. Last, Section 7 provides the concluding remarks.

2. PROPOSED DISTRIBUTION AND ITS PROPERTIES

2.1. DEFINITION AND MOTIVATION

At the basis of the NEB distribution, there is the following analytical result.

Proposition 2.1 *[] Let $a \in [0,2]$ and $c, k > 0$. Then, the following function:*

$$F(x) = \begin{cases} 1 - \dfrac{(1+ax^c)^k}{(1+x^c)^{2k}}, & x > 0, \\ 0, & x \leq 0 \end{cases} \tag{2.1}$$

has the properties of a valid cumulative distribution function (cdf).

Proof: First, it is clear that $F(x) \leq 1$ and, by the Bernoulli inequality, we have $(1+x^c)^2 \geq 1 + 2x^c \geq 1 + ax^c$, implying that $F(x) \geq 0$. Furthermore, $\lim_{x \to 0} F(x) = 0 = F(0)$ implying that $F(x)$ is continuous in 0 and, a fortiori, in $\mathbb{R}$. It is clear that $\lim_{x \to +\infty} F(x) = 1$. Now, for $x > 0$, since $a \in [0,2]$, we have

$$F'(x) = ckx^{c-1}(ax^c + 2 - a)\frac{(1+ax^c)^{k-1}}{(1+x^c)^{2k+1}} \geq 0,$$

implying that $F(x)$ is non-decreasing. The required properties are fulfilled; the function $F(x)$ is a valid cdf. □

Based on Proposition 2.1, we are now in the position to explicit the NEB distribution. The NEB distribution with parameters a, c and k, also denoted as NEB(a, c, k), is defined either with the cdf $F(x)$ given in (2.1) or the pdf specified as

$$f(x) = \begin{cases} ckx^{c-1}(ax^c + 2 - a)\dfrac{(1+ax^c)^{k-1}}{(1+x^c)^{2k+1}}, & x > 0, \\ 0, & x \leq 0. \end{cases} \tag{2.2}$$

It is worth mentioning that c and k are shape parameters, whereas a is a scale parameter.

Basically, a random variable X with the NEB distribution satisfies: $P(X \in D) = \int_D f(x)dx$ for any univariate real domain D and, for any function $\phi(x)$, the expectation of the transformed variable $\phi(X)$, denoted by $E(\phi(X))$, can be expressed in the following integral form: $E(\phi(X)) = \int_{-\infty}^{+\infty} \phi(x)f(x)dx$, provided that it converges (in the integral sense). These two formulas are the basis of measures and known distributional functions based on the moments.

Thus defined, thanks to the parameter a, the NEB distribution constitutes a new lifetime distribution with three parameters extending both the BXII and PITL distributions. More precisely, a can be viewed as a 'shift parameter' that allows a slip between the BXII and PITL distributions in the following sense: when $a = 0$, the NEB distribution becomes the BXII distribution, when $a = 2$, the NEB distribution becomes the PITL distribution, naturally, when $a = 2$ and $c = 1$, the power transformation of the PITL distribution disappears and the NEB distribution becomes the ITL distribution, and, to our knowledge, all the intermediary cases $a \in (0,2)$ bring new distributions.

To realize the possibilities of the NEB distribution modeling, let us now investigate some analytical properties of its pdf. First, when $x \to 0$, the following equivalence holds: $f(x) \sim (2-a)ckx^{c-1}$ and, when $x \to +\infty$, we get $f(x) \sim cka^k x^{-ck-1}$. From these results, we derive the following nuanced limits:

$$\lim_{x\to 0} f(x) = \begin{cases} 0, & c > 1, \\ (2-a)k, & c = 1, \\ +\infty, & c \in (0,1), \end{cases}$$

and $\lim_{x\to+\infty} f(x) = 0$ for all the values of the parameters, the rate of convergence having a polynomial decay governed by the parameter c. Further investigations show that $f(x)$ is a decreasing function for $c \leq 1$ and is unimodal for $c > 1$. The mode can be determined numerically.

Furthermore, using the Riemann integral criteria, we get $\int_0^{+\infty} e^{tx} f(x) dx = +\infty$ for all $t > 0$, meaning that the NEB distribution is heavy right-tailed. It thus keeps the heavy-tailed nature of its parental distributions: the BXII and PITL distributions.

For more remarks, Figure 1 shows some possible shapes of the pdf with diverse values for a, c and k.

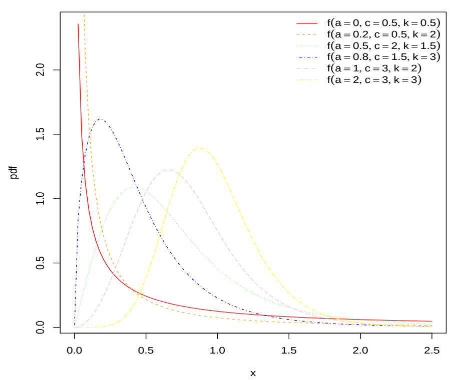

Figure 1: Panel of shapes of the pdf of the NEB distribution.

Figure 1 illustrates the decreasing and unimodal nature of $f(x)$. It is also shown that $f(x)$ has a versatile mode which is greatly affected by the parameter a. Almost symmetrical shapes can be seen, as in the yellow curve, also corresponding to the case $a = 2$ referring to the PTIL distribution. Moreover, Figure 1 illusrates the compromise that the NEB distribution made between the BXII and PITL distributions.

2.2. COMPLEMENTARY FUNCTIONS

We now focus on important reliability functions that may appear in various aspects of the NEB distribution analysis. The survival function (sf) and hrf of the NEB distribution are inscribed as

$$\bar{F}(x) = 1 - F(x) = \begin{cases} \dfrac{(1+ax^c)^k}{(1+x^c)^{2k}}, & x > 0, \\ 1, & x \le 0 \end{cases}$$

and

$$h(x) = \frac{f(x)}{\bar{F}(x)} = \begin{cases} ckx^{c-1}\dfrac{ax^c + 2 - a}{(1+ax^c)(1+x^c)}, & x > 0, \\ 0, & x \le 0, \end{cases}$$

respectively. An asymptotic study of $h(x)$ is now provided. First, when $x \to 0$, the following equivalence holds: $h(x) \sim (2-a)ckx^{c-1}$, and when $x \to +\infty$, we obtain $h(x) \sim ckx^{-1}$. From these results, we derive the following limits:

$$\lim_{x \to 0} h(x) = \begin{cases} 0, & c > 1, \\ (2-a)k, & c = 1, \\ +\infty, & c \in (0,1), \end{cases}$$

and $\lim_{x \to +\infty} h(x) = 0$ for the values of the parameters. Since the variety of shapes is an important indicator on the modeling flexibility of a distribution (see [1]), we provide a graphical analysis of $h(x)$ in Figure 2 with diverse values for a, c and k.

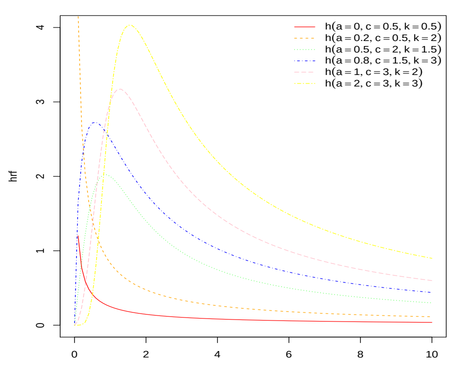

Figure 2: Panel of shapes of the hrf of the NEB distribution.

From Figure 2, we see that $h(x)$ has the same global shapes properties than $f(x)$, only varying on the weights of the tails: it is decreasing for $c \le 1$ and has only one maximal point for $c > 1$. The parameter a mainly affects the value of the maximal point. Hence, the so-called decreasing and bathtub upside-down hazard rates of survival data can be reached by the NEB model.

We complete the presentation of the NEB distribution by expressing its qf. The notion of qf is very useful on various aspects in probability and statistics; it is at the same level of importance as the cdf to define a distribution (see [14]). The expression of the qf of the NEB distribution follows through the solution of the following nonlinear equation: $F(x) = u$ with respect to x. After a step-by-step development, we come to

$$Q(u) = \frac{1}{2^{1/c}} \left\{ -[2 - a(1-u)^{-1/k}] + \sqrt{[2 - a(1-u)^{-1/k}]^2 - 4[1 - (1-u)^{-1/k}]} \right\}^{1/c},$$

where $u \in (0,1)$. As a basic application, the three quartiles of the NEB distribution are given by $Q_1 = Q(1/4)$, $Q_2 = Q(1/2)$ and $Q_3 = Q(3/4)$, respectively. Also, among the possible uses of this qf, one can use it to generate values from any random variable with the NEB distribution, define diverse distributional functions analogous to the pdf and hrf, and various measures on skewness and kurtosis.

2.3. STOCHASTIC DOMINANCE

The NEB distribution has several stochastic dominance properties involving $F(x)$ which are of interest in understanding the roles of the parameters a, c and k for distributional comparison. Here, we focus on the notion of first-order stochastic (fos) dominance as presented in [35].

Proposition 2.2 *[] The following stochastic order properties hold: if $a_2 \geq a_1$, the NEB distribution defined with $a = a_2$ fos dominates the NEB distribution defined with $a = a_1$; if $k_2 \geq k_1$, the NEB distribution defined with $k = k_1$ fos dominates the NEB distribution defined with $k = k_2$.*

Proof: The proof is based on the monotonicity of $F(x) = F(x; a, c, k)$ with respect to the parameters. We have

$$\frac{\partial}{\partial a} F(x; a, c, k) = -kx^c \frac{(1 + ax^c)^{k-1}}{(1 + x^c)^{2k}} \leq 0,$$

which means that $F(x)$ is a decreasing function with respect to a, implying that, if $a_2 \geq a_1$, the NEB distribution defined with $a = a_2$ fos the NEB distribution defined with $a = a_1$. Now, we have

$$\frac{\partial}{\partial k} F(x; a, c, k) = \frac{(1 + ax^c)^k}{(1 + x^c)^{2k}} [2 \log(1 + x^c) - \log(1 + ax^c)] \geq 0,$$

which means that $F(x)$ is an increasing function with respect to k, implying that, if $k_2 \geq k_1$, the NEB distribution defined with $k = k_1$ fos dominates the NEB distribution defined with $k = k_2$. This ends the proof of the three items of the proposition. □

Thus, based on Proposition 2.2, we see that the parameter c has the most complex role for the comparison of NEB distributions differing with their parameters. Moreover, the first result and the expression of $F(x)$ justify the naming of 'shift parameter' for a.

2.4. MOMENT PROPERTIES

The following result concerns the ordinary moments of the NEB distribution.

Proposition 2.3 *[] Let X be a random variable with the NEB distribution and r be an integer. Then, X admits an r^{th} ordinary moment, i.e., $\mu_r' = E(X^r)$, if and only if $r < ck$. In this case, μ_r' can be expressed as the following infinite sum expansion:*

$$\mu_r' = k\sum_{\ell=0}^{+\infty} \binom{k-1}{\ell}(a-1)^{\ell}\left[aB\left(\frac{r}{c}+\ell+2, k-\frac{r}{c}\right)+(2-a)B\left(\frac{r}{c}+\ell+1, k+1-\frac{r}{c}\right)\right],$$

where $B(u,v)$ is the beta function: $B(u,v)=\int_0^1 t^{u-1}(1-t)^{v-1}dt$ with $u,v>0$.

Proof: Provided that it exists, we have $\mu_r' = \int_{-\infty}^{+\infty} x^r f(x)dx$. In view of the definition of $f(x)$ in (2.2), only the neighborhoods of $x=0$ and $+\infty$ of the function $x^r f(x)$ need processing, and we can invoke the integral Riemann criteria in this regard. In the neighborhood of $x=0$, we have $x^r f(x) \sim ck(2-a)x^{r+c-1}$, which is the main term of a convergent integral over $x \in (0,d)$ with $d>0$ if and only if $r+c>0$, which is always fulfilled. Also, in the neighborhood of $x=+\infty$, we have $x^r f(x) \sim cka^k x^{r-ck-1}$ which is the main term of a convergent integral over $x \in (d,+\infty)$ if and only if $r-ck<0$, which is satisfied if $r<kc$. In the end, μ_r' exists if and only $r<ck$.

In this case, in order to express $\mu_r' = \int_{-\infty}^{+\infty} x^r f(x)dx$ as desired, for $x>0$, we set $f(x)=f_1(x)+f_2(x)$, where

$$f_1(x) = ckax^{2c-1}\frac{(1+ax^c)^{k-1}}{(1+x^c)^{2k+1}}, \quad f_2(x) = ck(2-a)x^{c-1}\frac{(1+ax^c)^{k-1}}{(1+x^c)^{2k+1}},$$

which can be also written as

$$f_1(x) = ckax^{2c-1}\frac{[1+(a-1)x^c/(1+x^c)]^{k-1}}{(1+x^c)^{k+2}}$$

and

$$f_2(x) = ck(2-a)x^{c-1}\frac{[1+(a-1)x^c/(1+x^c)]^{k-1}}{(1+x^c)^{k+2}}.$$

Since $a \in [0, 2]$ and $x > 0$, it is clear that $|(a-1)x^c/(1+x^c)| < 1$. Therefore, the generalized version of the binomial formula gives

$$\left[1 + (a-1)\frac{x^c}{1+x^c}\right]^{k-1} = \sum_{\ell=0}^{+\infty} \binom{k-1}{\ell} (a-1)^\ell \frac{x^{c\ell}}{(1+x^c)^\ell}.$$

Note that the limit $+\infty$ can be replaced by $k-1$ if k is an integer greater to 1. So $f_1(x)$ and $f_2(x)$ can be expressed as

$$f_1(x) = cka \sum_{\ell=0}^{+\infty} \binom{k-1}{\ell} (a-1)^\ell \frac{x^{c(\ell+2)-1}}{(1+x^c)^{\ell+k+2}}$$

and

$$f_2(x) = ck(2-a) \sum_{\ell=0}^{+\infty} \binom{k-1}{\ell} (a-1)^\ell \frac{x^{c(\ell+1)-1}}{(1+x^c)^{\ell+k+2}},$$

respectively. By invoking the dominated convergence theorem to justify the exchange of the signs $\sum$ and $\int$, we obtain

$$\begin{aligned} \mu'_r &= \int_0^{+\infty} x^r f_1(x)dx + \int_0^{+\infty} x^r f_2(x)dx \\ &= cka \sum_{\ell=0}^{+\infty} \binom{k-1}{\ell} (a-1)^\ell \int_0^{+\infty} \frac{x^{r+c(\ell+2)-1}}{(1+x^c)^{\ell+k+2}} dx \\ &+ ck(2-a) \sum_{\ell=0}^{+\infty} \binom{k-1}{\ell} (a-1)^\ell \int_0^{+\infty} \frac{x^{r+c(\ell+1)-1}}{(1+x^c)^{\ell+k+2}} dx. \end{aligned} \tag{2.3}$$

With the change of variable $y = x^c$, the two integral terms can be expressed as

$$\int_0^{+\infty} \frac{x^{r+c(\ell+2)-1}}{(1+x^c)^{\ell+k+2}} dx = \frac{1}{c} \int_0^{+\infty} \frac{y^{r/c+\ell+1}}{(1+y)^{\ell+k+2}} dy = \frac{1}{c} B\left(\frac{r}{c} + \ell + 2, k - \frac{r}{c}\right)$$

and

$$\int_0^{+\infty} \frac{x^{r+c(\ell+1)-1}}{(1+x^c)^{\ell+k+2}} dx = \frac{1}{c} \int_0^{+\infty} \frac{y^{r/c+\ell}}{(1+y)^{\ell+k+2}} dy = \frac{1}{c} B\left(\frac{r}{c} + \ell + 1, k + 1 - \frac{r}{c}\right).$$

By putting these equations into (2.3), we obtain the stated result. □

In any case, if $r < ck$, μ'_r can be evaluated in a numerical way by using any standard mathematical software.

With one of these approaches, we are able to evaluate standard moment measures such as the mean of X specified by $\mu = \mu'_1$ and the variance of X given as $V = \mu'_2 - \mu^2$, as well as moment measures of skewness and kurtosis.

The incomplete moments of X taken at a specific value $t \geq 0$ is also of interest. It can be expanded as described in the new result.

Proposition 2.4 *[] Let X be a random variable with the NEB distribution, r be an integer and $t \geq 0$. Then, X admits incomplete moments of all orders and the r^{th} incomplete moment of X at the level t, i.e., $\mu'_r(t) = E(X 1_{\{X \leq t\}})$, can be expressed as the following infinite sum expansion:*

$$\mu'_r(t) = k \sum_{\ell=0}^{+\infty} \binom{k-1}{\ell} (a-1)^{\ell} \times$$
$$\left[a B_{t^c/(1+t^c)} \left(\frac{r}{c} + \ell + 2, k - \frac{r}{c} \right) + (2-a) B_{t^c/(1+t^c)} \left(\frac{r}{c} + \ell + 1, k + 1 - \frac{r}{c} \right) \right],$$

where $B_x(u,v)$ denotes the incomplete beta function taken at x: $B_x(u,v) = \int_0^x t^{u-1}(1-t)^{v-1} dt$ with $x \in [0,1]$ and $u, v > 0$.

Proof: The proof is almost identical to the one of Proposition 2.3, we thus omit it. □

Following the spirit of [2], we can use the incomplete moments of X to define several inequality measures, and various residual life functions, as well as the related moments. We end this part with a generalization of the ordinary moments by investigating the weighted probability moments.

Proposition 2.5 *[] Let X be a random variable with the NEB distribution, and r and s be integers. Then, X admits an $(r,s)^{th}$ probability weighted moment, i.e., $\mu'_{r,s} = E(X^r \bar{F}(X)^s)$, if and only if $r < ck$. In this case, $\mu'_{r,s}$ can be expressed as the following infinite sum expansion:*

$$\mu'_{r,s} = \frac{k}{1+s} \sum_{\ell=0}^{+\infty} \binom{k(1+s)-1}{\ell} (a-1)^{\ell} \times$$
$$\left[a B \left(\frac{r}{c} + \ell + 2, k(1+s) - \frac{r}{c} \right) + (2-a) B \left(\frac{r}{c} + \ell + 1, k(1+s) + 1 - \frac{r}{c} \right) \right].$$

Proof: First, let us notice that, for $x > 0$,

$$f(x)\bar{F}(x)^s = \frac{1}{1+s} ck(1+s) x^{c-1} (ax^c + 2 - a) \frac{(1+ax^c)^{k(1+s)-1}}{(1+x^c)^{2k(1+s)+1}} = \frac{1}{1+s} f_\circ(x),$$

where $f_\circ(x)$ denotes the pdf of the NEB distribution with parameters a, c and $k(1+s)$. Therefore, we have

$$\mu'_{r,s} = \int_{-\infty}^{+\infty} x^r f(x) \bar{F}(x)^s dx = \frac{1}{1+s} \int_0^{+\infty} x^r f_\circ(x) dx = \frac{1}{1+s} \mu'^{\circ}_r,$$

where μ'°_r denotes the r^{th} ordinary moment of a random variable with the NEB distribution with parameters a, c and $k(1+s)$. Hence, the desired result follows from Proposition 2.3 with adjustment on the definition of the parameters. □

Probability-weighted moments can be considered as extended versions of the ordinary moments. Also, they appear in the theory of order statistics, and remain standard in several branches of statistics. On this topic, we may refer to [18].

2.5. STRESS-STRENGTH RELIABILITY COEFFICIENT

Let X and Y be two independent random variables following the NEB distributions with parameters a, c and k_1, and a, c and k_2, respectively. We are interested in the determination of the common stress-strength reliability coefficient defined by

$$R = P(Y < X). \tag{2.4}$$

This coefficient is a measure of reliability of a component with strength modeled by X, subject to a stress modeled by Y. Further details on this special coefficient can be found in [12].

Proposition 2.6 *[] The coefficient R precised in (2.4) is*

$$R = \frac{k_2}{k_1 + k_2}.$$

Proof: Let $F_2(x)$ be the cdf of Y and $f_1(x)$ be the pdf of X. Then, based on (2.1) and (2.2), after a linear integral development, we get

$$R = \int_{-\infty}^{+\infty} F_2(x) f_1(x) dx = \int_0^{+\infty} \left[1 - \frac{(1 + ax^c)^{k_2}}{(1 + x^c)^{2k_2}}\right] \times$$
$$ck_1 x^{c-1}(ax^c + 2 - a)\frac{(1 + ax^c)^{k_1 - 1}}{(1 + x^c)^{2k_1+1}} dx$$
$$= 1 - \int_0^{+\infty} ck_1 x^{c-1}(ax^c + 2 - a)\frac{(1 + ax^c)^{k_1+k_2-1}}{(1 + x^c)^{2(k_1+k_2)+1}} dx$$
$$= 1 - \frac{k_1}{k_1 + k_2}\int_0^{+\infty} c(k_1 + k_2)x^{c-1}(ax^c + 2 - a)\frac{(1 + ax^c)^{k_1+k_2-1}}{(1 + x^c)^{2(k_1+k_2)+1}} dx.$$

Note that the last integral term is equal to one since it corresponds to the integral of a pdf over its whole support; it is the pdf of the NEB distribution with parameters a, c and $k_1 + k_2$. Hence $R = 1 - k_1/(k_1 + k_2) = k_2/(k_1 + k_2)$. This ends the proof. □

Thus, in the configuration of Proposition 2.6, R has a quite simple expression. It is decreasing with respect to k_1, whereas it is increasing with respect to k_2. If $k_1 = k_2$, we get $R = 1/2$ meaning that there is a equal chance of Y to be greater than X, and vice-versa.

The rest of the article is devoted to the inferences of the NEB model, beginning with the estimation of the model parameters through the maximum likelihood approach.

3. MAXIMUM LIKELIHOOD ESTIMATION

3.1. ESTIMATION OF THE PARAMETERS

Let n be a positive parameter. Let us denote by $x_1, \ldots, x_n$ n independent observations from the NEB distribution. Then, the maximum likelihood method proposes to used the maximum likelihood estimates (MLEs) $\hat{a}$, $\hat{c}$ and $\hat{k}$ of a, c and k, respectively, defined by $(\hat{a}, \hat{c}, \hat{k}) = \text{argmax}_{(a,c,k)\in[0,2]\times(0,+\infty)^2}\, \ell(a, c, k)$, where $\ell(a, c, k)$ denotes the log-likelihood function defined by

$$\ell(a, c, k) = n \log c + n \log k + (c - 1) \sum_{i=1}^{n} \log(x_i) + \sum_{i=1}^{n} \log(a x_i^c + 2 - a) \\ + (k - 1) \sum_{i=1}^{n} \log(1 + a x_i^c) - (2k + 1) \sum_{i=1}^{n} \log(1 + x_i^c).$$

The MLEs $\hat{a}$, $\hat{c}$ and $\hat{k}$ can be determined through the score equations. Now let $\hat{V}_a$, $\hat{V}_c$ and $\hat{V}_k$ defined by $(\hat{V}_a, \hat{V}_c, \hat{V}_k) = \text{diag}\left[I(a, c, k)^{-1}\right] |_{a=\hat{a},c=\hat{c},k=\hat{k}}$, where

$$I(a, c, k) = \left(-\frac{\partial^2}{\partial u \partial v} \ell(a, c, k)\right)_{(u,v)=(a,c,k)^2}.$$

By applying a the well-known asymptotic property of the MLEs, as m and n tends to $+\infty$, the underlying distribution of

$$\left\{(1/\sqrt{\hat{V}_a})(\hat{a} - a), (1/\sqrt{\hat{V}_c})(\hat{c} - c), (1/\sqrt{\hat{V}_k})(\hat{k} - k)\right\}$$

can be approximated by the standard trivariate normal distribution. As an immediate consequence, a two-sided asymptotic $100(1 - \alpha)\%$ confidence interval of a with $\alpha \in (0, 1)$ is given as $I_a = \left[\hat{a} - u_\alpha \sqrt{\hat{V}_a}, \hat{a} + u_\alpha \sqrt{\hat{V}_a}\right]$, where $u_\alpha = Q_U(1 - \alpha/2)$, $Q_U(x)$ denoting the qf of the standard univariate normal distribution. Analogous two-sided asymptotic $100(1 - \alpha)\%$ confidence intervals for c and k can be presented in a similar way. The general theory and formulas of the maximum likelihood approach can be found in [11].

3.2. ESTIMATION OF R

We now focus on the estimation of the stress-strength reliability coefficient R as described in Subsection 2.5, recalling that $R = k_2/(k_1+k_2)$. Such estimation problem is of interest in various applied studied, as motivated in [27], [22], [33], [21], [9], [23] and [5]. We follow the same methodology as the one employed in [5].

Let n and m be two positive integers. Let us denote by $x_1, \ldots, x_n$ n independent observations from the NEB distribution with parameters a, c and k_1, and $y_1, \ldots, y_m$ m independent observations from the NEB distribution with parameters a, c and k_2, assuming that a and c are known. Then, the log-likelihood function based on these two samples is given by

$$\ell(k_1, k_2) = (n+m)\log c + n\log k_1 + (c-1)\sum_{i=1}^{n}\log(x_i) + \sum_{i=1}^{n}\log(ax_i^c + 2 - a)$$
$$+(k_1-1)\sum_{i=1}^{n}\log(1+ax_i^c) - (2k_1+1)\sum_{i=1}^{n}\log(1+x_i^c) + m\log k_2 + (c-1)\sum_{i=1}^{m}\log(y_i)$$
$$+\sum_{i=1}^{m}\log(ay_i^c + 2 - a) + (k_2-1)\sum_{i=1}^{m}\log(1+ay_i^c) - (2k_2+1)\sum_{i=1}^{m}\log(1+y_i^c).$$

The MLEs $\hat{k}_1$ and k_2 of k_1 and k_2, respectively, are obtained as

$$(\hat{k}_1, \hat{k}_2) = \operatorname{argmax}_{(k_1,k_2)\in(0,+\infty)^2}\, \ell(k_1, k_2).$$

Classically, they satisfy the score equations defined by $\partial\ell(k_1,k_2)/\partial k_1 \mid_{k_1=\hat{k}_1, k_2=\hat{k}_2} = 0$ and $\partial\ell(k_1,k_2)/\partial k_2 \mid_{k_1=\hat{k}_1, k_2=\hat{k}_2} = 0$, which give

$$\hat{k}_1 = \left\{-\frac{1}{n}\sum_{i=1}^{n}\log\left(\frac{1+ax_i^c}{(1+x_i^c)^2}\right)\right\}^{-1}, \quad \hat{k}_2 = \left\{-\frac{1}{m}\sum_{i=1}^{m}\log\left(\frac{1+ay_i^c}{(1+y_i^c)^2}\right)\right\}^{-1}.$$

Now, we have

$$\frac{\partial^2}{\partial k_1^2}\ell(k_1,k_2) = -\frac{n}{k_1^2}, \quad \frac{\partial^2}{\partial k_2^2}\ell(k_1,k_2) = -\frac{m}{k_2^2}, \quad \frac{\partial^2}{\partial k_1 k_2}\ell(k_1,k_2) = 0.$$

By applying a the well-known asymptotic property of the MLEs, as m and n tends to $+\infty$, the underlying distribution of $\left\{(\hat{k}_1/\sqrt{n})(\hat{k}_1 - k_1), (\hat{k}_2/\sqrt{m})(\hat{k}_2 - k_2)\right\}$ can be approximated by the standard bivariate normal distribution. On the other side, by substitution, a point estimate for R is obtained as

$$\hat{R} = \frac{\hat{k}_2}{\hat{k}_1 + \hat{k}_2}. \tag{3.1}$$

By applying the multivariate delta method (see [20]), since the underlying random estimates of k_1 and k_2 are independent, an estimate for the variance of the underlying random estimate of R is inscribed as

$$\hat{V}_R = \left(-\frac{\partial^2}{\partial k_1^2}\ell(k_1,k_2)\right)^{-1}\left(\frac{\partial}{\partial k_1}R\right)^2 + \left(-\frac{\partial^2}{\partial k_2^2}\ell(k_1,k_2)\right)^{-1}\left(\frac{\partial}{\partial k_2}R\right)^2\Bigg|_{k_1=\hat{k}_1, k_2=\hat{k}_2}$$
$$= \frac{\hat{k}_1^2\hat{k}_2^2}{(\hat{k}_1+\hat{k}_2)^4}\left(\frac{1}{n}+\frac{1}{m}\right).$$

Therefore, as m and n tends to $+\infty$, the underlying distribution of $(1/\sqrt{\hat{V}_R})(\hat{R} - R)$ can be approximated by the standard univariate normal distribution. As an immediate consequence, a two-sided asymptotic $100(1-\alpha)\%$ confidence interval of R is given as

$$I_R = \left[\hat{R} - u_\alpha \frac{\hat{k}_1 \hat{k}_2}{(\hat{k}_1+\hat{k}_2)^2}\sqrt{\frac{1}{n}+\frac{1}{m}}, \hat{R} + u_\alpha \frac{\hat{k}_1 \hat{k}_2}{(\hat{k}_1+\hat{k}_2)^2}\sqrt{\frac{1}{n}+\frac{1}{m}}\right].$$

The rest of the study focuses on the Bayesian inferences of the NEB model, with applications.

4. BAYESIAN INFERENCE

In the Bayesian framework, not only data but also prior information about the unknown parameter is used to analyze the data and draw conclusions. In this way, Bayesian inference incorporates the prior distribution of the model parameters with the likelihood function to produce the posterior distribution that gathers more quality inferences and controls the uncertainty. However, the choice of a suitable prior has a significant role in changing the result. If sufficient information is available about the parameter, then an informative prior is considered; otherwise, one can use a non-informative prior.

Here, we consider both informative and non-informative priors for the Bayesian analysis of the unknown model parameters and stress-strength reliability coefficient of the NEB distribution. Since the shape of the proposed distribution is skewed to the right, we use a gamma prior as a skewed distribution for the independent parameters k_1, k_2 and c, whereas a follows a uniformly distributed prior. Indeed, we know that the gamma distribution is very flexible and is used frequently everywhere. A slight change in the parameters is also observed, as are changes in the shape of the distributions. So, we consider this prior for the Bayesian computation in our manuscript. Because a is the scale parameter, it has little effect on the distribution's shape. As a result, we can easily consider the improper prior in place for uniform distribution. The description of the said priors can be summarized as follows: $\pi(k_1) = \text{Gamma}(r_1, s_1)$, $r_1 > 0, s_1 > 0$, $\pi(k_2) = \text{Gamma}(r_2, s_2)$, $r_2 > 0, s_2 > 0$, $\pi(c) = \text{Gamma}(r_3, s_3)$, $r_3 > 0, s_3 > 0$ and $\pi(a) \propto 1$, $a \in [0, 2]$, where $\text{Gamma}(r, s)$ denotes the standard gamma distribution with 'shape parameter' r and 'scale parameter' s, and $(r_1, s_1, r_2, s_2, r_3, s_3)$ are called the hyper-parameters. One can notice that, if $r_1 = s_1 = r_2 = s_2 = r_3 = s_3 = 0$, the prior is reduced to a non-informative form of gamma prior. Consequently, the joint prior $\pi(\Theta = (c, a, k_1, k_2))$ is defined as follows:

$$\pi(\Theta) = \frac{s_1^{r_1} s_2^{r_2} s_3^{r_3}}{\Gamma(r_1)\Gamma(r_2)\Gamma(r_3)} k_1^{r_1-1} k_2^{r_2-1} c^{r_3-1} e^{-k_1 s_1 - k_2 s_2 - c s_3},$$

where $\Gamma(u)$ is the gamma function, i.e., $\Gamma(u) = \int_0^{+\infty} t^{u-1} e^{-t} dt$, $u > 0$.

The posterior distribution $\pi(\Theta|\mathbf{data})$ of the parametric space (Θ) is obtained by incorporation of likelihood function $(L(\Theta|\mathbf{data}))$ with the joint prior distribution $\pi(\Theta)$, that is

$$\pi(\Theta|\mathbf{data}) = KL(\Theta|\mathbf{data})\,\pi(\Theta)$$
$$= Kc^{n+m+r_3-1}k_1^{n+r_1-1}k_2^{m+r_2-1}\frac{s_1^{r_1}s_2^{r_2}s_3^{r_3}e^{-k_1s_1-k_2s_2-cs_3}}{\Gamma(r_1)\Gamma(r_2)\Gamma(r_3)}\times$$
$$\prod_{i=1}^{n}\frac{x_i^{c-1}(ax_i^c+2-a)(1+ax_i^c)^{k_1-1}}{(1+x_i^c)^{2k_1+1}}\prod_{j=1}^{m}\frac{y_j^{c-1}(ay_j^c+2-a)(1+ay_j^c)^{k_2-1}}{(1+y_j^c)^{2k_2+1}},$$

where K is a constant such that $K^{-1}=\int L(\Theta|\mathbf{data})\,\pi(\Theta)\,d\Theta$.

Based on decision theory, it is a well known discussion that the best estimate decision depends on the pattern of the loss function adopted for a particular situation and the resulting outcome may be under or / and over estimation. If the amount of loss is equal in under and over estimation then the symmetric loss function is considered. On the other situations, the asymmetric loss function is useful when positive loss may be more serious than a given negative loss of the same magnitude or vice-versa. Here, we employ both asymmetric and symmetric loss functions to investigate the suitability of the loss functions for the model. More precisely, we use the squared error (symmetric) loss function (SELF) and entropy (asymmetric) loss function (ELF). The SELF and ELF are inscribed as $L_{SELF}(\theta,\hat{\theta})=\left(\hat{\theta}-\theta\right)^2$ and $L_{ELF}(\theta,\hat{\theta})=\hat{\theta}/\theta-\log\left(\hat{\theta}/\theta\right)-1$, respectively. Under the SELF and ELF, for any parametric function, say $\phi(\Theta)$, the Bayes estimate is obtained as follows:

$$\phi^{\star}_{self}(\Theta|\mathbf{data}) = K\int\phi(\Theta)\,\pi(\Theta|\mathbf{data})\,d\Theta \tag{4.1}$$

and

$$\phi^{\star}_{elf}(\Theta|\mathbf{data}) = \left(K\int\phi^{-1}(\Theta)\,\pi(\Theta|\mathbf{data}\;)\,d\Theta\right)^{-1}, \tag{4.2}$$

respectively. To obtain (4.1) and (4.2), we get the Bayesian estimates of the model parameters as well as the stress-strength reliability coefficient, where the Bayes estimate under the SELF is the posterior mean and under the ELF is the inverse of the harmonic mean. Due to the presence of multiple integrations in equations (4.1) and (4.2), they are very difficult to solve in an exact manner. Therefore, an iterative numerical procedure is required to solve these equations. For this situation, the Markov Chain Monte Carlo (MCMC) technique is suggested to generate a sequence of random draws from posteriors of interest. Using the MCMC method, a stochastic chain is produced that contains a sequence of random samples. The Gibbs sampling and the Metropolis-Hastings (MH) algorithm are two approaches in MCMC to computing the posterior distribution. To implement these approaches, the full conditional posterior distribution is derived for the study parameters. By putting $\underline{\mathbf{x}}=(x_1,\ldots,x_n)$ and $\underline{\mathbf{y}}=(y_1,\ldots,y_n)$, they are given as follows:

$$\pi_1\left(c|a,k_1,k_2,\underline{\mathbf{x}},\underline{\mathbf{y}}\right) \propto c^{n+m+r_3-1}e^{-cs_3}\prod_{i=1}^{n}\frac{x_i^{c-1}(ax_i^c+2-a)(1+ax_i^c)^{k_1-1}}{(1+x_i^c)^{2k_1+1}}\times \prod_{j=1}^{m}\frac{y_j^{c-1}(ay_j^c+2-a)(1+ay_j^c)^{k_2-1}}{(1+y_j^c)^{2k_2+1}},$$

$$\pi_2\left(a|c,k_1,k_2,\underline{\mathbf{x}},\underline{\mathbf{y}}\right) \propto \prod_{i=1}^{n}(ax_i^c+2-a)(1+ax_i^c)^{k_1-1}\prod_{j=1}^{m}(ay_j^c+2-a)(1+ay_j^c)^{k_2-1},$$

$$\pi_3\left(k_1|a,c,\underline{\mathbf{x}},\underline{\mathbf{y}}\right) \propto k_1^{n+r_1-1}e^{-k_1s_1}\prod_{i=1}^{n}\frac{(1+ax_i^c)^{k_1-1}}{(1+x_i^c)^{2k_1+1}}$$

and

$$\pi_4\left(k_2|a,c,\underline{\mathbf{x}},\underline{\mathbf{y}}\right) \propto k_2^{m+r_2-1}e^{-k_2s_2}\prod_{j=1}^{m}\frac{(1+ay_j^c)^{k_2-1}}{(1+y_j^c)^{2k_2+1}}.$$

Based on estimated values of the parameters $(\hat{k}_1,\hat{k}_2)$, the estimated value of stress-strength reliability coefficient is obtained. To evaluate the above conditional posterior distribution, the following steps are considered:

Step 1. Starting with an initial value vector $\Theta^0=(c^0,a^0,k_1^0,k_2^0)$ and set $l=1$.

Step 2. Generate the point vector $\Theta^p=(c^p,a^p,k_1^p,k_2^p)$ from the candidate proposal density $q\left(\Theta^p|\Theta^0\right)$ where $q\left(\Theta^p|\Theta^0\right)$ proposes a probability with a move Θ^p, having conditional probability density given Θ^0.

Step 3. Determine the Hastings-ratio using Θ^p and Θ^0 as specified by

$$\rho\left(\Theta^p|\Theta^0\right)=\frac{\pi_1\left(c^p|a^0,k_1^0,k_2^0,\mathbf{data}\right)q\left(\Theta^p|\Theta^0\right)}{\pi_1\left(c^0|a^0,k_1^0,k_2^0,\mathbf{data}\right)q\left(\Theta^0|\Theta^p\right)}.$$

Similarly for the remaining parameters, the Hastings-ratio is obtained.

Step 4. Take into account Θ^p with probability $\gamma\le\min\left[1,\rho\left(\Theta^p|\Theta^0\right)\right]$, otherwise $\Theta=\Theta^0$ with rejection probability $1-\gamma$, where γ is generated from the uniform $U(0,1)$ distribution.

Step 5. Repeat Steps 2-4, $K=5000$ times and record the sequence of parameter observations. Next, we get the Bayes estimate under different loss functions.

5. SIMULATION STUDY

This section performs a simulation experiment to determine the effectiveness of the proposed method in the model parameters as well as the stress-strength reliability coefficient for the NEB distribution. For this, various sample sizes, along with different sets of parameter values, are considered for making better inferences. We take the following sample size combinations, namely, $(n, m) = \{(20, 20), (30, 50), (50, 30), (50, 50), (40, 60), (60, 40), (40, 40)\}$ and different sets of stress-strength reliability coefficient values, namely $(k_1, k_2) = \{(2, 1), (2, 2), (1, 2)\}$ so that the true reliability parameter values are small (0.33), moderate (0.50) and high (0.67), respectively. The remaining parameter values are $a = 2.5$ and $c = 1.5$. We evaluate the performance of the stress-strength reliability coefficient on the basis of simulated samples with diverse sample sizes and combinations using the R software. To this end, we simulate a random sample of different sizes from the NEB distribution. In this regard, we use the Newton steps to generate a sample of size n from the NEB(a, c, k) distribution by following the steps below:

Step 1. Set n, a, c and k.

Step 2. Set initial value x_0.

Step 3. Set $j = 1$.

Step 4. Generate a value u from the uniform $U(0, 1)$ distribution.

Step 5. Update x_0 through the Newton formula for solving $F(x) = u$ such as $x_{new} = x_0 - \dfrac{F(x_0) - u}{f(x_0)}$, with the defined with the used parameters a, c and k.

Step 6. If $|x_0 - x_{new}| \leq \epsilon$ with $\epsilon > 0$ chosen as small, then x_{new} will be the desired value from $F(x)$.

Step 7. If $|x_0 - x_{new}| > \epsilon$, then, set $x_0 = x_{new}$ and go to Step 5.

Step 8. Repeat Steps 4-7, for $j = 1, 2, \ldots, n$ and obtained $x_1, x_2, \ldots, x_n$.

Using the generated samples, the maximum likelihood and Bayes estimates are obtained based on derived estimates of the parameters and reliability function. For the Bayes estimates, we use different loss functions under different priors and the hyper-parameters of the gamma prior are taken as follows:

1. When $r_1 = s_1 = r_2 = s_2 = r_3 = s_3 = 0$ (the non-informative prior case), the Bayes estimates are denoted as $SELF_0$ and ELF_0.
2. When the prior means are equal to the true value of parameters and the prior variances are equal to 1, the Bayes estimates are denoted as $SELF_1$ and ELF_1.

The results are based on 5000 replications. We vary the sample sizes with fixed values of the stress-strength reliability coefficient and for various combinations of the model parameters with fixed samples sizes. For different parameter values, different sample sizes and different priors under both SELF and ELF, we report the average estimates (AVs) and the corresponding mean squared errors (MSEs) of the MLEs and Bayes estimates of the model parameters and stress-strength reliability coefficient. The simulation results are postponed in Tables 1-5.

Table 1: AVs and MSEs of the estimates of R with varying n and m

			MLE		$SELF_0$		ELF_0		$SELF_1$		ELF_1	
(k_1, k_2)	R	(n, m)	AV	MSE	AV	MSE	AV	MSE	AV	MSE	AV	MSE
(2,1)	0.33	(20,20)	0.3919	0.0031	0.3656	0.0027	0.3772	0.0026	0.3559	0.0026	0.3711	0.0025
		(30,50)	0.3674	0.0024	0.3381	0.0016	0.3597	0.0019	0.3351	0.0014	0.3332	0.0014
		(50,30)	0.3670	0.0025	0.3568	0.0023	0.3592	0.0021	0.3427	0.0013	0.3378	0.0011
		(50,50)	0.3710	0.0022	0.3441	0.0016	0.3417	0.0015	0.3560	0.0015	0.3651	0.0018
		(40,60)	0.3677	0.0023	0.3458	0.0015	0.3616	0.0019	0.3436	0.0014	0.3407	0.0014
		(60,40)	0.3683	0.0019	0.3585	0.0016	0.3543	0.0014	0.3584	0.0014	0.3621	0.0015
		(40,40)	0.3727	0.0029	0.3439	0.0015	0.3399	0.0014	0.3550	0.0021	0.3654	0.0024
(2,2)	0.5	(20,20)	0.4715	0.0036	0.4789	0.0033	0.4795	0.0033	0.4843	0.0029	0.4859	0.003
		(30,50)	0.4783	0.0021	0.4827	0.0020	0.4851	0.0018	0.4954	0.0011	0.4893	0.0012
		(50,30)	0.4892	0.0031	0.4977	0.0030	0.4960	0.0029	0.5128	0.0020	0.5070	0.0019
		(50,50)	0.4827	0.0020	0.5042	0.0011	0.5041	0.0010	0.4848	0.0016	0.4800	0.0018
		(40,60)	0.5064	0.0021	0.4916	0.0019	0.4926	0.0018	0.5082	0.0017	0.5035	0.0017
		(60,40)	0.4886	0.0015	0.4951	0.0014	0.4940	0.0014	0.5014	0.0009	0.4965	0.0009
		(40,40)	0.4889	0.0036	0.5045	0.0016	0.4983	0.0016	0.4913	0.0026	0.4855	0.0028
(1,2)	0.67	(20,20)	0.6238	0.0029	0.6324	0.0027	0.6373	0.0026	0.6405	0.0029	0.6467	0.0027
		(30,50)	0.6428	0.0018	0.6434	0.0016	0.6412	0.0015	0.6568	0.0015	0.6637	0.0014
		(50,30)	0.6316	0.0028	0.6587	0.0026	0.6526	0.0022	0.6459	0.0025	0.6550	0.0021
		(50,50)	0.6416	0.0019	0.6536	0.0018	0.6507	0.0016	0.6548	0.0013	0.6602	0.0011
		(40,60)	0.6443	0.0016	0.6516	0.0015	0.6594	0.0015	0.6575	0.0015	0.6625	0.0013
		(60,40)	0.6419	0.0017	0.6547	0.0015	0.6502	0.0013	0.6552	0.0015	0.6624	0.0013
		(40,40)	0.6277	0.0028	0.6561	0.0022	0.6526	0.0020	0.6318	0.0025	0.6488	0.0020

Table 2: AVs and MSEs of the estimates of k_1 with varying n and m

		MLE		$SELF_0$		ELF_0		$SELF_1$		ELF_1	
k_1	(n, m)	AV	MSE	AV	MSE	AV	MSE	AV	MSE	AV	MSE
1	(20,20)	1.3978	0.3245	1.3142	0.2019	1.2756	0.1689	1.2314	0.1632	1.2249	0.1552
	(30,50)	1.3297	0.2966	1.2527	0.1728	1.2239	0.1588	1.1914	0.1527	1.1977	0.1290
	(50,30)	1.3540	0.2815	1.2803	0.1997	1.2533	0.1354	1.1591	0.1617	1.1445	0.1403
	(50,50)	1.3446	0.2448	1.2695	0.1590	1.2397	0.1370	1.1554	0.1400	1.1465	0.1498
	(40,60)	1.3789	0.2644	1.2936	0.1723	1.2681	0.1488	1.1498	0.1577	1.1430	0.1211
	(60,40)	1.3394	0.2394	1.2499	0.1634	1.2831	0.1584	1.1549	0.1329	1.1459	0.1238
	(40,40)	1.3080	0.2467	1.2877	0.1541	1.2020	0.1276	1.1594	0.1386	1.1485	0.1265
2	(20,20)	2.4732	0.312	2.2916	0.2235	2.2478	0.1481	2.2655	0.1567	2.2213	0.1235
	(30,50)	2.3296	0.3117	2.2513	0.2102	2.1981	0.1522	2.1634	0.1250	2.0348	0.0483
	(50,30)	2.4364	0.2752	2.1203	0.1041	1.9262	0.1312	2.1488	0.0906	2.1041	0.0378
	(50,50)	2.4488	0.2502	2.2601	0.2098	2.2046	0.1383	2.2330	0.1419	2.0907	0.1185
	(40,60)	2.2553	0.2214	2.2078	0.1740	2.1927	0.1556	2.1828	0.1167	2.1766	0.0825
	(60,40)	2.4525	0.2404	2.1537	0.0815	2.1444	0.1025	2.1106	0.0694	2.0210	0.0758
	(40,40)	2.4793	0.2340	2.2378	0.1422	2.2142	0.1149	2.1208	0.1073	1.9659	0.1097

Table 3: AVs and MSEs of the estimates of k_2 with varying n and m

		MLE		$SELF_0$		ELF_0		$SELF_1$		ELF_1	
k_2	(n,m)	AV	MSE	AV	MSE	AV	MSE	AV	MSE	AV	MSE
1	(20,20)	1.4710	0.2121	1.3760	0.1867	1.3076	0.1203	1.3129	0.1325	1.2739	0.1091
	(30,50)	1.4542	0.1936	1.3164	0.1181	1.2064	0.0564	1.2502	0.0761	1.1671	0.0459
	(50,30)	1.4152	0.1874	1.3290	0.1259	1.1942	0.0524	1.2036	0.0604	1.0937	0.0340
	(50,50)	1.4042	0.1855	1.3683	0.1581	1.2797	0.0952	1.3013	0.1046	1.2181	0.0665
	(40,60)	1.4653	0.2018	1.2885	0.1717	1.2183	0.1044	1.3210	0.1129	1.2558	0.0763
	(60,40)	1.4996	0.2322	1.3121	0.2061	1.2150	0.1168	1.2499	0.0843	1.1669	0.0532
	(40,40)	1.4183	0.1778	1.2856	0.1583	1.1743	0.0628	1.2454	0.0865	1.1539	0.0533
2	(20,20)	2.5262	0.3023	2.3506	0.2678	2.3008	0.2022	2.3265	0.2231	2.2753	0.1347
	(30,50)	2.3960	0.2065	2.3144	0.1845	2.2561	0.1528	2.2813	0.1113	2.1318	0.0483
	(50,30)	2.4262	0.2112	2.3415	0.2393	2.2966	0.1500	2.3161	0.1927	2.2163	0.1116
	(50,50)	2.4151	0.2259	2.3351	0.1966	2.2297	0.1231	2.3016	0.1286	2.1652	0.0655
	(40,60)	2.4631	0.3068	2.3378	0.2370	2.3048	0.1917	2.3079	0.2030	2.2521	0.1085
	(60,40)	2.4371	0.2929	2.2694	0.1812	2.2513	0.1236	2.2637	0.1621	2.2191	0.0777
	(40,40)	2.4664	0.3095	2.2665	0.1609	2.2678	0.1553	2.2918	0.1370	2.1326	0.0678

Table 4: AVs and MSEs of the estimates of c with varying n and m

		MLE		$SELF_0$		ELF_0		$SELF_1$		ELF_1	
c	(n,m)	AV	MSE	AV	MSE	AV	MSE	AV	MSE	AV	MSE
2.5	(20,20)	2.5632	0.1342	2.4183	0.1268	2.4678	0.1176	2.5105	0.1192	2.4747	0.1081
	(30,50)	2.5316	0.1100	2.4638	0.0992	2.4925	0.0821	2.4755	0.0955	2.4908	0.0848
	(50,30)	2.5489	0.1006	2.4300	0.1235	2.4681	0.0980	2.4732	0.0944	2.4830	0.0843
	(50,50)	2.5660	0.1182	2.4469	0.1140	2.4548	0.1110	2.4781	0.1060	2.4843	0.0961
	(40,60)	2.5683	0.1052	2.4835	0.0905	2.4977	0.0881	2.4930	0.0895	2.4971	0.0894
	(60,40)	2.5479	0.0984	2.4797	0.0997	2.5170	0.9567	2.4908	0.0894	2.4809	0.0803
	(40,40)	2.5647	0.1067	2.5247	0.0990	2.5625	0.1135	2.4862	0.0973	2.4858	0.0865

Table 5: AVs and MSEs of the estimates of a with varying n and m

		MLE		$SELF_0$		ELF_0		$SELF_1$		ELF_1	
a	(n,m)	AV	MSE	AV	MSE	AV	MSE	AV	MSE	AV	MSE
1.5	(20,20)	1.5378	0.0231	1.5442	0.0156	1.5256	0.0162	1.5114	0.0148	1.5249	0.0151
	(30,50)	1.4814	0.0189	1.5063	0.0102	1.5108	0.0115	1.4938	0.0103	1.4970	0.0100
	(50,30)	1.4918	0.0171	1.4958	0.0104	1.5009	0.0108	1.4952	0.0101	1.4947	0.0126
	(50,50)	1.4894	0.0184	1.4937	0.0116	1.5205	0.0127	1.5221	0.0113	1.4845	0.0120
	(40,60)	1.5205	0.0173	1.4836	0.0116	1.5386	0.0124	1.5028	0.0102	1.5155	0.0102
	(60,40)	1.5075	0.0178	1.5340	0.0128	1.4709	0.0147	1.4918	0.0106	1.4979	0.0106
	(40,40)	1.4955	0.0188	1.5366	0.0120	1.4865	0.0123	1.4996	0.0108	1.4987	0.0108

We deduce the following findings from the results:

1. The MSE of all estimates, obtained with different parameter values, decreases as the sample sizes increase.

2. For the distribution parameters and reliability function, the MSE based on the MLEs is higher as compared to the one of the Bayes estimates.

3. For gamma priors in comparison with informative and non-informative forms, the MSE of informative priors is smaller.

4. For reliability function and stress-strength reliability coefficients, the ELF performs better than the SELF in terms of the lesser value of the MSE.

5. For varying n and m, the MSE of k_1 is mostly greater than k_2 when $k_1 < k_2$ and $k_2 < k_1$.

6. APPLICATION

In this section, we work with two engineering data sets, initially reported in [10] and [24], to demonstrate that the proposed methodologies can be used in practice quite effectively. These data sets represent two different algorithms, called SC16 and P3, used by the electric utility industry to compare and estimate unit capacity factors. More precisely, SC16 represents the Southern Company's program using a piecewise linear representation of equivalent charging duration (ELDC) curves in 16 megawatt increments to represent the original charging duration curve. On his side, P3 represents the ELDC using the Gram-Charlier series involving all cumulative power in megawatts. The data sets considered are detailed as follows:

SC16(X), $n = 23$: 0.853, 0.759, 0.866, 0.809, 0.717, 0.544, 0.492, 0.403, 0.344, 0.213, 0.116,0.116, 0.092, 0.070, 0.059, 0.048, 0.036, 0.029, 0.021, 0.014, 0.011, 0.008, 0.006.

P3(Y), $m = 22$: 0.853, 0.759, 0.874, 0.800, 0.716, 0.557, 0.503, 0.399, 0.334, 0.207, 0.118,0.118, 0.097, 0.078, 0.067, 0.056, 0.044, 0.036, 0.026, .019, 0.014, 0.010.

We remove the value 0.000 from the P3 algorithm so that it does not make the parameter likelihood estimates meaningless. First, we check the validity of the proposed distribution using the negative log-likelihood (-logL), Kolmogorov-Smirnov (K-S) statistic, the Akaike information criterion (AIC) and the Bayesian information criterion (BIC). We compare the fits of the NEB distribution with those of the Topp-Leone BXII (TLBXII) distribution, Marshall-Olkin Extended BXII (MOEBXII) distribution, Weibull BXII (WBXII) distribution, and transmuted BXII (TBXII) distribution, as referenced in the introductory part. The expressions of the pdfs of the competitor distributions are briefly presented below.

$$\text{TLBXII} : f(x; l, c, k) = 2lckx^{c-1}(1 + x^c)^{-(2k+1)} \left[1 - (1 + x^c)^{-2k}\right]^{l-1},$$

$$\text{MOEBXII} : f(x; a, c, k) = ack\frac{x^{c-1}(1 + x^c)^{-(k+1)}}{[1 - (1 - a)(1 + x^c)^{-k}]^2},$$

$$\text{WBXII} : f(x; a, l, c, k) = \frac{alckx^{c-1}}{1 + x^c} \{k \log(1 + x^c)\}^{l-1} \exp\left[-a \{k \log(1 + x^c)\}^l\right],$$

$$\text{TBXII} : f(x; c, k, \theta, \lambda) = \frac{ck}{\theta^c}x^{c-1} \left[1 + \left(\frac{x}{\theta}\right)^c\right]^{-(k+1)} \left[1 - \lambda + 2\lambda \left\{1 + \left(\frac{x}{\theta}\right)^c\right\}^{-k}\right],$$

All the involved parameters are supposed to be strictly positive, except $\lambda \in [-1, 1]$ for the last distribution. It is supposed that $x > 0$, the standard completion applied on these pdfs for $x \leq 0$. We use the maximum likelihood estimation and the K-S test to fit the two data sets separately for the proposed and the above competitor distributions. We discover that the NEB distribution provides a better fit. We also use both information criteria to find the best model in two data sets that have a good fit based on the minimum values of AIC and BIC, and conclude that the NEB distribution fits both data sets better than the others distributions. The values of MLE, K-S test, AIC, and BIC are collected in Tables 6 and 7.

Table 6: MLEs, AIC, BIC and KS statistic for the SC16 data

Model	MLEs	-logL	K-S	AIC	BIC
NEB(a, c, k)	(1.3907,0.7741,5.3351)	-7.0094	0.9400	-8.0187	-4.6123
TLBXII(l, c, k)	(234.8565,0.1510,5.4960)	-6.1992	0.9739	-6.3985	-2.9920
MOEBXII(a, c, k)	(0.8894,0.8332,3.7239)	-6.9174	0.9433	-7.8348	-4.4283
WBXII(a, l, c, k)	(1.1192,13.1646,0.0756,1.582)	-7.4218	0.9500	-6.8437	-2.3017
TBXII(c, k, θ, λ)	(0.7533,337.0553,573.6415,0.0924)	-7.4215	1.0206	-6.8308	-2.2888

Table 7: MLEs, AIC, BIC and KS statistic for the P3 data

Model	MLEs	-logL	K-S	AIC	BIC
NEB(a, c, k)	(1.3630,0.8704,5.2907)	-4.4913	0.9407	-2.9825	0.2906
TLBXII(l, c, k)	(255.2371,0.1688,5.5850)	-3.9032	0.9767	-1.8063	1.4668
MOEBXII(a, c, k)	(0.7887,0.9515,3.6233)	-4.4305	0.9448	-2.8611	0.4120
WBXII(a, l, c, k)	(4.4466,9.8049,0.1123,1.3798)	-4.8735	0.9413	-1.7470	2.6171
TBXII(c, k, θ, λ)	(0.8467,474.7816,428.0671,0.1096)	-4.8649	1.0368	-1.7298	2.6344

From Tables 6 and 7, we can note that the parameter a is estimated in an intermediate way between 0 and 2, justifying the alternative identity of the distribution NEB compared to the BXII and PITL distributions.

For both data sets, the MLEs and Bayes estimates of the model parameters are given along with their standard errors (SEs), and the stress-strength reliability coefficient values are obtained in Table 8. As we had no prior information apart from a few observations, we only use non-informative values for the gamma prior.

Table 8: Maximum likelihood and Bayes estimates of R and distribution parameters with SEs based on the considered data-sets

Estimates	R	a		c		k_1		k_2	
		AV	SE	AV	SE	AV	SE	AV	SE
MLE	0.4861	1.1371	0.3581	0.8254	0.1494	4.1941	0.8725	3.9586	0.9018
$SELF_0$	0.4878	1.0790	0.2190	0.8488	0.1502	4.5912	0.7115	4.3966	0.1093
ELF_0	0.4747	0.9715	0.3127	0.8448	0.1684	3.9384	0.8198	3.7017	0.1121

Based on Table 8, an estimate of R is approximately obtained as 0.48. We conclude that the P3 algorithm has slightly more storage capacity for the electric utility industry.

7. CONCLUSION

This article emphasized a new three-parameter heavy right-tailed distribution that consolidates, in a certain sense, the "popular Burr type XII distribution" and the "promising power inverted Topp-Leone distribution". The slip between these two well-established distributions was made by a special shift parameter. The new distribution benefits from notable advantages, including a flexible decreasing and unimodal probability density function, a decreasing upside-down bathtub-shaped hazard rate function, as well as a manageable quantile function, (first-order) stochastic ordering properties, moments, incomplete moments, and probability weighted moments. The classical and Bayesian approaches were developed to estimate the model and stress-strength reliability parameters. The effectiveness and potential of the new model were highlighted using both simulated and actual data, demonstrating that it can be a superior replacement for other lifetime models in the literature.

ACKNOWLEDGMENTS

We thank the two referees for the constructive comments on the paper.

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