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Bootstrapping Order Statistics with Variable Rank

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Abstract:

- This work investigates the strong consistency of bootstrapping central and intermediate order statistics for an appropriate choice of re-sample size for known and unknown normalizing constants. We show that when the normalizing constants are estimated from the data, the bootstrap distribution for central and intermediate order statistics may be weakly or strongly consistent. A simulation study is conducted to show numerically how to choose the bootstrap sample size to give the best approximation of the bootstrapping distribution for the central and intermediate quantiles.

Keywords:

- *Bootstrap technique; central order statistics; intermediate order statistics; weak consistency; strong consistency.*

AMS Subject Classification:

- 62G30; 62F40.

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1. INTRODUCTION

Let $X_1, X_2, ..., X_n$ be iid random variables (RVs) with a common distribution function (DF) $F(x)$ and let $X_{1:n}, X_{2:n}, ..., X_{n:n}$ be the corresponding order statistics. The DF of the kth order statistic $X_{k:n}$, $1 \leq k \leq n$, is given by

$$F_{k:n}(x) = P(X_{k:n} \leq x) = B_{F(x)}(k, n-k+1), \tag{1.1}$$

where $B_x(a, b)$ is the usual incomplete beta function with the shape parameters $a, b > 0$ (cf. David and Nagaraja [19]). A sequence $\{X_{k:n}\}$ is called a sequence of order statistics with variable rank (cf. [3]) if $1 < k_n < n$ and $\min\{k_n, n-k_n\} \to \infty$, as $n \to \infty$ (denoted by $\min\{k_n, n-k_n\} \underset{n}{\longrightarrow} \infty$), where we have the following two cases:

1. If $\frac{k_n}{n} \underset{n}{\longrightarrow} 0$ (or $\frac{k_n}{n} \underset{n}{\longrightarrow} 1$), then $X_{k_n:n}$ is called the lower intermediate order statistic (or the upper intermediate order statistics).
2. If $\frac{k_n}{n} \underset{n}{\longrightarrow} p$ $(0 < p < 1)$, then $X_{k_n:n}$ is called the central order statistic.

A prominent example for the central order statistics is the *pth* sample quantile (including the median, quartiles, percentiles etc.), where $k_n = [np] + 1$ and $[.]$ is the greatest integer function (see David and Nagaraja [19]). On the other hand, the intermediate order statistics also have many applications, e.g., they can be used to estimate the probabilities of the future extremes and tail quantiles of the underlying distribution that are extremely relative to the available sample size, cf. [33]. Moreover, many authors, e.g., Mason [30] and Teugels [36] have also found estimates that are based, in part, on intermediate order statistics.

The literature abounds with many different results for intermediate and central order statistics and their applications. Interested readers may refer to Balkema and de Haan [3, 4], Barakat [5, 6], Barakat and El-Shandidy [7], Barakat and Omar [8, 9], Chibisov [18], Falk [21], Falk and Wisheckel [22], Frey and Zhang [23], Ho and Lee [27], Nagaraja and Nagaraja [31], Peng and Yang [32], Smirnov [35], and Wu [37]. The bootstrap method introduced in Efron [20] is a general procedure for approximating the sampling distributions of statistics based on re-sampling from the data at hand. There are several forms of the bootstrap and additionally several other re-sampling methods that are related to it, such as jackknifing, cross-validation, randomization tests, and permutation tests. The bootstrap method is shown to be successful in many situations and is accepted as an alternative to the asymptotic methods (for more details, see [14] and [31]). Let $X_n = (X_1, X_2, ..., X_n)$ be a random sample from an unknown DF $F(x)$. For $m = m(n) \underset{n}{\longrightarrow} \infty$, assume that $Y_i, i = 1, 2, ..., m$, are conditionally iid RVs with distribution

$$P(Y_i = X_j | X_n) = \frac{1}{n}, j = 1, 2, ..., n, \; i \in \{1, 2, ..., m\},$$

then $(Y_1, Y_2, ..., Y_m)$ is a random re-sample of size m from the empirical DF

$$F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I_{(-\infty,x)}(X_i) = \frac{1}{n}Q_n(x),$$

where $I_A(x)$ is the indicator function and $Q_n(x)$ is an RV distributed as a binomial distribution with parameters n and $F(x)$, denoted by $Q_n(x) \sim \mathrm{B}(n, F(x))$. Furthermore, let the extreme value theory (see [14]) be applicable to the extreme order statistic $X_{k:n}$, which means that there exist normalizing constants $a_n > 0$ and b_n such that $F_{k:n}(a_n x + b_n) = B_{F(a_n x+b_n)}(k, n-k+1)$ weakly converges, as $n \to \infty$ (denoted by $\xrightarrow[n]{w}$) to a non-degenerate DF $G(x)$, where $G(x)$ is one of the extreme value distributions. Now, let $Y_{1:m}, Y_{2:m}, ..., Y_{m:m}$ be the corresponding order statistics of $Y_1, Y_2, ..., Y_m$, and define

$$H_{n,m}(a_m x + b_m) = P(Y_{k:m} \le a_m x + b_m | X_n) = B_{F_n(a_m x+b_m)}(k, m-k+1).$$

$H_{n,m}(a_m x + b_m)$ is called the bootstrap distribution of $a_n^{-1}(X_{k:n} - b_n)$, where n and m are the sample size and re-sample size, respectively. A full-sample bootstrap is the case when $m = n$. In contrast, m out of n bootstrap technique is the case when $m < n$. One of the bootstrap's desired properties is consistency; namely, the bootstrap's limit distribution is the same as the original statistic distribution. For a long time, it has been known that a full-sample bootstrap does not work for order statistics. This seminal result was apparently first revealed for extremes by Athreya and Fukuchi [1] and Fukuchi [24]. Moreover, it was proved for intermediate order statistics by Geluk and de Haan [25] and Barakat et al. [16]. Finally, for central order statistics, this result was proved by Barakat et al. [16]. Athreya and Fukuchi [1] and Fukuchi [24] (see also Athreya and Fukuchi [2]) studied the consistency of bootstrapping extremes for known and unknown normalizing constants and they showed that the bootstrap DF fails to be consistent in the full-sample bootstrap case. Moreover, they showed that the bootstrap DF is a weakly consistent estimate if $m = o(n)$ and it is strongly consistent if $m = o(\frac{n}{\log n})$. Barakat et al. [11] extended this result to the extreme generalized order statistics. Later, Barakat et al. [16] have got some similar results for the order statistics with variable ranks. Namely, they showed that the bootstrapping central and intermediate quantiles fail to be consistent in the full-sample bootstrap case. Moreover, they also showed that when the normalizing constants are known, the bootstrap DFs for central and intermediate order statistics are weakly consistent when $m = o(n)$ (see, Theorems 4.1 and 4.2 in [16]). Barakat et al. [13] extended this result to the case where we use the bootstrap for estimating a central, or an intermediate quantile under power normalization.

The main aim of the present work is to extend the results of [16] by investigating the strong consistency of bootstrapping central and intermediate order statistics for an appropriate choice of re-sample size for known and unknown normalizing constants. A simulation study is conducted to illustrate how to choose the re-sample's size. Sections 2 and 3 are devoted respectively to the intermediate and central order statistics, while the simulation study is conducted in Section 4. Finally, we conclude the paper in Section 5. The rest of this introductory section will be devoted to review some basic results pertaining to the asymptotic behaviour of the central and intermediate order statistics, which are the essential pillars of our study.

1.1. Some important aspects of the asymptotic theory of order statistics with variable rank

The following lemma (Lemma 1.1 in Barakat [6]) is a cornerstone of the asymptotic theory of order statistics with variable rank.

Lemma 1.1 (cf. [6], see also [28]). *For any sequence of variable ranks $\{k_n\}$, let $\{u_n, n \geq 1\}$ be a sequence of real numbers and let $-\infty \leq \tau \leq \infty$. Then,*

$$F_{k_n:n}(u_n) = P(X_{k_n:n} \leq u_n) \underset{n}{\longrightarrow} \mathcal{N}(\tau), \tag{1.2}$$

if and only if

$$\frac{nF(u_n) - k_n}{\sqrt{k_n(1 - \frac{k_n}{n})}} \underset{n}{\longrightarrow} \tau, \tag{1.3}$$

where $\mathcal{N}(.)$ is the standard normal DF and $F_{k_n:n}$ is defined in (1.1).

Since the variable ranks are classified into central and intermediate ranks, we will consider each of the two cases separately.

Asymptotic theory of the intermediate order statistics

If $\frac{k_n}{n} \underset{n}{\longrightarrow} 0$ (i.e., the lower intermediate case), then by using the linear parametrization's transformation $u_n = a_n x + b_n$ and $\tau = U(x)$, (1.3) will be reduced to

$$\frac{nF(a_n x + b_n) - k_n}{\sqrt{k_n}} \underset{n}{\longrightarrow} U(x), \tag{1.4}$$

(cf. [18]). A sequence of intermediate rank $\{k_n\}$ is said to satisfy the Chibisov's condition ([18]), if

$$\sqrt{k_{n+z_n(\nu)}} - \sqrt{k_n} \underset{n}{\longrightarrow} \frac{\alpha \nu l}{2}, \tag{1.5}$$

for any sequence of integer values $z_n(\nu)$, with $\frac{z_n(\nu)}{n^{1-\frac{\alpha}{2}}} \underset{n}{\longrightarrow} \nu$, where $0 < \alpha < 1$, $l > 0$, and ν is any real number. Chibisov [18] showed that, whenever $\{k_n\}$ satisfies the condition (1.5), the only possible non-degenerate forms for $\mathcal{N}(U(x))$ in (1.2) are $\mathcal{N}(U_{i;\beta}(x))$, $i = 1, 2, 3$, where $U_{3;\beta}(x) = U_3(x) = x,\ \forall x$,

$$U_{2;\beta}(x) = \begin{cases} -\beta \log \mid x \mid, & x \le 0, \\ \infty, & x > 0, \end{cases} \qquad U_{1;\beta}(x) = \begin{cases} -\infty, & x \le 0, \\ \beta \log x, & x > 0, \end{cases}$$

and β is a positive constant depending only on α, l and the type of the DF $F(x)$. Chibisov [18] noted that, the condition (1.5) implies $\frac{k_n}{n^\alpha} \xrightarrow[n]{} l^2$. On the other hand, Barakat and Omar [8] showed that the last condition implies the Chibisov's condition, which means that the Chibisov rank sequences are widely-used and the Chibisov's limit types are vastly applicable. Recently, Barakat et al. [12] characterized the asymptotic behaviour of the scale normalizing constant a_n.

Lemma 1.2 ([12]). *Let $L(n) = \exp(\sqrt{n})$. Furthermore, let $F(a_n x + b_n) \in D^{(l,\alpha)}(\mathcal{N}(U(x)))$ mean that (1.4) is satisfied for $k_n \sim l^2 n^\alpha$. Then, for any $\varepsilon > 0$,*

1. $a_n L^{\frac{1}{\beta}+\varepsilon}(k_n) \xrightarrow[n]{} \infty$ and $a_n L^{\frac{1}{\beta}-\varepsilon}(k_n) \xrightarrow[n]{} 0$, if $F(a_n x + b_n) \in D^{(l,\alpha)}(\mathcal{N}(U_{1;\beta}(x)))$;
2. $a_n L^{\frac{-1}{\beta}+\varepsilon}(k_n) \xrightarrow[n]{} \infty$ and $a_n L^{\frac{-1}{\beta}-\varepsilon}(k_n) \xrightarrow[n]{} 0$, if $F(a_n x + b_n) \in D^{(l,\alpha)}(\mathcal{N}(U_{2;\beta}(x)))$;
3. $a_n L^{+\varepsilon}(k_n) \xrightarrow[n]{} \infty$ and $a_n L^{-\varepsilon}(k_n) \xrightarrow[n]{} 0$, if $F(a_n x + b_n) \in D^{(l,\alpha)}(\mathcal{N}(U_3(x)))$.

1.1.1. Asymptotic theory of the central order statistics

Smirnov [35] revealed that it is possible to find two rank sequences $\{k_n\}$ and $\{k_n^\star\}$ with $\frac{k_n}{n} \sim \frac{k_n^\star}{n} \sim p$, $0 < p < 1$, to lead to different non-degenerate limiting DFs for $X_{k_n:n}$ and $X_{k_n^\star:n}$. However, this is not possible if $k_n \sim k_n^\star \sim pn + o(\sqrt{n})$. Under this condition, Smirnov [35] showed that with $u_n = a_n x + b_n$ and $\tau = V(x)$, (1.3) will be reduced to

$$\sqrt{n}\frac{F(a_n x + b_n) - p}{C_p} \xrightarrow[n]{} V(x), \tag{1.6}$$

where $C_p = \sqrt{p(1-p)}$. Smirnov [35] showed that, whenever $\{k_n\}$ satisfies the condition $k_n \sim pn + o(\sqrt{n})$, the only possible non-degenerate forms for $\mathcal{N}(V(x))$ in (1.2) are $\mathcal{N}(V_{i;\beta}(x))$, $i = 1, 2, 3, 4$, where

$$V_{1;\beta}(x) = \begin{cases} -\infty, & x \le 0, \\ c x^\beta, & x > 0, \end{cases} \qquad V_{2;\beta}(x) = \begin{cases} -c \mid x \mid^\beta, & x \le 0, \\ \infty, & x > 0, \end{cases}$$

$$V_{3;\beta}(x) = \begin{cases} -c_1 \mid x \mid^\beta, & x \le 0, \\ c_2 x^\beta, & x > 0, \end{cases} \qquad V_{4;\beta}(x) = W_4(x) = \begin{cases} -\infty, & x \le -1, \\ 0, & -1 < x \le 1, \\ \infty, & x > 1, \end{cases}$$

$c = c_1 = \frac{1}{\sqrt{p(1-p)}}$, $c_2 = \frac{c_1}{A}$, and $A > 0$. In this case, we say that the DF F belongs to the domain of normal $p-$attraction of the limit type $V_{i;\beta}(x),\ i \in \{1,2,3,4\}$, and we write $F \in D^{(p)}(V_{i;\beta}(x))$.

2. Bootstrapping intermediate order statistics

In this section, we investigate the strong consistency of the bootstrap distribution $H_{n,m}(a_m x + b_m) = P(X_{k_m:m} \le a_m x + b_m | X_n)$, where k_n is the Chibisov rank sequence, which satisfies the condition (1.5), and the condition (1.4) is satisfied with $U(x) = U_{i;\beta}(x),\ i \in \{1,2,3\}$, for some suitable normalizing constants $a_n > 0$ and b_n.

2.1. Almost sure consistency of bootstrapping intermediate for known normalizing constants

Barakat et al. [16] proved the weak limit relation $\sup_{x\in\Re}|H_{n,m}(a_m x + b_m) - \mathcal{N}(U_{i;\beta}(x))| \xrightarrow[n]{p} 0$, if $m = o(n)$, where "$\xrightarrow[n]{p}$" stands for convergence in probability, as $n \to \infty$. The following theorem extends this result.

Theorem 2.1. *Let m be chosen such that $\sum_{n=1}^{\infty} \lambda^{\sqrt{\frac{n}{m}}} < \infty$, for each $\lambda \in (0,1)$. Then*

$$\sup_{x\in\Re} |H_{n,m}(a_m x + b_m) - \mathcal{N}(U_{i;\beta}(x))| \xrightarrow[n]{w.p.1} 0,$$

where the symbol "$\xrightarrow[n]{w.p.1}$" denotes the convergence with probability one (almost surely convergence) as $n \to \infty$.

Proof: Let $\bar{k}_n = \frac{k_n}{n}$. Then, we have

$$\frac{mF_n(a_m x + b_m) - k_m}{\sqrt{k_m}} = \sqrt{m}\frac{F_n(a_m x + b_m) - \bar{k}_m}{\sqrt{\bar{k}_m}} = \sqrt{\frac{m}{n}}\left(\frac{nF_n(a_m x + b_m) - n\bar{k}_m}{\sqrt{n\bar{k}_m}}\right)$$
$$= \sqrt{\frac{m}{n}}\left(\frac{nF_n(a_m x+b_m) - nF(a_m x + b_m)}{\sqrt{n\bar{k}_m}}\right) + \frac{mF(a_m x+b_m) - k_m}{\sqrt{k_m}}.$$

On the other hand, from the assumptions of the theorem, we get

$$\frac{mF(a_m x + b_m) - k_m}{\sqrt{k_m}} \xrightarrow[n]{} U_{i;\beta}(x),\ i \in \{1,2,3\}.$$

Thus, to prove the theorem, we only need to show that

$$\sqrt{\frac{m}{n}}\left(\frac{nF_n(a_mx+b_m)-nF(a_mx+b_m)}{\sqrt{n\bar{k}_m}}\right)\xrightarrow[n]{w.p.1}0.$$

By Borel-Cantelli lemma, it is enough to prove that

$$\sum_{n=1}^{\infty}P\left(\sqrt{\frac{m}{n}}\left|\frac{nF_n(a_mx+b_m)-nF(a_mx+b_m)}{\sqrt{n\bar{k}_m}}\right|>\epsilon\right)<\infty,$$

for every $\epsilon>0$. Now for each $\theta>0$ we get

$$\sqrt{\frac{m}{n}}\log P\left(\sqrt{\frac{m}{n}}\left(\frac{nF_n(a_mx+b_m)-nF(a_mx+b_m)}{\sqrt{n\bar{k}_m}}\right)>\epsilon\right)=$$

$$\sqrt{\frac{m}{n}}\log P\left(\frac{nF_n(a_mx+b_m)-nF(a_mx+b_m)}{\sqrt{n\bar{k}_m}}>\sqrt{\frac{n}{m}}\,\epsilon\right)=\sqrt{\frac{m}{n}}\log P\left(e^{\theta T_{n,m}}>e^{\theta\sqrt{\frac{n}{m}}\,\epsilon}\right),$$

where

$$T_{n,m}=\frac{nF_n(a_mx+b_m)-nF(a_mx+b_m)}{\sqrt{n\bar{k}_m}}.$$

By using the Markov inequality, we get

$$\sqrt{\frac{m}{n}}\log P\left(e^{\theta T_{n,m}}>e^{\theta\sqrt{\frac{n}{m}}\,\epsilon}\right)\leq\sqrt{\frac{m}{n}}\log\left(e^{-\theta\sqrt{\frac{n}{m}}\,\epsilon}E\left(e^{\theta T_{n,m}}\right)\right)=$$

$$\sqrt{\frac{m}{n}}\left(-\sqrt{\frac{n}{m}}\theta\epsilon+\log\varphi_m(\theta)\right)=-\theta\epsilon+\sqrt{\frac{m}{n}}\log\varphi_m(\theta)\xrightarrow[n]{}-\theta\epsilon,$$

where $\varphi_m(\theta)$ is the moment generating function for the standard normal DF. Therefore, for sufficiently large n, we get the following relation:

$$\sum_{n=1}^{\infty}P\left(\sqrt{\frac{m}{n}}\left(\frac{nF_n(a_mx+b_m)-nF(a_mx+b_m)}{\sqrt{n\bar{k}_m}}\right)>\epsilon\right)=$$

$$\sum_{n=1}^{\infty}\exp\left\{\log P\left(\sqrt{\frac{m}{n}}\left(\frac{nF_n(a_mx+b_m)-nF(a_mx+b_m)}{\sqrt{n\bar{k}_m}}\right)>\epsilon\right)\right\}\leq\sum_{n=1}^{\infty}e^{-\theta\epsilon\sqrt{\frac{n}{m}}}<\infty,$$

for every $\epsilon>0$, since the condition $\sum_{n=1}^{\infty}\lambda^{\sqrt{\frac{n}{m}}}<\infty$, for each $\lambda\in(0,1)$, guarantees the convergence of the infinite series $\sum_{n=1}^{\infty}\exp\left\{-\theta\epsilon\sqrt{\frac{n}{m}}\right\}$, for every $\epsilon>0$. By similar reasoning we can show that

$$\sum_{n=1}^{\infty}P\left(\sqrt{\frac{m}{n}}\left(\frac{nF_n(a_mx+b_m)-nF(a_mx+b_m)}{\sqrt{n\bar{k}_m}}\right)<-\epsilon\right)<\infty,$$

for every $\epsilon>0$. The theorem is proved. $\square$

2.2. Almost sure consistency of bootstrapping intermediate for unknown normalizing constants

If the DF F is unknown, the normalizing constants a_m and b_m need to be estimated from the sample data for $H_{n,m}(.)$ to be of use. Let $\hat{a}_m$ and $\hat{b}_m$ be estimators of a_m and b_m based on $X_n = (X_1, X_2, ..., X_n)$. Define the bootstrap distribution for the normalized intermediate order statistic $a_n^{-1}(X_{k_n:n} - b_n)$ with the estimated normalizing constants by

$$\hat{H}_{n,m}(\hat{a}_m x + \hat{b}_m) = P\left(Y_{k_m:m} \leq \hat{a}_m x + \hat{b}_m | X_n\right).$$

Fukuchi [24] provided some sufficient conditions for the bootstrap distribution of the maximum order statistics to be consistent. The following theorem extends the Fukuchi's result by providing sufficient conditions for $\hat{H}_{n,m}(\hat{a}_m x + \hat{b}_m)$ to be consistent.

Theorem 2.2. *Let $m = m(n)$. Then*

$$\sup_{x\in\Re}|\hat{H}_{n,m}(\hat{a}_m x + \hat{b}_m) - \mathcal{N}(U_{i;\beta}(x))| \xrightarrow[n]{w.p.1} 0,\ i = 1, 2, 3,$$

if (i) $H_{n,m}(x) \xrightarrow[n]{w.p.1} \mathcal{N}(U_{i;\beta}(x))$, (ii) $\frac{\hat{a}_m}{a_m} \xrightarrow[n]{w.p.1} 1$, and (iii) $\frac{\hat{b}_m - b_m}{a_m} \xrightarrow[n]{w.p.1} 0$. Moreover, this theorem holds if "$\xrightarrow[n]{w.p.1}$" is replaced by "$\xrightarrow[n]{p}$".

Proof: First, we note that (i) is equivalent to

$$\sqrt{m}\frac{F_n(a_m x + b_m) - \bar{k}_m}{\sqrt{\bar{k}_m}} \xrightarrow[n]{w.p.1} U_{i;\beta}(x).$$

Moreover, for every $\epsilon > 0$, the relations (ii) and (iii) imply

$$(1 - \epsilon)a_m < \hat{a}_m < (1 + \epsilon)a_m \tag{2.1}$$

and

$$b_m - \epsilon a_m < \hat{b}_m < b_m + \epsilon\, a_m, \tag{2.2}$$

respectively. Now, fix $x > 0$, the relations (2.1) and (2.2) yield

$$\begin{aligned}\limsup_{n\to\infty} \sqrt{m}\frac{F_n(\hat{a}_m x + \hat{b}_m) - \bar{k}_m}{\sqrt{\bar{k}_m}} &\leq \limsup_{n\to\infty} \sqrt{m}\frac{F_n((1+\epsilon)x + \epsilon)a_m + b_m) - \bar{k}_m}{\sqrt{\bar{k}_m}} \\ &\leq U_{i;\beta}((1+\epsilon)x + \epsilon).\end{aligned}$$

By a similar way we can prove that

$$\liminf_{n\to\infty} \sqrt{m}\frac{F_n(\hat{a}_m x + \hat{b}_m) - \bar{k}_m}{\sqrt{\bar{k}_m}} \geq U_{i;\beta}((1-\epsilon)x - \epsilon).$$

Since $U_{i;\beta}(x)$ is continuous, we get

$$\lim_{n\to\infty}\sqrt{m}\frac{F_n(\hat{a}_m x+\hat{b}_m)-\bar{k}_m}{\sqrt{\bar{k}_m}}=U_{i;\beta}(x).$$

By a similar argument, the same limit relation can easily be proved for $x<0$. Thus, $H_{n,m}(\hat{a}_m x+\hat{b}_m)\xrightarrow[n]{w.p.1}\mathcal{N}(U_{i;\beta}(x))$. Now, suppose that the conditions (i), (ii), and (iii) hold in probability. Then for any subsequence $\{n_i\}_{i=1}^{\infty}$ of $\{n\}_{n=1}^{\infty}$, there exists a further subsequence $\{n_i^\star\}_{i=1}^{\infty}$ such that (i), (ii), and (iii) hold w.p.1. Then, by applying the first part of the theorem, we get

$$\sup_{x\in\Re}|\hat{H}_{n_i^\star,m(n_i^\star)}(\hat{a}_{m(n_i^\star)}x+\hat{b}_{m(n_i^\star)})-\mathcal{N}(U_{i;\beta}(x))|\xrightarrow[n]{w.p.1}0.$$

The theorem is established. □

Now, for the bootstrap distribution $\hat{H}_{n,m}(\hat{a}_m x+\hat{b}_m)$ to be consistent, we need to choose $\hat{a}_m$ and $\hat{b}_m$ satisfying the conditions (ii) and (iii) in Theorem 2.2. Since a_m and b_m are functionals of F then, the natural choices of $\hat{a}_m$ and $\hat{b}_m$ are the empirical counter parts of a_m and b_m. In the next theorem, we give appropriate choices for $\hat{a}_m$ and $\hat{b}_m$ for each domain of attraction of $\mathcal{N}(U_{i;\beta}(x)), i=1,2,3$.

Theorem 2.3. *Let $k_n^{'}=\frac{n}{m}k_m$, $k_n^{''}=\frac{n}{m}(k_m+\sqrt{k_m})$, and x_0 be the left endpoint of F (i.e., $x_0=\inf\{x:F(x)>0)$. Then*

(i) if $F(a_nx+b_n)\in D^{(l,\alpha)}(\mathcal{N}(U_{1;\beta}(x)))$, $\hat{a}_m=F_n^{-1}\left(\frac{k_m}{m}\right)-\hat{x}_0=X_{k_n^{'}:n}-X_{k_n:n}$, and $\hat{b}_m=X_{k_n:n}$, where $\hat{x}_0=X_{k_n:n}$ is an estimator for x_0;

(ii) if $F(a_nx+b_n)\in D^{(l,\alpha)}(\mathcal{N}(U_{2;\beta}(x)))$, $\hat{a}_m=-F_n^{-1}\left(\frac{k_m}{m}\right)=-X_{k_n^{'}:n}$, and $\hat{b}_m=0$;

(iii) if $F(a_nx+b_n)\in D^{(l,\alpha)}(\mathcal{N}(U_3(x)))$, $\hat{a}_m=F_n^{-1}\left(\frac{k_m+\sqrt{k_m}}{m}-\frac{k_m}{m}\right)=X_{k_n^{''}:n}-X_{k_n^{'}:n}$, and $\hat{b}_m=F_n^{-1}\left(\frac{k_m}{m}\right)=X_{k_n^{'}:n}$.

If $m=o(n)$, then

$$\sup_{x\in\Re}\left|\hat{H}_{n,m}(\hat{a}_m x+\hat{b}_m)-\mathcal{N}(U_{i;\beta}(x))\right|\xrightarrow[n]{p}0. \tag{2.3}$$

Moreover, if $\sum_{n=1}^{\infty}\lambda^{\sqrt{\frac{n}{m}}}<\infty$, for each $\lambda\in(0,1)$, then (2.3) holds w.p.1.

Proof: First, let $F(a_nx+b_n)\in D^{(l,\alpha)}(\mathcal{N}(U_{1;\beta}(x)))$. Therefore, in view of the result of Chibisov [18], we have $b_n=b_m=x_0>-\infty$. In order to apply Parts (ii) and (iii) in Theorem 2.2, it suffices to show that

$$\frac{\hat{a}_m}{a_m}=\frac{X_{k_n^{'}:n}-X_{k_n:n}}{a_m}\xrightarrow[n]{}1 \tag{2.4}$$

and

$$\frac{\hat{b}_m - b_m}{a_m} = \frac{X_{k_n:n} - x_0}{a_m} \xrightarrow[n]{} 0, \tag{2.5}$$

both in probability or w.p.1. First, let us focus on the case of convergence in probability. Now, we have

$$\frac{\hat{a}_m}{a_m} = \frac{X_{k'_n:n} - x_0}{a_m} - \frac{a_n}{a_m} \times \frac{X_{k_n:n} - x_0}{a_n}, \ \frac{\hat{b}_m - b_m}{a_m} = \frac{a_n}{a_m} \times \frac{X_{k_n:n} - x_0}{a_n},$$

and $P(\frac{X_{k_n:n}-x_0}{a_n} \leq x) \xrightarrow[n]{w} \mathcal{N}(U_{1;\beta}(x))$, where $\mathcal{N}(U_{1;\beta}(x))$ is a non-degenerate DF. Therefore, to prove (2.4) and (2.5), it is sufficient to show that

$$\frac{X_{k'_n:n} - x_0}{a_m} \xrightarrow[n]{p} 1 \tag{2.6}$$

and

$$\frac{a_n}{a_m} \xrightarrow[n]{} 0. \tag{2.7}$$

First we prove (2.6). Clearly,
(2.8)

$$\frac{nF(a_m x + b_m) - k'_n}{\sqrt{k'_n}} = \sqrt{\frac{n}{m}}\left(\sqrt{m}\frac{F(a_m x + b_m) - \bar{k}_m}{\sqrt{\bar{k}_m}}\right) \xrightarrow[n]{} \begin{cases} \infty, & \text{if } x > 1, \\ -\infty, & \text{if } x < 1. \end{cases}$$

Thus, from (2.8), we get

$$P\left(\frac{X_{k'_n:n} - x_0}{a_m} < \epsilon + 1\right) \xrightarrow[n]{} \mathcal{N}(\infty) = 1,$$

which in turn implies

$$P\left(\frac{X_{k'_n:n} - x_0}{a_m} > \epsilon + 1\right) \xrightarrow[n]{} 0. \tag{2.9}$$

Similarly we have

$$P\left(\frac{X_{k'_n:n} - x_0}{a_m} < -\epsilon + 1\right) \xrightarrow[n]{} \mathcal{N}(-\infty) = 0. \tag{2.10}$$

From (2.9) and (2.10), we get

$$P\left(\left|\frac{X_{k'_n:n} - x_0}{a_m} - 1\right| > \epsilon\right) \xrightarrow[n]{} 0.$$

Hence (2.6) is proved. Turning now to prove (2.7). By using Lemma 1.2 and the condition $m = o(n)$, we get

$$\frac{a_n}{a_m} \sim \frac{L^{\frac{-l}{\beta}}(k_n)}{L^{\frac{-l}{\beta}}(k_m)} = \frac{e^{\frac{-l}{\beta}n^{\frac{\alpha}{2}}}}{e^{\frac{-l}{\beta}m^{\frac{\alpha}{2}}}} = e^{\frac{-l}{\beta}n^{\frac{\alpha}{2}}\left(1-\left(\frac{m}{n}\right)^{\frac{\alpha}{2}}\right)} \xrightarrow[n]{} 0,$$

which proves (2.7). Finally, in order to switch from convergence in probability to convergence w.p.1, we argue by the same way as in the end of the proof of Theorem 2.1. This completes the proof of Part (i). Now, assume that $F(a_nx+b_n)\in D^{(l,\alpha)}(\mathcal{N}(U_{2;\beta}(x)))$. Therefore, in view of the result of Chibisov [18], we have $x_0=-\infty$ and $b_n=b_m=0$ (this legitimates the choice $\hat{b}_m=0$). On the other hand, by Theorem 2.3 in order to prove Part (ii) of the theorem, it suffices to show that

$$\frac{\hat{a}_m}{a_m}=\frac{-X_{k_n':n}}{a_m}\xrightarrow[n]{}1 \tag{2.11}$$

and

$$\frac{\hat{b}_m-b_m}{a_m}\xrightarrow[n]{}0, \tag{2.12}$$

both in probability or w.p.1. First, let us focus on the case of convergence in probability. It is clear that (2.12) is satisfied (actually $\frac{\hat{b}_m-b_m}{a_m}=0,\quad$ for all m). Therefore, we have only to prove (2.11). Clearly, we have

$$\frac{nF(a_mx+b_m)-k_n'}{\sqrt{k_n'}}=\sqrt{\frac{n}{m}}\left(\sqrt{m}\frac{F(a_mx)-\bar{k}_m}{\sqrt{\bar{k}_m}}\right)\xrightarrow[n]{}\begin{cases}-\infty, & \text{if } x<-1,\\ \infty, & \text{if } x>-1.\end{cases} \tag{2.13}$$

Thus, from (2.13), we get

$$P\left(\frac{X_{k_n':n}}{a_m}<-(\epsilon+1)\right)\xrightarrow[n]{}\mathcal{N}(-\infty)=0,$$

which implies to

$$P\left(\frac{-X_{k_n':n}}{a_m}>\epsilon+1\right)\xrightarrow[n]{}0. \tag{2.14}$$

Similarly we have

$$P\left(\frac{X_{k_n':n}}{a_m}<-(1-\epsilon)\right)\xrightarrow[n]{}\mathcal{N}(\infty)=1,$$

which in turn is equivalent to

$$P\left(\frac{-X_{k_n':n}}{a_m}<1-\epsilon\right)\xrightarrow[n]{}0. \tag{2.15}$$

From (2.14) and (2.15), we get $P\left(\left|\frac{-X_{k_n':n}}{a_m}-1\right|>\epsilon\right)\xrightarrow[n]{}0$, which proves (2.11), as well as Part (ii), when the convergence in the probability. In order to switch to the convergence w.p.1, we again argue by the same way as in the end of the proof of Theorem 2.1. Finally, assume that $F(a_nx+b_n)\in D^{(l,\alpha)}(\mathcal{N}(U_3(x)))$. By Theorem 2.2, in order to prove Part (iii), it suffices to show that

$$\frac{\hat{a}_m}{a_m}=\frac{X_{k_n'':n}-X_{k_n':n}}{a_m}\xrightarrow[n]{}1 \tag{2.16}$$

and

$$\frac{\hat{b}_m - b_m}{a_m} = \frac{X_{k_n':n} - b_m}{a_m} \xrightarrow[n]{} 0, \tag{2.17}$$

both in probability or w.p.1. First, let us again focus on the case of convergence in probability and write

$$\frac{\hat{a}_m}{a_m} = \frac{X_{k_n'':n} - X_{k_n':n}}{a_m} = \frac{X_{k_n'':n} - b_m}{a_m} - \frac{X_{k_n'} - b_m}{a_m}.$$

Hence, to prove (2.16) and (2.17), it is sufficient to show that

$$\frac{X_{k_n'':n} - b_m}{a_m} \xrightarrow[n]{p} 1 \tag{2.18}$$

and

$$\frac{X_{k_n':n} - b_m}{a_m} \xrightarrow[n]{p} 0. \tag{2.19}$$

First, we prove (2.18). One can write

$$\frac{nF(a_m x + b_m) - k_n^{''}}{\sqrt{k_n^{''}}} = \frac{nF(a_m x + b_m) - \frac{n}{m}(k_m + \sqrt{k_m})}{\sqrt{\frac{n}{m}(k_m + \sqrt{k_m})}}$$

$$= \sqrt{\frac{n}{m}}\left(\frac{mF(a_m x + b_m) - (k_m + \sqrt{k_m})}{\sqrt{(k_m + \sqrt{k_m})}}\right) = \sqrt{\frac{n}{m}}\left(\frac{mF(a_m x + b_m) - (k_m + \sqrt{k_m})}{\sqrt{k_m(1 + \frac{1}{\sqrt{k_m}})}}\right)$$

$$= \sqrt{\frac{n}{m}}\left(\frac{mF(a_m x + b_m) - (k_m + \sqrt{k_m})}{\sqrt{k_m(1 + o(1))}}\right) = \sqrt{\frac{n}{m}}\left(\frac{mF(a_m x + b_m) - k_m}{\sqrt{k_m(1 + o(1))}} - \frac{\sqrt{k_m}}{\sqrt{k_m(1 + o(1))}}\right).$$

On the other hand, the assumption of the theorem yields

$$\frac{mF(a_m x + b_m) - k_m}{\sqrt{k_m(1 + o(1))}} \xrightarrow[n]{} x.$$

Therefore, we get

$$\frac{nF(a_m x + b_m) - k_n^{''}}{\sqrt{k_n^{''}}} \xrightarrow[n]{} \begin{cases} \infty, & \text{if } x > 1, \\ -\infty, & \text{if } x < 1. \end{cases} \tag{2.20}$$

Thus, for every $\epsilon > 0$, we get

$$P\left(\frac{X_{k_n'':n} - b_m}{a_m} < \epsilon + 1\right) \xrightarrow[n]{} \mathcal{N}(\infty) = 1,$$

which implies

$$P\left(\frac{X_{k_n'':n} - b_m}{a_m} > \epsilon + 1\right) \xrightarrow[n]{} 0. \tag{2.21}$$

Moreover, by the same way we get

$$P\left(\frac{X_{k''_n:n}-b_m}{a_m}<1-\epsilon\right)\underset{n}{\longrightarrow}\mathcal{N}(-\infty)=0. \tag{2.22}$$

Thus, (2.21) and (2.22) lead to

$$P\left(\left|\frac{X_{k''_n:n}-b_m}{a_m}-1\right|>\epsilon\right)\underset{n}{\longrightarrow}0,$$

which proves (2.18). Next, we prove (2.19). We have

$$\frac{nF(a_mx+b_m)-k'_n}{\sqrt{k'_n}}=\frac{nF(a_mx+b_m)-\frac{n}{m}k_m}{\sqrt{\frac{n}{m}k_m}}$$

$$=\sqrt{\frac{n}{m}}\left(\sqrt{m}\frac{F(a_mx+b_m)-\bar{k}_m}{\sqrt{\bar{k}_m}}\right)\underset{n}{\longrightarrow}\begin{cases}\infty, & \text{if } x>0,\\ -\infty, & \text{if } x<0.\end{cases} \tag{2.23}$$

Thus, (2.23) yields $P\left(\frac{X_{k'_n:n}-b_m}{a_m}<\epsilon\right)\underset{n}{\longrightarrow}\mathcal{N}(\infty)=1$, which implies that

$$P\left(\frac{X_{k'_n:n}-b_m}{a_m}>\epsilon\right)\underset{n}{\longrightarrow}0. \tag{2.24}$$

Similarly we have

$$P\left(\frac{X_{k'_n:n}-b_m}{a_m}<-\epsilon\right)\underset{n}{\longrightarrow}\mathcal{N}(-\infty)=0. \tag{2.25}$$

Therefore, the relations (2.24) and (2.25) imply

$$P\left(\left|\frac{X_{k'_n:n}-b_m}{a_m}\right|>\epsilon\right)\underset{n}{\longrightarrow}0,$$

and this proves (2.19). In order to switch to the convergence w.p.1, we argue by the same way as in the end of the proof of Theorem 2.1. This completes the proof of the theorem. □

3. Bootstrapping central order statistics

In this section, we discuss the strong consistency of the bootstrap distribution $H^{\star}_{n,m}(c_mx+d_m)=P(X_{k_m:m}\le c_mx+d_m|X_n)$, where k_n is the central rank sequence, which satisfies the condition $k_n\sim pn+o(\sqrt{n})$, and (1.6) is satisfied with $V(x)=V_{i;\beta}(x),\ i=1,2,3,4$, for some suitable normalizing constants $c_n>0$ and d_n.

3.1. Almost sure consistency of bootstrapping central for known normalizing constants

Barakat et al. [16] proved the weak limit relations $\sup_{x\in\Re}|H^{\star}_{n,m}(c_m x+d_m)-\mathcal{N}(V_{i;\beta}(x))| \xrightarrow[n]{p} 0,\ i=1,2,3,4$, if $m=o(n)$. The following theorem extends this result.

Theorem 3.1. *If* $\sum_{n=1}^{\infty}\lambda^{\sqrt{\frac{n}{m}}}<\infty$, *for each* $\lambda\in(0,1)$, *then*

$$\sup_{x\in\Re}|H^{\star}_{n,m}(c_m x+d_m)-\mathcal{N}(V_{i;\beta}(x))| \xrightarrow[n]{w.p.1} 0,\ i=1,2,3,4.$$

Proof: On one hand, we have

$$\sqrt{m}\frac{F_n(c_m x+d_m)-p}{C_p}=\sqrt{\frac{m}{n}}\left(\frac{nF_n(c_m x+d_m)-nF(c_m x+d_m)}{\sqrt{np(1-p)}}\right)+\sqrt{m}\frac{F(c_m x+d_m)-p}{C_p}.$$

On the other hand, the assumption of the theorem guarantees that $F(c_m x+d_m)\sim p$, as $n\to\infty$, and

$$\sqrt{m}\frac{F(c_m x+d_m)-C_p}{C_p}\xrightarrow[n]{} V_{i;\beta}(x).$$

Thus, to prove

$$\sqrt{m}\frac{F_n(c_m x+d_m)-C_p}{C_p}\xrightarrow[n]{w.p.1} V_{i;\beta}(x),$$

we need only to show that

$$\sqrt{\frac{m}{n}}\left(\frac{nF_n(c_m x+d_m)-nF(c_m x+d_m)}{\sqrt{nF(c_m x+d_m)(1-F(c_m x+d_m))}}\right)\xrightarrow[n]{w.p.1} 0.$$

By Borel-Cantelli lemma, it is enough to prove that

$$\sum_{n=1}^{\infty}P\left(\sqrt{\frac{m}{n}}\left|\frac{nF_n(c_m x+d_m)-nF(c_m x+d_m)}{\sqrt{nF(c_m x+d_m)(1-F(c_m x+d_m))}}\right|>\epsilon\right)<\infty,$$

for every $\epsilon>0$. Now, for each $\theta>0$ we have

$$\sqrt{\frac{m}{n}}\log P\left(\sqrt{\frac{m}{n}}\frac{nF_n(c_m x+d_m)-nF(c_m x+d_m)}{\sqrt{nF(c_m x+d_m)(1-F(c_m x+d_m))}}>\epsilon\right)=\sqrt{\frac{m}{n}}$$

$$\times\log P\left(\frac{nF_n(c_m x+d_m)-nF(c_m x+d_m)}{\sqrt{nF(c_m x+d_m)(1-F(c_m x+d_m))}}>\sqrt{\frac{n}{m}}\,\epsilon\right)=\sqrt{\frac{m}{n}}\log P\left(e^{\theta T_{n,m}}>e^{\theta\sqrt{\frac{n}{m}}\,\epsilon}\right),$$

where $T_{n,m}$ is defined as

$$T_{n,m} = \frac{nF_n(c_m x + d_m) - nF(c_m x + d_m)}{\sqrt{nF(c_m x + d_m)(1 - F(c_m x + d_m))}}.$$

By using Markov inequality we get

$$\begin{aligned}\sqrt{\frac{m}{n}} \log P\left(e^{\theta T_{n,m}} > e^{\theta\sqrt{\frac{n}{m}}\,\epsilon}\right) &\le \sqrt{\frac{m}{n}} \log\ \left(e^{-\theta\sqrt{\frac{n}{m}}\,\epsilon} E\left(e^{\theta T_{n,m}}\right)\right)\\ &= -\theta\epsilon + \sqrt{\frac{m}{n}} \log \varphi_m(\theta) \xrightarrow[n]{} -\theta\epsilon.\end{aligned}$$

Therefore, for sufficiently large n, we get

$$\sum_{n=1}^{\infty} P\left(\sqrt{\frac{m}{n}}\left(\frac{nF_n(c_m x + d_m) - nF(c_m x + d_m)}{\sqrt{nF(c_m x + d_m)(1 - F(c_m x + d_m))}}\right) > \epsilon\right) =$$

$$\sum_{i=1}^{\infty} \exp\left\{\log P\left(\sqrt{\frac{m}{n}}\left(\frac{nF_n(c_m x + d_m) - nF(c_m x + d_m)}{\sqrt{nF(c_m x + d_m)(1 - F(c_m x + d_m))}}\right) > \epsilon\right)\right\} \le \sum_{i=1}^{\infty} e^{-\theta\epsilon\sqrt{\frac{n}{m}}} < \infty,$$

for every $\epsilon > 0$, since the condition $\sum_{n=1}^{\infty} \lambda^{\sqrt{\frac{n}{m}}} < \infty$, for each $\lambda \in (0,1)$, guarantees the convergence of the infinite series $\sum_{n=1}^{\infty} \exp\left\{-\theta\epsilon\sqrt{\frac{n}{m}}\right\}$, for every $\epsilon > 0$. By similar reasoning we can show that

$$\sum_{n=1}^{\infty} P\left(\sqrt{\frac{m}{n}}\left(\frac{nF_n(c_m x + d_m) - nF(c_m x + d_m)}{\sqrt{nF(c_m x + d_m)(1 - F(c_m x + d_m))}}\right) < -\epsilon\right) < \infty,$$

for every $\epsilon > 0$. The theorem is proved. □

3.2. Limits of bootstrap distribution for central order statistics when normalizing constants are unknown

Let $\hat{c}_m$ and $\hat{d}_m$ be estimators of c_m and d_m based on $X_n = (X_1, X_2, ..., X_n)$, respectively. Define the bootstrap distribution for the normalized central order statistic $c_n^{-1}(X_{k_n:n} - d_n)$ with the estimated normalizing constants by $\hat{H}^{\star}{}_{n,m}(\hat{c}_m x + \hat{d}_m) = P(Y_{k_m:m} \le \hat{c}_m x + \hat{d}_m | X_n)$. In order to study the limit of bootstrap distribution for central order statistics when normalizing constants are unknown, we start with the following essential theorem.

Theorem 3.2. *Let $m = m(n)$. Then, for all the continuity points of $V_{i;\beta}(x)$, $i = 1, 2, 3$ (see Remark 3.1), we have*

$$\sup_{x\in\Re}\left|\hat{H}^{\star}{}_{n,m}(\hat{c}_m x + \hat{d}_m) - \mathcal{N}(V_{i;\beta}(x))\right| \xrightarrow[n]{w.p.1} 0,$$

if (i) $H^{\star}_{n,m}(x) \xrightarrow[n]{w.p.1} \mathcal{N}(V_{i;\beta}(x))$, (ii) $\frac{\hat{c}_m}{c_m} \xrightarrow[n]{w.p.1} 1$, and (iii) $\frac{\hat{d}_m - d_m}{c_m} \xrightarrow[n]{w.p.1} 0$. Moreover, this theorem holds if "$\xrightarrow[n]{w.p.1}$" is replaced by "$\xrightarrow[n]{p}$".

Proof: The proof of the theorem is similar to the proof of Theorem 2.2. □

Remark 3.1. A quick look at the possible non-degenerate limit laws $\mathcal{N}(V_{i;\beta}(x))$, $i = 1,2,3$, reveals that each of these limit laws has at most one discontinuity point.

For the bootstrap distribution $\hat{H}^{\star}{}_{n,m}(\hat{c}_m x + \hat{d}_m)$ to be consistent, we need to choose $\hat{c}_m$ and $\hat{d}_m$ satisfying the conditions (ii) and (iii) in Theorem 3.2. In the next theorem, we suggest choices for $\hat{c}_m$ and $\hat{d}_m$ as a functional of the empirical distribution for the domains of attraction $F(c_n x + d_n) \in D^{(p)}\mathcal{N}(V_{i;\beta}(x)), i = 1,2,3$.

Theorem 3.3. *Let* $k_n^{'} = [pn] + 1$, $k_n^{''} = [\frac{n}{\sqrt{m}} + pn] + 1$, *and* $k_n^{'''} = [pn - \frac{n}{\sqrt{m}}] + 1$. *Then*

1. *if* $F(c_n x + d_n) \in D^{(p)}(\mathcal{N}(V_{1;\beta}(x)))$, $\hat{c}_m = F_n^{-1}\left(p + \frac{1}{\sqrt{m}}\right) - F_n^{-1}(p) = X_{k_n^{''}:n} - X_{k_n^{'}:n}$, and $\hat{d}_m = F_n^{-1}(p) = X_{k_n^{'}:n}$;
2. *if* $F(c_n x + d_n) \in D^{(p)}(\mathcal{N}(V_{2;\beta}(x)))$, $\hat{c}_m = F_n^{-1}(p) - F_n^{-1}\left(p - \frac{1}{\sqrt{m}}\right) = X_{k_n^{'}:n} - X_{k_n^{'''}:n}$, and $\hat{d}_m = F_n^{-1}(p) = X_{k_n^{'}:n}$;
3. *if* $F(c_n x + d_n) \in D^{(p)}(\mathcal{N}(V_{3;\beta}(x)))$, $\hat{c}_m = F_n^{-1}\left(p + \frac{1}{\sqrt{m}}\right) - F_n^{-1}(p) = X_{k_n^{''}:n} - X_{k_n^{'}:n}$, and $\hat{d}_m = F_n^{-1}(p) = X_{k_n^{'}:n}$.

If $m = o(n)$, *then*

$$\sup_{x \in \Re} \left| \hat{H}^{\star}{}_{n,m}(\hat{c}_m x + \hat{d}_m) - \mathcal{N}(V_{i;\beta}(x)) \right| \xrightarrow[n]{p} 0. \tag{3.1}$$

Moreover, if $\sum_{n=1}^{\infty} \lambda^{\sqrt{\frac{n}{m}}} < \infty$ *for each* $\lambda \in (0,1)$ *then (3.1) holds w.p.1.*

Proof: Let $F(c_n x + d_n) \in D^{(p)}(\mathcal{N}(V_{1;\beta}(x)))$. From Theorem 3.2, it suffices to show that

$$\frac{\hat{c}_m}{c_m} = \frac{X_{k_n^{''}:n} - X_{k_n^{'}:n}}{c_m} \xrightarrow[n]{} 1 \tag{3.2}$$

and

$$\frac{\hat{d}_m - d_m}{c_m} = \frac{X_{k_n^{'}:n} - d_m}{c_m} \xrightarrow[n]{} 0, \tag{3.3}$$

both in probability or w.p.1. First, let us focus on the case of convergence in probability. We start with

$$\frac{\hat{c}_m}{c_m} = \frac{X_{k''_n:n} - X_{k'_n:n}}{c_m} = \frac{X_{k''_n:n} - d_m}{c_m} - \frac{X_{k'_n:n} - d_m}{c_m}.$$

Thus, to prove (3.2) and (3.3), it is sufficient to show that

$$\frac{X_{k''_n:n} - d_m}{c_m} \xrightarrow[n]{p} 1 \tag{3.4}$$

and

$$\frac{X_{k'_n:n} - d_m}{c_m} \xrightarrow[n]{p} 0. \tag{3.5}$$

We start with the proof of (3.4). By using the relations $[\frac{n}{\sqrt{m}} + pn] = \frac{n}{\sqrt{m}} + pn - \delta$, where $0 \leq \delta < 1$, and $\frac{1}{\sqrt{m}} + p + \frac{1-\delta}{n} \sim p$, as $n \to \infty$, we get

$$\frac{nF(c_m x + d_m) - k''_n}{\sqrt{k''_n(1 - \frac{k''_n}{n})}} = \sqrt{n}\frac{F(c_m x + d_m) - \left(\frac{1}{\sqrt{m}} + p + \frac{1-\delta}{n}\right)}{\sqrt{\left(\frac{1}{\sqrt{m}} + p + \frac{1-\delta}{n}\right)\left(1 - \left(\frac{1}{\sqrt{m}} + p + \frac{1-\delta}{n}\right)\right)}}$$

(3.6)

$$\sim \sqrt{\frac{n}{m}}\left(\sqrt{m}\frac{F(c_m x + d_m) - p}{C_p} - \frac{1 + \frac{\sqrt{m}}{n}(1-\delta)}{C_p}\right) \xrightarrow[n]{} \begin{cases} \infty, & \text{if } x > 1, \\ -\infty, & \text{if } x < 1. \end{cases}$$

Relation (3.6) follows from the two obvious relations

$$\frac{1 + \frac{\sqrt{m}}{n}(1-\delta)}{C_p} \xrightarrow[n]{} \frac{1}{C_p} = c \quad \text{and} \quad \sqrt{m}\frac{F(c_m x + d_m) - p}{C_p} \xrightarrow[n]{} cx^{\beta}.$$

The relation (3.6) yields $P(\frac{X_{k''_n:n} - d_m}{c_m} < \epsilon + 1) \xrightarrow[n]{} \mathcal{N}(\infty) = 1$, which is equivalent to

$$P\left(\frac{X_{k''_n:n} - d_m}{c_m} > \epsilon + 1\right) \xrightarrow[n]{} 0. \tag{3.7}$$

Similarly we have

$$P\left(\frac{X_{k''_n:n} - d_m}{c_m} < -\epsilon + 1\right) \xrightarrow[n]{} \mathcal{N}(-\infty) = 0. \tag{3.8}$$

From (3.7) and (3.8) we get $P(|\frac{X_{k''_n:n} - d_m}{c_m} - 1| > \epsilon) \xrightarrow[n]{} 0$, which proves (3.4). Now, we prove (3.5). One can easily deduce that

(3.9)

$$\sqrt{n}\frac{F(c_m x + d_m) - p}{C_p} = \sqrt{\frac{n}{m}}\left(\sqrt{m}\frac{F(c_m x + d_m) - p}{C_p}\right) \xrightarrow[n]{} \begin{cases} \infty, & \text{if } x > 0, \\ -\infty, & \text{if } x < 0. \end{cases}$$

Thus, from (3.9), we have $P(\frac{X_{k'_n:n}-d_m}{c_m}<\epsilon)\underset{n}{\longrightarrow}\mathcal{N}(\infty)=1$, which is equivalent to

$$P\left(\frac{X_{k'_n:n}-d_m}{c_m}>\epsilon\right)\underset{n}{\longrightarrow}0. \tag{3.10}$$

Similarly we obtain

$$P\left(\frac{X_{k'_n:n}-d_m}{c_m}<-\epsilon\right)\underset{n}{\longrightarrow}0. \tag{3.11}$$

Therefore, by combining the relations (3.10) and (3.11), we get $P(|\frac{X_{k'_n:n}-d_m}{c_m}|>\epsilon)\underset{n}{\longrightarrow}0$, which proves (3.5). In order to switch to convergence w.p.1, we proceed as in the end of the proof of Theorem 2.1. This completes the proof of Part (i).

Now, let $F(c_nx+d_n)\in D^{(p)}(\mathcal{N}(V_{2;\beta}(x)))$. From Theorem 3.2, it suffices to show that

$$\frac{\hat{c}_m}{c_m}=\frac{X_{k'_n:n}-X_{k'''_n:n}}{c_m}\underset{n}{\longrightarrow}1 \tag{3.12}$$

and

$$\frac{\hat{d}_m-d_m}{c_m}=\frac{X_{k'_n:n}-d_m}{c_m}\underset{n}{\longrightarrow}0, \tag{3.13}$$

both in probability or w.p.1. Again, we first focus on the case of the convergence in probability and we start with

$$\frac{\hat{c}_m}{c_m}=\frac{X_{k'_n:n}-X_{k'''_n:n}}{c_m}=\frac{X_{k'_n:n}-d_m}{c_m}-\frac{X_{k'''_n:n}-d_m}{c_m}.$$

Hence, to prove (3.12) and (3.13), it is sufficient to show that

$$\frac{X_{k'''_n:n}-d_m}{c_m}\overset{p}{\underset{n}{\longrightarrow}}-1 \tag{3.14}$$

and

$$\frac{X_{k'_n:n}-d_m}{c_m}\overset{p}{\underset{n}{\longrightarrow}}0. \tag{3.15}$$

We prove (3.14). By applying the relations $[pn-\frac{n}{\sqrt{m}}]=pn-\frac{n}{\sqrt{m}}-\delta,\ 0\leq\delta<1$, and $p-\frac{1}{\sqrt{m}}+\frac{1-\delta}{n}\sim p$, as $n\to\infty$, we can deduce that

$$\frac{nF(c_mx+d_m)-k'''_n}{\sqrt{k'''_n(1-\frac{k'''_n}{n})}}=\sqrt{n}\frac{F(c_mx+d_m)-\left(p-\frac{1}{\sqrt{m}}+\frac{1-\delta}{n}\right)}{\sqrt{\left(p-\frac{1}{\sqrt{m}}+\frac{1-\delta}{n}\right)\left(1-\left(p-\frac{1}{\sqrt{m}}+\frac{1-\delta}{n}\right)\right)}}$$

(3.16)

$$\sim \sqrt{\frac{n}{m}}\left(\sqrt{m}\frac{F(c_mx+d_m)-p}{C_p}-\frac{-1+\frac{\sqrt{m}}{n}(1-\delta)}{C_p}\right)\xrightarrow[n]{}\begin{cases}-\infty, & \text{if } |x|>1,\\ \infty, & \text{if } |x|<1.\end{cases}$$

Thus, on account (3.16), we get $P(\frac{X_{k_n''':n}-d_m}{c_m}<\epsilon-1)\xrightarrow[n]{}\mathcal{N}(\infty)=1$, which is equivalent to

$$P\left(\frac{X_{k_n''':n}-d_m}{c_m}>\epsilon-1\right)\xrightarrow[n]{}0. \tag{3.17}$$

In the same manner, we have

$$P\left(\frac{X_{k_n''':n}-d_m}{c_m}<-\epsilon-1\right)\xrightarrow[n]{}\mathcal{N}(-\infty)=0. \tag{3.18}$$

From (3.17) and (3.18), we get $P(|\frac{X_{k_n''':n}-d_m}{c_m}+1|>\epsilon)\xrightarrow[n]{}0$. Hence (3.14) is proved. We turn now to prove (3.15). We start with the obvious limit relation

(3.19)

$$\sqrt{n}\frac{F(c_mx+d_m)-p}{C_p}=\sqrt{\frac{n}{m}}\left(\sqrt{m}\frac{F(c_mx+d_m)-p}{C_p}\right)\xrightarrow[n]{}\begin{cases}\infty, & \text{if } x>0,\\ -\infty, & \text{if } x<0,\end{cases}$$

which in turn implies that $P(\frac{X_{k_n':n}-d_m}{c_m}<\epsilon)\xrightarrow[n]{}\mathcal{N}(\infty)=1$ and hence

$$P\left(\frac{X_{k_n':n}-d_m}{c_m}>\epsilon\right)\xrightarrow[n]{}0. \tag{3.20}$$

Moreover, the limit relation (3.19) yields

$$P\left(\frac{X_{k_n':n}-d_m}{c_m}<-\epsilon\right)\xrightarrow[n]{}0. \tag{3.21}$$

By combining (3.20) and (3.21), we get $P(|\frac{X_{k_n':n}-d_m}{c_m}|>\epsilon)\xrightarrow[n]{}0$, which proves (3.15). Finally, the fact that the convergence in (3.14) and (3.15) is w.p.1 can be easily proved by the same way as in the end of the proof of Theorem 2.1. This completes the proof of Part (ii).

Finally, consider the case $F(c_nx+d_n)\in D^{(p)}(\mathcal{N}(V_{3;\beta}(x)))$. From Theorem 3.2, it suffices to show that

$$\frac{\hat{c}_m}{c_m}=\frac{X_{k_n'':n}-X_{k_n':n}}{c_m}\xrightarrow[n]{}1 \tag{3.22}$$

and

$$\frac{\hat{d}_m-d_m}{c_m}=\frac{X_{k_n':n}-d_m}{c_m}\xrightarrow[n]{}0, \tag{3.23}$$

both in probability or w.p.1. We first focus on the case of the convergence in probability and we start with

$$\frac{\hat{c}_m}{c_m}=\frac{X_{k_n'':n}-X_{k_n':n}}{c_m}=\frac{X_{k_n'':n}-d_m}{c_m}-\frac{X_{k_n':n}-d_m}{c_m}.$$

Therefore, to prove (3.22) and (3.23), it is sufficient to show that

$$\frac{X_{k_n':n}-d_m}{c_m}\xrightarrow[n]{p}1 \tag{3.24}$$

and

$$\frac{X_{k_n'':n}-d_m}{c_m}\xrightarrow[n]{p}0. \tag{3.25}$$

By proceeding as we did in Parts (i) and (ii), we can easily show that

$$\frac{nF(c_mx+d_m)-k_n''}{\sqrt{k_n''(1-\frac{k_n''}{n})}}\xrightarrow[n]{}\begin{cases}\infty, & \text{if } x>1,\\ -\infty, & \text{if } x<1.\end{cases} \tag{3.26}$$

Again, by proceeding as we did in Parts (i) and (ii), the relation (3.26) yields $P(|\frac{X_{k_n'':n}-d_m}{c_m}-1|>\epsilon)\xrightarrow[n]{}0$, which in turn proves (3.24). On the other hand, the proof of the relation (3.25) follows also by proceeding as we did in Parts (i) and (ii). Finally, we can prove that the convergence in both the relations (3.24) and (3.25) is w.p.1, by the same way as in the end of the proof of Theorem 2.1. The proof is complete. □

3.3. Bootstrapping sample quantiles when the DFs of these quantiles weakly converge to $\mathcal{N}(x)$ and F is unknown

It has been known for a long time that the DF of the sample quantile $X_{k_n:n}=X_{[pn]+1:n}$, $0<p<1$, based on a continuous DF $F(x)$ with positive probability density (PDF) $f(x)$ in a neighborhood of the pth population quantile $x_o=F^{-1}(p)$, weakly converges to the standard normal DF (e.g., see [35]). In the present subsection, we will study the limit bootstrapping sample quantiles when the PDF f is unknown. We start with a classical result; its proof can be found in many known references among them [35].

Lemma 3.1. *Let $X_{k_n:n}=X_{[pn]+1:n}$, $0<p<1$, be a sample quantile, which is based on a continuous DF $F(x)$ with a positive PDF $f(x)$ in a neighborhood of the pth population quantile $x_0=F^{-1}(p)$. Then*

$$P(X_{k_n:n}<c_nx+d_n)=P\left(\sqrt{n}f(F^{-1}(p))\frac{X_{k_n:n}-F^{-1}(p)}{\sqrt{p(1-p)}}\leq x\right)\xrightarrow[n]{w}\mathcal{N}(x). \tag{3.27}$$

where $c_n=\frac{\sqrt{p(1-p)}}{\sqrt{n}f(x_0)}$ and $d_n=x_0=F^{-1}(p)$.

It is known (cf. [34]) that $X_{k_n:n}$ is a consistent estimator of $F^{-1}(p)$. Moreover, the relation (3.27) can be used to construct an approximate confidence interval for $F^{-1}(p)$, if either the form of f is completely specified around $F^{-1}(p)$ or a good estimator for $f(F^{-1}(p))$ is available. Siddiqui [34] proposed an estimator for $\frac{1}{f(x_0)} = \frac{1}{f(F^{-1}(p))}$ in the form $S_{rn} = \frac{n}{2r}(X_{[np]+r:n} - X_{[np]-r+1:n})$. Moreover, Siddiqui [34] showed that this estimator is asymptotically normal DF, when r is chosen to be of order $n^{\frac{1}{2}}$. Bloch and Gastwirth [17] showed that, if $r = o(n)$ and $r \xrightarrow[n]{} \infty$ then, S_{rn} is a consistent estimator for $\frac{1}{f(F^{-1}(p))}$. Now, we study the bootstrap distribution of $X_{k_m:m}$, $k_m = [mp] + 1$, which defined for unknown normalizing constants by $H^*_{n,m}(\hat{c}_m x + \hat{d}_m) = P(X_{k_m:m} < \hat{c}_m x + \hat{d}_m \mid X_n)$, where $\hat{c_m}$ and $\hat{d_m}$ are some estimators of c_m and d_m, respectively.

Theorem 3.4. *Let $\hat{c}_m = \frac{\sqrt{p(1-p)}}{\sqrt{m}} S_{rm}$, $\hat{d}_m = X_{[np]+1:n}$, where $r = o(m)$. Then*

$$\sup_{x\in\Re} |H^*_{n,m}(\hat{c}_m x + \hat{d}_m) - \mathcal{N}(x))| \xrightarrow[n]{p} 0, \text{ if } m = o(n).$$

Moreover, if there exist $\lambda \in (0,1)$ such that $\sum_{n=1}^{\infty} \lambda^{\sqrt{\frac{n}{m}}} < \infty$ then

$$\sup_{x\in\Re} |H^*_{n,m}(\hat{c}_m x + \hat{d}_m) - \mathcal{N}(x))| \xrightarrow[n]{w.p.1} 0.$$

Proof: In order to the bootstrap distribution $H^*_{n,m}(\hat{c}_m x + \hat{d}_m)$ to be consistent, we have to prove that $\hat{c}_m$ and $\hat{d}_m$ satisfy the conditions (ii) and (iii) in Theorem 3.2, respectively. Since S_{rm} is a consistent estimator for $\frac{1}{f(x_0)}$ (cf. [17]), we get

$$\frac{\hat{c}_m}{c_m} = \frac{\sqrt{p(1-p)}/\sqrt{m} S_{rm}}{\sqrt{p(1-p)}/\sqrt{m} f(x_o)} = S_{rm} f(x_0) \xrightarrow[n]{p} 1.$$

On the other hand, we have $\frac{c_n}{c_m} = \frac{\sqrt{p(1-p)}/\sqrt{n} f(x_0)}{\sqrt{p(1-p)}/\sqrt{m} f(x_0)} = \sqrt{\frac{m}{n}} \xrightarrow[n]{} 0$. Thus, on account of Lemma 3.3, we get

$$\frac{\hat{d}_m - d_m}{c_m} = \frac{X_{[np]+1:n} - F^{-1}(p)}{c_n} \frac{c_n}{c_m} \xrightarrow[n]{p} 0.$$

Therefore, the conditions (ii) and (iii) in Theorem 3.2 are proved when the convergence is in probability. The proof of these conditions when the convergence is w.p.1 is achieved by the same way as in the end of the proof of Theorem 2.1. The proof is complete. □

4. Simulation study

In this section, we address two applications of the earlier theoretical findings. Firstly, we provide a p-value-based method for choosing m. We present a simulation study in Example 4.1 that is carried out using Mathematica 11 to explain how we choose numerically the values of m to give the best approximation of the bootstrapping DFs for the central and intermediate quantiles. In Example 4.1, we choose normality to highlight the key issue that pertains to the selection of m. On the other hand, under typical circumstances, the majority of the practical issues that any researcher faces result in the asymptotic normality of the quantiles (e.g., see Lemma 3.1). Consequently, based on the Kolmogorov-Smirnov test of normality and the corresponding p-values, the best value of m (that corresponds to the largest p-value) should be chosen such that $\sum_{n=1}^{\infty}\lambda^{\sqrt{\frac{n}{m}}} < \infty$, for each $\lambda \in (0,1)$ (see Remark 4.1). Although this method is applied when the quantiles being bootstrapped are asymptotically normal, other possible asymptotically laws given in Theorems 2.1, 2.2, 3.1, and 3.2 can be considered by applying a similar algorithm. Secondly, in Example 4.2, based on several large samples from a logistic distribution, we construct confidence intervals for the median using the bootstrapping methodology and the approach provided in Example 4.1. Additionally, predicted coverage probabilities are included with each computed confidence interval.

4.1. Examples

Example 4.1. This example relies on the fact that the sample median $\mathcal{S}_{1;n} = X_{[\frac{n}{2}+1]:n}$, and the sample intermediate quantiles $\mathcal{S}_{2;n} = X_{[2\sqrt{n}]:n}$, $\mathcal{S}_{3;n} = X_{[\sqrt{n}]:n}$, $\mathcal{S}_{4;n} = X_{2[\sqrt[3]{n}]:n}$, and $\mathcal{S}_{5;n} = X_{[\sqrt[3]{n}]:n}$ based on the standard normal DF weakly converge to the normal DF. Let $\hat{\mathcal{S}}_{i;m},\ i = 1,2,...,5$, be the corresponding bootstrapping statistics of $\mathcal{S}_{i;n},\ i = 1,2,...,5$, respectively, where each of these bootstrapping statistics is based on a sub-sample with replacement of size m (a bootstrap sample of size m). According, to the results of Sections 2 and 3, we expect that the bootstrapping DFs of the statistics $\hat{\mathcal{S}}_{i;m},\ i = 1,2,...,5$, converge to the normal DF provided that $m \ll n$ (i.e., $m = o(n)$).

This study, shown in Table 1, is achieved via the following algorithm:

(i) Generate a random sample (parent sample) of size $n = 100,000$ from $\mathcal{N}(.)$;

(ii) Determine a value of m (100, 200, ..., 5000, as shown in Table 1) and generate a sub-sample with a replacement of size m (a bootstrap sample) from the parent sample;

(iii) Determine each of the sample bootstrapping statistics $\hat{\mathcal{S}}_{i;m},\ i = 1,2,...,5$;

(iv) Repeat the steps (ii) and (iii) 1000 times to obtain the observed sample bootstrapping statistics $\hat{\mathcal{S}}_{ij;m},\ i = 1,2,...,5;\ j = 1,2,...,1000$;

(v) By using the Kolmogorov-Smirnov test, check the normality of the data sets $\{\hat{\mathcal{S}}_{ij;m},\ i = 1,2,...,5;\ j = 1,2,...,1000\}$ and determine the corresponding p-values (see Remark 4.2);

(vi) Repeat the steps (ii)-(v) 100 times for each chosen m and compute the average p-values (denoted by $\overline{p}$) for each chosen m and each of the five statistics. These averages, $\overline{p}$, are written as entries in Table 1, where the best $\overline{p}$ is distinguished by an asterisk.

It is noted that for $n = 100,000$, the best choice of m falls in the interval $[200, 400]$, i.e., the values 200 to 400 are $0.2 - 0.4\%$ of the value of n. (see Remark 4.1). Moreover, the $\overline{p}$ for the central case are higher than those for the intermediate case.

Remark 4.1. According, to the results of Sections 2 and 3, the best performance of the bootstrapping DFs of the central and intermediate order statistics occurs at the values of m for which $\sum_{n=1}^{\infty} \lambda^{\sqrt{\frac{n}{m}}} < \infty$, for each $\lambda \in (0,1)$. On the other hand, according, to [1] the condition $\sqrt{m} = o(\frac{2\sqrt{n}}{\log n})$ is a sufficient condition for $\sum_{n=1}^{\infty} \lambda^{\sqrt{\frac{n}{m}}} < \infty$, which implies that the best performance of the bootstrapping DFs of the central and intermediate order statistics occurs when $m \ll 3000$. Therefore, the simulation output endorses this anticipated result.

Remark 4.2. In the earlier version of this paper, in order to implement Part (v) of the given algorithm, we fitted the data sets $\{\hat{\mathcal{S}}_{ij;m},\ i = 1, 2, ..., 5;\ j = 1, 2, ..., 1000\}$ to the normal DF by using the Kolmogorov-Smirnov test after calculating the sample mean and standard deviation. However, one referee point out to an important issue that the Kolmogorov-Smirnov test can be used to fit the normal DF only when parameters are not estimated from the data (cf. [29]). Since our focus here is only on checking the normality of the bootstrap samples, we apply the Kolmogorov-Smirnov test to check the normality of the given sample bootstrapping statistics without estimating any parameters. Namely, in Mathematica 11, there are two ways to fit any data to the normal DF. The first is to provide the mean and variance values; if not, estimate them based on the data. The second choice is to examine the data's normality without figuring out what the fitted normal distribution's parameter values should be. The second choice was adopted.

Example 4.2. In this example, we generate three samples of sizes $n = 100,000$, $n = 50,000$, and $n = 30,000$, from the logistic distribution with location and scale parameters 0=mean=median and 1, respectively. We construct a confidence interval for each median, which pertains to the three samples, using the bootstrapping technique. We first apply the p-value-based method for choosing m, which is given in Example 4.1, where 100 bootstrap runs are taken into consideration. In Table 2, the three best values of m and the corresponding best average values of p-values are given. We currently have a sample of 100 observed medians for each of the three initial samples. These median samples

$k_n \rightarrow$	Central	Intermediate			
$m \downarrow$	$k_n = [\frac{n}{2}] + 1$	$k_n = [\sqrt{n}]$	$k_n = 2[\sqrt{n}]$	$k_n = [\sqrt[3]{n}]$	$k_n = 2[\sqrt[3]{n}]$
100	0.424534	0.241040	0.374819	0.0471849	0.221942
200	0.445344	0.35888*	0.354039	0.0759685	0.136431
300	0.47536*	0.326927	0.422355	0.135734	0.28982*
400	0.461186	0.294167	0.43096*	0.18197*	0.213884
500	0.415695	0.254239	0.396061	0.160607	0.229961
600	0.413815	0.145734	0.231875	0.171207	0.206003
700	0.423271	0.165254	0.275231	0.141310	0.206607
800	0.447738	0.095997	0.249245	0.140825	0.154863
900	0.396874	0.104246	0.248103	0.082727	0.137661
1000	0.416074	0.136514	0.134745	0.094409	0.099289
2000	0.388266	0.104154	0.135125	0.020539	0.149453
3000	0.389145	0.002338	0.207131	0.001534	0.009307
4000	0.356465	0.003308	0.113068	0.000084	0.050480
5000	0.338578	0.024744	0.014881	0.000383	0.049058

Table 1: $\overline{p}$ corresponding to the checking normality of different bootstrap central and intermediate quantiles for various values of m.

follow a normal DF with unknown parameters. Use these median samples to estimate these unknown parameters. Finally, construct a 99% confidence interval of each median pertaining to the three original samples (of sizes 100,000, 50,000, and 30,000). For each of these three samples, we get one constructed confidence interval. Therefore, according to the algorithm given below, we have 10000 confidence intervals to be checked whether each of them contains zero. Due to the use of the bootstrapping technique and also estimating the unknown parameters, we anticipate that the significant levels (SL) of these confidence intervals are smaller than 99%. We estimate the average lower limit ($\overline{L}$), average upper limit ($\overline{U}$), and coverage probability (CP) of the estimated confidence intervals. By doing this, we can estimate the quality of these confidence intervals and subsequently the quality of the suggested approaches. These findings are presented in Table 2, which demonstrates that the SL is not less than 96%, which endorses the results of Theorems 3.3 and 3.4. Moreover, the results presented in Table 2 is achieved via the following algorithm:

(i) Generate a random sample (parent sample) of size n ($n = 100,000; n = 50,0000$, and $n = 30,000$) from the standard logistic distribution;

(ii) Apply the $p-$value-based method which is given in Example 4.1 and choose the best m corresponding to the largest $p-$value (e.g., for the $n = 100,000$ case, we have $m = 300$);

(iii) Generate M ($M = 100$) sub-samples of size m with replacement from the parent sample and calculate the median for each sample;

(iv) Calculate the mean μ_B and standard deviation σ_B for the set of the sample medians (100 medians) in step (iii). In addition, calculate a 99% confidence interval for the population median according to the usual law $\mu_B \pm z_{\alpha/2}\frac{\sigma_B}{\sqrt{M}}$, where $\alpha = 0.01$, and z_p is the pth quantile of the standard normal DF;

(v) Repeat steps (iii) and (iv) 100 times. Determine how many times (say $0 \leq n_1 \leq 100$), the population median (i.e., zero) falls within the constructed confidence intervals.

(vi) Repeat steps (i)-(v) 100 times. In each of those times, we get in step (v), $0 \leq n_i \leq 100, i = 1, 2, ..., 100$;

(vii) Compute $\overline{L}$, $\overline{U}$, and CP$=\frac{\sum_{i=1}^{100} n_i}{10000}$.

$n \downarrow$	$m \downarrow$	p-value	$\overline{L}$	$\overline{U}$	CP
$n = 100,000$	300	0.480606	-0.03	0.03	97.21
$n = 50,000$	200	0.435146	-0.04	0.03	96.69
$n = 30,000$	150	0.420365	-0.04	0.04	96.39

Table 2: The average p-values, $\overline{L}$, $\overline{U}$, and CP for median for three different samples from logistic distribution.

4.2. Discussion

In the light of the preceding simulation study given in Examples 4.1 and 4.2, we consider a virtual case study to show how the developed bootstrap technique in this paper saves time and cost. Suppose our purpose is modeling (i.e., to detect its asymptotic distribution) the sample median of some random phenomenon that is governed by a DF that satisfies the conditions given in Lemma 3.3 (i.e., the sample median from this DF weakly converges to the normal DF).

The usual way to achieve this purpose is to get a large number N of independent random samples, and from each of them, we determine the median. By finding a suitable DF (a normal DF with specified mean and variance) that fits this median-data set (the set of the collected sample medians) we can achieve our aim. As an example, if $N = 1000$ and each sample has a size of 200, we will need 200,000 observations. On the other hand, if we had one large sample of size 100,000 (say) and apply the bootstrap technique, we can achieve our aim by choosing $m \in [200, 400]$ (as the simulation study shows). In this case, bearing in mind that obtaining a large number of independent samples, even of moderated sample sizes, is more difficult and costly than obtaining one sample of a large size, we find that the bootstrap technique is very beneficial. Moreover, regarding the natural question that which of the usual way and bootstrap technique allows us to make better inference on the population median, the theoretical results concerning the bootstrap technique, and especially the result of this paper, guarantees both ways are asymptotically the same. Therefore, one of the most important advantages of the given bootstrap technique is that it enables us to model the different quantiles via one large sample instead of a large number of independent samples.

Undoubtedly the cornerstone of the bootstrap technique given in this section is determining the best value of m. The theoretical result of the paper stipulates that m is small concerning n. One reviewer of this paper provided an elegant intuition about why one wants m to be small, namely, "it is because of discreteness. When m is big, the bootstrap distribution will have big chunks of probability, which can make the distribution less normal than when m is small". The given algorithm to determine m depends on four determinants, which are the parent DF of the given large sample, the sample size n, the number of replications of the p-value, and the number of bootstrap runs. Of course, when any researcher applies the given algorithm he should consider his determinants. However, we repeated the preceding simulation study with different determinants to shed some light on the influence of these determinants on the choice of m.

1. Figures 1-4 suggest that one essentially wants to make m as small as possible, with respect to n, as long as the sufficient condition given in Remark 4.1 is satisfied (and of course, we preserve the necessary requirements that $m \underset{n}{\longrightarrow} \infty$ and $\frac{m}{n} \underset{n}{\longrightarrow} 0$).

2. When the sample size n becomes smaller than 100,000 (with fixing the other determinants), the range of m (the ratio of the best value of m to n) changes by a small amount as shown in Figures 1 and 2. Namely, at $n = 50,000$, the best value of m is about 200 with $\overline{p}$=0.45, while at $n = 30,000$, the best value of m is about 150 with $\overline{p}$=0.43.

3. The change in the number of bootstrap runs (with fixing the other determinants) does not influence the range of the choice of m ($0.2 - 0.4\%$ of the value of n). On the other hand, increasing this number makes $\overline{p}$ decrease and become more stable, see Figure 3.

4. The change in the number of replications of the p-value (with fixing the other determinants) has no influence on the choice of m. On the other hand, increasing this number makes $\overline{p}$ more stable, see Figure 4.

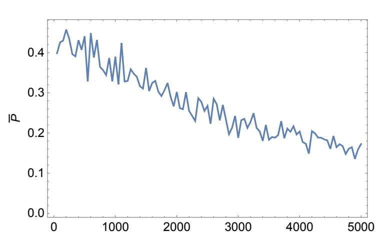

Figure 1: $\overline{p}$ vs. m at $n = 50,000$.

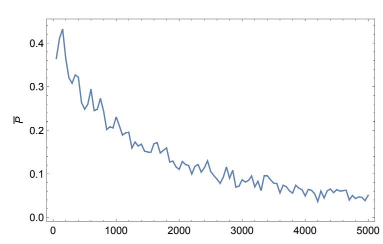

Figure 2: $\overline{p}$ vs. m at $n = 30,000$.

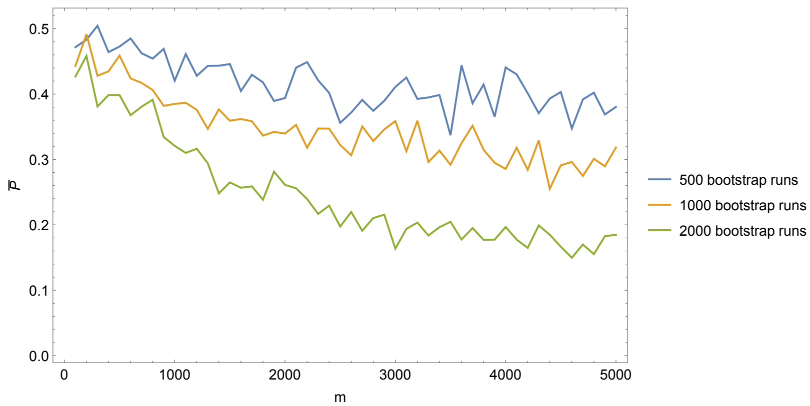

Figure 3: $\overline{p}$ vs. m at different values of the bootstrap runs.

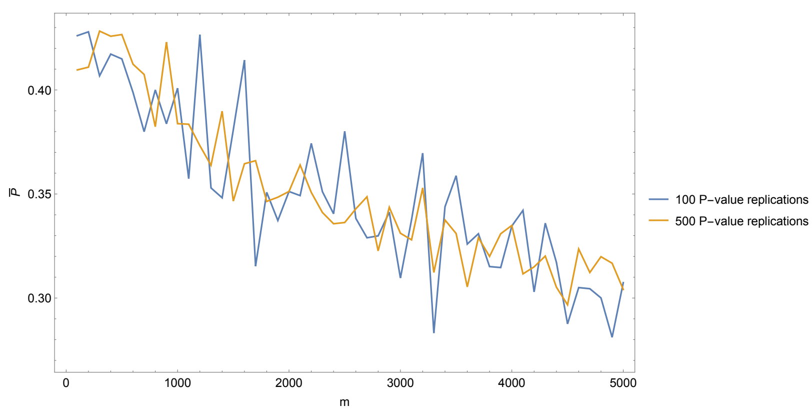

Figure 4: $\overline{p}$ vis m at different values of p-value replications.

5. Concluding remarks

The bootstrap is an extremely flexible technique that can be applied to a wide variety of problems. One of the desired properties of the bootstrapping method is consistency, which guarantees that the limit of the bootstrap distribution is the same as that of the distribution of the given statistic.

In this paper, we investigated the strong consistency of bootstrapping central and intermediate order statistics for an appropriate choice of re-sample size for known and unknown normalizing constants. Consequently, inference concerning quartiles can now be performed by applying the bootstrap technique. For central order statistics, one can use the bootstrap to obtain a confidence interval for the pth population quantile. On the other hand, it is well known that the asymptotic behavior of intermediate quantiles is one of the pillar factors in choosing a suitable value of threshold in the peak over threshold (POT) approach and the constructing related estimators (the Hill estimators) of the tail index (cf. [10, 14, 15, 26]. Therefore, the study of bootstrapping intermediate order statistics will pave the way touse and improve the modeling of extreme values via the POT approach. This potential application of the bootstrapping intermediate order statistics will be the subject of future studies.

The implemented simulation study in this paper aims to show how we choose numerically the values of m to give the best approximation (performance) of the bootstrapping DF for the central and intermediate quantiles. To our best knowledge, there is no such study was done in the literature even for extreme order statistics.

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