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Kernel Estimation of the Dynamic Cumulative Past Inaccuracy Measure for Right Censored Dependent Data

Authors: K. V. Viswakala
– Department of Statistics, University of Kerala,
Thiruvananthapuram, Kerala, India
viswakalakv@gmail.com

E. I. Abdul Sathar ✉
– Department of Statistics, University of Kerala,
Thiruvananthapuram, Kerala, India
sathare@gmail.com

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Abstract:

- This paper proposes a nonparametric estimator for the lifetime distribution's dynamic cumulative past inaccuracy measure based on censored dependent data. The asymptotic properties of the estimator are discussed under suitable regularity conditions. We use Monte-Carlo simulations to compare the estimator's performance to that of an empirical estimator using mean squared errors to test the estimator's properties numerically. The methods are demonstrated using two different real data sets.

Keywords:

- *dynamic cumulative past inaccuracy measure; alpha-mixing; mean squared error (MSE); mean integrated squared error (MISE).*

AMS Subject Classification:

- 62B10, 62G20.

✉ Corresponding author

1. INTRODUCTION

Let $f(x)$ and $g(x)$ be the probability density functions (pdfs) of the failure times of two systems X and Y, with distribution functions $F(x) = P(X \leq x)$ and $G(x) = P(Y \leq x)$ respectively. Kerridge's [12] measure of inaccuracy between X and Y is given by

$$I(X,Y) = -\int_0^\infty f(x) \log g(x) dx. \tag{1.1}$$

It has been known for a long time as a helpful tool for determining the degree of error in experimental results. It can also be interpreted as an error that occurred when an experiment's true density function, $f(x)$, was assigned to $g(x)$ by the experimenter. Kerridge [12] discussed the application of inaccuracy measures in statistical inference. This measure is also applied in the field of economics. International demand or cross-country demand analysis estimates the demand for goods or services for a group of countries. James and Anita [10] address the outlier problem in the international demand analysis, which can be remedied using inaccuracy measures. Kayal and Sunoj [11] introduced a generalized dynamic conditional Kerridge's inaccuracy measure, which can be represented as the sum of conditional Renyi's divergence and Renyi's entropy. Rajesh *et al.* [17] and Sathar *et al.* [20] suggested nonparametric estimator for inaccuracy measure in the reliability context, such as residual life distributions and past life distributions, respectively, and found their properties under some regularity conditions.

Hooda and Tuteja [9] defined some nonadditive measures of relative information and inaccuracy. Using reversible symmetry, Bhatia and Taneja [2] defined the quantitative-qualitative measure of inaccuracy. Straightforwardly, Gur Dial [8] established the noiseless coding theorems for subjective probability codes for nonadditive measures of inaccuracy. Goel *et al.* [7] introduce and discuss a measure of inaccuracy between the distributions of n^{th} record value. Although this measure is inapplicable when the random variables' pdfs are void, Kundu *et al.* [16] proposed an alternative measure of inaccuracy called dynamic cumulative past inaccuracy between random variables X and Y, which is represented as

$$\bar{C}I(X,Y) = -\int_0^\infty F(x) \log G(x) dx. \tag{1.2}$$

Kundu *et al.* [16] investigated general results for this measure. Relying on various applications of stochastic classes in reliability and information theory fields, Khorashadizadeh [13] studied new classes of the lifetime in terms of cumulative inaccuracy along with their relations with other famous aging classes. Also, some characterization results are obtained under the proportional reversed hazard rate model. Di Crescenzo and Longobardi [4] defined the empirical expression of cumulative inaccuracy in connection with empirical cumulative entropy.

In many realistic situations, if a system is found to be down at time t, the random variable $[t - X|X \leq x]$ describes the time elapsed between the failure of a system and the time. Based on this idea, the cumulative inaccuracy measure between two past lifetimes, analogous to the measure, (1.2), is defined by Kumar and Taneja [15] and Kundu *et al.* [16] independently as

$$\bar{C}I(X,Y,t) = -\int_0^t \frac{F(x)}{F(t)} \log\left[\frac{G(x)}{G(t)}\right] dx, \tag{1.3}$$

and so called dynamic cumulative past inaccuracy measure. Clearly when $t = 0$, (1.3) becomes (1.2). (1.3) equivalently can be written as

$$\begin{aligned} \bar{C}I(X,Y,t) &= -\frac{1}{F(t)}\int_0^t F(x)\log G(x)dx + \frac{\log G(t)}{F(t)}\int_0^t F(x)dx \\ &= \bar{\mathcal{A}}_t + \bar{\mathcal{B}}_t, \end{aligned} \tag{1.4}$$

where

$$\bar{\mathcal{A}}_t = -\frac{1}{F(t)}\int_0^t F(x)\log G(x)dx \quad \text{and} \quad \bar{\mathcal{B}}_t = \frac{\log G(t)}{F(t)}\int_0^t F(x)dx.$$

Ghosh and Kundu [6] introduced the notion of cumulative past inaccuracy of order α and study the proposed measure for conditionally specified models of two components failed at different time instants, called generalized conditional cumulative past inaccuracy, and their properties are discussed.

Example 1.1. Let the random variables X and Y have the following distribution functions $F(x) = 2x - x^2$, and $G(x) = x^\lambda$ respectively, $x \in [0,1]$. Then for $t \in [0,1]$, the dynamic cumulative past inaccuracy measure, $\bar{C}I_n(t)$ is obtained as

$$\bar{C}I_n(t) = \frac{\lambda t(2t-9)}{18(t-2)}.$$

Figure 1 depicts the dynamic cumulative past inaccuracy measure for for $t \in [0,1]$ and for $\lambda \in \{4,\ 6,\ ...,\ 12\}$. According to the graph, the dynamic cumulative past inaccuracy measure is an increasing function in λ and t.

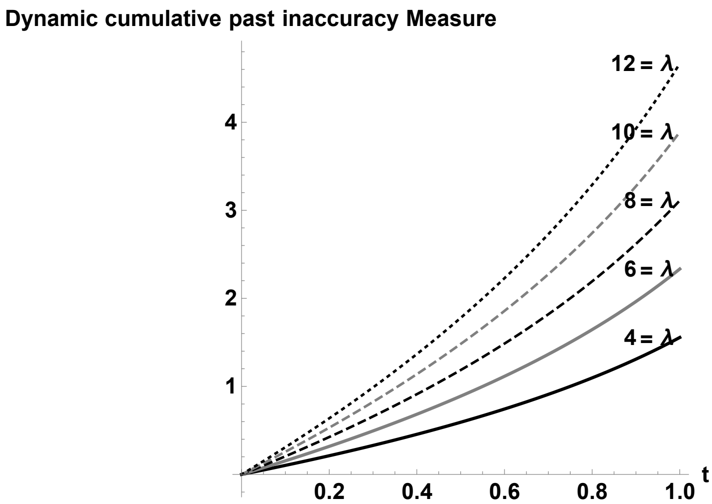


Figure 1: Plot of $\bar{C}I(X,Y,t)$ against $t \in [0,1]$ for different parameter λ.

From a practical standpoint, it appears more reasonable to forego independence in favour of some dependency. For example, if a family's income is exclusively dependent on the salary of one of its members, an accident or the death of that individual will have a negative influence on the family's performance. However, this will not be the case when examined from the perspective of society as a whole. Random variables are derived from specific types of mixing conditions that have already been defined in the literature. Alpha-mixing is a strong mixing condition with many practical applications among the various mixing conditions used in the literature.

Censorship is either desirable or unavoidable in life testing and can take several forms. Withdrawals from a clinical trial, death unrelated to the condition under study, and a person still alive at the end of the follow-up period are all examples of random censoring. Right censorship is one of the most common types of censorship. Right censoring is appropriate in studies of electrical equipment failure, the occurrence of a specific disease, and so on.

Motivated by the emerging work and the importance of (1.3), we intend to develop a kernel function-based estimation technique for this measure in practical situations. This paper considers the nonparametric estimation of (1.3) under right censoring and discusses some of its properties. Throughout this paper, we assume that the random variables are alpha-mixing (Rosenblatt [19]).

This paper's outline is as follows: In Section 2, we present a nonparametric estimator for (1.4) in censored samples. Section 3 looks into the asymptotic properties of the estimator. Section 4 contains a simulation study to demonstrate the estimator's behaviour and a comparison to an empirical estimator. Furthermore, they are compared to two different real-world data sets.

2. KERNEL ESTIMATION

In this section, we propose a nonparametric estimator for the cumulative past inaccuracy measure for right censored data sets. Consider $\{X_i\}$, $\{Y_i\}$, $i = 1, 2, ..., n$ be identically distributed random samples have distribution functions be $F(x) = Pr(X_i \le x)$ and $G(x) = Pr(Y_i \le x)$ respectively. We use independent and identically distributed random variable R_{1i} and R_{2i} with corresponding distribution functions $P_1(x)$ and $P_2(x)$ for creating right-censored data from X_i and Y_i respectively. Note that R_{1i} and R_{2i} are independent of X_i and Y_i respectively. Let $C_i = \min(X_i, R_{1i})$, $C_i^* = \min(Y_i, R_{2i})$, $\delta_i = I(X_i \le R_{1i})$ and $\delta_i^* = I(Y_i \le R_{2i})$. Then the kernel density estimator of (1.4) under right censoring is as follows:

$$
\begin{aligned}
\bar{CI}_n(t) &= \bar{\mathcal{A}}_{nt} + \bar{\mathcal{B}}_{nt}, \\
&= -\frac{1}{F_n(t)} \int_0^t F_n(x) \log G_n(x) dx + \frac{\log G_n(t)}{F_n(t)} \int_0^t F_n(x) dx, \qquad (2.1)
\end{aligned}
$$

where

$$
F_n(t) = \int_0^t \frac{1}{nh} \sum_{i=1}^n \frac{K\left(\frac{x-C_i}{h}\right)\delta_i dx}{1 - P_1(C_i)} \quad \text{and} \quad G_n(t) = \int_0^t \frac{1}{nh} \sum_{i=1}^n \frac{K\left(\frac{x-C_i^*}{h}\right)\delta_i^* dx}{1 - P_2(C_i^*)},
$$

respectively are the nonparametric density estimator for $F(t)$ and $G(t)$ under censoring and $K(\cdot)$ be the kernel function. For the positive integers, i and j $h \to 0$, $nh \to \infty$ and the following assumptions hold:

(**i**) $f^{(k)}(x), 1 \leq k \leq 2j$, exists and $f^{(2j)}(x)$ is bounded

and if $K(\cdot)$ satisfies,

(**ii**) $K(s) \geq 0,\ -\infty < s < \infty$, $\int_{\mathbb{R}+} K(s)ds = 1$, $\int_{\mathbb{R}+} s^a K^i(s)ds = 0$ for positive odd integer a, $\int_{\mathbb{R}+} s^b K^i(s)ds < \infty$ for positive even integer b.

Denote $Q^*(t) = P(C_1 \leq t, \delta_1 = 1)$ the sub distribution function for the uncensored observations, and $q^*(t) = [1 - P_1(t)]f(t)$ the corresponding density, then a reasonable estimate of $f(t)$ can be obtained from Cai [3] as $q^*(t)/[1 - P_1(t)]$. Consider the transformation $\alpha = \frac{x - C_i}{h}$, then we get

$$
\begin{aligned}
E\left[\frac{1}{h}\frac{K\left(\frac{x-C_i}{h}\right)\delta_i}{1-P_1(C_i)}\right] &= \frac{1}{h}\int_{\mathbb{R}+} \frac{K\left(\frac{x-C_i}{h}\right)}{1-P_1(C_i)} q^*(C_i)dC_i, \\
&= \int_{\mathbb{R}+} K(\alpha) f(x - \alpha h)d\alpha, \\
&= \int_{\mathbb{R}+} K(\alpha)\left[f(x) - f^{(1)}(x)\alpha h + \frac{f^{(2)}(x)}{2!}\alpha^2 h^2 - \ldots\right], \\
&= f(x) + \frac{h^2}{2}\int_{\mathbb{R}+} \alpha^2 K(\alpha)d\alpha f^{(2)}(x) + O(h_n^2), \qquad (2.2)
\end{aligned}
$$

and using Lemma 2 in Elias Masry [5], we get

$$
E\left[\frac{1}{h}\frac{K\left(\frac{x-C_i}{h}\right)\delta_i}{1-P_1(C_i)}\right]^2 = \frac{C_k}{h}\frac{f(x)}{1-P_1(x)}, \qquad (2.3)
$$

where $C_k = \int_{\mathbb{R}+} K^2(\alpha)d\alpha$. Let $K_1 = \frac{K\left(\frac{x-C_i}{h}\right)\delta_i}{1-P_1(C_i)}$, then using (2.2) and (2.3), we get

$$
\begin{aligned}
\text{Bias}[F_n(t)] &= \int_0^t E\left(\frac{K_1}{h}\right)dx - F(x), \\
&= \frac{h^2}{2}\int_{\mathbb{R}+} \alpha^2 K(\alpha)d\alpha \int_0^t f^{(2)}(x)dx + O(h^4), \\
\text{Var}[F_n(t)] &\approx \frac{1}{n}\left\{\int_0^t E\left(\frac{K_1}{h}\right)^2 dx - \int_0^t \left[E\left(\frac{K_1}{h}\right)\right]^2 dx\right\} \\
&\quad + \left\{\int_0^t E\left(\frac{K_1}{h}\right)dx - F(x)\right\}^2, \\
&= \frac{C_k}{nh}\int_0^t \frac{f(x)}{1-P_1(x)}dx.
\end{aligned}
$$

2.1. Estimation of $\bar{\mathcal{A}}_t$ and $\bar{\mathcal{B}}_t$

Using Taylor's series expansion, we have

$$
\log G_n(x) = \log G(x) + \frac{G_n(x) - G(x)}{G(x)} + R_n,
$$

where

$$R_n = \int_0^1 \frac{2(1-\tau)}{\{G(x)+\tau[G_n(x)-G(x)]\}^2}[G_n(x)-G(x)]^2 d\tau.$$

Hence,

$$\begin{aligned} F_n(x)\log G_n(x) - F(x)\log G(x) \ &= \\ &= \log G(x)[F_n(x)-F(x)] + \frac{1}{G(x)}[F_n(x)-F(x)][G_n(x)-G(x)] \\ &+ \frac{F(x)}{G(x)}[G_n(x)-G(x)] + R_n[F_n(x)-F(x)] + F(x)R_n. \end{aligned} \tag{2.4}$$

Next, we need to find $E|R_n|^j$, for any positive integer j. For this, consider $V_1 = \Big\{x : |G_n(x) - G(x)| \le \frac{G(x)}{2}\Big\}$ and V_1^c is the compliment of V_1. Clearly, for $\theta \in V_1$ and for every $0 \le \epsilon \le 1$, we have

$$0 < G(x)\Big(1-\frac{\epsilon}{2}\Big) \le G(x) + \epsilon[G_n(x)-G(x)] < G(x)\Big(1+\frac{\epsilon}{2}\Big).$$

Equivalently, we get

$$0 < \frac{(1-\epsilon)[G_n(x)-G(x)]^2}{\{G(x)+\epsilon[G_n(x)-G(x)]\}^2} \le \frac{(1-\epsilon)[G_n(x)-G(x)]^2}{\big[\big(1-\frac{\epsilon}{2}\big)G(x)\big]^2}.$$

Let $I(\cdot)$ denotes the indicator function, since $\int_0^1 \frac{(1-\epsilon)}{\big(1-\frac{\epsilon}{2}\big)^2} d\epsilon < 1$, then for every positive integer j we get

$$E|R_n|^j I(V_1) \le \frac{1}{[G(x)]^{2j}} E[G_n^*(x) - G(x)]^{2j}.$$

Also, we have

$$E|R_n|^j I(V_1^c) \le E\left[\left|\frac{1}{G_n(x)} - \frac{1}{G(x)} - \frac{G_n(x)-G(x)}{G(x)}\right|^j I(V_1^c)\right].$$

For $1 \le i \le n$, we have $K(x - Y_i) \ne 0$ and $m < K(\alpha) < N$ so that $G_n(x) \ge \frac{m}{nh}$, or equivalently, $\frac{1}{G_n(x)} \le \frac{nh}{m}$. Also, $G_n(x) \le \frac{N}{h}$ and $nh^2 \to \infty$ implies for sufficiently large n,

$$\begin{aligned} E|R_n|^j I(V_1^c) &\le \left|\frac{nh^2}{m} + h - \frac{h}{G(x)} - \frac{N}{G(x)}\right|^j \frac{1}{h^j} E[I(V_1^c)], \\ &= O(n^j h^j) P\left[|G_n^*(x) - G(x)| \ge \frac{G(x)}{2}\right], \\ &\le O(n^j h^j)\Big\{P\left[|G_n^*(x) - E[G_n^*(x)]| \ge \frac{G(x)}{4}\right] \\ &\quad + P\left[|E[G_n^*(x)] - G(x)| \ge \frac{G(x)}{4}\right]\Big\}. \end{aligned}$$

For sufficiently large n,

$$P\left[|E[G_n^*(x)] - G(x)| \geq \frac{G(x)}{4}\right] = 0,$$

and

$$P\left[|G_n(x) - E[G_n(x)]| \geq \frac{G(x)}{4}\right] \leq 2\exp\{-Cnh\},$$

for some constant C, (see Rao [18]), we obtain for sufficiently large n

$$E|R_n|^j I(V_1^c) \leq 2\exp\{-Cnh\}.$$

Also, we have

$$\begin{aligned} E|R_n|^j &= E|R_n|^j I(V_1) + E|R_n|^j I(V_1^c), \\ &\leq \frac{1}{[G(x)]^{2j}} E[G_n(x) - G(x)]^{2j} + O(n^j h^j)\exp\{-Cnh\}. \end{aligned}$$

In particular for $j = 1, 2$ in the above inequality, we get

$$\begin{aligned} E|R_n| &\leq \frac{1}{[G(x)]^2} E[G_n(x) - G(x)]^2 + O(nh)\exp\{-Cnh\}, \\ &= O\left(\frac{1}{nh}\right) + O(h^4) + O(nh), \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} E|R_n|^2 &\leq \frac{1}{[G(x)]^4} E[G_n(x) - G(x)]^4 + O(n^2h^2)\exp\{-Cnh\}, \\ &= O\left(\frac{h^3}{n}\right) + O\left(\frac{1}{n^2h^2}\right) + O(h^8) + O(n^2h^2), \end{aligned} \tag{2.6}$$

since nh goes to infinity, $O(nh)\exp\{-Cnh\}$ and $O(n^2h^2)\exp\{-Cnh\}$ have smaller orders than that of $E[G_n(x) - G(x)]$ and $E[G_n(x) - G(x)]^2$ respectively.

In order to simplify the notation define $\mathfrak{h}_{\mathfrak{n}}(t) = \int_0^t F_n(x)\log G_n(x)dx$, $\mathfrak{g}_{\mathfrak{n}}(t) = \int_0^t F_n(x)dx$, $\mathfrak{h}(t) = \int_0^t F(x)\log G(x)dx$ and $\mathfrak{g}(t) = \int_0^t F(x)dx$ so that we can easily prove that

$$\frac{\mathfrak{h}_{\mathfrak{n}}(t)}{F_n(t)} - \frac{\mathfrak{h}(t)}{F(t)} \approx \frac{\mathfrak{h}_{\mathfrak{n}}(t) - \frac{\mathfrak{h}(t)}{F(t)}F_n(t)}{F(t)}, \tag{2.7}$$

and

$$\frac{\log G_n(t)\mathfrak{g}_{\mathfrak{n}}(t)}{F_n(t)} - \frac{\log G(t)\mathfrak{g}(t)}{F(t)} \approx \frac{\log G_n(t)\mathfrak{g}_{\mathfrak{n}}(t) - \frac{\log G(t)\mathfrak{g}(t)}{F(t)}F_n(t)}{F(t)}. \tag{2.8}$$

Hence using (2.4)–(2.8), we get the following:

$$\begin{aligned} \mathrm{Bias}[\bar{\mathcal{A}}_{nt}] &= -\mathrm{Bias}\left[\frac{\mathfrak{h}_{\mathfrak{n}}(t)}{F_n(t)} - \frac{\mathfrak{h}(t)}{F(t)}\right], \\ &= \frac{-h^2}{2}\int_{\mathbb{R}+} \alpha^2 K(\alpha)d\alpha\left\{\frac{1}{F(t)}\int_0^t \left[\log G(x)\int_0^x f^{(2)}(y)dy\right.\right. \\ &\quad \left.\left. + \frac{F(x)}{G(x)}\int_0^x g^{(2)}(y)dy\right]dx - \frac{\mathfrak{h}(t)}{F^2(t)}\int_0^t f^{(2)}(x)dx\right\} + O(h^4), \end{aligned} \tag{2.9}$$

and

$$\begin{aligned}\text{Bias}[\bar{\mathcal{B}}_{nt}] &= \frac{1}{F(t)}\text{Bias}[\mathfrak{g}_{\mathfrak{n}}(t)\log G_n(t)] - \frac{\log G(t)\mathfrak{g}(t)}{F^2(t)}\text{Bias}[F_n(t)],\\ &= \frac{h^2}{2}\int_{\mathbb{R}^+}\alpha^2K(\alpha)d\alpha\left\{\frac{\log G(t)}{F(t)}\int_0^t\int_0^x f^{(2)}(y)dydx + \frac{\mathfrak{g}(t)}{F(t)G(t)}\int_0^t g^{(2)}(x)dx\right.\\ &\quad\left. - \frac{\log G(t)\mathfrak{g}(t)}{F^2(t)}\int_0^t f^{(2)}(x)dx\right\} + O(h^4). \end{aligned} \tag{2.10}$$

Moreover,

$$\begin{aligned}\text{Var}[\bar{\mathcal{A}}_{nt}] &= \text{Var}\left[\frac{\mathfrak{h}_{\mathfrak{n}}(t)}{F_n(t)} - \frac{\mathfrak{h}(t)}{F(t)}\right],\\ &\approx \frac{C_k}{nh}\frac{1}{F^2(t)}\left\{\int_0^t\log^2 G(x)\int_0^x\frac{f(y)}{1-P_1(y)}dydx\right.\\ &\quad\left.+\int_0^t\left[\frac{F(x)}{G(x)}\right]^2\int_0^x\frac{g(y)}{1-P_2(y)}dydx + \frac{\mathfrak{h}^2(t)}{F^2(t)}\int_0^t\frac{f(x)}{1-P_1(x)}dx\right\}, \end{aligned} \tag{2.11}$$

and

$$\begin{aligned}\text{Var}[\bar{\mathcal{B}}_{nt}] &= \frac{1}{F^2(t)}\text{Var}[\mathfrak{g}_{\mathfrak{n}}(t)\log G_n(t)] + \left[\frac{\log G(t)\mathfrak{g}(t)}{F^2(t)}\right]^2\text{Var}[F_n(t)],\\ &\approx \frac{C_k}{nh}\frac{\log^2 G(t)}{F^2(t)}\left\{\int_0^t\int_0^x\frac{f(y)}{1-P_1(y)}dydx + \frac{\mathfrak{g}^2(t)}{F^2(t)}\int_0^t\frac{f(x)}{1-P_1(x)}dx\right\}\\ &\quad+\frac{C_k}{nh}\frac{\mathfrak{g}^2(t)}{G^2(t)F^2(t)}\int_0^t\frac{g(x)}{1-P_2(x)}dx. \end{aligned} \tag{2.12}$$

The following theorem gives bias and variance of the proposed estimator.

Theorem 2.1. *Under the assumptions given in Section 2, bias and variance of $\bar{C}I_n(t)$ is given as*

$$\begin{aligned}\text{Bias}\left[\bar{C}I_n(t)\right] &= \frac{h^2}{2}\int_{\mathbb{R}^+}\alpha^2K(\alpha)d\alpha\left\{\frac{\log G(t)}{F(t)}\int_0^t\int_0^x f^{(2)}(y)dydx + \frac{\mathfrak{g}(t)}{F(t)G(t)}\int_0^t g^{(2)}(x)dx\right.\\ &\quad-\frac{\log G(t)\mathfrak{g}(t)}{F^2(t)}\int_0^t f^{(2)}(x)dx - \frac{1}{F(t)}\int_0^t\log G(x)\int_0^x f^{(2)}(y)dydx\\ &\quad\left.-\frac{1}{F(t)}\int_0^t\frac{F(x)}{G(x)}\int_0^x g^{(2)}(y)dydx + \frac{\mathfrak{h}(t)}{F^2(t)}\int_0^t f^{(2)}(x)dx\right\}, \end{aligned}$$

and

$$\begin{aligned}\text{Var}\left[\bar{C}I_n(t)\right] &\approx \frac{C_k}{nh}\frac{1}{F^2(t)}\left\{\int_0^t\log^2 G(x)\int_0^x\frac{f(y)}{1-P_1(y)}dydx + \int_0^t\left[\frac{F(x)}{G(x)}\right]^2\int_0^x\frac{g(y)}{1-P_2(y)}dydx\right.\\ &\quad+\frac{\mathfrak{h}^2(t)}{F^2(t)}\int_0^t\frac{f(x)}{1-P_1(x)}dx + \log^2 G(t)\int_0^t\int_0^x\frac{f(y)}{1-P_1(y)}dydx\\ &\quad\left.+\frac{\log^2 G(t)\mathfrak{g}^2(t)}{F^2(t)}\int_0^t\frac{f(x)}{1-P_1(x)}dx + \frac{\mathfrak{g}^2(t)}{G^2(t)}\int_0^t\frac{g(x)}{1-P_2(x)}dx\right\}. \end{aligned}$$

Proof: Using the equations (2.9), (2.10), (2.11) and (2.12), the result follows. □

The following example shows the application of Theorem 2.1.

Example 2.1. Consider the two non-negative random variables X and Y have the pdfs $f(x)$ and $g(x)$ respectively, so that for $x \in (0,1)$

$$\begin{aligned} f(x) &= 2x \text{ and } F(x) = P(X \leq x) = x^2, \\ g(x) &= 3x^2 \text{ and } G(x) = P(Y \leq x) = x^3. \end{aligned}$$

Let the random variables X and Y be right censored by uniform random variables with parameters $(0, 0.5)$ and $(0.5, 1)$, respectively. Then we get

$$\text{Bias}\left[\bar{CI}_n(t)\right] = \frac{-2h^2}{t}\int_{\mathbb{R}+} \alpha^2 K(\alpha)d\alpha,$$

and

$$\begin{aligned} \text{Var}\left[\bar{CI}_n(t)\right] \approx\ & \frac{C_k}{nh}\frac{1}{t^4}\left\{\int_0^t \frac{9\log^2 x}{2}[\log(1-2x)-2x]dx - \frac{3(t-2)t+6(t-1)\log(1-t)}{4t}\right. \\ & - \frac{t^2}{18}[(3\log t-1)^2+9\log^2 t][2t+\log(1-2t)] - \frac{1}{12}[t(2+t)+2\log(1-t)] \\ & \left. + \frac{9\log^2 t}{2}[t-t^2-(t-\frac{1}{2})\log(1-2t)]\right\}. \end{aligned}$$

3. ASYMPTOTIC PROPERTIES

In this section, we discuss some asymptotic properties of (1.4). The following theorem reveals the consistency property of the estimator.

Theorem 3.1. *Under the assumptions given in Section 2, $\bar{CI}_n(t)$ is a consistent estimator of $\bar{CI}(X,Y,t)$.*

Proof: We have

$$\bar{CI}_n(t) = \frac{\log G_n(t)\mathfrak{g}_n(t)}{F_n(t)} - \frac{\mathfrak{h}_n(t)}{F_n(t)}.$$

$\text{MSE}[\mathfrak{h}_n(t)] \to 0$, $\text{MSE}[\mathfrak{g}_n(t)] \to 0$, $\text{MSE}[F_n(t)] \to 0$, $\text{MSE}[\log G_n(t)] \to 0$, when $n \to \infty$, and using Slutsky's theorem we obtain desired result. □

In the following theorem, we check the asymptotic nature of the estimator's mean integrated squared error (MISE).

Theorem 3.2. *Under the assumptions given in Section 2, the MISE of $\bar{CI}_n(t)$ tends to zero as $n \to \infty$.*

Proof:

$$\begin{aligned} \text{MISE}[\bar{CI}_n(t)] &= E\int\left[\bar{CI}_n(t) - \bar{CI}(X,Y,t)\right]^2 dt, \\ &= \text{MISE}[\bar{\mathcal{A}}_{nt}] + \text{MISE}[\bar{\mathcal{B}}_{nt}] + 2E\int\left[\bar{\mathcal{A}}_{nt} - \bar{\mathcal{A}}_t\right]\left[\bar{\mathcal{B}}_{nt} - \bar{\mathcal{B}}_t\right]dt. \end{aligned}$$

Also

$$\begin{aligned}\text{MISE}[\bar{\mathcal{A}}_{nt}] \ \leq \ \int \frac{1}{F^2(t)}\Bigg\{&\text{MSE}[\mathfrak{h}_{\mathfrak{n}}(t)] + \left[\frac{\mathfrak{h}(t)}{F(t)}\right]^2 \text{MSE}[F_n(t)] \\ &-2\frac{\mathfrak{h}(t)}{F(t)}\text{MSE}^{\frac{1}{2}}[\mathfrak{h}_{\mathfrak{n}}(t)]\text{MSE}^{\frac{1}{2}}[F_n(t)]\Bigg\}dt \to 0,\end{aligned}$$

as $n \to \infty$. Using similar steps and applying Holder's inequality, we get the proof. □

The following theorem states the asymptotic normal distribution of the proposed estimator.

Theorem 3.3. *Let $\bar{C}I_n(t)$ be nonparametric estimator of $\bar{C}I(X,Y,t)$, $K(x)$ be a kernel and h satisfying the conditions for bandwidth. Then for fixed t*

$$(nh)^{\frac{1}{2}}\left[\frac{\bar{C}I_n(t) - \bar{C}I(X,Y,t)}{\sigma_{\bar{C}I_n}}\right]$$

follows normal distribution with mean zero and variance 2, as $n \to \infty$ with

$$\begin{aligned}\sigma^2_{\bar{C}I_n} \ = \ \frac{C_k}{F^2(t)}\Bigg\{&\int_0^t \log^2 G(x)\int_0^x \frac{f(y)}{1-P_1(y)}dydx + \int_0^t\left[\frac{F(x)}{G(x)}\right]^2\int_0^x \frac{g(y)}{1-P_2(y)}dydx \\ &+ \frac{\mathfrak{h}^2(t)}{F^2(t)}\int_0^t \frac{f(x)}{1-P_1(x)}dx + \log^2 G(t)\int_0^t\int_0^x \frac{f(y)}{1-P_1(y)}dydx \\ &+ \frac{\log^2 G(t)\mathfrak{g}^2(t)}{F^2(t)}\int_0^t \frac{f(x)}{1-P_1(x)}dx + \frac{\mathfrak{g}^2(t)}{G^2(t)}\int_0^t \frac{g(x)}{1-P_2(x)}dx\Bigg\}.\end{aligned}$$

Proof: We have

$$\begin{aligned}\bar{\mathcal{A}}_{nt} - \bar{\mathcal{A}}_t \ &= \ -\frac{\mathfrak{h}_{\mathfrak{n}}(t)}{F_n(t)} + \frac{\mathfrak{h}(t)}{F(t)}, \\ &= \ -\frac{[\mathfrak{h}_{\mathfrak{n}}(t) - \mathfrak{h}(t)]}{F_n(t)} + \frac{\mathfrak{h}(t)[F_n(t) - F(t)]}{F(t)F_n(t)}, \\ &= \ -\frac{1}{F_n(t)}\int_0^t\left\{\log G(x)\Big[F_n(x) - F(x)\Big] + \frac{F(x)}{G(x)}\Big[G_n(x) - G(x)\Big]\right\}dx \\ &\quad + \frac{\mathfrak{h}(t)}{F(t)F_n(t)}\Big[F_n(x) - F(x)\Big].\end{aligned}$$

Using asymptotic normality and almost sure convergence properties of of $F_n(t)$ given in Cai [3], we get

$$(nh)^{\frac{1}{2}}\left[\frac{\bar{\mathcal{A}}_{nt} - \bar{\mathcal{A}}_t}{\sigma_{\bar{\mathcal{A}}}}\right]$$

asymptotically follows standard normal distribution with

$$\begin{aligned}\sigma^2_{\bar{\mathcal{A}}} \ = \ \frac{C_k}{F^2(t)}\Bigg\{&\int_0^t \log^2 G(x)\int_0^x \frac{f(y)}{1-P_1(y)}dydx + \int_0^t\left[\frac{F(x)}{G(x)}\right]^2\int_0^x \frac{g(y)}{1-P_2(y)}dydx \\ &+ \frac{\mathfrak{h}^2(t)}{F^2(t)}\int_0^t \frac{f(x)}{1-P_1(x)}dx\Bigg\}.\end{aligned}$$

Similarly, we get

$$(nh)^{\frac{1}{2}}\left[\frac{\bar{\mathcal{B}}_{nt}-\bar{\mathcal{B}}_t}{\sigma_{\bar{\mathcal{B}}}}\right]$$

asymptotically follows standard normal distribution with

$$\begin{aligned}\sigma^2_{\bar{\mathcal{B}}} &= \frac{C_k}{F^2(t)}\Bigg\{\log^2 G(t)\Bigg[\int_0^t\int_0^x \frac{f(y)}{1-P_1(y)}dydx+\frac{\mathfrak{g}^2(t)}{F^2(t)}\int_0^t \frac{f(x)}{1-P_1(x)}dx\Bigg]\\ &+\frac{\mathfrak{g}^2(t)}{G^2(t)}\int_0^t\frac{g(x)}{1-P_2(x)}dx\Bigg\}.\end{aligned}$$

Hence the proof. □

In the following theorem, we check the almost sure convergence property of the suggested estimator.

Theorem 3.4. *Let $\bar{C}I_n(t)$ be a nonparametric estimator of $\bar{C}I(X,Y,t)$, suppose that $F(\cdot),G(\cdot),f(\cdot)$ and $g(\cdot)$ satisfy the Lipschitz conditions and the kernel $K(\cdot)$ satisfies the requirements and for $0<\tau<\infty$, the marginal distribution function of R satisfies $L(\tau)<1$ (see Cai [3]) then*

$$\sup_{0\le t\le\tau}\left|\bar{C}I_n(t)-\bar{C}I(X,Y,t)\right|\to 0 \quad \textit{almost surely.}$$

Proof: We have

$$\left|\bar{C}I_n(t)-\bar{C}I(X,Y,t)\right|\le\left|\bar{\mathcal{A}}_{nt}-\bar{\mathcal{A}}_t\right|+\left|\bar{\mathcal{B}}_{nt}-\bar{\mathcal{B}}_t\right|.$$

Also,

$$\begin{aligned}\left|\bar{\mathcal{A}}_{nt}-\bar{\mathcal{A}}_t\right| &= \left|\frac{\mathfrak{h}_{\mathfrak{n}}(t)}{F_n(t)}-\frac{\mathfrak{h}(t)}{F(t)}\right|\\ &\le \left|\frac{\mathfrak{h}_{\mathfrak{n}}(t)-\mathfrak{h}(t)}{F_n(t)}\right|+\left|\frac{\mathfrak{h}(t)}{F_n(t)F(t)}\right|\left|F_n(t)-F(t)\right|\\ &\le \frac{1}{F_n(t)}\int_0^t\left\{|\log G(x)||F_n(x)-F(x)|+\left|\frac{F(x)}{G(x)}\right||G_n(x)-G(x)|\right\}dx\\ &\quad+\left|\frac{\mathfrak{h}(t)}{F_n(t)F(t)}\right|\left|F_n(t)-F(t)\right|.\end{aligned}$$

Similarly, we get

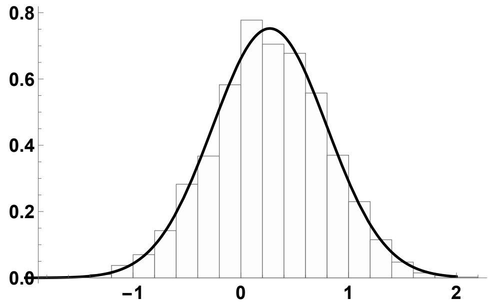
$$\begin{aligned}|\bar{\mathcal{B}}_{nt}-\bar{\mathcal{B}}_t| &\le \frac{\log G(t)}{F_n(t)}\left\{\int_0^t|F_n(x)-F(x)|dx+\left|\frac{\mathfrak{g}(t)}{F(t)}\right|\left|F_n(t)-F(t)\right|\right\}\\ &\quad+\frac{\mathfrak{g}(t)}{G(t)F_n(t)}|G_n(t)-G(t)|.\end{aligned}$$

By using the almost sure convergence of $G_n(t)$, $F_n(t)$ given in Cai [3], the proof immediately follows. □

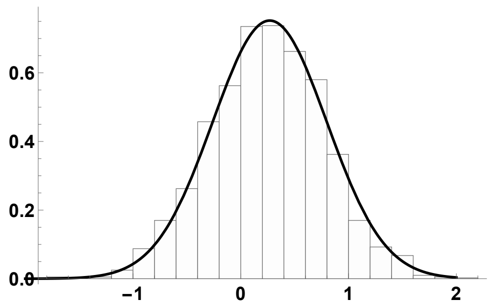
4. NUMERICAL ANALYSIS

In this section, we simulate to evaluate the performance of the proposed estimators. We are interested in random variables of the form $U = \sqrt{(1-\rho)}|V|$, where V are generated from the AR(1) model to obtain dependent samples.

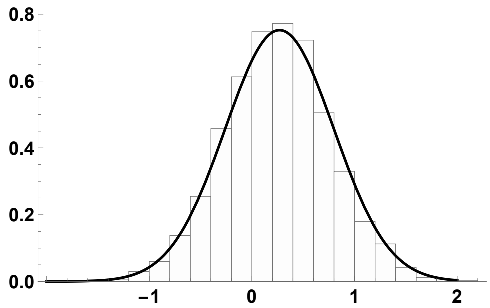
We generate two sets of 2000 simulated samples with white noise distributed as normal density and parameters of $(0, 1)$ and $(0, 2)$, respectively, to test the proposed estimator's asymptotic normality. Exponential distributions with parameters 1 and 2 are used for right censoring observations. The kernel function, in this case, is Epanechnikov, and it has the form $0.75(1-u^2)I(|u| < 1)$. The process is repeated 250 times, and the estimator's histogram with a normal curve is shown in Figure 2 for $t = 1.5$, 1.6, 1.7, and 1.8. We passed the AIC and BIC tests, indicating that the estimator has asymptotic normality.



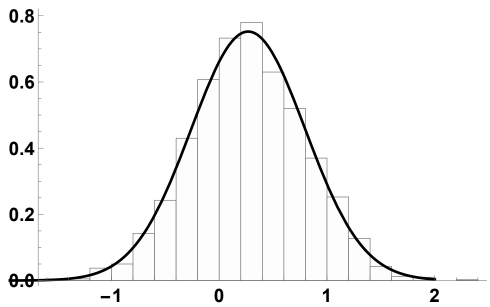
(a) $\bar{C}I_n(1.5)$



(b) $\bar{C}I_n(1.6)$



(c) $\bar{C}I_n(1.7)$



(d) $\bar{C}I_n(1.8)$

Figure 2: Histogram of $\bar{C}I_n(t)$ with normal density curve from sample of size 250.

Table 1 shows the MISE of the estimator for varying parameter ρ, and the table values show that as sample size increases, the MISE approaches zero. In Table 1, we also compute the estimator's 95% confidence interval when $t = 0.6$ and ρ varies. We can conclude from the table values that the confidence interval width for these data sets decreases as the sample size increases.

Table 1: Comparison of MISE of $\bar{CI}_n(x)$ and 95% confidence interval of $\bar{CI}_n(0.6)$.

n	50		100	
ρ	MISE	95% CI	MISE	95% CI
−0.9	0.27742	(0.26248, 0.89153)	0.18867	(0.45588, 0.54162)
−0.6	0.17139	(0.41373, 0.60759)	0.12089	(0.48320, 0.57749)
−0.3	0.25134	(0.41558, 0.71311)	0.17854	(0.43049, 0.64481)
0	0.24360	(0.37707, 0.73257)	0.19241	(0.47052, 0.59493)
0.3	0.24152	(0.44217, 0.66990)	0.19816	(0.46348, 0.55320)
0.6	0.19935	(0.47671, 0.54945)	0.17724	(0.48481, 0.53651)
0.9	0.24180	(0.48576, 0.62959)	0.20584	(0.49064, 0.56981)

n	200		300	
ρ	MISE	95% CI	MISE	95% CI
−0.9	0.09656	(0.48291, 0.53108)	0.00513	(0.48801, 0.51252)
−0.6	0.08262	(0.49566, 0.55799)	0.00476	(0.49782, 0.53547)
−0.3	0.08787	(0.45501, 0.55358)	0.00449	(0.48234, 0.54069)
0	0.09106	(0.48313, 0.54717)	0.00346	(0.48922, 0.53103)
0.3	0.10085	(0.49187, 0.53199)	0.00132	(0.49807, 0.52636)
0.6	0.07573	(0.48991, 0.52993)	0.00648	(0.49629, 0.51122)
0.9	0.10222	(0.49436, 0.55430)	0.00670	(0.49809, 0.54510)

Figure 3 shows the proposed estimator's value and its upper and lower confidence bounds when $t = 1.5$ and $\rho = 0.5$. It is observed that $\bar{CI}_n(1.5)$ is not monotone for n for fixed values $t = 1.5$ and $\rho = 0.5$. Also, as sample size n increases, the width of the confidence intervals narrows, and both bounds approach the kernel estimator, which means the kernel estimator is more precise when the sample size becomes large.

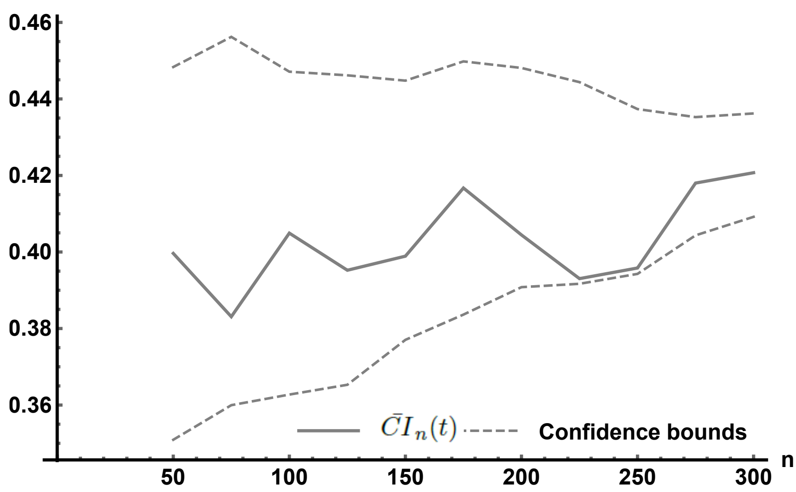


Figure 3: $\bar{CI}_n(1.5)$ and confidence bounds when $\rho = 0.5$.

4.1. Comparison with an empirical estimator

Consider $\{X_i\}$, $\{Y_i\}$, $i = 1, 2, ..., n$ be identically distributed random samples have survival functions be $F(x) = Pr(X_i \geq x)$ and $G(x) = Pr(Y_i \geq x)$ respectively. We use independent and identically distributed random variable R_{1i} and R_{2i} with corresponding distribution functions $P_1(x)$ and $P_2(x)$ for creating right censored data from X_i and Y_i respectively. Note that R_{1i} and R_{2i} are independent of X_i and Y_i respectively. Let $C_i = \min(X_i, R_{1i})$ and $C_i^* = \min(Y_i, R_{2i})$. In this censoring scheme one can observe (C_i, δ_i) and (C_i^*, δ_i^*), where $\delta_i = I(X_i \leq R_{1i})$ and $\delta_i^* = I(Y_i \leq R_{2i})$. Denote $\{C_{i:n}\}_{1\leq i\leq n}$ and $\{C^*_{i:n}\}_{1\leq i\leq n}$, be the sample order statistics, where ties within lifetimes or within censoring times are ordered arbitrarily and ties among lifetimes and censoring times are treated as if the former precedes the latter. $\{\delta_{i:n}\}_{1\leq i\leq n}$ and $\{\delta^*_{i:n}\}_{1\leq i\leq n}$ are the concomitant of i^{th} order statistic of each sample. Let

$$Z_j = \sum_{r=1}^{n} I(C_i \leq C^*_{j:n}), \quad i = 1, 2, ..., n, \tag{4.1}$$

the number of random variables of the first censored sample that are less than or equal to j^{th} order statistics of the second censored sample. Also we rename by $C_{(j,1)} < C_{(j,2)} < ...$ the random sample of the first censored sample belonging to $(C^*_{j:n}, C^*_{(j+1):n}]$, if any. Then in the context of right censoring, we get the estimator of cumulative inaccuracy measure owing to Di Crescenzo and Longobardi (2013), as follows:

$$\begin{aligned}\bar{CI}_n^{\text{cen}}(t) \;=\; -\frac{1}{n}\sum_{j=1}^{n-1}&\left[\frac{Z_{j+1}C^*_{(j+1):n} - Z_jC^*_{j:n} - \sum_{k=1}^{Z_{j+1}-Z_j} C_{(j,k)}}{F_*(t)\sum_{r=1}^{n} I(R_{1r} > C^*_{j:n})}\right]\\ &\times \ln\left(\frac{j}{G_*(t)\sum_{r=1}^{n} I(R_{2r} > Y_{2j:n})}\right) I(Y_{1j:n} \leq t),\end{aligned} \tag{4.2}$$

where $F_*(t)$ and $G_*(t)$ are the Kaplan–Meier estimators of distribution functions $F(t)$ and $G(t)$ respectively defined as

$$F_*(x) = 1 - \prod_{1\leq i\leq n}\left(1 - \frac{\delta_{i:n}}{n-i+1}\right)^{I(C_{i:n}\leq t)} \quad \text{and} \quad G_*(x) = 1 - \prod_{1\leq i\leq n}\left(1 - \frac{\delta^*_{i:n}}{n-i+1}\right)^{I(C^*_{i:n}\leq t)}. \tag{4.3}$$

Table 2 shows the results of comparing the proposed estimator with the empirical estimator using bias and MSE for varying t. We can conclude from these data sets that bias and MSE decrease with increasing sample size n and are inversely proportional to t.

Table 2: Comparison of |Bias| and MSE of the estimators $\bar{C}I_n(t)$ and $\bar{C}I_n^{\text{cen}}(t)$.

|Bias|						
n	100		200		300	
t	$\bar{C}I_n(t)$	$\bar{C}I_n^{\text{cen}}(t)$	$\bar{C}I_n(t)$	$\bar{C}I_n^{\text{cen}}(t)$	$\bar{C}I_n(t)$	$\bar{C}I_n^{\text{cen}}(t)$
1.1	0.17026	0.33860	0.09752	0.28616	0.02315	0.10148
1.3	0.19966	0.56255	0.11951	0.34930	0.03831	0.13926
1.5	0.22769	0.79647	0.14579	0.41002	0.05833	0.16947
1.7	0.28286	1.03660	0.16794	0.46752	0.08318	0.19762
1.9	0.29928	1.27998	0.18381	0.52152	0.11240	0.21901

MSE						
n	100		200		300	
t	$\bar{C}I_n(t)$	$\bar{C}I_n^{\text{cen}}(t)$	$\bar{C}I_n(t)$	$\bar{C}I_n^{\text{cen}}(t)$	$\bar{C}I_n(t)$	$\bar{C}I_n^{\text{cen}}(t)$
1.1	0.02933	0.11467	0.01043	0.09070	0.00157	0.01826
1.3	0.04012	0.31647	0.01442	0.13500	0.00350	0.05177
1.5	0.05189	0.63436	0.02134	0.18629	0.00552	0.08249
1.7	0.08305	1.07454	0.02827	0.24301	0.00761	0.11512
1.9	0.09261	1.63838	0.03387	0.30381	0.00973	0.13043

4.1.1. Real data Analysis

Example 4.1. We use 101 data points from Andrews and Herzberg [1], representing the stress-rupture life of kevlar 49/epoxy strands subjected to constant sustained pressure at 90% stress until all fail, giving us complete data with exact failure times. The fitting details for this data set are given in Table 3. We generated 100 bootstrap samples from the data sets, which were right censored by exponential models with parameters 0.09 and 0.03, respectively. Under the assumption that $F(x)$ is distributed as an Extreme value and $G(x)$ is distributed as a Weibull model, we plotted the mean values of the kernel and empirical estimators in Figure 4 using the Epanechnikov kernel. We also find the kernel estimator's relative efficiencies compared to the empirical estimator, which is plotted in Figure 6(a).

Example 4.2. We consider the 20 failure times of equal-load share samples from Table 1's sample 1, as investigated by Kim and Kvam [14]. We discovered better models for the data sets using AIC and BIC, shown in Table 3. We assumed that $F(x)$ is an exponential distribution and $G(x)$ is a Weibull distribution. We generated 100 bootstrap samples from the data sets, right censored by exponential models with parameters of 0.2 and 0.1, respectively. The Epanechnikov kernel is used as the kernel form. The mean values of empirical and kernel estimators are calculated and plotted in Figure 5. The relative efficiencies of the kernel estimator to the empirical estimator are also determined and plotted in Figure 6(b).

In Figures 4 and 5, we plot $\bar{C}I(X,Y,t)$, $\bar{C}I_n(t)$ and $\bar{C}I_n^{\text{cen}}(t)$ with respect to Examples 4.1 and 4.2 respectively. We can observe from Figure 4, $\bar{C}I(X,Y,t)$ and $\bar{C}I_n(t)$ are non decreasing and monotonic, while $\bar{C}I_n^{\text{cen}}(t)$ is non increasing in t. In Figure 5, $\bar{C}I(X,Y,t)$ and $\bar{C}I_n(t)$ are monotonic. Furthermore, in both cases, the kernel estimator outperforms the empirical estimator. We can conclude from Figures 6 that the relative efficiencies of kernel estimators in comparison to empirical estimators are decreasing functions in t. Furthermore, the graph shows that relative efficiencies are greater than one, indicating that the kernel estimator outperforms the empirical estimator in the Examples 4.1 and 4.2.

Table 3: Fitting details of real data.

	Distribution	Parameters	AIC	BIC
Example 4.1:	Extreme value	(0.60869, 0.70011)	−2.48095	−2.49513
	Weibull	(0.94876, 1.12481, −0.07909)	−2.23536	−2.25600
	Pareto	(7, 5.61164,1.19461,−0.07909)	−2.13630	−2.16296
	Frechet	(2.50547, 1.36598, −0.89583)	−2.05503	−2.07566
Example 4.2:	Exponential	0.19053	−5.25114	−5.19792
	Weibull	(0.74337, 4.36222, 0.13999)	−4.86134	−4.66002
	Pareto	(7, 1.86990, 2.96583, 0.14000)	−4.402009	−4.11923
	Gamma	(0.83604, 6.27797)	−4.39639	−4.27688

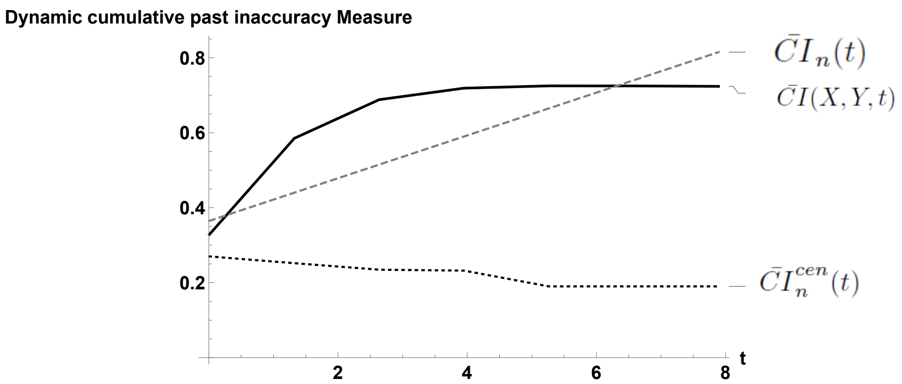


Figure 4: Comparison of $\bar{CI}(X,Y,t)$, $\bar{CI}_n(t)$ and $\bar{CI}_n^{\text{cen}}(t)$ for the stress-rupture lives of kevlar 49/epoxy strands.

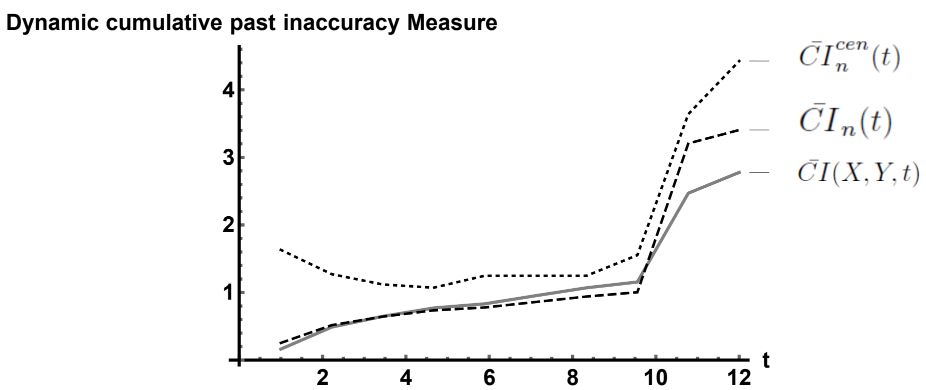
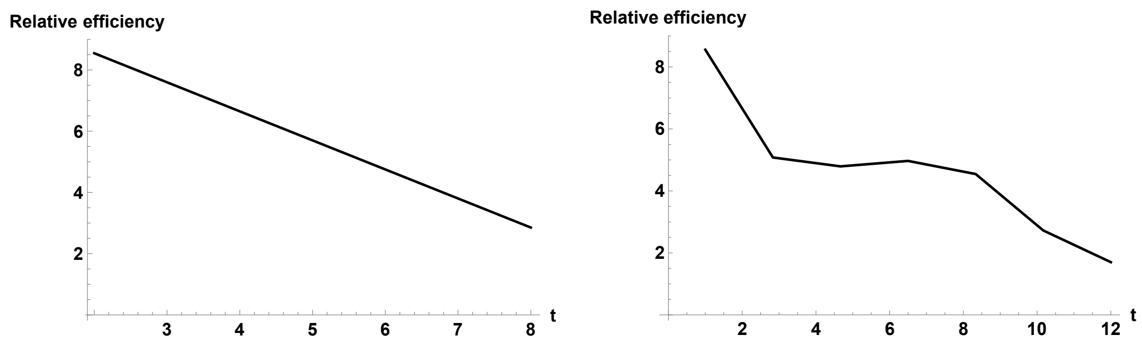


Figure 5: Comparison of $\bar{CI}(X,Y,t)$, $\bar{CI}_n(t)$ and $\bar{CI}_n^{\text{cen}}(t)$ for the 20 failure times of equal-load share samples.



(a) Example 4.1 (b) Example 4.2

Figure 6: Comparison of $\bar{C}I_n(t)$'s relative efficiency with respect to $\bar{C}I_n^{\text{cen}}(t)$.

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