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Estimation for Inverse Burr Distribution under Generalized Progressive Hybrid Censored Data with an Application to Wastewater Engineering Data

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Abstract:

- The inverse Burr distribution is a significant and commonly used lifetime distribution, which plays an important role in reliability engineering. In this article, the estimation of parameters of the inverse Burr distribution based on generalized Type II progressive hybrid censored sample is studied. The expectation-maximization (EM) algorithm is employed for computing the maximum likelihood estimates of the unknown parameters. It is shown that the maximum likelihood estimates exist uniquely. The asymptotic confidence intervals for the parameters are constructed using the missing value principle. Under Bayesian framework, the Bayes estimators are developed based on Lindley's technique and Metropolis–Hastings algorithm. Furthermore, the highest posterior density (HPD) credible intervals are successively constructed. Finally, simulation experiments are implemented to access performance of several proposed methods in this article, and sewer invert trap real data is presented to exemplify the theoretical outcomes.

Keywords:

- *Bayes estimators; EM algorithm; generalized Type II progressive hybrid censoring; HPD credible interval; inverse Burr distribution; separation of sewer solids.*

AMS Subject Classification:

- 62N02, 62P12.

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1. INTRODUCTION

Discharge of industrial and domestic effluent wastes, leakage from water tanks, marine dumping and atmospheric deposition are major causes of pollution. The removal of suspended solids from any sewer plays an important part in its overall waste treatment program. There are several methods for separating suspended particles from sewers. One of these methods is the use of invert traps. Several researcher have obtained experiment and simulation results on the invert trap.

For instance, Buxton *et al.* [7] presented the results from a laboratory investigation comparing the trapping performance of three slot size configurations of a laboratory-scale invert trap. Thinglas [25] studied flow field prediction and optimization of invert trap configuration using three-dimensional computational fluid dynamics (CFD) modeling. Mohsin and Kaushal [21] considered the experimental and discrete phase modeling for sediment retention ratio for invert traps. Moreover, in invert trap data analysis, the complete information is generally difficult to acquire on account of experimental cost and time-consuming of simulation. Therefore, censored data is more common whose censoring schemes are mainly divided into Type I and Type II censoring.

Furthermore, if Type I and Type II censored schemes are mixed together, it is hybrid censoring scheme (Epstein [13]). In Type I hybrid censored sample, the experiment stops at time $T^* = \min\{x_{m:m:n}, T\}$, where $X_{m:m:n}$ means the m-th failure time from n units, and T is the predetermined experiment time. Based on this censoring, it is a possibility that very few failures may occur before time T^*. So, Childs *et al.* [10] introduced the Type II hybrid censoring scheme that would terminate the experiment at $T^* = \max\{x_{m:m:n}, T\}$. Based on these censoring schemes, many statistical inferences have been carried out by several authors, see for example, Balakrishnan *et al.* [5], Banerjee and Kundu [4], Kundu and Howlader [17], Gupta and Singh [14].

A progressive censoring scheme (PCS) was then proposed to permit more flexibility in the conduct of the experiment, where individuals can be removed at several stages of the experiment rather than at the end. It can be classified into progressive Type I (PICS) and progressive Type II censoring schemes (PIICS). In PICS, let the number of items used in a life testing experiment be n. In this scheme, $R_1, R_2, ..., R_m$ items are randomly withdrawn at pre-specified time points $T_1, T_2, ..., T_m$, respectively. The test will be terminated at prefixed time point T_m in this scheme. Now, we describe PIICS. Consider n number of total units at initial time on an experiment. We remove randomly R_1 number of survival units when first failure time $X_{1:m:n}$ is observed. This process continues till m-th failure occurs. We assume that the m-th failure takes place at time $X_{m:m:n}$ and the remaining number of surviving units is $R_m = n - (m + \sum_{i=1}^{m-1} R_i)$.

Today, due to the high lifespan of many products, the total experimental time can be very long if PCS is used. Consequently, with the aim of enhancing the experimental efficiency and accuracy, it was further proposed as a progressive hybrid censoring scheme (PHCS). For various applications of the progressive hybrid sampling schemes in life testing experiments, we refer to Panahi [22] and El-Sherpieny *et al.* [12]. The main limitation of PHCS is that the number of observed failures is random and it can turn out to be a very small number, thus, any inference procedure will be invalid or its accuracy will be extremely low.

To overcome this drawback, a new hybrid censoring scheme has been proposed by Cho *et al.* [16] and is referred to generalized progressive hybrid censoring scheme (GPHCS), which maintains the experimental time in an acceptable range for researchers and guarantees a sufficient number of failed individuals. This scheme provides not only time and cost savings but also promotes more efficient statistical inference based on more observable data. The procedure of generalized Type II progressive hybrid (GIIPH) censoring scheme can be described as follows:

Suppose that n units are put on a test and the number of failures m, two time points T_1 and T_2 $(0 < T_1 < T_2 < \infty)$ and also the progressive censoring scheme $R_1, R_2, ..., R_m$ $(\sum_{j=1}^{m} R_j + m = n)$ are fixed beforehand. At the first failure time, (say $X_{1:m:n}$), R_1 number of live items are selected and randomly removed from the experiment. At the second failure time $(X_{2:m:n})$, R_2 units are removed from the remaining test items and so on, until the termination time $T^* = \max\{T_1,\ \min(x_{m:m:n}, T_2)\}$ failure observed and then all the remaining units are removed from the experiment. Let Q_1 and Q_2 denoted the number of observed failures up to time T_1 and T_2, respectively. Therefore,

- If $X_{m:m:n} < T_1$, then the experiment continue to observe failures until times T_1. In this case the failure times are denoted by $x_{1:m:n}, ..., x_{m:m:n}, x_{m+1:m:n}, ..., x_{q_1:n}$ (say Case I).
- If $T_1 < X_{m:m:n} < T_2$, then the experiment terminate at the m-th failure. In this case the failure times are represented by $x_{1:m:n}, ..., x_{q_1:m:n}, ..., x_{m:m:n}$ (say Case II).
- If $X_{m:m:n} > T_2$, then the experiment terminate at time T_2. In this case the failure times are denoted by $x_{1:m:n}, ..., x_{q_2:m:n}, ..., x_{m:m:n}$ (say Case III).

Where, q_1 and q_2 are the observed values of Q_1 and Q_2 respectively.

There are some authors studying this scheme under different lifetime distributions, see for example, Chan *et al.* [9], Gorny *et al.* [15], Koley and Cramer [18]. Based on the observed GIIPH censored sample, the likelihood function can be written as:

$$L(\alpha, \beta) = \begin{cases} \Im_i \prod_{j=1}^{Q_1} f(x_{j:m:n})[1 - F(x_{j:m:n})]^{R_j} & \text{Case I,} \\ \Im_i \prod_{j=1}^{m} f(x_{j:m:n})[1 - F(x_{j:m:n})]^{R_j} & \text{Case II,} \\ \Im_i \prod_{j=1}^{Q_2} f(x_{j:m:n})[1 - F(x_{j:m:n})]^{R_j} & \text{Case III,} \end{cases} \tag{1.1}$$

$$\Im_i = \begin{cases} [1 - F(T_1)]^{\tilde{R}_{Q_1+1}} \prod_{j=1}^{Q_1} \sum_{k=j}^{m} (1 + R_k), & \text{Case I,} \\ \prod_{j=1}^{m} \sum_{k=j}^{m} (1 + R_k) & \text{Case II,} \\ [1 - F(T_2)]^{\tilde{R}_{Q_2+1}} \prod_{j=1}^{Q_2} \sum_{k=j}^{m} (1 + R_k) & \text{Case III.} \end{cases}$$

In our work, estimation problems of unknown parameters of the inverse Burr (Burr III) distribution under GIIPH censoring scheme gets discussed. The Burr III distribution is one from twelve distributions was explored by using the method of differential equation (Burr [6]). This distribution has the following probability density function and the cumulative distribution function as:

$$f(x; \alpha, \beta) = \alpha\beta x^{-\beta-1}(1 + x^{-\beta})^{-(\alpha+1)};\ \ x > 0, \alpha > 0, \beta > 0, \tag{1.2}$$

and

$$F(x; \alpha, \beta) = (1 + x^{-\beta})^{-\alpha};\ \ x > 0, \alpha > 0, \beta > 0. \tag{1.3}$$

From Figure 1, it can be noticed that the inverse Burr distribution has two important shapes of its hazard rate function: decreasing and upside-down bathtub (or unimodal). It is worth mentioning that in reliability engineering, biology and several statistical modelling, different shaped hazard rate functions are used with different interpretations. We would like to mention that because of various shapes of the hazard rate function of inverse Burr distribution, it can be applied in many areas of research. Further, for fitting various lifetime data, inverse Burr distribution can be treated as an alternative model to other distributions such as gamma, Weibull and log-normal. Moreover, there are various real engineering data sets, for which inverse Burr (Burr III) distribution fits better than Weibull distribution.

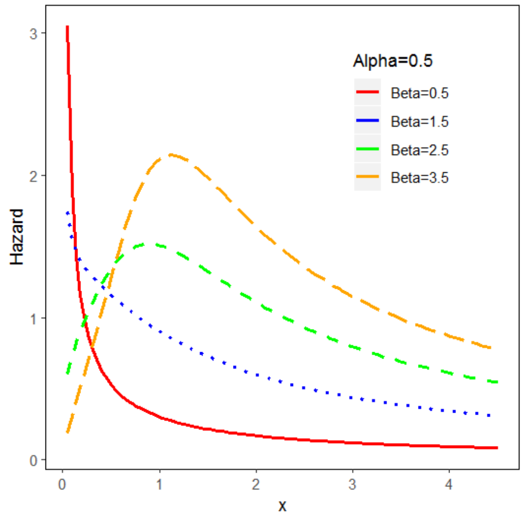

Figure 1: Graphs of the hazard rate function of the B-III distribution for different sets of parameters.

For example, the inverse Burr distribution fits the nano droplet dispersion data set (see Panahi and Asadi [23]).

The inverse Burr distribution has been studied by many researchers based on different censoring schemes. Abd-Elfattah and Alharbey [1] discussed the parameter estimations of this distribution based on a trimmed samples. Singh *et al.* [24] considered statistical inferences for the unknown parameters based on Type II progressive censoring scheme. Altindag *et al.* [2] studied the estimation and prediction problems for the inverse Burr distribution under Type II censored data. Panahi and Asadi [23] studied the application of this distribution on the Nano droplet censored data.

To the best of our knowledge, nobody has considered the inverse Burr distribution for the purpose of statistical inference based on GIIPH censoring scheme. Thus, our objectives in this study to close this gap are: First, estimating the parameters of the inverse Burr distribution using the EM algorithm. Using the Fisher information matrix, the approximate confidence intervals (ACIs) for unknown parameters are obtained.

Second objective is to obtain the Bayes estimates of the unknown inverse Burr parameters using independent gamma priors. Since the Bayes estimates cannot be obtained in closed expressions, Lindley's approximation and Markov chain Monte Carlo technique are considered to compute the complex posterior functions and in turn calculating Bayes estimates and the associated highest posterior density (HPD) credible intervals. Using various choices of the censoring schemes, the performance of the proposed methods is compared through an extensive simulation study in terms of their simulated mean squared-error (MSE) and average confidence lengths.

Also, third objective is to show the practical application of this distribution in separation of sewer solids data which are obtained by the authors using the computational fluid dynamics (CFD) method. The rest of the paper is organized as follows.

In Section 2, it is represented how the EM algorithm is utilized to obtain the maximum likelihood estimators (MLEs) of the unknown parameters as well as Fisher information matrix of the inverse Burr distribution under GIIPH censored sample. The existence and uniqueness properties of the MLEs have also been studied graphically. In Section 3, we derive the approximate explicit expressions for the Bayesian estimates using Lindley's approximation and Markov chain Monte Carlo technique. The Markov chain Monte Carlo samples are also used to construct the HPD credible intervals of the unknown parameters. Section 4 is devoted a simulation study to compare the proposed point and interval estimators. One real data set is analyzed for illustration in Section 5. Conclusions are given in Section 6.

2. MAXIMUM LIKELIHOOD ESTIMATORS

In this Section, the maximum likelihood method is carried out on the model based on the GIIPH censoring scheme. By (1.2), the likelihood function without additive constant is presented as follows:

$$L(\alpha,\beta) = \begin{cases} \prod_{j=1}^{q_1} \sum_{k=j}^{m} (1+R_k)(\alpha\beta)^{q_1} \prod_{j=1}^{q_1} x_{j:m:n}^{-\beta-1} A_j^{-(\alpha+1)} B_j^{R_j} D^{\tilde{R}_{Q_1+1}} & \text{case I} \\ \prod_{j=1}^{m} \sum_{k=j}^{m} (1+R_k)(\alpha\beta)^{m} \prod_{j=1}^{m} x_{j:m:n}^{-\beta-1} A_j^{-(\alpha+1)} B_j^{R_j} & \text{case II} \\ \prod_{j=1}^{q_2} \sum_{k=j}^{m} (1+R_k)(\alpha\beta)^{q_2} \prod_{j=1}^{q_2} x_{j:m:n}^{-\beta-1} A_j^{-(\alpha+1)} B_j^{R_j} D^{\tilde{R}_{Q_2+1}} & \text{case III,} \end{cases} \tag{2.1}$$

where $A_j = (1 + x_{j:m:n}^{-\beta})$, $B_j = (1 - (1 + x_{j:m:n}^{-\beta})^{-\alpha})$, $\tilde{R}_{Q_1+1} = n - q_1 - \sum_{j=1}^{m-1} R_j$, $\tilde{R}_{Q_2+1} = n - q_2 - \sum_{j=1}^{q_2} R_j$ and

$$D = \begin{cases} 1 - (1 + T_1^{-\beta})^{-\alpha} & \text{for case I} \\ 1 - (1 + T_2^{-\beta})^{-\alpha} & \text{for case III.} \end{cases}$$

The corresponding log-likelihood function is given by:

$$\begin{aligned} l(\alpha,\beta) &= \ln L(\alpha,\beta) \\ &= \iota(\ln\alpha + \ln\beta) - (\beta+1)\sum_{j=1}^{\iota} \ln x_{j:m:n} - (\alpha+1)\sum_{j=1}^{\iota} \ln A_j + \sum_{j=1}^{\iota} R_j \ln B_j + (\tilde{R}_{Q_j+1}) \ln D, \end{aligned}$$

where

$$(\iota, \tilde{R}_{Q_j+1}) = \begin{cases} (q_1, \tilde{R}_{Q_1+1}), & \text{Case I,} \\ (m, 0) & \text{Case II,} \\ (q_2, \tilde{R}_{Q_2+1}), & \text{Case III.} \end{cases}$$

After differentiating the function $l(\alpha,\beta)$ with respect to α and β, we have

$$\hat{\alpha} = \iota\left(\sum_{j=1}^{\iota}\ln A_j - \sum_{j=1}^{\iota} R_j \frac{A_j^{-\alpha}\ln A_j}{B_j} - \varpi_1\right)^{-1}, \tag{2.2}$$

(2.3)
$$\hat{\beta} = \iota\left(\sum_{j=1}^{\iota}\ln x_{j:m:n} + \sum_{j=1}^{\iota} R_j\alpha \frac{x_{j:m:n}^{-\beta}\ln x_{j:m:n} A_j^{-\alpha-1}}{B_j} - \sum_{j=1}^{\iota}(\alpha+1)\frac{x_{j:m:n}^{-\beta}\ln x_{j:m:n}}{A_j} + \varpi_2\right)^{-1},$$

where

$$(\iota,\varpi_1) = \begin{cases} (q_1, \tilde{R}_{Q_1+1}\frac{(1+T_1^{-\beta})^{-\alpha}\ln(1+T_1^{-\beta})}{D}), & \text{Case I,}\\ (m,0) & \text{Case II,}\\ (q_2, \tilde{R}_{Q_2+1}\frac{(1+T_2^{-\beta})^{-\alpha}\ln(1+T_2^{-\beta})}{D}), & \text{Case III,}\end{cases}$$

$$\varpi_2 = \begin{cases} \tilde{R}_{Q_1+1}\frac{\alpha T_1^{-\beta}\ln T_1(1+T_1^{-\beta})^{-\alpha-1}}{D}, & \text{Case I,}\\ 0 & \text{Case II,}\\ \tilde{R}_{Q_2+1}\frac{\alpha T_2^{-\beta}\ln T_2(1+T_2^{-\beta})^{-\alpha-1}}{D}, & \text{Case III,}\end{cases}$$

respectively. Now, we show the existence and uniqueness of the maximum likelihood estimates of the parameters of the inverse Burr distribution under GIIPH censored data using the graphical method (Ateya [3]) as:

- A sample of size 50 from the inverse Burr distribution are generated.
- Based on certain case of censored data ($T_1 = .8$, $T_2 = 2$, $m = 30$, $R_{15} = 20$, $R_j = 0$, $j \neq 15$), the curves of the equations $\partial(l(\alpha,\beta))/\partial(\alpha)$ and $\partial(l(\alpha,\beta))/\partial(\beta)$ are presented in Figure 2.
- The curve of $l(\hat{\alpha},\beta)$ and $l(\alpha,\hat{\beta})$ are also drawn in Figures 2 and 3, respectively.
- It is easy to see from Figure 1 that there exist one intersection point (1.1922,1.6455) which indicates that the solution of $\frac{\partial l(\alpha,\beta)}{\partial\alpha} = 0$ and $\frac{\partial l(\alpha,\beta)}{\partial\beta} = 0$, exists and is unique. This concludes that the maximum likelihood estimates of the parameters α and β exist and are unique.
- The Figure 3 shows that the previous intersection point maximizes the $l(\alpha,\hat{\beta})$.
- Similarly from Figure 4, it is observed the intersection point is the maximization point of the $l(\hat{\alpha},\beta)$.
- An important implication is that the maximum likelihood estimates of the parameters α and β exist and are unique for other generalized Type II progressive hybrid censored cases.

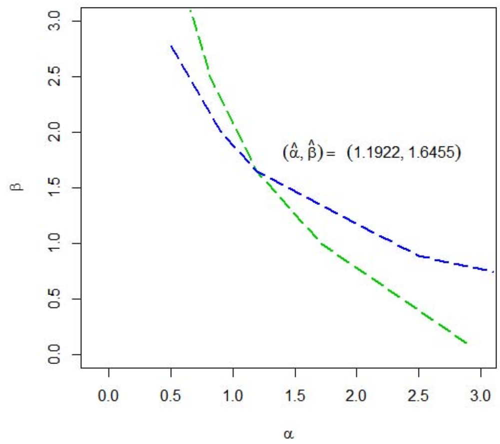

Figure 2: The plot of the ML estimates of α and β graphically.

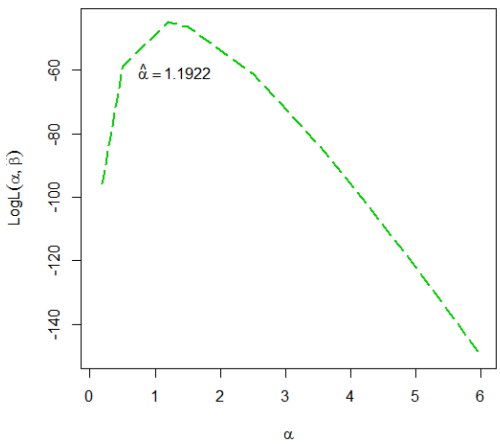

Figure 3: The curve of the log-likelihood function $l(\alpha, \hat{\beta})$.

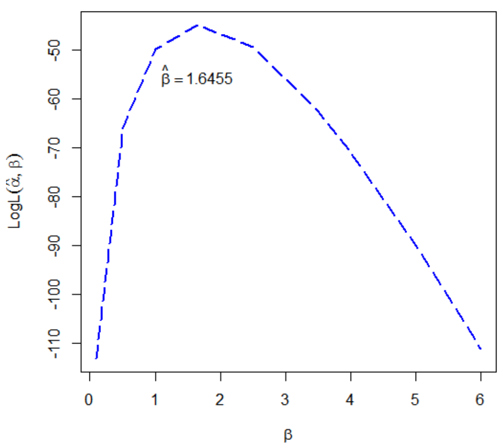

Figure 4: The curve of the the log-likelihood function $l(\hat{\alpha}, \beta)$.

2.1. EM algorithm

It is found that there is no explicit solution of (2.2) and (2.3), making them incredibly difficult to get the exact form of their solutions, thus utilizing the EM algorithm to work out these equations. Suppose that $\mathbf{X} = (X_1, X_2, ..., X_\iota)$ denotes the observed and $(\mathbf{Z}_j, \mathbf{Z}')$ represent the censored data. Where, $\mathbf{Z}_j = (Z_{j_1}, Z_{j_2}, ..., Z_{jR_j})$ and

$$\mathbf{Z}' = \begin{cases} (Z_1, Z_2, ..., Z_{\tilde{R}_{Q_1+1}}) & \text{Case I} \\ (Z_1, Z_2, ..., Z_{\tilde{R}_{Q_2+1}}) & \text{Case III}, \end{cases}$$

The log-likelihood function of (α, β) under the complete data is:

$$l_{\text{Complete}}(\alpha,\beta) = \begin{cases} \Delta + \mathtt{C}_1 & \text{Case I} \\ \Delta & \text{Case II} \\ \Delta + \mathtt{C}_2 & \text{Case III}, \end{cases} \tag{2.4}$$

$$\Delta = n\ln(\alpha) + n(\ln\beta) - (\beta+1)\sum_{j=1}^{\iota} \ln x_{j:m:n} - (\alpha+1)\sum_{j=1}^{\iota} \ln(1 + x_{j:m:n}^{-\beta})$$
$$-(\beta+1)\sum_{j=1}^{\iota}\sum_{k=1}^{R_j} E[\ln Z_{jk}|Z_{jk} > x_{j:m:n}] - (\alpha+1)\sum_{j=1}^{\iota}\sum_{k=1}^{R_j} E[\ln(1+Z_{jk}^{-\beta})|z_{jk} > x_{j:m:n}],$$

$$\mathtt{C}_1 = -(\beta+1)\sum_{p=1}^{\tilde{R}_{Q_1+1}} E[\ln(Z'_p)|Z'_p > T_1] - (\alpha+1)\sum_{p=1}^{\tilde{R}_{Q_1+1}} E[\ln(1+(Z'_p)^{-\beta})|Z'_p > T_1],$$

and

$$\mathtt{C}_2 = -(\beta+1)\sum_{p=1}^{\tilde{R}_{Q_2+1}} E[\ln Z'_p|Z'_p > T_2] - (\alpha+1)\sum_{p=1}^{\tilde{R}_{Q_2+1}} E[\ln(1+(Z'_p)^{-\beta})|Z'_p > T_2].$$

The E-step of the EM-iteration needs the following conditional expectations:

$$E[\ln Z_{jk}|Z_{jk} > c] = \frac{\alpha\beta}{1 - F_X(c;\alpha,\beta)}\int_c^{\infty} x^{-\beta-1}(1+x^{-\beta})^{-(\alpha+1)}\ln x dx = H_1(x_j,\alpha,\beta),$$

$$E[\ln(1+Z_{jk}^{-\beta})|Z_{jk} > c] = \frac{\alpha\beta}{1 - F_X(c;\alpha,\beta)}\int_c^{\infty} x^{-\beta-1}(1+x^{-\beta})^{-(\alpha+1)}\ln(1+x^{-\beta}) dx$$
$$= H_2(x_j,\alpha,\beta),$$

$$E[\ln Z'_p|Z'_p > T_1] = \frac{\alpha\beta}{1 - F_X(T_1;\alpha,\beta)}\int_{T_1}^{\infty} x^{-\beta-1}(1+x^{-\beta})^{-(\alpha+1)}\ln x dx = H_3(x_j,\alpha,\beta),$$

$$E[\ln Z'_p|Z'_p > T_2] = \frac{\alpha\beta}{1 - F_X(T_2;\alpha,\beta)}\int_{T_2}^{\infty} x^{-\beta-1}(1+x^{-\beta})^{-(\alpha+1)}\ln x dx = H_4(x_j,\alpha,\beta),$$

$$E[\ln(1+(Z'_p)^{-\beta})|Z'_p > T_1] = \frac{\alpha\beta}{1 - F_X(T_1;\alpha,\beta)}\int_{T_1}^{\infty} x^{-\beta-1}(1+x^{-\beta})^{-(\alpha+1)}\ln(1+x^{-\beta}) dx$$
$$= H_5(T_1,\alpha,\beta),$$

and

$$E[\ln(1+(Z'_p)^{-\beta})|Z'_p > T_2] = \frac{\alpha\beta}{1-F_X(T_2;\alpha,\beta)}\int_{T_2}^{\infty} x^{-\beta-1}(1+x^{-\beta})^{-(\alpha+1)}\ln(1+x^{-\beta})dx$$
$$= H_6(T_2,\alpha,\beta),$$

The M-step in a EM-iteration is maximizing the likelihood under complete sample over (α,β), with the missing values replaced by their conditional expectations.

2.2. Approximate Confidence Interval

For each unknown parameter, the approximate confidence intervals (ACIs) are presented by utilizing the observed Fisher information matrix. We have

(2.5) $$I_{\mathbf{X}}(\theta) = I_{\mathbf{W}}(\theta) - I_{\mathbf{Z}|\mathbf{X}}(\theta).$$

Where,

(2.6) $$I_{\mathbf{W}}(\theta) = -E_\theta\Big[\frac{\partial^2 l_{\text{Complete}}(\theta)}{\partial\theta^2}\Big]; \quad \theta = (\alpha,\beta),$$

and

$$\hat{l}_{\alpha\alpha} = \frac{\partial^2 l}{\partial\alpha^2}|_{\alpha=\hat{\alpha},\beta=\hat{\beta}} = -\frac{\iota}{\hat{\alpha}^2} - \sum_{j=1}^{\iota} R_j \frac{A_j^\alpha \ln^2(A_j)}{(A_j^\alpha - 1)^2} - \varpi_3,$$

Also, ϖ_3 is equal to $\tilde{R}_{Q_1+1}\{\frac{S_1^\alpha \ln^2(S_1)}{(S_1^\alpha-1)^2}\}$, 0 and $\tilde{R}_{Q_2+1}\{\frac{S_2^\alpha \ln^2 S_2}{(S_2^\alpha-1)^2}\}$ for cases I, II and III respectively.

$$\hat{l}_{\beta\beta} = \frac{\partial^2 l}{\partial\beta^2}|_{\alpha=\alpha,\beta=\hat{\beta}} = -\frac{\iota}{\beta^2} - (\alpha+1)\sum_{i=1}^{\iota}\frac{x_j^\beta \ln^2 x_j}{(1+x_j^\beta)^2} + \sum_{j=1}^{\iota}\alpha R_j \frac{x_j^\beta \ln^2 x_j (A_j^{\alpha+1}-1)}{(x_j^\beta(A_j^{\alpha+1}-1)-1)^2}$$
$$- \sum_{j=1}^{\iota}\alpha(\alpha+1)R_j \frac{\ln^2 x_j (A_j)^\alpha}{(x_j^\beta(A_j^{\alpha+1}-1)-1)^2} + \varpi_4,$$

$$\varpi_4 = \tilde{R}_{Q_1+1}\Big\{\frac{\alpha T_1^\beta \ln^2 T_1 (S_1^{\alpha+1}-1)}{(T_1^\beta(S_1^{\alpha+1}-1)-1)^2} - \frac{\alpha(\alpha+1)\ln^2 T_1 S_1^\alpha}{(T_1^\beta(S_1^{\alpha+1}-1)-1)^2}\Big\} \quad \text{for case I}$$
$$\varpi_4 = 0 \quad \text{for case II}$$
$$\varpi_4 = \tilde{R}_{Q_2+1}\Big\{\frac{\alpha T_2^\beta \ln^2 T_2 (S_2^{\alpha+1}-1)}{(T_2^\beta(S_2^{\alpha+1}-1)-1)^2} - \frac{\alpha(\alpha+1)\ln^2 T_2 (S_2)^\alpha}{(T_2^\beta(S_2^{\alpha+1}-1)-1)^2}\Big\} \quad \text{for case III}$$

and

$$\hat{l}_{\beta\alpha} = \hat{l}_{\alpha\beta} = \frac{\partial^2 l}{\partial\beta\partial\alpha}|_{\alpha=\hat{\alpha},\beta=\hat{\beta}} = \sum_{j=1}^{\iota}\frac{x_j}{1+x_j^\beta} - \sum_{j=1}^{\iota} R_j \frac{\ln x_j}{x_j^\beta(A_j^{\alpha+1}-1)-1}$$
$$+ \sum_{j=1}^{\iota}\alpha R_j \frac{A_j^{\alpha+1} x_j^\beta \ln x_j ln(A_j)}{(x_j^\beta(A_j^{\alpha+1}-1)-1)^2} - \varpi_5$$

$$\varpi_5 = \tilde{R}_{Q_1+1}\Big\{\frac{\ln T_1}{T_1^{\beta}(S_1^{\alpha+1}-1)-1} - \frac{\alpha T_1^{\beta}\ln T_1 S_1^{\alpha+1}\ln(S_1)}{(T_1^{\beta}(S_1^{\alpha+1}-1)-1)^2}\Big\} \quad \text{for case I}$$
$$\varpi_5 = 0 \quad \text{for case II}$$
$$\varpi_5 = \tilde{R}_{Q_2+1}\Big\{\frac{\ln T_2}{T_2^{\beta}(S_2^{\alpha+1}-1)-1} - \frac{\alpha T_2^{\beta}\ln T_1 S_2^{\alpha+1}\ln(S_2)}{(T_2^{\beta}(S_2^{\alpha+1}-1)-1)^2}\Big\} \quad \text{for case III.}$$

Note that, we consider $x_{j:m:n} = x_j$, $S_1 = (1+T_1^{-\beta})$, $S_2 = (1+T_2^{-\beta})$, and $A_j = (1+x_j^{-\beta})$. Based on the conditional distribution, the Fisher information in the j-th observation can be evaluated as

$$I^{(j)}_{\mathbf{Z}|\mathbf{X}}(\theta) = -E\big[\frac{\partial^2}{\partial\theta^2}\ln(f(z_{jk}|x_{j:m:n},\theta))\big]. \tag{2.7}$$

Therefore, we have

$$I_{\mathbf{Z}|\mathbf{X}}(\theta) = \begin{cases} \sum_{j=1}^{q_1} R_i I^{(j)}_{\mathbf{Z}|\mathbf{X}}(\theta) + \tilde{R}_{Q_1+1} I^{*}_{\mathbf{Z}|\ \mathbf{X}}(\theta), & \text{Case I} \\ \sum_{j=1}^{m} R_i I^{(j)}_{\mathbf{Z}|\mathbf{X}}(\theta), & \text{Case II} \\ \sum_{j=1}^{q_2} R_i I^{(j)}_{\mathbf{Z}|\mathbf{X}}(\theta) + \tilde{R}_{Q_2+1} I^{*}_{\mathbf{Z}|\mathbf{X}}(\theta) & \text{Case III.} \end{cases}$$

Where, $I^{(j)}_{\mathbf{Z}|\mathbf{X}}(\theta)$ and $I^{*}_{\mathbf{Z}|\mathbf{X}}(\theta)$ are the information matrix of a single observation for the truncated inverse Burr distribution. Therefore, the $100(1-\gamma)\%$ ACIs for the parameters are given by:

$$\Big(\hat{\alpha} - Z_{\gamma/2}\sqrt{\text{Var}(\hat{\alpha})}\ ,\ \hat{\alpha} + Z_{\gamma/2}\sqrt{\text{Var}(\hat{\alpha})}\Big)\ \ and\ \ \Big(\hat{\beta} - Z_{\gamma/2}\sqrt{\text{Var}(\hat{\beta})}\ ,\ \hat{\beta} + Z_{\gamma/2}\sqrt{\text{Var}(\hat{\beta})}\Big).$$

3. BAYESIAN ETIMATION

In contrast to traditional frequentist methods, the Bayesian approaches take advantage of available data information and incorporate prior information of parameters, thereby attracting much attention in statistical inference. For obtaining the Bayesian estimates, we consider independent gamma prior distributions for α and β with hyper-parameters (a_1, b_1) and (a_2, b_2) respectively, that reflect prior beliefs. Hence the PDF of the joint prior distribution takes the following expression:

$$\pi(\alpha,\beta) \propto \alpha^{b_1-1}e^{-a_1\alpha}\beta^{b_2-1}e^{-a_2\beta};\ \alpha>0,\ \beta>0,\ a_1>0,\ a_2>0,\ b_1>0,\ b_2>0, \tag{3.1}$$

In prior distributions, hyper parameters a_i and $b_i, i = 1, 2$ are assumed as non-negative and known. In the case of noninformative priors, very small non-negative values of the hyper-parameters , *i.e.* $a_1 = a_2 = b_1 = b_2 = 0.0001$, are used as suggested by Congdon [8] which are almost like Jeffrey's priors, but they are proper, inversely. As more informative priors, different cases of the hyperparameters can be evaluated. Therefore, Bayes estimation of a general function of parameters $(\Upsilon(\alpha,\beta))$ with the square error loss function can be derived as

$$\tilde{\Upsilon}(\alpha,\beta) = E(\Upsilon(\alpha,\beta)|Data) = \top^{-1}\iint \Upsilon(\alpha,\beta)\pi(\alpha,\beta|Data)d\alpha d\beta, \tag{3.2}$$

where $\top = \iint \pi(\alpha, \beta|Data)d\alpha d\beta$ and
(3.3)

$$\pi(\alpha,\beta|Data) = \begin{cases} \psi\alpha^{q_1+b_1-1}\beta^{q_1+b_2-1}\prod_{j=1}^{q_1} x_{j:m:n}^{-\beta-1}A_j^{-(\alpha+1)}e^{-\alpha a_1-\beta a_2}B_j^{R_j}D^{\tilde{R}_{Q_1+1}} & \text{case I},\\ \psi\alpha^{m+b_1-1}\beta^{m+b_2-1}\prod_{j=1}^{m} x_{j:m:n}^{-\beta-1}A_j^{-(\alpha+1)}e^{-\alpha a_1-\beta a_2}B^{R_j} & \text{case II},\\ \psi\alpha^{q_2+b_1-1}\beta^{q_2+b_2-1}\prod_{j=1}^{q_2} x_{j:m:n}^{-\beta-1}A_j^{-(\alpha+1)}e^{-\alpha a_1-\beta a_2}B_j^{R_j}D^{\tilde{R}_{Q_2+1}} & \text{case III}.\end{cases}$$

Here, A_j, B_j, D are introduced previously and ψ and Υ can be written as:

$$\psi = \begin{cases} \prod_{j=1}^{q_1}\sum_{k=j}^{m}(1+R_k) & \text{for case I},\\ \prod_{j=1}^{m}\sum_{k=j}^{m}(1+R_k) & \text{for case II},\\ \prod_{j=1}^{q_2}\sum_{k=j}^{m}(1+R_k) & \text{for case III},\end{cases}$$

and

$$\Upsilon(\alpha,\beta) = \alpha^{\upsilon_1}\beta^{\upsilon_2} \rightsquigarrow \begin{cases} \upsilon_1 = 1, \upsilon_2 = 0 & \text{for estimating } \alpha\\ \upsilon_1 = 0, \upsilon_2 = 1 & \text{for estimating } \beta.\end{cases}$$

It is clear that the Bayes estimator in (3.2) cannot be obtained analytically. Therefore, some approximation methods are required in order to compute the approximate Bayes estimates. We adopt the Lindley's method and Metropolis–Hastings algorithm to solve the problem.

3.1. Lindley's Approximation

In the above Section, we see that the proposed Bayes estimates are in the form of the ratio of two integrals. These integrals can not be evaluated in terms of some closed-form expressions. So, we developed the Bayesian estimates using the Lindley's approximation (Lindley [19]). Based on the Lindley's method, the Bayes estimations of parameters have the following expression:

$$\tilde{\alpha} = \hat{\alpha} + \frac{1}{2}\big[2\hat{\rho}_\alpha\hat{\upsilon}_{\alpha\alpha} + 2\hat{\rho}_\beta\hat{\upsilon}_{\alpha\beta} + \hat{\upsilon}_{\alpha\alpha}^2\hat{l}_{\alpha\alpha\alpha} + \hat{\upsilon}_{\alpha\alpha}\hat{\upsilon}_{\beta\beta}\hat{l}_{\beta\beta\alpha} + 2\hat{\upsilon}_{\alpha\beta}\hat{\upsilon}_{\beta\alpha}\hat{l}_{\alpha\beta\beta} + \hat{\upsilon}_{\alpha\beta}\hat{\upsilon}_{\beta\beta}\hat{l}_{\beta\beta\beta}\big],$$

and

$$\tilde{\beta} = \hat{\alpha} + \frac{1}{2}\big[2\hat{\rho}_\alpha\hat{\upsilon}_{\beta\beta} + 2\hat{\rho}_\beta\hat{\upsilon}_{\beta\alpha} + \hat{\upsilon}_{\beta\beta}^2\hat{l}_{\beta\beta\beta} + 3\hat{\upsilon}_{\beta\beta}\hat{\upsilon}_{\alpha\beta}\hat{l}_{\alpha\beta\beta} + \hat{\upsilon}_{\alpha\alpha}\hat{\upsilon}_{\beta\alpha}\hat{l}_{\alpha\alpha\alpha}\big].$$

Here, $\hat{\rho}_\alpha = \frac{b_1-1}{\hat{\alpha}} - a_1$, $\hat{\rho}_\beta = \frac{b_2-1}{\hat{\beta}} - a_2$, $\hat{l}_{\alpha^n\beta^m} = \partial^{n+m}l(\alpha,\beta)/\partial\alpha^n\partial\beta^m; n, m = 0, 1, \ldots$ and $\hat{\upsilon}_{ij}$ are the (ij)-th elements of matrix $\big[-\partial^2 l(\alpha,\beta)/\partial\alpha\partial\beta\big]^{-1}; i, j = 1, 2$. Also, we have

$$\hat{l}_{\alpha\alpha\alpha} = \frac{\partial^3 l}{\partial\alpha^3} = \frac{2\iota}{\alpha^3} - \sum_{j=1}^{\iota} R_j\frac{A_j^\alpha\ln^3(A_j)}{(A_j^\alpha-1)^2} + 2\sum_{j=1}^{\iota} R_j\frac{A_j^{2\alpha}\ln^3(A_j)}{(A_j^\alpha-1)^3} - \varpi_6,$$

$$\begin{aligned} \varpi_6 &= \tilde{R}_{Q_1+1}\Big\{\frac{S_1^\alpha\ln^3(S_1)}{(S_1^\alpha-1)^2} - 2\frac{S_1^{2\alpha}\ln^3(S_1)}{(S_1^\alpha-1)^3}\Big\} && \text{for case I},\\ \varpi_6 &= 0 && \text{for case II},\\ \varpi_6 &= \tilde{R}_{Q_2+1}\Big\{\frac{S_2^\alpha\ln^3(S_2)}{(S_2^\alpha-1)^2} - 2\frac{S_2^{2\alpha}\ln^3(S_2)}{(S_2^\alpha-1)^3}\Big\} && \text{for case III},\end{aligned}$$

$$\hat{l}_{\alpha\alpha\beta} = \hat{l}_{\alpha\beta\alpha} = \hat{l}_{\beta\alpha\alpha} = \frac{\partial^3 l}{\partial\alpha^2\partial\beta}|_{\alpha=\alpha,\beta=\beta} + \sum_{j=1}^{\iota} R_j \frac{\alpha A_j^{\alpha-1}\ln x_j x_j^{-\beta}\ln^2(A_j)}{(A_j^{\alpha}-1)^2}$$
$$+ 2\sum_{j=1}^{\iota} R_j \frac{A_j^{\alpha-1}\ln x_j x_j^{-\beta}\ln(A_j)}{(A_j^{\alpha}-1)^2} - 2\sum_{j=1}^{\iota} R_j \frac{\alpha A_j^{2\alpha-1}\ln x_j x_j^{-\beta}\ln^2(A_j)}{(A_j^{\alpha}-1)^3} + \varpi_7,$$

$$\varpi_7 = \tilde{R}_{Q_1+1}\Big\{\frac{\alpha S_1^{\alpha-1}T_1^{-\beta}\ln T_1\ln^2(S_1)}{(S_1^{\alpha}-1)^2} + 2\frac{\ln(S_1)T_1^{-\beta}\ln T_1 S_1^{\alpha-1}}{(S_1^{\alpha}-1)^2}$$
$$- 2\frac{\alpha S_1^{2\alpha-1}T_1^{-\beta}\ln T_1\ln^2(S_1)}{(S_1^{\alpha}-1)^3}\Big\} \qquad \text{for case I,}$$
$$\varpi_7 = 0 \qquad \text{for case II,}$$
$$\varpi_7 = \tilde{R}_{Q_2+1}\Big\{\frac{\alpha S_2^{\alpha-1}T_2^{-\beta}\ln T_2\ln^2(S_2)}{(S_2^{\alpha}-1)^2} + 2\frac{\ln(S_2)T_2^{-\beta}\ln T_2 S_2^{\alpha-1}}{(S_2^{\alpha}-1)^2}$$
$$- 2\frac{\alpha S_2^{2\alpha-1}T_2^{-\beta}\ln T_2\ln^2(S_2)}{(S_2^{\alpha}-1)^3}\Big\} \qquad \text{for case III,}$$

$$\hat{l}_{\beta\beta\beta} = \frac{\partial^3 l}{\partial\beta^3}|_{\alpha=\hat{\alpha},\beta=\hat{\beta}} = \frac{2\iota}{\beta^3} - \sum_{j=1}^{\iota}(\alpha+1)\frac{x_j^{\beta}\ln^3 x_j}{(1+x_j^{\beta})^2} + \sum_{j=1}^{\iota}2(\alpha+1)\frac{x_j^{2\beta}\ln^3 x_j}{(1+x_j^{\beta})^3}$$
$$+ \sum_{j=1}^{\iota}\alpha R_j\frac{x_j^{\beta}\ln^3 x_j(A_j^{\alpha+1}-1)}{(x_j^{\beta}(A_j^{\alpha+1}-1)-1)^2} - \sum_{j=1}^{\iota}\alpha(\alpha+1)R_j\frac{\ln^3 x_j A_j^{\alpha}}{(x_j^{\beta}(A_j^{\alpha+1}-1)-1)^2}$$
$$- \sum_{j=1}^{\iota}2\alpha R_j\frac{x_j^{2\beta}\ln^3 x_j(A_j^{\alpha+1}-1)^2}{(x_j^{\beta}(A_j^{\alpha+1}-1)-1)^3} + \sum_{j=1}^{\iota}4\alpha(\alpha+1)R_j\frac{x_j^{\beta}A_j^{\alpha}\ln^3 x_j(A_j^{\alpha+1}-1)}{(x_j^{\beta}(A_j^{\alpha+1}-1)-1)^3}$$
$$+ \sum_{j=1}^{\iota}\alpha^2(\alpha+1)R_j\frac{x_j^{-\beta}\ln^3 x_j A_j^{\alpha-1}}{(x_j^{\beta}(A_j^{\alpha+1}-1)-1)^2} - \sum_{j=1}^{\iota}2\alpha(\alpha+1)^2R_j\frac{\ln^3 x_j A_j^{2\alpha}}{(x_j^{\beta}(A_j^{\alpha+1}-1)-1)^3}$$
$$+ \varpi_8,$$

$$\varpi_8 = \tilde{R}_{Q_1+1}\Big\{\frac{\alpha T_1^{\beta}\ln^3 T_1(S_1^{\alpha+1}-1)}{(T_1^{\beta}(S_1^{\alpha+1}-1)-1)^2} - \frac{\alpha(\alpha+1)\ln^3 T_1 S_1^{\alpha}}{(T_1^{\beta}(S_1^{\alpha+1}-1)-1)^2}$$
$$- \frac{2\alpha T_1^{2\beta}\ln^3 T_1(S_1^{\alpha+1}-1)^2}{(T_1^{\beta}(S_1^{\alpha+1}-1)-1)^3} + \frac{4\alpha(\alpha+1)T_1^{\beta}\ln^3 T_1 S_1^{\alpha}(S_1^{\alpha+1}-1)}{(T_1^{\beta}(S_1^{\alpha+1}-1)-1)^3}$$
$$+ \frac{\alpha^2(\alpha+1)T_1^{-\beta}\ln^3 T_1 S_1^{\alpha-1}}{(T_1^{\beta}(S_1^{\alpha+1}-1)-1)^2} - \frac{2\alpha(\alpha+1)^2\ln^3 T_1 S_1^{2\alpha}}{(T_1^{\beta}(S_1^{\alpha+1}-1)-1)^3}\Big\} \quad \text{for case I,}$$
$$\varpi_8 = 0 \qquad \text{for case II,}$$
$$\varpi_8 = \tilde{R}_{Q_2+1}\Big\{\frac{\alpha T_2^{\beta}\ln^3 T_2(S_2^{\alpha+1}-1)}{(T_2^{\beta}(S_2^{\alpha+1}-1)-1)^2} - \frac{\alpha(\alpha+1)\ln^3 T_2 S_2^{\alpha}}{(T_2^{\beta}(S_2^{\alpha+1}-1)-1)^2}$$
$$- \frac{2\alpha T_2^{2\beta}\ln^3 T_2(S_2^{\alpha+1}-1)^2}{(T_2^{\beta}(S_2^{\alpha+1}-1)-1)^3} + \frac{4\alpha(\alpha+1)T_2^{\beta}\ln^3 T_2 S_2^{\alpha}(S_2^{\alpha+1}-1)}{(T_2^{\beta}(S_2^{\alpha+1}-1)-1)^3}$$
$$+ \frac{\alpha^2(\alpha+1)T_2^{-\beta}\ln^3 T_2 S_2^{\alpha-1}}{(T_2^{\beta}(S_2^{\alpha+1}-1)-1)^2} - \frac{2\alpha(\alpha+1)^2\ln^3 T_2 S_2^{2\alpha}}{(T_2^{\beta}(S_2^{\alpha+1}-1)-1)^3}\Big\} \quad \text{for case III,}$$

$$\hat{l}_{\beta\beta\alpha} = \hat{l}_{\beta\alpha\beta} = \hat{l}_{\alpha\beta\beta} = \frac{\partial^3 l}{\partial\beta^2\partial\alpha}|_{\alpha=\hat{\alpha},\beta=\hat{\beta}} = -\sum_{i=1}^{\iota} \frac{x_j^{\beta}\ln^2 x_j}{(1+x_j^{\beta})^2}$$
$$+ \sum_{j=1}^{\iota} R_j \frac{\ln^2 x_j x_j^{\beta}(A_j^{\alpha+1}-1)}{(x_j^{\beta}(A_j^{\alpha+1}-1)-1)^2} + \sum_{j=1}^{\iota} \alpha R_j \frac{A_j^{\alpha+1} x_j^{\beta}\ln(A_j)\ln^2 x_j}{(x_j^{\beta}(A_j^{\alpha+1}-1)-1)^2}$$
$$- \sum_{j=1}^{\iota} 2\alpha R_j \frac{A_j^{\alpha+1}(A_j^{\alpha+1}-1)x_j^{2\beta}\ln^2 x_j \ln(A_j)}{(x_j^{\beta}(A_j^{\alpha+1}-1)-1)^3}$$
$$- \sum_{j=1}^{\iota}(2\alpha+1)R_j \frac{A_j^{\alpha}\ln^2 x_j}{(x_j^{\beta}(A_j^{\alpha+1}-1)-1)^2} - \sum_{j=1}^{\iota}(\alpha^2+\alpha)R_j \frac{A_j^{\alpha}\ln^2 x_j \ln(A_j)}{(x_j^{\beta}(A_j^{\alpha+1}-1)-1)^2}$$
$$+ \sum_{j=1}^{\iota} 2\alpha(\alpha+1)R_j \frac{A_j^{2\alpha+1}x_j^{\beta}\ln^2 x_j \ln(A_j)}{(x_j^{\beta}(A_j^{\alpha+1}-1)-1)^3} + \varpi_9,$$

$$\varpi_9 = \tilde{R}_{Q_1+1}\Big\{ \frac{T_1^{\beta}\ln^2 T_1(S_1^{\alpha+1}-1)}{(T_1^{\beta}(S_1^{\alpha+1}-1)-1)^2} + \alpha\frac{\ln^2 T_1 S_1^{\alpha+1}\ln(S_1)T_1^{\beta}}{(T_1^{\beta}(S_1^{\alpha+1}-1)-1)^2}$$
$$- 2\alpha\frac{T_1^{2\beta}\ln^2 T_1(S_1^{\alpha+1}-1)S_1^{\alpha+1}\ln S_1}{(T_1^{\beta}(S_1^{\alpha+1}-1)-1)^3} - (2\alpha+1)\frac{\ln^2 T_1(S_1)^{\alpha}}{(T_1^{\beta}(S_1^{\alpha+1}-1)-1)^2}$$

$$- (\alpha+\alpha^2)\frac{\ln^2 T_1\ln(S_1)S_1^{\alpha}}{(T_1^{\beta}(S_1^{\alpha+1}-1)-1)^2} + 2\alpha(\alpha+1)\frac{\ln^2 T_1\ln(S_1)T_1^{\beta}S_1^{2\alpha+1}}{(T_1^{\beta}(S_1^{\alpha+1}-1)-1)^3}\Big\} \quad \text{for case I,}$$
$$\varpi_9 = 0 \qquad \text{for case II,}$$

$$\varpi_9 = \tilde{R}_{Q_2+1}\Big\{ \frac{T_2^{\beta}\ln^2 T_2(S_2^{\alpha+1}-1)}{(T_2^{\beta}(S_2^{\alpha+1}-1)-1)^2} + \alpha\frac{\ln^2 T_2 S_1^{\alpha+1}\ln(S_2)T_2^{\beta}}{(T_2^{\beta}(S_2^{\alpha+1}-1)-1)^2}$$
$$- 2\alpha\frac{T_2^{2\beta}\ln^2 T_2(S_2^{\alpha+1}-1)S_2^{\alpha+1}\ln S_2}{(T_2^{\beta}(S_2^{\alpha+1}-1)-1)^3} - (2\alpha+1)\frac{\ln^2 T_2 S_2^{\alpha}}{(T_2^{\beta}(S_2^{\alpha+1}-1)-1)^2}$$
$$- (\alpha+\alpha^2)\frac{\ln^2 T_2\ln(S_2)S_2^{\alpha}}{(T_2^{\beta}(S_2^{\alpha+1}-1)-1)^2} + 2\alpha(\alpha+1)\frac{\ln^2 T_2\ln(S_2)T_2^{\beta}S_2^{2\alpha+1}}{(T_2^{\beta}(S_2^{\alpha+1}-1)-1)^3}\Big\} \quad \text{for case III.}$$

3.2. Metropolis–Hastings Algorithm

In the previous Subsection, we obtain the Bayes estimates using Lindley's approximation method. One disadvantage of this method is that it requires higher order partial derivatives of the log-likelihood function. Further, the Lindley's approximation can not be used to construct HPD credible intervals. Moreover, it is observed that the conditional posterior distribution of unknown parameters cannot be reduced to any well-known distribution. To overcome this problem, we propose to apply the Metropolis–Hastings (Metropolis *et al.* [20]) algorithm for generating samples from the respective posterior distributions. This algorithm is the most popular example of the Markov chain Monte Carlo (MCMC) method and it is free from the higher order partial derivatives. The basic scheme of the Metropolis–Hastings (M-H) is given as follows:

Step 1: Use the MLEs of (α, β) as the initial point of the iteration, denoted by (α_0, β_0).

Step 2: Generate α_j and β_j from the normal proposal distributions $N(\alpha_{j-1}, \sigma^2)$ and $N(\beta_{j-1}, \sigma^2)$, respectively, for $j = 1, ..., N$.

Step 3: Compute $h = \dfrac{\pi(\alpha_j, \beta_j | Data)}{\pi(\alpha_{j-1}, \beta_{j-1} | Data)}$.

Step 4: Accept the new sample with probability $\min(1, h)$.

Step 5: Set $j = j + 1$.

Step 6: Repeat Step 2–5, up to N times.

So, the Bayes estimates of α and β are respectively obtained as below:

$$\tilde{\alpha} = \frac{1}{N - N_0} \sum_{i=N_0+1}^{N} \alpha_i, \qquad \text{and} \qquad \tilde{\beta} = \frac{1}{N - N_0} \sum_{i=N_0+1}^{N} \beta_i.$$

In order to guarantee the convergence and to remove the affection of the selection of initial values, the first N_0 simulated varieties are discarded (burn-in-period of Markov chain). Also, for computing the confidence interval based on MCMC samples, we first order the samples $\alpha_{1:N}, \alpha_{2:N}, ..., \alpha_{N:N}$ and $\beta_{1:N}, \beta_{2:N}, ..., \beta_{N:N}$, then a $(1 - \gamma) \times 100\%$ HPD credible interval for α and β are obtained as:

$$[\alpha_{N\gamma}, \alpha_{N(1-\gamma)}] \qquad \text{and} \qquad [\beta_{N\gamma}, \beta_{N(1-\gamma)}].$$

Finally, choose the interval which has the smallest width as a HPD credible interval.

4. SIMULATION STUDY

To evaluate the behavior of the theoretical results obtained in the previous Sections, including the classical and Bayesian estimators and the associated confidence/credible intervals, an extensive Monte Carlo simulation study is performed. We simulate GIIPH censored samples for different combinations of (n, m, T_1, T_2) from the inverse Burr (α, β) distribution. We adopted the true values of unknown parameters as $\alpha = 1.2$ and $\beta = 1.6$. Note that all the computations have been performed using R software. Through the sample data, we evaluate the MLEs by employing an EM algorithm. Approximate expressions for the Bayesian estimators have been obtained using the Lindley's approximation and Metropolis–Hastings algorithm. Using the M-H sampler algorithm described in Subsection 3.2, 10000 MCMC samples and discard the first 2000 values as 'burn-in' are generated. In Bayesian paradigm, the choice of the hyper-parameter values is the main issue. For this propose, both non-informative prior (NIP) and informative prior (IP) are taken into account in the Bayesian approach, where all hyper-parameters in the NIP are chosen to be 0.0001 instead of 0, which is more appropriate since the hyper-parameters are greater than 0, and the hyper-parameters in the IP are selected according to this manner: the means of prior (PR) distributions are equal to original parameters($a_1 = 1.2, a_2 = 1.6, b_1 = 1, b_2 = 1$). The %95 approximate confidence (AC) and Bayesian (HPD) intervals for the parameters are also constructed. The HPD credible intervals are computed based on 10000 MCMC samples. We take three different censoring schemes as follows:

Scheme 1: $R_m = n - m$ and $R_j = 0$ for $j \neq m$;

Scheme 2: $R_1 = R_m = (n - m)/2$ and $R_j = 0$ for $j \neq 1, m$;

Scheme 3: $R_{m/2} = n - m$ and $R_j = 0$ for $j \neq m/2$.

Based on these set up assumptions, we show the numerical results in the Table 1, Table 2, Table 3 and Table 4.

Tables 1 and 2 (also, Figure 5 and Figure 6) present the average ML and Bayes estimates and the corresponding MSEs based on 10000 replications. Moreover, the average lower and upper bounds of the AC and HPD intervals are displayed in Tables 3 and 4.

The following conclusions are found from Tables 1–4 and Figures 5–6:

- For fixed n, T_1 and T_2 as m increases, the average estimates and the MSEs of the parameters decreases. Also, with increasing m, the average lengths of all intervals mostly decrease.
- For fixed m, T_1 and T_2 as sample size n increases the MSEs of all the estimators decreases (Figure 6). Similar trend is observed (Figure 6) for fixed n, T_1 and T_2 as m increases.
- The MSEs have a downward trend for fixed n, m, T_1 and increasing T_2 (Figure 5).
- For fixed n, m and T_2 as T_1 increases, the MSEs decreases (Figure 5).
- To evaluate the effect of the proposed estimation methods with respect to the smallest MSE, it is observed that the Bayes estimates work efficiently and provide better performance as compared to those obtained based on MLEs. For the parameters α and β, the MSEs of the maximum likelihood estimates are larger than the Bayes estimates.
- The Bayesian MCMC estimation using M-H algorithm sampler for the unknown parameters under GIIPH censoring is recommended for all values of n, m, T_1 and T_2.
- As expected, the Bayesian estimation with IP tends to be preferable to that with NIP.
- The average lengths of the ACI for α and β are relatively large compared to those of Bayesian credible intervals.
- As for the Bayes method, similar to the findings for the point estimates, the Bayesian intervals under non-informative prior are slightly worse than those under informative prior.

Table 1: The MSEs of the MLEs ($\hat{\alpha}$), Lindleys ($\tilde{\alpha}_{LIN}$) and M-Hs ($\tilde{\alpha}_{MH}$) for $T_1 = 1$, $T_2 = 2.5$.

n	m	*Scheme*	$\hat{\alpha}(MSE)$	*PR*	$\tilde{\alpha}_{LIN}(MSE)$	$\tilde{\alpha}_{MH}(MSE)$
30	15	R_1	1.4754(0.2256)	*IP*	1.4897(0.1945)	1.4270(0.0979)
				NIP	1.5023(0.2094)	1.4328(0.1006)
		R_2	1.4427(0.1789)	*IP*	1.4475(0.1581)	1.3665(0.0733)
				NIP	1.4596(0.1637)	1.3709(0.0769)
		R_3	1.4246(0.1005)	*IP*	1.3937(0.0897)	1.3508(0.0621)
				NIP	1.4214(0.0942)	1.3700(0.0699)
50	24	R_1	1.2982(0.1785)	*IP*	1.3687(0.1586)	1.2843(0.0464)
				NIP	1.3278(0.1669)	1.3199(0.0497)
		R_2	1.3064(0.0939)	*IP*	1.3000(0.0900)	1.1674(0.0175)
				NIP	1.3087(0.0911)	1.1721(0.0244)
		R_3	0.9172(0.0439)	*IP*	0.9450(0.0382)	0.9913(0.0230)
				NIP	0.9608(0.0410)	0.9725(0.0277)
	30	R_1	1.2982(0.1316)	*IP*	1.2786(0.1212)	1.2028(0.0388)
				NIP	1.2865(0.1283)	1.2536(0.0449)
		R_2	1.1464(0.0476)	*IP*	1.1397(0.0455)	0.9741(0.0091)
				NIP	1.1471(0.0462)	01.9932(.0105)
		R_3	1.006(0.0295)	*IP*	0.9995(0.0261)	1.001(0.0141)
				NIP	1.002(0.0269)	1.009(0.0169)
100	48	R_1	1.3875(0.1632)	*IP*	1.3768(0.1544)	1.2434(0.0336)
				NIP	1.3811(0.1598)	1.2709(0.0390)
		R_2	1.1687(0.0283)	*IP*	1.1666(0.0277)	1.1019(0.0100)
				NIP	1.1679(0.0281)	1.1052(0.0133)
		R_3	0.9414(0.0221)	*IP*	0.9551(0.0205)	1.1174(0.0188)
				NIP	0.9532(0.0215)	1.1168(0.0209)
	60	R_1	1.3212(0.1226)	*IP*	1.3105(0.1152)	1.2190(0.0327)
				NIP	1.3176(0.1174)	1.2187(0.0328)
		R_2	0.9028(0.0090)	*IP*	0.9162(0.0070)	0.9786(0.0004)
				NIP	0.9200(0.0084)	0.9921(0.0013)
		R_3	1.0090(0.0145)	*IP*	1.0060(0.0136)	1.0400(0.0016)
				NIP	1.008(0.0140)	1.0488(0.0021)

Table 2: The MSEs of the MLEs ($\hat{\beta}$), Lindleys ($\tilde{\beta}_{LIN}$) and M-Hs ($\tilde{\beta}_{MH}$) for $T_1 = 1$, $T_2 = 2.5$.

n	m	*Scheme*	$\hat{\beta}(MSE)$	*PR*	$\tilde{\beta}_{LIN}(MSE)$	$\tilde{\beta}_{MH}(MSE)$
30	15	R_1	1.3675(0.2354)	*IP*	1.3954(0.1738)	1.3700(0.1614)
				NIP	1.4186(0.1845)	1.3961(0.1823)
		R_2	1.6325(.1535)	*IP*	1.5885(0.1366)	1.5143(0.1122)
				NIP	1.5899(0.1389)	1.5357(0.1308)
		R_3	1.7859(0.1673)	*IP*	1.7654(0.1611)	1.4998(0.1066)
				NIP	1.7865(0.1738)	1.5403(0.1231)
50	24	R_1	1.2763(0.1807)	*IP*	1.3029(0.1322)	1.3122(0.1316)
				NIP	1.3043(0.1493)	1.3145(0.1387)
		R_2	1.5917(.1068)	*IP*	1.5643(0.1031)	1.5457(0.0900)
				NIP	1.5749(0.1054)	1.5499(0.0938)
		R_3	1.7431(0.1673)	*IP*	1.7183(0.1527)	1.5125(0.0760)
				NIP	1.7327(0.1602)	1.5226(0.0811)
	30	R_1	1.3846(0.1155)	*IP*	1.4220(0.1136)	1.7034(0.0921)
				NIP	1.4165(0.1140)	1.6993(0.0976)
		R_2	1.5075(0.0860)	*IP*	1.5064(0.0850)	1.6737(0.0540)
				NIP	1.5069(0.0857)	1.3199(0.0497)
		R_3	1.6514(0.0897)	*IP*	1.6094(0.0741)	1.5509(0.0240)
				NIP	1.6328(0.0809)	1.5499(0.0296)
100	48	R_1	1.2681(0.1699)	*IP*	1.2796(0.1242)	1.6234(0.1045)
				NIP	1.2765(0.1374)	1.6309(0.1089)
		R_2	1.7893(0.0558)	*IP*	1.7674(0.0480)	1.4064(0.0540)
				NIP	1.7763(0.0518)	1.4078(0.0616)
		R_3	1.6676(0.0573)	*IP*	1.6559(0.0540)	1.4797(0.0244)
				NIP	1.6621(0.0564)	1.4859(0.0289)
	60	R_1	1.3435(0.1047)	*IP*	1.3586(0.0905)	1.6112(0.0446)
				NIP	1.3488(0.0977)	1.6269(0.0535)
		R_2	1.5587(0.0460)	*IP*	1.54974(0.0430)	1.6850(0.0070)
				NIP	1.5546(0.0451)	1.6589(0.0087)
		R_3	1.5973(0.0397)	*IP*	1.5784(0.0370)	1.6405(0.0046)
				NIP	1.5884(0.0383)	1.6712(0.0065)

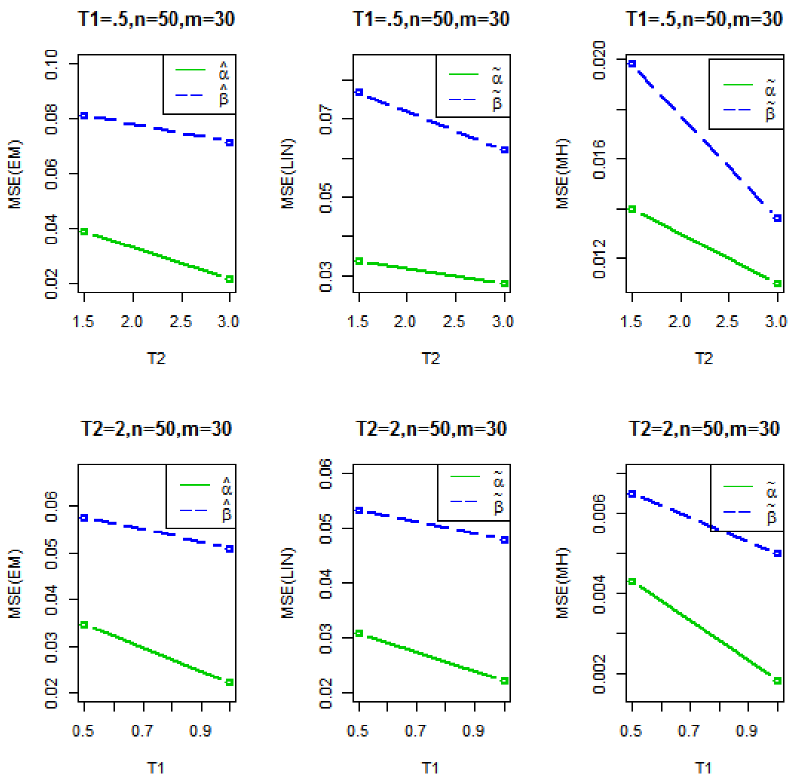

Figure 5: The MSEs of the estimators for different choices of T_1 and T_2.

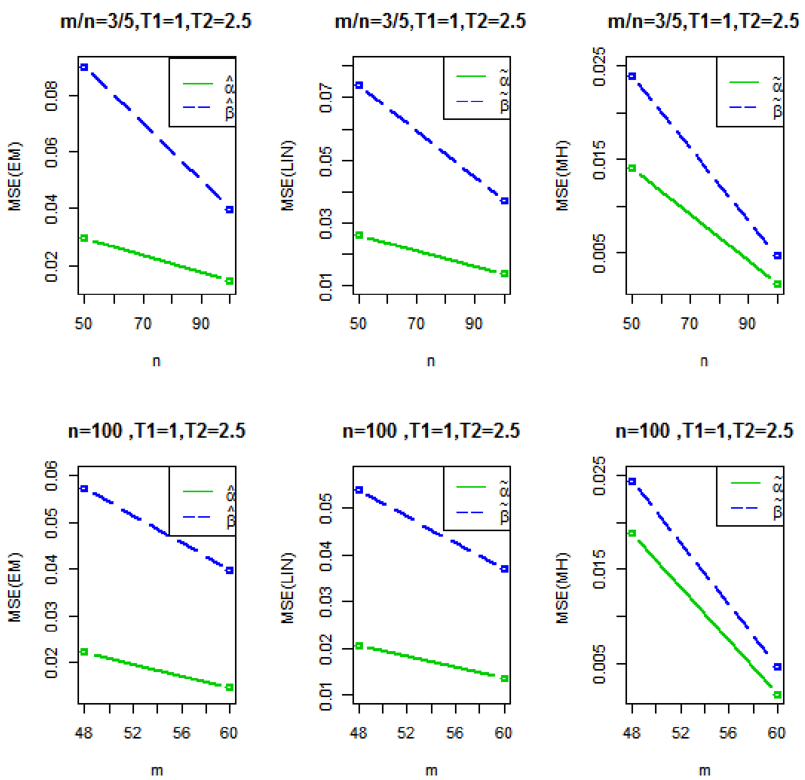

Figure 6: The MSEs of the estimators for different choices of n and m.

Table 3: The average upper and lower bounds for α when $T_1 = 1$, $T_2 = 2.5$.

n	m	$Scheme$	$LACI$	$UACI$	PR	$LHPD$	$UHPD$
30	15	R_1	1.1432	1.8865	IP	1.1675	1.8536
					NIP	1.1604	1.8653
		R_2	0.6884	1.4893	IP	0.7476	1.4280
					NIP	0.7421	1.4452
		R_3	0.7189	1.3966	IP	0.7728	1.3609
					NIP	0.7496	1.3648
50	24	R_1	1.1820	1.7020	IP	1.1224	1.5726
					NIP	1.1148	1.5921
		R_2	0.8440	1.3540	IP	0.9980	1.3932
					NIP	0.9972	1.4266
		R_3	0.7850	1.2893	IP	0.9782	1.3780
					NIP	0.9760	1.3882
	30	R_1	0.9334	1.4357	IP	0.9855	1.3921
					NIP	0.9599	1.3989
		R_2	0.9233	1.3585	IP	0.9315	1.2667
					NIP	0.9035	1.2833
		R_3	0.9540	1.3585	IP	0.8993	1.2264
					NIP	0.8556	1.2345
100	48	R_1	0.9420	1.4441	IP	1.1434	1.5134
					NIP	1.1139	1.5173
		R_2	0.8021	1.2721	IP	1.0019	1.3431
					NIP	1.0011	1.3564
		R_3	0.8527	1.2427	IP	1.0674	1.3174
					NIP	1.0221	1.3308
	60	R_1	0.9378	1.3876	IP	1.0319	1.3108
					NIP	0.9881	1.3288
		R_2	0.8937	1.2437	IP	0.9735	1.2899
					NIP	0.9711	1.3004
		R_3	0.7860	1.2334	IP	0.9989	1.2206
					NIP	0.9366	1.2371

Table 4: The average upper and lower bounds for β when $T_1 = 1$, $T_2 = 2.5$.

n	m	$Scheme$	$LACI$	$UACI$	IP	$LHPD$	$UHPD$
30	15	R_1	0.5265	1.7832	IP	0.6643	1.7548
					NIP	0.6532	1.7802
		R_2	0.9568	2.1005	IP	1.0944	1.9136
					NIP	0.9867	1.9197
		R_3	1.2147	2.2715	IP	0.6957	1.6257
					NIP	0.6809	1.6441
50	24	R_1	0.5669	1.6790	IP	0.5997	1.6212
					NIP	0.5911	1.63211
		R_2	1.0828	2.0995	IP	0.8532	1.6834
					NIP	0.8498	1.6980
		R_3	1.2147	2.2715	IP	0.6957	1.6257
					NIP	0.6709	1.6299
	30	R_1	0.7412	1.8433	IP	1.5999	2.3618
					NIP	1.5799	1.3654
		R_2	1.0339	1.9803	IP	1.4956	2.1097
					NIP	1.4832	2.1217
		R_3	1.1319	2.1708	IP	1.0911	1.6639
					NIP	1.0783	1.6823
100	48	R_1	0.9202	1.6397	IP	1.1000	2.0211
					NIP	1.0906	2.0466
		R_2	1.3920	2.1866	IP	1.2016	1.6581
					NIP	1.1923	1.6734
		R_3	0.9736	1.6248	IP	1.2079	1.6297
					NIP	1.1996	1.6500
	60	R_1	0.9922	1.6392	IP	1.4573	1.9612
					NIP	1.4524	1.9903
		R_2	1.1871	1.9002	IP	1.5417	1.8247
					NIP	1.5289	1.8357
		R_3	1.3432	1.9515	IP	1.4902	1.8750
					NIP	1.4599	1.8786

5. APPLICATIONS OF BIII DISTRIBUTION TO SEPARATION OF SEWER SOLIDS

A real set of experimental data contains the invert trap efficiency. The invert traps are used to separate suspended solids in the sewers and storm water drainage channels. The solid particles are deposited in the bottom of the sewer drainage channel and decreases the channel cross section and thus reduces the hydraulic efficiency. Therefore, increasing invert trap efficiency directly affects the hydraulic efficiency. For computational convenience we divided each data point by 70. Figure 7(a) shows the velocity stream lines of water in channel. The color of the velocity stream lines shows that the velocity decreases in the trap, so the particles entering the low-velocity zone of the invert trap settle in the bottom of the trap. Figure 7(b) shows 3D view of an open rectangular channel fitted with an invert trap at the bottom of the channel. Before we carry out numerical calculations and give way to an advanced point in the analysis of this data, we compute the Kolmogorov-Smirnov (K-S) distances between the empirical distribution and the fitted distribution functions based on MLEs, it is 0.1189, and the associated p-value is 0.8312. We also presented the P-P and CDF (the empirical function and the fitted function) plots for the fitted inverse Burr distribution in Figures 8 and 9 respectively. The result indicates that considered distribution can be used to to obtain inferential results from the considered data set. We have obtained the MLEs by using EM algorithm by taking initial values with the help of contour and 3D profile plot given in Figure 10.

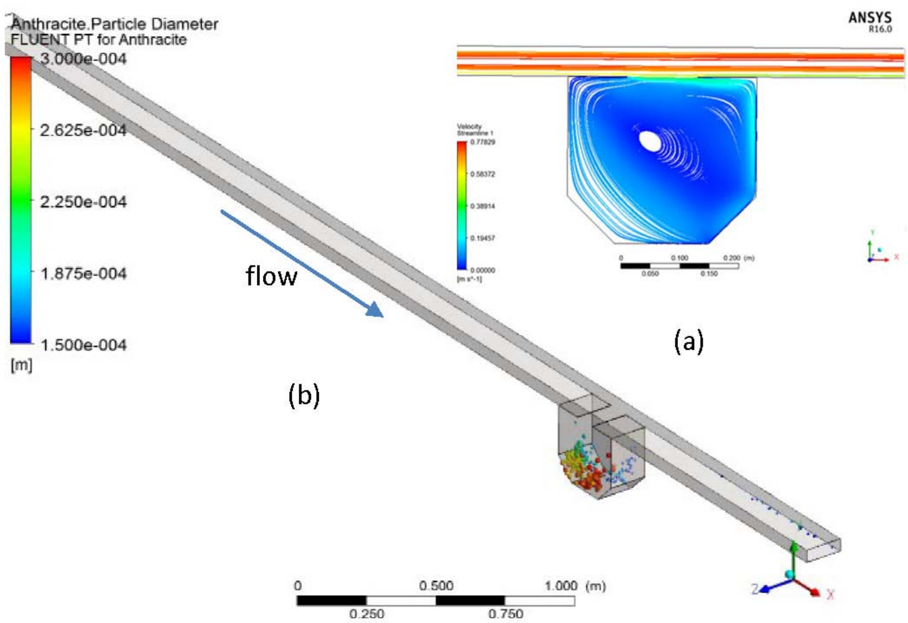

Figure 7: (a) stream lines of water in invert trap. (b) trapping of sewer solids, flowing into a sewer drainage system. Particle traces coloured according to the particle size of 150–300 micron.

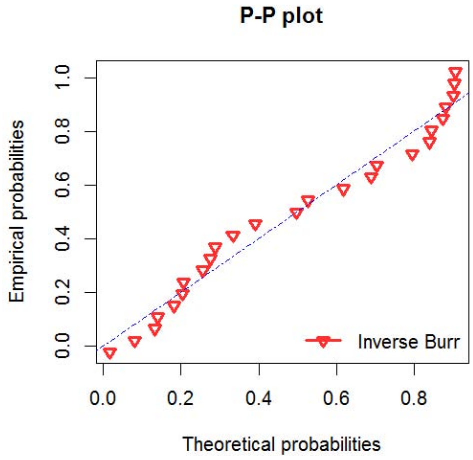

Figure 8: The P-P plot.

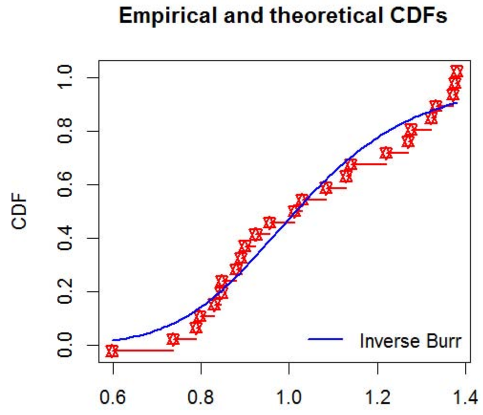

Figure 9: The CDF plot.

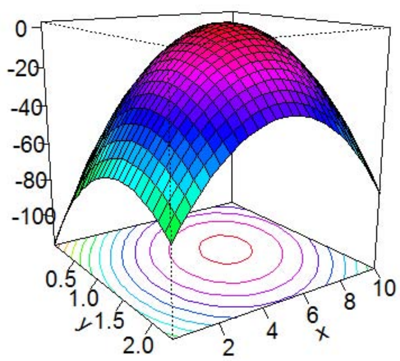

Figure 10: Contour plot and 3D profile plot of log likelihood for invert trap data (x=α and y=β).

We shall use these data to consider three different GIIPH censoring schemes:

Case I: $n = 25, m = 20$ $R = (5, 0 * 19)$,$T_1 = 1.4$ and $T_2 = 1.6$;

Case II: $n = 25, m = 20$ $R = (5, 0 * 19)$,$T_1 = 1.2$ and $T_2 = 1.6$;

Case III: $n = 25, m = 20$ $R = (5, 0 * 19)$,$T_1 = 1.2$ and $T_2 = 1.35$.

Based on the following censoring schemes, the MLEs and Bayes estimates of both the unknown parameters are reported in Table 5.

The length of approximate intervals (LAC) and HPD intervals (LHPD) are also calculated individually and presented in Table 5. For Bayesian aspect, we use non-informative Gamma priors ($a_1 = 0.0001; a_2 = 0.0001; b_1 = 0.0001; b_2 = 0.0001$) due to the lack prior information.

As seen in Table 5, two types of point estimates of parameters are observed: MLEs and Bayes estimates are quite similar. Comparing approximate and credible intervals derived from Bayesian method, the latter are noticeably smaller in interval lengths than the former.

Table 5: Different point and interval estimates of α and β for $(n, m) = (25, 20)$.

Cases	T_1	T_2	$\hat{\alpha}$	$\tilde{\alpha}_{LIN}$	$\tilde{\alpha}_{MH}$	*LAC*	*LHPD*
Case I	1.4	1.6	0.99817	1.03622	0.94491	1.2327	0.8574
Case II	1.2	1.6	1.20987	1.23875	1.15643	1.3734	0.8897
Case III	1.2	1.35	1.21124	1.24054	1.15991	1.3798	0.8687
Cases	T_1	T_2	$\hat{\beta}$	$\tilde{\beta}_{LIN}$	$\tilde{\beta}_{MH}$	*LAC*	*LHPD*
Case I	1.4	1.6	7.51460	7.74952	7.46857	2.7632	1.2864
Case II	1.2	1.6	7.66071	7.79034	7.65335	2.8395	1.3323
Case III	1.2	1.35	7.62534	7.75890	7.44881	2.8567	1.3682

6. CONCLUSIONS

In this paper, we derived the different point and interval estimators of the inverse Burr distribution based on a newly proposed censoring scheme known as generalized progressive hybrid censoring, where experimenters are allowed more flexibility in designing the test, leading to shorter experimental periods and higher efficiency. We obtained the maximum likelihood estimates using the EM algorithm. The observed Fisher information matrix is used to construct the asymptotic confidence intervals of the unknown parameters. Moreover, the Bayesian approach is investigated with a flexible prior distribution, since Bayesian estimation cannot be derived in closed form, two approximations say Lindley's approximation and Metropolis–Hastings algorithm are utilized to achieve approximate point estimates. Using these MCMC samples, the HPD credible intervals are also constructed. The numerical experiments are carried out to evaluate the performance of proposed point and interval estimators, and some conclusions can be drawn from the results that the Bayesian method is comparatively favorable compared to considered classical method. The applicability of the inverse Burr distribution in real situation has been illustrated based on the separation of sewer solids data and it was observed that the proposed distribution can be utilized for analyzing this data well.

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