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Stochastic comparison for extreme order statistics arising from PHR, PRHR or location model

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Abstract:

- We have considered series and parallel systems with independent set of random variables belonging from PHR, PRHR family of distributions to study dispersive and star ordering between two systems such that the number of components in both the system are different. This will help us in identifying the deviation of lifetime of a product from the warranty of the product. Moreover, we have also considered series and parallel systems with dependent set of random variables each belonging from location based models, such that the baseline distribution for both the sets are different. The Archimedean copula generators used here are ψ_1 and ψ_2 such that the condition "$\phi_1 \cdot \psi_2$ or $\phi_2 \cdot \psi_1$ is super-additive" holds. Earlier researchers have studied location-scaled or resilience-scaled models for independent or dependent set of random variables. Our study is an addition to the existing research.

Keywords:

- *Archimedean Copula; Dispersive order; Proportional hazard rate distribution; Proportional reversed hazard rate distribution; Star order.*

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1. INTRODUCTION

The series and parallel systems are the most frequent and maximum encountered systems in nature. These systems are statistically referred to as the minimum and the maximum order statistic respectively. Let $X_1, X_2, \ldots, X_n$ be n independent and non-identical random variables from a particular population. Then arranging the random variables according to their magnitude or strength we observe that $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$, where $X_{k:n}$ is known as the k-th order statistic. $X_{k:n}$ represents the lifetime of a $(n-k+1)$-out-of-n system. In this paper, we focus only on the minimum and maximum order statistic. A great deal of literature is available on the stochastic relationship among the order statistics for various probability distributions.

In particular, our problem deals with the Proportional hazard rate (PHR) model and the proportional reversed hazard rate (PRHR) model. We consider $X_1, X_2, \ldots, X_n$ as independent random variables, where the survival function of each random variable X_i follows the PHR model for $i = 1, \ldots, n$. Then reliability or survival probability of X_i is:

$$P(X_i > x) = \overline{F}_i(x) = [\overline{F}_0(x)]^{\lambda_i}, \lambda_i > 0, i = 1, 2, \ldots, n,$$

where λ_i is the proportionality parameter. Here let X_0 be the baseline random variable with the baseline distribution $F_0(x)$ and the baseline survival function $\overline{F}_0(x) = 1 - F_0(x)$. Exponential, Weibull, Pareto, Lomax, Kumaraswamy's distributions are some examples of PHR model distribution. [24] pioneered the study of stochastic ordering (details about stochastic ordering are given in the next section) for k-out-of-n systems which included usual stochastic ordering results for PHR model. [25] studied dispersive and star ordering for general distributions in detail. Later on, many researchers have continued the study and found many results for PHR model. [5] discussed that the existing results for exponential distribution which can be extended for PHR models, this was possible as the random variable corresponding to the cumulative hazard rate function of a PHR family of distribution follows exponential distribution with the proportionality constant as the parameter i.e., if X follows $[\overline{F}(x)]^{\lambda}$, then the cumulative hazard rate function follows Exp(λ) distribution. [15] demonstrated dispersive ordering between the maximum order statistics of two PHR populations. [23] and [28] observed dispersive ordering between the 2^{nd} order statistics (also known as fail-safe systems) from two different populations and derived bounds on the corresponding parameters. Considering the parallel systems having PHR distributed components, [16] studied the dispersive ordering between them. A comprehensive review of the various stochastic ordering between the order statistics for random variables belonging from the PHR model has been done by [5]. Recently [11] observed stochastic ordering for series and parallel systems with Kumaraswamy's and Frechet distributed components. Now we shall observe what is meant by a multiple outlier model.

Let $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ be n-independent PHR samples having the same baseline distribution but the parameter vectors are given by $(\underbrace{\alpha_1,\ldots,\alpha_1}_{p},\underbrace{\alpha_2,\ldots,\alpha_2}_{q})$ and $(\underbrace{\beta_1,\ldots,\beta_1}_{p},\underbrace{\beta_2,\ldots,\beta_2}_{q})$ respectively, where $p+q=n$. Such an arrangement is described as the multiple-outlier model. [4] and [29] discussed the hazard rate and the likelihood ratio ordering for parallel systems with multiple-outlier PHR model. For a similar model, [2] derived conditions on the distribution function for the dispersive ordering of k-th order statistic where the parameter vectors follow majorization relation. [9] found the necessary and sufficient conditions for the hazard rate ordering among the second order statistics. [30] examined the stochastic comparison between series and parallel systems where the component lifetimes are dependent, heterogeneous and resilience scaled. [13] and [14] found several conditions for stochastic ordering of maximum and minimum order statistics from a location-scale family of distributions. [3] observed stochastic ordering between the sample ranges where component lifetimes (number of components are different) are independent and follows multiple-outlier exponential distribution and PHR models.

In contrast to the PHR model, proportional reversed hazard rate (PRHR) model was developed. Let X_i follows PRHR model then the distribution function of X_i is given by

$$P(X_i < x) = F_i(x) = [F_0(x)]^{\theta_i}, \theta_i > 0, i = 1, 2, \ldots, n,$$

where θ_i is the proportionality constant. Some known examples of PRHR model are exponentiated Weibull, exponentiated exponential, exponentiated Gamma, etc. [1] observed dispersive ordering for the series systems with components following the PRHR model.

Here we consider two sets of independent PHR and PRHR models where the baseline distribution of both the sets are different and the sample sizes are also different i.e., the first set of random variables $X_i \sim \overline{F}_i(x) = (\overline{F}_0(x))^{\alpha_i}$ for $i = 1, 2, \ldots, n_1$ and the second set $Y_i \sim \overline{G}_i(x) = (\overline{G}_0(x))^{\beta_i}$ for $i = 1, 2, \ldots, n_2$. Considering the same baseline distribution $F_0 = G_0$, we study dispersive and star ordering for series/parallel models. A similar kind of study being conducted for series/parallel system made up of PRHR distributed components.

We have also considered a general model as, $X_1, \ldots, X_{p_1}$ that has survival function $[\overline{F}(x)]^{\alpha_i}$ and $X_{p_1+1}, \ldots, X_{n_1}$ has survival function $[\overline{G}(x)]^{\alpha_i}$. And $Y_1, \ldots, Y_{p_2}$ has survival function as $[\overline{F}(x)]^{\beta_i}$ whereas the components $Y_{p_2+1}, \ldots, Y_{n_2}$ has survival function $[\overline{G}(x)]^{\beta_i}$. We have proved that the hazard rate ordering for sample minimum exists for such models, analogously reversed hazard rate ordering for sample maximums exist for PRHR model. A reversed hazard rate ordering for sample maximum (with equal sample sizes) for Pareto distributed random variables has been observed when only the shape parameter varies.

Lastly, we study some results for series system having dependent components, where the dependence among components has been considered as having Archimedean type of copula. These studies include the results when the location

parameter is varied along with a comparison between two generating functions (super-additive property) and usual stochastic ordering among baseline distributions.
The paper has been constructed as follows: Section 2 includes all the definitions used in the paper, Section 3 contains results and discussion where Subsection 3.1 contains dispersive ordering results for PHR and PRHR model with unequal sample sizes, Subsection 3.2 contains star ordering result for unequal sample sizes and Subsection 3.2 contains result for the dependent model. The various well-known lemmas that have been used in proving the results are discussed under Section 2.

2. Definitions

Let X and Y be two absolutely continuous random variables with distribution functions $F(x)$ and $G(x)$; reliability functions as $\overline{F}(x)$ and $\overline{G}(x)$; probability density functions as $f(x)$ and $g(x)$; hazard rate functions as $r(x) = \dfrac{f(x)}{\overline{F}(x)}$ and $s(x) = \dfrac{g(x)}{\overline{G}(x)}$; reversed hazard rate functions as $\tilde{r}(x) = \dfrac{f(x)}{F(x)}$ and $\tilde{s}(x) = \dfrac{g(x)}{G(x)}$, respectively. Let F^{-1} and G^{-1} be the right continuous quantiles of X and Y respectively. A real valued function ψ is super-additive when $\psi(x_1+x_2) \geq \psi(x_1)+\psi(x_2)$ for all $x_1, x_2 \in Domain(\psi)$. This concept is valid even when the summation is over n-variables. For details on the above definitions we refer the reader to [6]. One can note that a random variable has decreasing reversed hazard rate(DRHR) if and only if the distribution function is log-concave. It is known that there exists no distribution which is log convex or increasing reversed hazard rate(IRHR) over the entire domain $[0, \infty)$. An IRHR distribution can be constructed if the domain is taken as $(-\infty, \alpha)$ for some finite α (see [8]), an example of that is Truncated Normal distribution with domain as $(-\infty, 0]$. Next we discuss some of the various stochastic orders available in literature. We refer the reader to [26] for the detail of these orderings.

Definition 2.1. X is smaller than Y in

(a) usual stochastic order ($X \leq_{st} Y$) if and only if

$$\overline{F}(x) \leq \overline{G}(x),\ \forall x \in (-\infty, \infty).$$

(b) hazard rate order ($X \leq_{hr} Y$) if $r(x) \geq s(x), x \in \mathbb{R}$. Equivalently, if $\dfrac{\overline{G}(x)}{\overline{F}(x)}$ is increasing in x over the union of the supports of X and Y.

(c) reversed hazard rate order ($X \leq_{rh} Y$) if $\tilde{r}(x) \leq \tilde{s}(x), x \in \mathbb{R}$. Equivalently, if $\dfrac{G(x)}{F(x)}$ is increasing in x over the union of the supports of X and Y.

(d) likelihood ratio order ($X \leq_{lr} Y$) if $\dfrac{g(x)}{f(x)}$ is increasing in x over the union of the supports of X and Y.

(e) dispersive order ($X \leq_{disp} Y$) if

$$F^{-1}(\alpha_2) - F^{-1}(\alpha_1) \leq G^{-1}(\alpha_2) - G^{-1}(\alpha_1) \text{ whenever } 0 < \alpha_1 \leq \alpha_2 < 1.$$

Equivalently, ($X \leq_{disp} Y$) if and only if

$$G^{-1}(\alpha) - F^{-1}(\alpha) \text{ increases in } \alpha \in (0, 1).$$

(f) star order ($X \leq_{*} Y$) if $\dfrac{G^{-1}(t)}{F^{-1}(t)}$ increases in $t \in (0, 1)$.

Here,

$$X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{st} Y.$$

Similarly,

$$X \leq_{lr} Y \Rightarrow X \leq_{rh} Y \Rightarrow X \leq_{st} Y.$$

The detailed description about the inter-relationship between each of the stochastic orders can be seen from the book [26].

Definition 2.2. Majorization:

Let $\underline{a} = (a_1, \ldots, a_n)$ and $\underline{b} = (b_1, \ldots, b_n)$ be two real valued vectors then $\underline{a}$ is majorized by $\underline{b}$ ($\underline{a} \prec \underline{b}$) if $\sum_{i=1}^{n} a_{i:n} = \sum_{i=1}^{n} b_{i:n}$ and $\sum_{i=1}^{k} a_{i:n} \geq \sum_{i=1}^{k} b_{i:n}$ $\forall\ k = 1, \ldots, n-1$;

$\underline{a}$ is weakly submajorized by $\underline{b}$ ($\underline{a} \prec_w \underline{b}$) if $\sum_{i=1}^{k} a_{n-i+1:n} \leq \sum_{i=1}^{k} b_{n-i+1:n}$ $\forall\ k = 1, \ldots, n$; $\underline{a}$ is weakly supermajorized by $\underline{b}$ ($\underline{a} \prec^w \underline{b}$) if $\sum_{i=1}^{k} a_{i:n} \geq \sum_{i=1}^{k} b_{i:n}$ $\forall\ k = 1, \ldots, n$; where $a_{1:n} \leq \ldots \leq a_{n:n}$ ($b_{1:n} \leq \ldots \leq b_{n:n}$) is the increasing arrangement of $a_1, \ldots, a_n$ ($b_1, \ldots, b_n$).

For $\underline{a}$ and $\underline{b}$, we have $\underline{a} \prec^w \underline{b} \Leftarrow \underline{a} \prec \underline{b} \Rightarrow \underline{a} \prec_w \underline{b}$.

Definition 2.3. Schur-convexity (Schur-concavity): A real valued function ψ defined on a subset of $\mathbb{R}^n$ is *Schur-convex (Schur-concave)* if

$$\underline{a} \prec \underline{b} \Rightarrow \psi(\underline{a}) \ \leq (\geq)\ \psi(\underline{b}), \tag{2.1}$$

where $\underline{a} = (a_1, \ldots, a_n)$ and $\underline{b} = (b_1, \ldots, b_n)$ are two real valued vectors.

Throughout the paper, the notation $a \overset{\text{sgn}}{=} b$ has been used to represent sign of a is same as b. The results and lemmas that are used in obtaining the proofs are mentioned in the following subsection.

2.1. Useful results

Lemma 2.1 (Theorem 3.A.4, see [19]). *Let*

$$\Delta = (a_i - a_j)\left(\frac{\partial\psi(\underline{a})}{\partial a_i} - \frac{\partial\psi(\underline{a})}{\partial a_j}\right),$$

for an open interval $\mathbb{A} \subset \mathbb{R}$, a continuously differentiable function $\psi : \mathbb{A}^n \to \mathbb{R}$ is Schur-convex (Schur-concave) if and only if it is symmetric on $\mathbb{A}^n$ and for all $i \neq j$, $\Delta \geq (\leq) 0$.

Lemma 2.2 (Proposition 3.C.1, see [19]). *If $\mathbb{A} \subset \mathbb{R}$ is an interval and $h : \mathbb{A} \to \mathbb{R}$ is convex (concave), then $\psi(\underline{a}) = \sum_{i=1}^{n} h(a_i)$ is Schur-convex (Schur-concave) on $\mathbb{A}^n$, where $\underline{a} = (a_1, \ldots, a_n)$.*

Lemma 2.3 (Theorem 3.A.8, see [19]). *Let $S \subset \mathbb{R}^n$, a function $f : S \to \mathbb{R}$ satisfying*

$$\underline{a} \prec_w \underline{b} \ (\underline{a} \prec^w \underline{b}) \text{ on } S \ \Rightarrow f(\underline{a}) \leq f(\underline{b})$$

if and only if f is increasing (decreasing) and Schur-convex on S.

Lemma 2.4 (see [25]). *Let F_α, $\alpha \in \mathbb{R}$ be a class of distribution functions such that the support of F_α is given by some interval $(x_0, x_1) \subset \mathbb{R}^+$ and has a non-vanishing density $f_\alpha(x)$ on any subinterval of (x_0, x_1), where x_0 and x_1 are the left and right end points respectively. Then*

$$F_\alpha \leq_{disp} F_{\alpha^*}, \ \alpha, \alpha^* \in \mathbb{R}, \ \alpha \leq \alpha^*, \tag{2.2}$$

if and only if $\frac{F'_\alpha(x)}{f_\alpha(x)}$ is decreasing in x, where F'_α is the derivative of F_α with respect to α.
And

$$F_\alpha \leq_{*} F_{\alpha^*}, \ \alpha, \alpha^* \in \mathbb{R}, \ \alpha \leq \alpha^*, \tag{2.3}$$

if and only if $\frac{F'_\alpha(x)}{x f_\alpha(x)}$ is decreasing in x, where F'_α is the derivative of F_α with respect to α.
The first inequalities in (2.2) and (2.3) reverses as the quantity $\frac{F'_\alpha(x)}{f_\alpha(x)}$ and *$\frac{F'_\alpha(x)}{x f_\alpha(x)}$ respectively increases in x.*

3. Results and Discussion

3.1. Dispersive ordering results for unequal sample sizes

In this section we compare minimum and maximum order statistics arising from taking random variables having general proportional hazard rate and proportional reversed hazard rate distribution. As a corollary some results for multiple-outlier models has also been obtained. The multiple-outlier model has been explained in [4, 29] as an independent set of random variables $X_1, X_2, \ldots, X_n$, where $F_{X_i} = F_X$ for $i = 1, \ldots, p$ and $F_{X_i} = F_Y$ for $i = p+1, \ldots, n$, necessarily $1 \leq p < n$. When the value of $p = n-1$, this becomes a single-outlier model. Earlier many researchers have studied various results for the comparison of order statistics from multiple-outlier models. [2] considered the following model

$$(X_1, X_2, \ldots, X_n) \sim (\underbrace{(\overline{F}(x))^{\alpha_1}, \ldots, (\overline{F}(x))^{\alpha_1}}_{p}, \underbrace{(\overline{F}(x))^{\alpha_2}, \ldots, (\overline{F}(x))^{\alpha_2}}_{q})$$

and

$$(Y_1, Y_2, \ldots, Y_n) \sim (\underbrace{(\overline{F}(x))^{\alpha_1^*}, \ldots, (\overline{F}(x))^{\alpha_1^*}}_{p}, \underbrace{(\overline{F}(x))^{\alpha_2^*}, \ldots, (\overline{F}(x))^{\alpha_2^*}}_{q}).$$

They observed star and dispersive ordering for the kth order statistic by imposing majorization properties over the parameters. In the following paper they primarily discussed hazard rate ordering for exponentially distributed components and derived similar hazard rate ordering results for maximum order statistic with some additional conditions over the parameters. Under the same conditions [9] observed hazard rate ordering for second order statistic. Moreover they found hazard rate orderings when the number of components and number of outliers were different. Whereas [21] studied maximum order statistic for PHR model (survival function of X_i is $\overline{F}_{X_i}(x) = (\overline{F}(x))^{\alpha_i}$ for $i = 1, \ldots, n$) such that the distribution function of $\max_{i \in P} X_i$, $P \subset \{1, 2, \ldots, n\}$ is

$$F_{\max}(x) = Q_P(F(x))$$

where Q_P is a distortion function (continuous and increasing in [0,1], also Q(0)=0, Q(1)=1) and it depends on the underlying copula and the proportionality parameters. Few results were observed for different subsets of $\{1, 2, \ldots, n\}$. Further they have also discussed some results corresponding to multiple outlier model, PHR distributions using the aforementioned distortion function and results for the independent cases. We have considered various models in our study which includes model where the baseline distributions are same but the shape parameter varies, the baseline distributions are different and the shape parameters are also different. Several researchers have studied multiple-outlier models extensively as it helps in dealing with outliers. Recently, [31] studied some results where the n-component lifetimes of both the systems are dependent with multiple-outlier proportional hazard rates. [10] studied stochastic ordering for two types of models: Modified proportional hazard rate scale model and Modified proportional reversed hazard rate scale model. In our present study we first observe results for series systems where the component lifetimes are independent and follows different proportional hazard rates (the number of components in both the systems are not necessarily same) and the results for multiple-outlier models can be derived subsequently.

The following theorem has been observed for series systems with components following PHR family of distributions such that the baseline distribution for both the sets are different.

Theorem 3.1. *Let $X_1, X_2, \ldots, X_{n_1}$ be a set of n_1-independent random variables each belonging from a particular PHR family with parameters $(\alpha_1, \alpha_2, \ldots, \alpha_{n_1})$. We assume that $X_i \sim \overline{F}_i(x) = (\overline{F}_0(x))^{\alpha_i}$ for $i = 1, 2, \ldots, n_1$. Also, let $Y_1, Y_2, \ldots, Y_{n_2}$ be another set of n_2-independent random variables each following PHR family of distributions with a different distribution function and the parameter set is $(\beta_1, \beta_2 \ldots, \beta_{n_2})$. Let $Y_i \sim \overline{G}_i(x) = (\overline{G}_0(x))^{\beta_i}$ for $i = 1, 2, \ldots, n_2$. Under the assumption that $\sum_{i=1}^{n_2} \beta_i \geq \sum_{i=1}^{n_1} \alpha_i$, the baseline distribution function F_0 is DFR, and $G_0 \leq_{hr} F_0$ then $Y_{1:n_2} \leq_{disp} X_{1:n_1}$.*

Proof: The distribution function of $X_{1:n_1}$ and $Y_{1:n_2}$ are

$$\overline{F}_{1:n_1}(x) = \left(\overline{F}_0(x)\right)^{\sum_{i=1}^{n_1} \alpha_i}, \tag{3.1}$$

and,

$$\overline{G}_{1:n_2}(x) = \left(\overline{G}_0(x)\right)^{\sum_{i=1}^{n_2} \beta_i} \tag{3.2}$$

respectively. For simplicity we replace $\sum_{i=1}^{n_1} \alpha_i$ by α and $\sum_{i=1}^{n_2} \beta_i$ by β. Let

$$\begin{aligned} \psi_1(y) &= F_{1:n_1}^{-1}(y) - G_{1:n_2}^{-1}(y) \\ &= \overline{F}_0^{-1}\left((1-y)^{1/\alpha}\right) - \overline{G}_0^{-1}\left((1-y)^{1/\beta}\right) \end{aligned}$$

We are required to prove $Y_{1:n_2} \leq_{disp} X_{1:n_1}$, i.e., $\psi_1(y)$ is increasing in $y \in (0, 1)$. Hence $Y_{1:n_2} \leq_{disp} X_{1:n_1}$ if and only if $\phi_1(t) = \overline{F}_0^{-1}(t) - \overline{G}_0^{-1}\left(t^{\frac{\alpha}{\beta}}\right)$

is decreasing in $t \in (0,1)$, where $t = (1-y)^{1/\alpha}$. Note that

$$\phi_1'(t) = -\frac{1}{f_0(\overline{F}_0^{-1}(t))} + \frac{\alpha}{\beta}\frac{t^{\frac{\alpha}{\beta}-1}}{g_0\left(\overline{G}_0^{-1}\left(t^{\frac{\alpha}{\beta}}\right)\right)}. \tag{3.3}$$

We need to show that $\phi_1'(t) \leq 0$, i.e.,

$$\frac{t}{f_0(\overline{F}_0^{-1}(t))} \geq \frac{\alpha}{\beta}\frac{t^{\frac{\alpha}{\beta}}}{g_0\left(\overline{G}_0^{-1}\left(t^{\frac{\alpha}{\beta}}\right)\right)}. \tag{3.4}$$

Let $\overline{F}_0^{-1}(t) = z_1$ and $\overline{G}_0^{-1}\left(t^{\frac{\alpha}{\beta}}\right) = z_2$,

$$\begin{aligned} &\frac{\overline{F}_0(z_1)}{f_0(z_1)} \geq \frac{\alpha}{\beta}\frac{\overline{G}_0(z_2)}{g_0(z_2)} \\ &\Rightarrow s_0(z_2)\frac{\beta}{\alpha} \geq r_0(z_1), \end{aligned} \tag{3.5}$$

where $r_0(z_1) = \dfrac{f_0(z_1)}{\overline{F}_0(z_1)}$ and $s_0(z_2) = \dfrac{g_0(z_2)}{\overline{G}_0(z_2)}$. Under the hypothesis of the theorem

$$\begin{gathered} \beta \geq \alpha \\ \Rightarrow t = \overline{F}_0(z_1) \leq t^{\left(\frac{\alpha}{\beta}\right)} = \overline{G}_0(z_2). \end{gathered}$$

and $G_0 \leq_{hr} F_0$ implies that $z_2 \leq z_1$ ($G_0 \leq_{st} F_0$ follows from $G_0 \leq_{hr} F_0$ subsequently we can derive that $\overline{G}_0(z_2) \geq \overline{F}_0(z_1) \geq \overline{G}_0(z_1)$. Finally the implication is possible as $\overline{G}_0$ is a decreasing function) and $s_0(z_2) \geq r_0(z_2)$. Also F_0 is DFR then, $z_2 \leq z_1 \Rightarrow r_0(z_2) \geq r_0(z_1)$. Combining all these we find that (3.5) holds true. Hence the result.

□

The above result provides a general outlook over the PHR distributions. Apart from the fact that the component lifetimes are independent, the result can be compared with Theorem 3.11 from [31]. Here the baseline distributions are different, also the number of components are not same. The theorem holds true when we encounter a multiple-outlier model. An example has been provided here that satisfies the condition given in the above theorem.

Example 3.1. Let (X_1, X_2, X_3) and (Y_1, Y_2, Y_3) be independent Transformed Pareto distributed random variables. The survival function of X_i is $\overline{F}_{k_i}(t)$ and corresponding to Y_i is $\overline{G}_{k_i^*}(t)$ for $i = 1, 2, 3$. Consider $k_1 = 1.7, k_2 = 2, k_3 = 0.9$ and $k_1^* = 1, k_2^* = 3, k_3^* = 2.3$, here $\sum_{i=1}^{3} k_i = 4.6$ and $\sum_{i=1}^{3} k_i^* = 6.3$.

Let us consider $\overline{F}_{k_i}(t) = (\overline{F}_0(t))^{k_i}$, where $\overline{F}_0(t) = \dfrac{1}{(1+t)^2}, t > 0$; $\overline{G}_{k_i^*}(t) = (\overline{G}_0(t))^{k_i^*}$, where $\overline{G}_0(t) = \dfrac{1}{(1+t)^3}, t > 0$.

Here, F_0 is DFR and the ratio $\dfrac{\overline{F}_0(t)}{\overline{G}_0(t)} = 1 + t$ is increasing in t $\forall t > 0$. Thus, $F_0 \geq_{hr} G_0$.

We observe that $\overline{F}_0^{-1}(u) = \left(\dfrac{1}{u}\right)^{\frac{1}{2}} - 1, 0 < u < 1$ and $\overline{G}_0^{-1}(u) = \left(\dfrac{1}{u}\right)^{\frac{1}{3}} - 1, 0 < u < 1$. Hence, as mentioned in Theorem 3.1, the expression

$$\psi_1(y) = \overline{F}_0^{-1}((1-y)^{1/\alpha}) - \overline{G}_0^{-1}((1-y)^{1/\beta})$$

$$= \frac{1}{(1-y)^{1/2\alpha}} - \frac{1}{(1-y)^{1/3\beta}}, \alpha = \sum_{i=1}^{3} k_i \text{ and } \beta = \sum_{i=1}^{3} k_i^*. \tag{3.6}$$

Plotting (3.6) with respect to y, for $0 < y < 1$, we observe that $\psi_1(y)$ is increasing in y, i.e. the theorem holds true in this case.

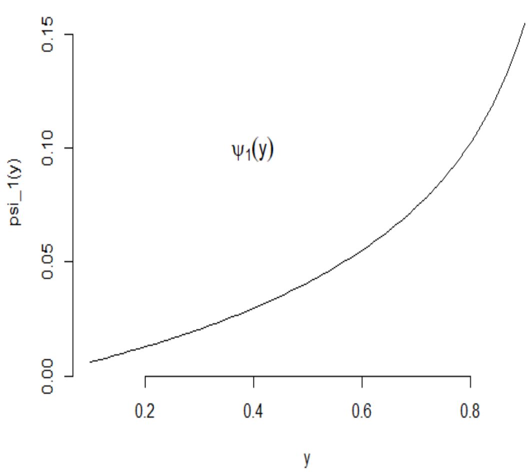

Figure 1: $\psi_1(y)$ is increasing for $0 < y < 1$.

The conditions "DFR" and hr order necessary in Theorem 3.1. Let us consider

$$\overline{F}_{1:n_1}(x) = \left(\overline{F}_0(x)\right)^{\sum_{i=1}^{n_1}\alpha_i}$$

and

$$\overline{G}_{1:n_2}(x) = \left(\overline{G}_0(x)\right)^{\sum_{i=1}^{n_2}\beta_i}$$

where $\overline{F}_0(x) = \exp(-x^2), x > 0$ and $\overline{G}_0(x) = \exp(-x), x > 0$; $\sum_{i=1}^{n_1}\alpha_i = 2$ and $\sum_{i=1}^{n_2}\beta_i = 2.5$.

Here $\frac{\overline{F}_0(x)}{\overline{G}_0(x)}$ is non-monotone and $\psi_1(y) = \overline{F}_0^{-1}\left((1-y)^{1/\sum_{i=1}^{n_1}\alpha_i}\right) - \overline{G}_0^{-1}\left((1-y)^{1/\sum_{i=1}^{n_2}\beta_i}\right)$ is also non-monotone. The condition for DFR and hr order are not satisfied here (see Figure 2 and Figure 3), also the dispersive order does not hold in this situation even though the conditions for the parameters are satisfied. The plots are shown below:

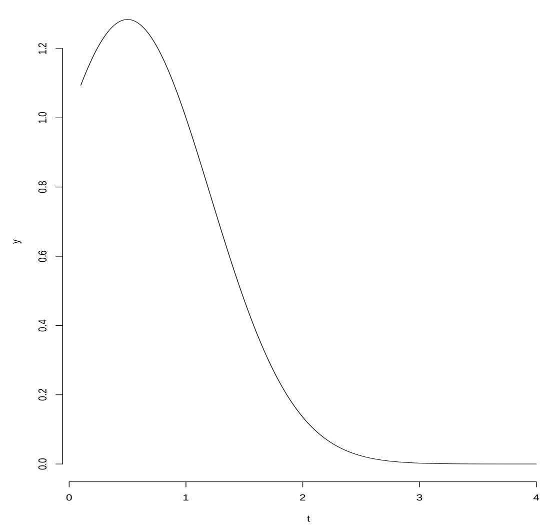

Figure 2: $\frac{\overline{F}_0(x)}{\overline{G}_0(x)}$ is non-monotone.

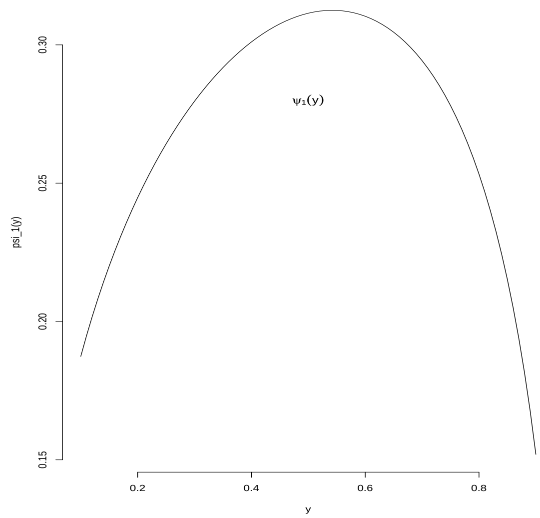

Figure 3: $\psi_1(y)$ is non-monotone.

Thus the conditions mentioned in Theorem 3.1 are necessary. Here we discuss the following result for a parallel system with the PRHR distributed components.

Theorem 3.2. *Let $X_1, X_2, \ldots, X_{n_1}$ be a n_1-independent set of random variables each belonging from PRHR family of distributions with parameters $(\alpha_1, \alpha_2 \ldots, \alpha_{n_1})$, such that $X_i \sim F_i(x) = (F_0(x))^{\alpha_i}$ for $i = 1, 2, \ldots, n_1$. Also $Y_1, Y_2, \ldots, Y_{n_2}$ be another set of n_2-independent random variables each following PRHR family of distributions with a different distribution function and the parameter set is $(\beta_1, \beta_2, \ldots, \beta_{n_2})$. Let $Y_i \sim G_i(x) = (G_0(x))^{\beta_i}$ for $i = 1, 2, \ldots, n_2$. Then $Y_{n_2:n_2} \leq_{disp} X_{n_1:n_1}$ if the baseline distribution F_0 follows IRHR model, $\sum_{i=1}^{n_2} \beta_i \geq \sum_{i=1}^{n_1} \alpha_i$ and $F_0 \leq_{rh} G_0$.*

Proof: The distribution function of $X_{n_1:n_1}$ and $Y_{n_2:n_2}$ are

$$F_{n_1:n_1}(x) = [F_0(x)]^{\sum_{i=1}^{n_1} \alpha_i}, \text{ and,}$$
$$G_{n_2:n_2}(x) = [G_0(x)]^{\sum_{i=1}^{n_2} \beta_i}$$

respectively. Similar to the previous theorem, we take $\sum_{i=1}^{n_1} \alpha_i = \alpha$ and $\sum_{i=1}^{n_2} \beta_i = \beta$.

Let

$$\begin{aligned}\psi_2(y) &= F^{-1}_{n_1:n_1}(y) - G^{-1}_{n_2:n_2}(y)\\ &= F_0^{-1}(y^{1/\alpha}) - G_0^{-1}(y^{1/\beta}).\end{aligned}$$

We are required to prove that $Y_{n_2:n_2} \leq_{disp} X_{n_1:n_1}$, i.e., $\psi_2(y)$ is increasing in $y \in (0,1)$.

Hence $Y_{n_2:n_2} \leq_{disp} X_{n_1:n_1}$ if and only if $\phi_2(t) = F_0^{-1}(t) - G_0^{-1}\left(t^{\frac{\alpha}{\beta}}\right)$ is increasing in $t \in (0,1)$, where $t = y^{1/\alpha}$. Note that

$$\phi_2'(t) = \frac{1}{f_0(F_0^{-1}(t))} - \frac{\alpha}{\beta}\frac{t^{\frac{\alpha}{\beta}-1}}{g_0\left(G_0^{-1}\left(t^{\frac{\alpha}{\beta}}\right)\right)}.$$

We need to show that $\phi_2'(t) \geq 0$, i.e.,

$$\frac{t}{f_0(F_0^{-1}(t))} \geq \frac{\alpha}{\beta}\frac{t^{\frac{\alpha}{\beta}}}{g_0\left(G_0^{-1}\left(t^{\frac{\alpha}{\beta}}\right)\right)}. \tag{3.7}$$

Put $F_0^{-1}(t) = z_1$ and $G_0^{-1}\left(t^{\frac{\alpha}{\beta}}\right) = z_2$. From (3.7) it is sufficient to show

$$\begin{aligned}&\frac{F_0(z_1)}{f_0(z_1)} \geq \frac{\alpha}{\beta}\frac{G_0(z_2)}{g_0(z_2)}\\ &\Leftrightarrow \tilde{s}_0(z_2)\frac{\beta}{\alpha} \geq \tilde{r}_0(z_1).\end{aligned} \tag{3.8}$$

As

$$\begin{aligned}&\frac{\beta}{\alpha} \geq 1\\ &\Rightarrow t = F_0(z_1) \leq t^{\frac{\alpha}{\beta}} = G_o(z_2).\end{aligned}$$

Since $F_0 \leq_{rh} G_0$ implies $F_0 \leq_{st} G_0$, hence $G_0(z_1) \leq F_0(z_1) \leq G_0(z_2)$ i.e., $z_1 \leq z_2$. Again F_0 follows increasing reversed hazard rate (IRHR) model hence $z_1 \leq z_2 \Rightarrow \tilde{r}_0(z_1) \leq \tilde{r}_0(z_2)$. Lastly, $F_0 \leq_{rh} G_0 \Rightarrow \tilde{r}_0(x) \leq \tilde{s}_0(x)$ for all x, thus $\tilde{r}_0(z_2) \leq \tilde{s}_0(z_2)$. Combining these inequalities we obtain the required result. □

Example 3.2. We have observed an example of IRHR distribution from Example 3.4 of Oliveira and Torrado(2015). Let X be a random variable following Truncated Normal(μ, σ^2) distribution with distribution function as

$$F(x) = \frac{\Phi\left(\frac{x-\mu}{\sigma}\right)}{\Phi\left(\frac{-\mu}{\sigma}\right)}, x \in (-\infty, 0]$$

Let us consider a set of 3 independent random variables X_1, X_2, X_3 such that $X_i \sim F_i(x) = [F_0(x)]^{\alpha_i}, i = 1, 2, 3$ where F_0 corresponds to Truncated Normal$(0, 4)$ i.e., $F_0(x) = \frac{\Phi\left(\frac{x}{2}\right)}{0.5}$. Let us consider another set of 2 independent random variables Y_1, Y_2 such that $Y_i \sim G_i(x) = [G_0(x)]^{\beta_i}, i = 1, 2, 3$ where G_0 corresponds to Truncated Normal$(0, 1)$ i.e., $G_0(x) = \frac{\Phi(x)}{0.5}$. We can observe that the reversed hazard rate function of the baseline distributions F_0 and G_0 are

$$\tilde{h}_{F_0}(x) = \frac{1}{2}\frac{\phi(x)}{\Phi(x)},$$
$$\tilde{h}_{G_0}(x) = \frac{\phi(x)}{\Phi(x)}.$$

Thus, $F_0 \leq_{rh} G_0$. The distribution function of $X_{3:3}$ and $Y_{2:2}$ are

$$F_{3:3}(x) = \left(\frac{\Phi\left(\frac{x}{2}\right)}{0.5}\right)^{\sum_{i=1}^{3}\alpha_i} \text{ and } G_{2:2}(x) = \left(\frac{\Phi(x)}{0.5}\right)^{\sum_{i=1}^{2}\beta_i}.$$

Taking $\sum_{i=1}^{3}\alpha_i = \alpha = 2$ and $\sum_{i=1}^{2}\beta_i = \beta = 3$. Here all the conditions of Theorem 3.2 are satisfied, further we observe that

$$\begin{aligned}\psi_2(y) &= F_{3:3}^{-1}(y) - G_{2:2}^{-1}(y) \\ &= F_0^{-1}(y^{1/\alpha}) - G_0^{-1}(y^{1/\beta}) \\ &= 2\Phi^{-1}(0.5y^{1/2}) - \Phi^{-1}(0.5y^{1/3})\end{aligned}$$

is increasing in $y \in (0, 1)$.

As a corollary, we can obtain a result for multiple-outlier model from PRHR distributions. The condition IRHR is necessary here, we can understand this through an example. Let (X_1, X_2, X_3) and (Y_1, Y_2, Y_3) be independent random variables such that the distribution function of X_i is $(F_0(x))^{\alpha_i}$, where $F_0(x) = 1 - e^{-3x}, x > 0$ and distribution function of Y_i is $(G_0(x))^{\beta_i}$, where $G_0(x) = \frac{1}{(1+x)^2}, x > 0$. $F_0(x)$ is DRHR.

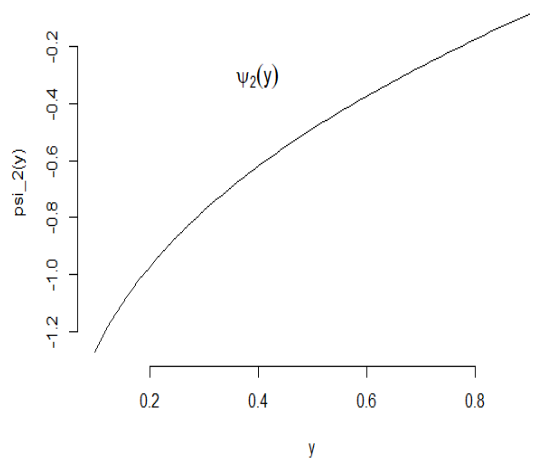

Figure 4: $\psi_2(y)$ is increasing for $0 < y < 1$.

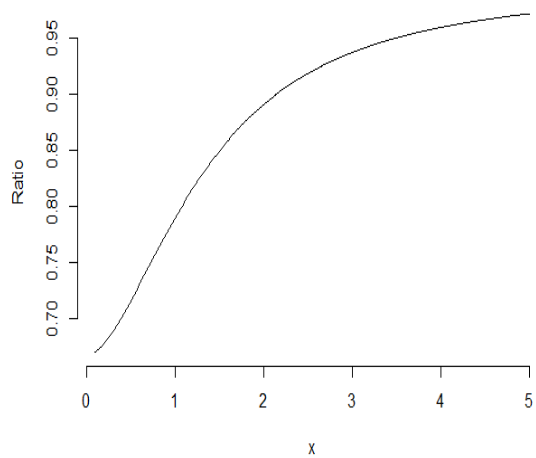

Figure 5: $\dfrac{G_0(x)}{F_0(x)}$ is increasing for $x > 0$.

Figure 5 represents that $F_0 \leq_{rh} G_0$. Let $\alpha = \sum_{i=1}^{n} \alpha_i$ and $\beta = \sum_{i=1}^{n} \beta_i$, then

$$\begin{aligned}\psi_2(y) &= F_0^{-1}(y^{1/\alpha}) - G_0^{-1}(y^{1/\beta})\\ &= 1 + \frac{1}{3} ln(1 - y^{1/4.6}) - \frac{1}{(1 - y^{1/6.3})^{1/2}}, 0 < y < 1\end{aligned}$$

Thus from figure 6, we observe that $Y_{3:3} \geq_{disp} X_{3:3}$, i.e. the inequality reverses.

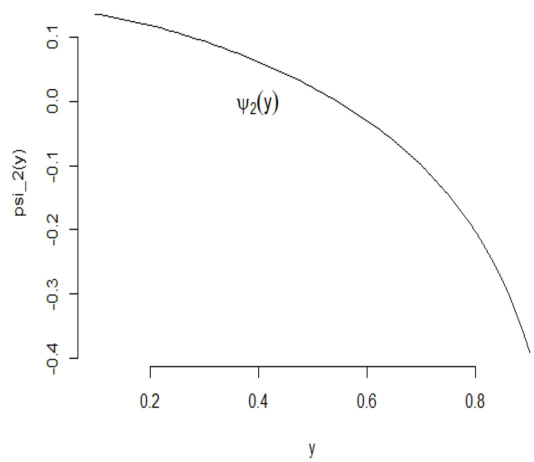

Figure 6: $\psi_2(y)$ is decreasing for $0 < y < 1$.

In the next theorem, we provide a result for series systems with unequal number of components following PHR models with different baseline distributions. It can be noted that there is some relationship between hazard rate ordering and dispersive ordering. If X and Y are two non negative random variables then

1. If $X \leq_{\text{hr}} Y$ and X or Y is DFR, then $X \leq_{\text{disp}} Y$.
2. If $X \leq_{\text{disp}} Y$ and X or Y is IFR, then $X \leq_{\text{hr}} Y$.

from theorem 3.B.20 of [26] and Corollary 4.3 of [7].

Theorem 3.3. *Consider a system of n_1 components, where the lifetime of each component is represented by the random variable $X_1, X_2, \ldots, X_{n_1}$ respectively such that each of $X_1, \ldots, X_{p_1}$ has survival function $[\overline{F}(x)]^{\alpha_i}$, $i = 1, 2, \ldots, p_1$ and $X_{p_1+1}, \ldots, X_{n_1}$ has survival function $[\overline{G}(x)]^{\alpha_i}$, $i = p_1+1, p_1+2, \ldots, n_1$. Similarly another system with n_2 components is considered where the components $Y_1, \ldots, Y_{p_2}$ has survival function as $[\overline{F}(x)]^{\beta_i}$, $i = 1, 2, \ldots, p_2$ whereas the components $Y_{p_2+1}, \ldots, Y_{n_2}$ has survival function $[\overline{G}(x)]^{\beta_i}$, $i = p_2+1, p_2+2, \ldots, n_2$. Then $X_{1:n_1} \leq_{hr} Y_{1:n_2}$ whenever $\sum_{i=1}^{p_1} \alpha_i > \sum_{i=1}^{p_2} \beta_i$ and $\sum_{i=p_1+1}^{n_1} \alpha_i > \sum_{i=p_2+1}^{n_2} \beta_i$.*

Proof: The survival function of $X_{1:n_1}$ is

$$\overline{F}_{1:n_1}(x) = [\overline{F}(x)]^{\sum_{i=1}^{p_1} \alpha_i} [\overline{G}(x)]^{\sum_{i=p_1+1}^{n_1} \alpha_i},$$

and the survival function of $Y_{1:n_2}$ is

$$\overline{G}_{1:n_2}(x) = [\overline{F}(x)]^{\sum_{i=1}^{p_2} \beta_i} [\overline{G}(x)]^{\sum_{i=p_2+1}^{n_2} \beta_i}.$$

Consider the ratio

$$\frac{\overline{F}_{1:n_1}(x)}{\overline{G}_{1:n_2}(x)} = [\overline{F}(x)]^{\left(\sum_{i=1}^{p_1} \alpha_i - \sum_{i=1}^{p_2} \beta_i\right)} [\overline{G}(x)]^{\left(\sum_{i=p_1+1}^{n_1} \alpha_i - \sum_{i=p_2+1}^{n_2} \beta_i\right)} \tag{3.9}$$

Differentiating (3.9) with respect to x,

$$\frac{d}{dx}\left(\frac{\overline{F}_{1:n_1}(x)}{\overline{G}_{1:n_2}(x)}\right) = -\frac{\overline{F}_{1:n_1}(x)}{\overline{G}_{1:n_2}(x)} \left(\left(\sum_{i=1}^{p_1} \alpha_i - \sum_{i=1}^{p_2} \beta_i \right) \frac{f(x)}{\overline{F}(x)} + \left(\sum_{i=p_1+1}^{n_1} \alpha_i - \sum_{i=p_2+1}^{n_2} \beta_i \right) \frac{g(x)}{\overline{G}(x)} \right)$$
$$< 0,$$

whenever $\sum_{i=1}^{p_1} \alpha_i > \sum_{i=1}^{p_2} \beta_i$ and $\sum_{i=p_1+1}^{n_1} \alpha_i > \sum_{i=p_2+1}^{n_2} \beta_i$. Hence the result follows. $\square$

When the random variables $X_{1:n_1}$ or $Y_{1:n_2}$ is DFR then $X_{1:n_1} \leq_{disp} Y_{1:n_2}$.

If we consider a similar problem wherein the random variables follows a PRHR distribution, we arrive at the following theorem.

Theorem 3.4. *Consider an independent set of n_1 random variables $X_1, X_2, \ldots, X_{p_1}, X_{p_1+1}, \ldots, X_{n_1}$ such that the distribution function of X_i, $F_{X_i}(x) = [F(x)]^{\alpha_i}$ for $i = 1, 2, \ldots, p_1$ and $F_{X_i}(x) = [G(x)]^{\alpha_i}$ for $i = p_1 + 1, \ldots, n_1$. Another set of n_2 independent components $Y_1, Y_2, \ldots, Y_{p_2}, Y_{p_2+1}, \ldots, Y_{n_2}$ are such that the distribution function of Y_i, $F_{Y_i}(x) = [F(x)]^{\beta_i}$, $i = 1, 2, \ldots, p_2$ and $F_{Y_i}(x) = [G(x)]^{\beta_i}$ for $i = p_2 + 1, \ldots, n_2$. Then $X_{n_1:n_1} \geq_{rh} Y_{n_2:n_2}$ whenever $\sum_{i=1}^{p_1} \alpha_i > \sum_{i=1}^{p_2} \beta_i$ and $\sum_{i=p_1+1}^{n_1} \alpha_i > \sum_{i=p_2+1}^{n_2} \beta_i$.*

Proof: The distribution functions of $X_{n_1:n_1}$ and $Y_{n_2:n_2}$ are

$$F_{n_1:n_1}(x) = [F(x)]^{\sum_{i=1}^{p_1} \alpha_i} [G(x)]^{\sum_{i=p_1+1}^{n_1} \alpha_i},$$

$$G_{n_2:n_2}(x) = [F(x)]^{\sum_{i=1}^{p_2} \beta_i} [G(x)]^{\sum_{i=p_2+1}^{n_2} \beta_i}$$

respectively. Differentiating the ratio $\dfrac{F_{n_1:n_1}(x)}{G_{n_2:n_2}(x)}$ with respect to x, we observe,

$$\frac{d}{dx}\left(\frac{F_{n_1:n_1}(x)}{G_{n_2:n_2}(x)}\right) = \frac{F_{n_1:n_1}(x)}{G_{n_2:n_2}(x)}\left(\left(\sum_{i=1}^{p_1}\alpha_i - \sum_{i=1}^{p_2}\beta_i\right)\frac{f(x)}{F(x)} + \left(\sum_{i=p_1+1}^{n_1}\alpha_i - \sum_{i=p_2+1}^{n_2}\beta_i\right)\frac{g(x)}{G(x)}\right)$$
$$> 0,$$

whenever $\sum_{i=1}^{p_1}\alpha_i > \sum_{i=1}^{p_2}\beta_i$ and $\sum_{i=p_1+1}^{n_1}\alpha_i > \sum_{i=p_2+1}^{n_2}\beta_i$ and the result follows. □

The above theorem deals with a different set of parameters and baseline distributions as compared to that of theorem 3.7 from [31] where the component lifetimes are dependent but the parameters are restricted and the baseline distributions are all same. If $X_{n_1:n_1}$ or $Y_{n_2:n_2}$ is IRFR then from the above theorem we can observe that $X_{n_1:n_1} \leq_{disp} Y_{n_2:n_2}$.
We can observe the inter-relationship between reversed hazard rate ordering and dispersive ordering, as mentioned in Corollary 4.4 of [7]. For two random variables X and Y,

1. If $X \leq_{\text{rh}} Y$ and X or Y is IRFR, then $Y \leq_{\text{disp}} X$.
2. If $X \leq_{\text{disp}} Y$ and X or Y is DRFR, then $Y \leq_{\text{rh}} X$.

It is interesting to note that in [21], Proposition 4.4 can be realized from Theorem 3.3 and 3.4. Such as, if $p_1 = n$ and $p_2 = n$ in theorem 3.3 then

$$X_{1:n} \leq_{hr} Y_{1:n} \text{ whenever } \sum_{i=1}^{n}\alpha_i > \sum_{i=1}^{n}\beta_i,$$

and if $p_1 = n$ and $p_2 = n$ in theorem 3.4 then

$$X_{n:n} \geq_{rh} Y_{n:n} \text{ whenever } \sum_{i=1}^{n}\alpha_i > \sum_{i=1}^{n}\beta_i.$$

Next we consider a reversed hazard rate ordering result for the parallel system having Pareto distributed components such that the sample sizes are equal. Pareto distribution is DRFR hence we have obtained a reversed hazard rate ordering for $X_{n:n}$ and $Y_{n:n}$.

Theorem 3.5. *Let $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ be two sets of n-independent Pareto distributed random variables such that the survival function of X_i is $\overline{F}_i(x) = \left(1+\frac{x}{\theta}\right)^{-\alpha_i}$, $x > 0$, $\theta > 0$, $\alpha_i > 0$ and that of Y_i is $\overline{G}_i(x) = \left(1+\frac{x}{\theta}\right)^{-\alpha_i^*}$, $x > 0$, $\theta > 0$, $\alpha_i^* > 0$. Let $\underline{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\underline{\alpha}^* = (\alpha_1^*, \alpha_2^*, \ldots, \alpha_n^*)$, then $\underline{\alpha} \prec^w \underline{\alpha}^* \Rightarrow X_{n:n} \leq_{rh} Y_{n:n}$.*

Proof: The distribution function of $X_{n:n}$ is

$$F_{X_{n:n}}(x) = \prod_{i=1}^{n} \left[1 - \left(1 + \frac{x}{\theta}\right)^{-\alpha_i}\right], \tag{3.10}$$

and the corresponding reversed hazard rate function is

$$\tilde{r}_{X_{n:n}}(x) = \frac{1}{x+\theta} \sum_{i=1}^{n} g(\alpha_i), \tag{3.11}$$

where $g(\alpha) = \dfrac{\alpha}{\left(\frac{x}{\theta}+1\right)^{\alpha} - 1}$. Let $u = \left(\frac{x}{\theta}+1\right)^{\alpha}$ and $u > 1$ such that $g(\alpha) = \dfrac{\alpha}{u^\alpha - 1}$. Now,

$$g'(\alpha) = \frac{u^\alpha(1 - \alpha \ln u) - 1}{(u^\alpha - 1)^2} \tag{3.12}$$

and

$$g''(\alpha) = \frac{u^\alpha \ln u((u^\alpha \ln u + \ln u)\alpha - 2u^\alpha + 2)}{(u^\alpha - 1)^3}. \tag{3.13}$$

$g''(\alpha) \overset{sgn}{=} u^\alpha (\ln u) \phi(u)$, where $\phi(u) = (u^\alpha \ln u + \ln u)\alpha - 2u^\alpha + 2$, $\phi(1) = 0$ Also,

$$\begin{aligned}\phi'(u) &= \alpha^2 u^{\alpha-1} \ln u + \frac{\alpha}{u} - \alpha u^{\alpha-1} \\ &= \frac{\alpha}{u}\phi_1(u),\end{aligned}$$

such that $\phi_1(u) = \alpha u^\alpha \ln u + 1 - u^\alpha$ and $\phi_1(1) = 0$. And

$$\begin{aligned}\phi_1'(u) &= \alpha^2 u^{\alpha-1} \ln u \\ &> 0.\end{aligned}$$

Hence it is observed that $g''(\alpha) > 0$ for $x > 0$ $(u > 1)$, i.e., $g(\alpha)$ is convex in α. Hence, using Lemma 2.2 we obtain, $\tilde{r}_{X_{n:n}}(x)$ is Schur convex w.r.t $\underline{\alpha}$. Moreover,

$$\begin{aligned}g'(\alpha) &\overset{sgn}{=} u^\alpha(1 - \alpha \ln u) - 1, \ u > 1 \\ &= h(u) \text{ say,}\end{aligned}$$

then $h'(u) = -\alpha^2 u^{\alpha-1} \ln u$. Also $h(1) = 0$, then $g'(\alpha) < 0$ for $x > 0$ $(u > 1)$. Thus $\tilde{r}_{X_{n:n}}(x)$ is decreasing in $\underline{\alpha}$ and Schur convex w.r.t $\underline{\alpha}$. Using Lemma 2.3, we infer that $\underline{\alpha} \prec^w \underline{\alpha}^* \Rightarrow \tilde{r}_{X_{n:n}}(x) \leq \tilde{r}_{Y_{n:n}}(x)$. Hence the result follows. □

3.2. Star ordering result for unequal sample sizes

In this section we present a comparison between two systems based on star ordering. Consider a series system with components following PHR model and have unequal sample sizes.

Theorem 3.6. *Let $X_1, X_2, \ldots, X_{n_1}$ be a n_1-independent set of non-negative random variables such that $X_i \sim [\overline{F}(x)]^{\alpha_i}$ for $i = 1, 2, \ldots, n_1$ and $Y_1, Y_2, \ldots, Y_{n_2}$ be another n_2-independent set of non-negative random variables such that $Y_i \sim [\overline{F}(x)]^{\beta_i}$ for $i = 1, 2, \ldots, n_2$, where n_1 and n_2 may or may not be the same. Then*

$$\sum_{i=1}^{n_1} \alpha_i \leq \sum_{i=1}^{n_2} \beta_i \Rightarrow X_{1:n_1} \geq_* Y_{1:n_2}, \text{ whenever } xr(x) \text{ is decreasing.}$$

Proof: The survival function of $X_{1:n_1}$ is

$$\overline{F}_{1:n_1}(x) = [\overline{F}(x)]^{\sum_{i=1}^{n_1} \alpha_i}. \tag{3.14}$$

Let $\sum_{i=1}^{n_1} \alpha_i = \alpha$, then $\overline{F}_{1:n_1}(x) = [\overline{F}(x)]^{\alpha} = \overline{F}_{\alpha}(x)$ (say).
The corresponding probability density function is

$$\begin{aligned} f_{X_{1:n_1}}(x) &= \alpha f(x)[\overline{F}(x)]^{\alpha-1} \\ &= f_{\alpha}(x). \end{aligned}$$

Note that the ratio

$$\frac{F'_{\alpha}(x)}{f_{\alpha}(x)} = -\frac{1}{\alpha} \frac{\ln \overline{F}(x)}{r(x)}, \tag{3.15}$$

where $F_{\alpha}(x) = 1 - [\overline{F}(x)]^{\alpha}$ and $F'_{\alpha}(x) = \frac{d}{d\alpha} F_{\alpha}(x)$.
The theorem follows by differentiating the ratio $\dfrac{F'_{\alpha}(x)}{x f_{\alpha}(x)}$ with respect to x.
Note that

$$\begin{aligned} \frac{d}{dx}\left(\frac{F'_{\alpha}(x)}{x f_{\alpha}(x)}\right) &= \frac{1}{\alpha}\left(\frac{x(r(x))^2 + (xr'(x) + r(x)) \ln \overline{F}(x)}{(xr(x))^2}\right) \\ &> 0, \end{aligned}$$

whenever $xr(x)$ is decreasing in x. Now using Lemma 2.4, we obtain $X_{1:n_1} \geq_* Y_{1:n_2}$ whenever $\sum_{i=1}^{n_1} \alpha_i \leq \sum_{i=1}^{n_2} \beta_i$. $\square$

We observe here that the hazard rate functions of $X_{1:n_1}$ and $Y_{1:n_2}$ are $r_{X_{1:n_1}}(x) = \sum_{i=1}^{n_1} \alpha_i r(x)$ and $r_{Y_{1:n_2}}(x) = \sum_{i=1}^{n_2} \beta_i r(x)$ respectively, where $r(x)$ is the hazard rate function of baseline distribution $F(x)$. Then

$$r_{X_{1:n_1}}(x) \leq r_{Y_{1:n_2}}(x) \text{ whenever } \sum_{i=1}^{n_1} \alpha_i \leq \sum_{i=1}^{n_2} \beta_i.$$

The class of decreasing proportional hazard rate has been studied by [22] where several examples are also provided. The above result is applicable for multiple-outlier models. Moreover this theorem can be considered as a more general form of theorem 3.9 from [31]. Here the parameters are all different and only a simple inequality exists between them.

3.3. Dependent model

In this section we have considered a dependent set of random variables instead of independent random variables as discussed in the earlier sections. [12] studied scaled samples with proportional hazard and proportional reversed hazard rate models whereas [30] studied stochastic ordering results of Resilience-scaled(RS) models ($X \sim RS(\alpha, \lambda)$ if $F_X(x) = F^{\alpha}(\lambda x)$, $\alpha > 0$, $\lambda > 0$) for series and parallel systems with dependent set of components. Moreover, [13] and [14] have discussed about the stochastic ordering between two systems where the component lifetimes are independent and each belongs from a location-scale family, necessarily with the same baseline distribution function. [18] discussed stochastic ordering results for series system from dependent and independent random variables following location-scale family of distributions. Thus it might be interesting to study the conditions under which a series (parallel) system can be compared with another series (parallel) system, where all the component lifetimes are dependent and each belonging from location family of distributions, the baseline distribution functions for both the sets are also different.
Hence we shall observe few definitions required especially to study the dependent models.

Definition 3.1. **Survival copula:** Let $(X_1, \ldots, X_n)$ be a n-dimensional random vector defined on a probability space $(\Omega, \mathbb{F}, \mathbb{P})$, the multivariate survival function is defined as

$$\begin{aligned}\overline{F}(x_1, \ldots, x_n) &= P[X_1 > x_1, \ldots, X_n > x_n] \\ &= \tilde{C}(\overline{F}_1(x_1), \ldots, \overline{F}_n(x_n)),\ x_1, \ldots, x_n \in \mathbb{R},\end{aligned}$$

where $\tilde{C}$ is the n-dimensional survival copula of the random vector $(X_1, \ldots, X_n)$. $\tilde{C}$ is a continuous function defined over the n-dimensional space as $\tilde{C} : [0,1]^n \mapsto [0,1]$, to develop multivariate survival functions from the marginal survival functions.

Archimedean copula is a very widely used class of survival copula because of its analytical tractability.

Definition 3.2. **Archimedean copula**

A n-dimensional *Archimedean copula* $\tilde{C} : [0,1]^n \mapsto [0,1]$ is represented as

$$\tilde{C}(u_1,\ldots,u_n) = \psi(\psi^{-1}(u_1) + \ldots + \psi^{-1}(u_n)),\ u_k \in [0,1] \text{ for } k = 1,\ldots,n,$$

where the survival copula $\tilde{C}$ is generated by the generator function (also known as Archimedean generator function) $\psi : [0,\infty) \mapsto [0,1]$, ψ is n -monotone $(n \geq 2)$ over an open interval $I \subset \mathbb{R}$ (where the end points of the interval I belongs to the limit point of $\mathbb{R}$) if ψ has derivatives upto order $n-2$ and

$$(-1)^r \psi^{(r)}(x) \geq 0 \ \text{ for } r = 0,1,2,\ldots,n-2$$

for any $x \in I$ and also $(-1)^{(n-2)}\psi^{(n-2)}$ is non-increasing and convex over I. $\phi = \psi^{-1}$ is the corresponding inverse function. Clayton copula, Frank copula are few archimedean copulas studied in the literature.

For a detailed discussion on Archimedean Copula one can refer to [20]. Recently, [27] have published results for systems with heterogeneous, dependent and distribution-free components. The following two propositions are mentioned here, the proofs of these propositions can be easily derived from the proof of propositions 3.16 and 3.7 from [27].

Proposition 3.1. *Let $Y_1, Y_2, \ldots, Y_n$ be n random variables such that $Y_i = X - \mu_i$, $(P[X > x] = \overline{F}(x))$ where μ_i for $i = 1,2,\ldots,n$ are the corresponding location parameters respectively, then the survival function of the minimum of $Y_1, Y_2, \ldots, Y_n$ $(P[\min\{Y_1, Y_2, \ldots, Y_n\} > x])$ is given by*

$$J_1(\underline{\mu}; \overline{F}(x), \psi_1) = \psi_1(\sum_{k=1}^{n} \phi_1(\overline{F}(x + \mu_k))),$$

ψ_1 is log-convex (log-concave) and F is IFR (DFR) distribution. If there exists another set of n random variables $Z_1, Z_2, \ldots, Z_n$ $(Z_i = W - \mu_i^*$ and $P[W > x] = \overline{G}(x))$ such that the survival function for the minimum of $Z_1, Z_2, \ldots, Z_n$ is

$$J_1(\underline{\mu}^*; \overline{G}(x), \psi_2) = \psi_2(\sum_{k=1}^{n} \phi_2(\overline{G}(x + \mu_k^*))),$$

then as $(\mu_1, \mu_2, \ldots, \mu_n) \prec_w (\prec^w)$ $(\mu_1^*, \mu_2^*, \ldots, \mu_n^*)$ we obtain $Y_{1:n} \geq_{st} (\leq_{st}) Z_{1:n}$ as ψ is log-convex (log-concave), $X \geq_{st} W$ and F is IFR (DFR) distribution and $\phi_1 \cdot \psi_2(\phi_2 \cdot \psi_1)$ is super-additive.

Proposition 3.2. *Let $Y_1, \ldots, Y_n$ and $Z_1, \ldots, Z_n$ be two n-dimensional random variables such that $Y_i = X - \mu_i$ and $Z_i = W - \mu_i*$, $i = 1,2,\ldots,n$. Then*

$$\underline{\mu} \prec_w \underline{\mu}^* \ (\underline{\mu} \prec^w \underline{\mu}^*) \ \Rightarrow Y_{n:n} \geq_{st} (\leq_{st}) Z_{n:n}$$

whenever $\psi_1$or ψ_2 is log-convex (log-concave), F is IRFR(DRFR) distribution, $X \geq_{st} (\leq_{st}) W$ and $\phi_2 \cdot \psi_1(\phi_1 \cdot \psi_2)$ is super additive.

Here the condition "$\phi_2 \cdot \psi_1$ is super-additive" is necessary. Let us consider 2 Archimedean copula generators as

$$\phi_1(t) = (-\ln t)^2, \ t \in (0, 1] \text{ and } \phi_2(t) = (1 - t)^3, \ t \in (0, 1].$$

The corresponding inverses are

$$\psi_1(t) = exp(-t^{1/2}) \text{ and } \psi_2(t) = 1 - t^{1/3}.$$

We can observe that ψ_1 is log-convex. We are ineterested in finding the sign of the difference term

$$\phi_2 \cdot \psi_1(t_1 + t_2) - \phi_2 \cdot \psi_1(t_1) - \phi_2 \cdot \psi_1(t_2).$$

Figure 7 shows that the generators are chosen such that $\phi_2 \cdot \psi_1$ is not super-additive. As mentioned in the proposition, we shall consider $Y_i = X - \mu_i$ and

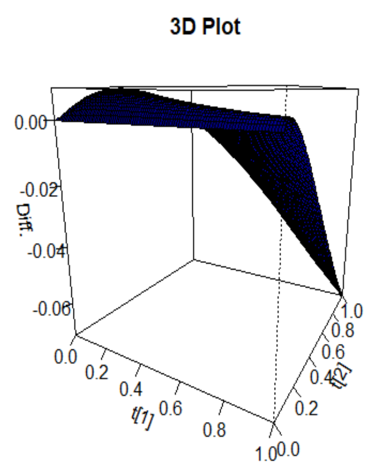

Figure 7: $\phi_2 \cdot \psi_1$ is not super additive.

$Z_i = W - \mu_i^*$ for $i = 1, 2, 3$. The location parameters are $\underline{\mu} = (0.5, 1, 2)$ and $\underline{\mu}^* = (1, 2, 3)$, thus $\underline{\mu} \prec_w \underline{\mu}^*$. The cdf of X and W are respectively given by

$$F(x) = \frac{\Phi\left(\frac{x}{2}\right)}{0.5}, x \in (-\infty, 0] \text{ and } G(x) = \frac{\Phi(x)}{0.5}, x \in (-\infty, 0].$$

The cdf of $Y_{3:3}$ is

$$J_2(\underline{\mu}; F(x), \psi_1) = 1 - \psi_1\left(\sum_{k=1}^{3}\left(-\ln\frac{\Phi\left(\frac{x+\mu_k}{2}\right)}{0.5}\right)^2\right).$$

The cdf of $Z_{3:3}$ is

$$J_2(\underline{\mu}^*; G(x), \psi_2) = 1 - \psi_2\left(\sum_{k=1}^{3}\left(1 - \frac{\Phi(x+\mu_k^*)}{0.5}\right)^3\right).$$

We shall observe the difference between the above 2 terms in Figure 8. Thus when $\phi_2 \cdot \psi_1$ is not super additive, usual stochastic ordering does not exist between $Y_{3:3}$ and $Z_{3:3}$.

$$\begin{aligned} y_1 &= J_2(\underline{\mu}; F(x), \psi_1) - J_2(\underline{\mu}^*; G(x), \psi_2) \\ &= 1 - \left(\sum_{k=1}^{3}\left(1 - \frac{\Phi(x+\mu_k^*)}{0.5}\right)^3\right)^{1/3} - \exp\left(-\left(\sum_{k=1}^{3}\left(-\ln\frac{\Phi\left(\frac{x+\mu_k}{2}\right)}{0.5}\right)^2\right)^{1/2}\right). \end{aligned}$$

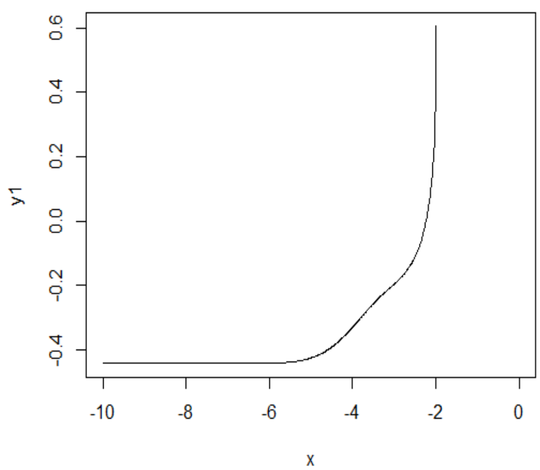

Figure 8: Usual stochastic ordering does not exist between $Y_{3:3}$ and $Z_{3:3}$.

When we take the generator function $\psi(x) = \exp(-x)$, $\phi(x) = -\ln x$. This generator indicates the independence copula (when the random variables are independent). Subsequently one can obtain the usual stochastic ordering between two sets of independent random variables.

Consider the Clayton copula generator function as

$$\psi_\theta(x) = \max((1+\theta x)^{-1/\theta}, 0),\ \ \theta >\ 0.$$

The above Archimedean generator is completely monotone (n-monotone for every $n \in \mathbb{N}$) for $\theta > 0$, and hence generates an Archimedean Copula. Here ψ_θ is a log-convex function, and hence the above theorems hold for this archimedean generator.

Examples: Let us consider $\phi(t) = (-\ln t)^{\theta}, \theta > 1, t \in (0, 1]$, the corresponding inverse function is $\psi(t) = e^{-t^{1/\theta}}, 0 \leq t < \infty$. $\ln \psi(t)$ and its corresponding derivatives with respect to t are

$$\begin{aligned} \ln \psi(t) &= -t^{1/\theta} \\ \frac{d}{dt}(\ln \psi(t)) &= -\frac{1}{\theta} t^{-1+1/\theta} \\ \frac{d^2}{dt^2}(\ln \psi(t)) &= \frac{\theta - 1}{\theta^2} t^{-2+1/\theta}. \end{aligned}$$

We can observe that $\frac{d^2}{dt^2}(\ln \psi(t))$ is non-negative. Hence $\psi(t)$ is log-convex.

Let us consider $\phi(t) = \ln(1 - \theta \ln t), \theta > 0, t \in (0, 1]$, the corresponding inverse function is $\psi(t) = e^{\frac{1 - e^t}{\theta}}, 0 \leq t < \infty$. $\ln \psi(t)$ and its corresponding derivatives with respect to t are

$$\begin{aligned} \ln \psi(t) &= \frac{1 - e^t}{\theta} \\ \frac{d}{dt}(\ln \psi(t)) &= -\frac{e^t}{\theta} \\ \frac{d^2}{dt^2}(\ln \psi(t)) &= -\frac{e^t}{\theta}. \end{aligned}$$

We can observe that $\frac{d^2}{dt^2}(\ln \psi(t))$ is non-positive. Hence $\psi(t)$ is log-concave.

4. Conclusion

Electronic devices, mechanical or electrical system consists of various units that are linked with one another either in series, parallel or any other combination, all of them are prone to failure at a certain point. We often refer to the warranty of the product to understand which system to purchase. Obviously any system which does not fail early is worth purchasing. If we are able to understand the dispersion of such a system compared to any other then we can compare two products. In order to understand the lifetime of any series or parallel system, we considered the random variables corresponding to the components. The results discussed in this paper can be divided into 3 subpart as Proportional Hazard rate (PHR) model, Proportional Reversed Hazard rate (PRHR) model, Dependent model. For PHR model we considered different models, a generalized situation where we consider two sets of independent PHR random variables

and the baseline distribution for both the sets are different ($X_1, X_2, \ldots, X_{n_1}$ such that $X_i \sim \overline{F}_i(x) = (\overline{F}_0(x))^{\alpha_i}$ for $i = 1, 2, \ldots, n_1$ and another set $Y_1, Y_2, \ldots, Y_{n_2}$, $Y_i \sim \overline{G}_i(x) = (\overline{G}_0(x))^{\beta_i}$ for $i = 1, 2, \ldots, n_2$). We have obtained conditions over the parameters and the baseline distributions so that a dispersive ordering exist between the minimum order statistics. Whereas when both the baseline distributions are same, star ordering occurs between these minimum order statistics provided $xr(x)$ is decreasing. Since Pareto distribution is also PHR model, a reversed hazard rate ordering occurs between the sample maximums (also known as parallel systems) when the shape parameter varies. Proceeding similarly we have observed a result for PRHR model too. Here the two sets of random variables follow different baseline distributions and the number of samples are also unequal ($X_i \sim F_i(x) = (F_0(x))^{\alpha_i}$ for $i = 1, 2, \ldots, n_1$ and $Y_i \sim G_i(x) = (G_0(x))^{\beta_i}$ for $i = 1, 2, \ldots, n_2$). All of these results are true for multiple-outlier models.
Another form of generalized model has been studied where $X_1, \ldots, X_{p_1}$ has survival function $[\overline{F}(x)]^{\alpha_i}$ and $X_{p_1+1}, \ldots, X_{n_1}$ has survival function $[\overline{G}(x)]^{\alpha_i}$. Similarly another system with n_2 components is considered where the components $Y_1, \ldots, Y_{p_2}$ has survival function as $[\overline{F}(x)]^{\beta_i}$ whereas the components $Y_{p+1}, \ldots, Y_{n_2}$ has survival function $[\overline{G}(x)]^{\beta_i}$ and hazard rate ordering results has been observed for series systems. A reversed hazard rate ordering result with PRHR components has been observed.
In the last section, dependent random variables have been studied. Here we obtained usual stochastic ordering results between two sample minimums and two sample maximums such that the location parameter corresponding to the random variables from two sets obeys a weak majorization ordering while the baseline distribution obeys a usual stochastic ordering and the generating functions follows super-additive property.

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REFERENCES

[1] Ali, D.; Towhidi, M. and Shekari, M. (2011). Stochastic and dependence comparisons between extreme order statistics in the case of proportional reversed hazard model, *Journal of the Iranian Statistical Society*, **10**, 1, 29–43.

[2] Amini-Seresht, E.; Qiao, J.; Zhang, Y. and Zhao, P. (2016). On the skewness of order statistics in multiple-outlier PHR models, *Metrika* **79**, 7, 817–836.

[3] Balakrishnan, N.; Chen, J; Zhang, Y. and Zhao, P. (2019). Comparisons of sample ranges arising from multiple-outlier models: In memory of Moshe Shaked, *Probability in the Engineering and Informational Sciences*, 28–49.

[4] Balakrishnan, N. and Torrado, N. (2016). Comparisons between largest order statistics from multiple-outlier models, *Statistics* **50**, 1, 176–189.

[5] Balakrishnan, N. and Zhao, P. (2013). Ordering properties of order statistics from heterogeneous populations: a review with an emphasis on some recent developments, *Probability in the Engineering and Informational Sciences* **27**, 4, 403–443.

[6] Barlow, R. E. and Proschan, F.(1975). *Statistical Theory of Reliability and Life Testing: Probability Models*, Florida State Univ Tallahassee.

[7] Bartoszewicz, J. (1997). Dispersive functions and stochastic orders, *Applicationes Mathematicae*, **24**, 429–444.

[8] Block, H. W.; Savits, T. H. and Singh, H. (1998). The reversed hazard rate function, *Probability in the Engineering and Informational Sciences*, **12**, 1, 69–90.

[9] Cai, X.; Zhang, Y. and Zhao, P. (2017). Hazard rate ordering of the second-order statistics from multiple-outlier PHR samples, *Statistics*, **51**, 3, 615–626.

[10] Das, S. and Kayal, S. (2021). Some ordering results for the Marshall and Olkin's family of distributions, *Communications in Mathematics and Statistics*, **9**, 2, 153–179.

[11] Datta, M. and Gupta, N. (2017). Stochastic comparison study of series and parallel systems having Kumaraswamy's and Frechet distributed components, *Journal of the Indian Statistical Association*, **55**, 2, 103–126.

[12] Fang, R. Li, C. and Li, X. (2018). Ordering results on extremes of scaled random variables with dependence and proportional hazards, *Statistics*, **52**, 2, 458–478.

[13] Hazra, N. K.; Kuiti, M. R.; Finkelstein, M. and Nanda, A. K. (2017). On stochastic comparisons of maximum order statistics from the location-scale family of distributions, *Journal of Multivariate Analysis*, **160**, 31–41.

[14] Hazra, N. K.; Kuiti, M. R.; Finkelstein, M. and Nanda, A. K. (2018). On stochastic comparisons of minimum order statistics from the location-scale family of distributions, *Metrika*, **81**, 2, 105–123.

[15] Jeon, J.; Kochar, S. and Park, C. G. (2006). Dispersive ordering – Some applications and examples, *Statistical Papers*, **47**, 2, 227–247.

[16] Kochar, S. C. and Xu, M. (2007). Some recent results on stochastic comparisons and dependence among order statistics in the case of PHR model, *Journal of Reliability and Statistical Studies*, **6**, 2, 125–140.

[17] Kochar, S. and Xu, M. (2009). Comparisons of parallel systems according to the convex transform order, *Journal of Applied Probability*, **46**, 2, 342–352.

[18] Kundu, A. and Chowdhury, S. (2020). On stochastic comparisons of series systems with heterogeneous dependent and independent location-scale family distributed components, *Operations Research Letters*, **48**, 1, 40–47.

[19] MARSHALL, A. W.; OLKIN, I. and ARNOLD, B. C. (1979). *Inequalities: Theory of Majorization and its Applications*, Springer.

[20] MCNEIL, A. J. and NESLEHOVA, J. (2009). Multivariate Archimedean copulas, d-monotone functions and l_1-norm symmetric distributions, *The Annals of Statistics*, **37**, 5B, 3059–3097.

[21] NAVARRO, J.; TORRADO, N. and DEL AGUILA, Y. (2018). Comparisons between largest order statistics from multiple-outlier models with dependence, *Methodology and Computing in Applied Probability*, **20**, 1, 411–433.

[22] OLIVEIRA, P. E. and TORRADO, N. (2015). On proportional reversed failure rate class, *Statistical Papers*, **56**, 4, 999–1013.

[23] PĂLTĂNEA, E. (2008). On the comparison in hazard rate ordering of fail-safe systems, *Journal of Statistical Planning and Inference*, **138**, 7, 1993–1997.

[24] PLEDGER, P. and PROSCHAN, F. (1971). Comparisons of order statistics and of spacings from heterogeneous distributions, *Optimizing Methods in Statistics*, 89–113.

[25] SAUNDERS, I. W. and MORAN, P. A. P. (1978). On the quantiles of the Gamma and F distributions, *Journal of Applied Probability*, **15**, 2, 426–432.

[26] SHAKED, M. and SHANTHIKUMAR, J.G. (2007). *Stochastic Orders*, Springer Science & Business Media.

[27] TORRADO, N. (2021). Comparing the reliability of coherent systems with heterogeneous, dependent and distribution-free components, *Quality Technology & Quantitative Management*, **18**, 6, 740–770.

[28] ZHAO, P. and BALAKRISHNAN, N. (2011). Dispersive ordering of fail-safe systems with heterogeneous exponential components, *Metrika*, **74**, 2, 203–210.

[29] ZHAO, P. and BALAKRISHNAN, N. (2012). Stochastic comparisons of largest order statistics from multiple-outlier exponential models, *Probability in the Engineering and Informational Sciences*, **26**, 2, 159–182.

[30] ZHANG, Y.; CAI, X.; ZHAO, P. and WANG, H. (2019). Stochastic comparisons of parallel and series systems with heterogeneous resilience-scaled components, *Statistics*, **53**, 1, 126–147.

[31] ZHANG, Y.; DING, W. and ZHAO, P. (2019). On variability of series and parallel systems with heterogeneous components, *Probability in the Engineering and Informational Sciences*, 1–20.