REVSTAT – Statistical Journal
Volume 0, Number 0, Month 0000, 000-000
https://doi.org/00.00000/revstat.v00i0.000

Generalizing the Heat Equation

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Received: Month 0000 Revised: Month 0000 Accepted: Month 0000

Abstract:

- The target of this paper is to generalize the Heat Equation, highly related with the Normal distribution. Therefore a generalization of the Normal distribution, the γ-order Normal distribution is introduced, which influences the entropy type information measures and offers the generalization of the Heat Equation.

Keywords:

- *Heat Equation; Wiener Process; Logarithm Sobolev Inequality; γ-order Normal.*

AMS Subject Classification:

- 60E05, 62B10, 62E99, 62P35

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1. INTRODUCTION

The most well known continuous Markov stochastic process $\{X(t); t \in [0, \infty)\}$ is the Brownian motion due to the name of the Scottish botanist Robert Brown (1773-1858), who in 1827 observed the phenomenon: minute particles, who were executing a continuous jittery and erratic motion. As Brownian motion was studied by Norbert Wiener (1894-1964) it is also known as Wiener process. The basic framework is that for the stochastic process, as above, $X(t)$ is considered the x component of a particle, always as a function of time. Let at the time $t_0, X(t_0) = x_0$ and let the conditional probability density of $X(t + t_0)$ given $X(t_0) = x_0$ to be $p(x, t|x_0)$. In principle we postulate the probability law governing the tradition, is stationary in time and therefore $p(x, t|x_0)$ does not depend on t_0. Therefore the density function $p(x, t|x_0)$ we stipulate that for "small t" $X(t+t_0) \approx X(t_0)$ i.e. we assume $\lim_{t\to 0} p(x, t|x_0) = 0$, see for details ([20], [11], [22]). The Brownian motion can be applied to continuous time optimization in Economics, see [24] among others.

Since Albert Einstein (1879-1955) explained the behavior of the stochastic process physically, [10], and he proved that eventually holds

$$\frac{\partial p}{\partial t} = D\frac{\partial^2 p}{\partial x^2} \tag{1.1}$$

with $D = 2RT/Nf$ the so called diffusion coefficient with R being the gas constant, T the temperature, N the Avogadro number and f the coefficient of friction, the diffusion equation (1.1) attracted a special interest. We mention that Schrodinger (1915) also investigated the Brownian Motion and worked with the Normal Inverse Gaussian (NIG), see [19]. From an Analysis point of view is considered as a partial differential equation, [32], among others, from physical point of view, known as Heat Equation (HE)-modeling the proportion of the amount of heat divided by the "amount" (precisely the mass) of the material, with a proportionality factor, which under a proper scale can be $D = 1/2$ i.e. (1.1) is reduced to

$$\frac{\partial^2 p}{\partial x^2} = 2\frac{\partial p}{\partial t} \tag{1.2}$$

Eventually Probability theory is also involved as we can easily verify that the unique solution of (1.2), under the boundary conditions

(a) $\lim_{t\to 0} p(x, t|x_0) = 0, \quad x \neq x_0$

(b) $p(x, t|x_0)$ is a density function in x thus $p(x, t|x_0) \geq 0$ and $\int_{-\infty}^{\infty} p(x, t|x_0)dx = 1$.

is:

$$p(x, t|x_0) = \frac{1}{\sqrt{2\pi t}} \exp\left\{-\frac{1}{2t}(x - x_0)^2\right\} \tag{1.3}$$

i.e. if, without loss of generality, we assume that $x_0 = 0$, $p(x, t|x_0)$ coincides with the (distribution function of the) standard Normal distribution

$$\phi(x) = \frac{1}{\sqrt{2\pi t}} \exp\left\{-\frac{1}{2}\left(\frac{x}{\sqrt{t}}\right)^2\right\} := \phi(x; 0, t) \tag{1.4}$$

The target of this paper is to work with a general form of (1.4) introducing an extra parameter, γ say, and form $\phi_\gamma(x; 0, t)$ which is discussed in Section 3, and evaluate a constant term $K = K(x, t; \gamma)$ so that (1.2) to be generalized as:

$$\frac{\partial^2 \phi_\gamma}{\partial x^2} = K\frac{\partial \phi_\gamma}{\partial t} \tag{1.5}$$

Relation (1.5) is the introduced Generalized Heat Equation (GHE). This is discussed in Section 4. We shall prove that for $\gamma = 2$, relation (1.2) is obtained. Section 2 is devoted to explain the line of thought of the introduction of the extra parameter γ as a background to Section 3 where the γ-order Normal distribution is introduced.

2. Background

The fact that we are facing the Heat Equation from a statistical point of view, provides evidence that no Physics is involved in this paper, but certainly an appropriate background from the Analysis point of view has been adopted to discuss the problem. Moreover it is emphasized that both in Physics and Analysis there is a strong theoretical insight with different approaches facing the Thermal or Heat equations. The pioneering work of Feller (1950) and Lévy (1948) covers the theoretical statistical background. Somebody might think that these are "old references", but they are the pioneering work in the field, the "back-bone" of the probability line of thought for the subject. We are referring to such pioneer work in this paper, as actually there are not that many from a Probability point of view. Moreover, we worked with the statistical oriented papers adopting an Analysis approach.

We shall use throughout this paper the statistical notation for example we let p to be the number of the involved parameters, and not n as n is the sample size in Statistics. We avoid to adopt a as a is the significance level. We are adopting γ instead, as the extra parameter of the γ-order Normal, see Section 3. Recall that in all entropy type research problems, as well, the three lines of thought: Statistics, Physics, Analysis are also met. Both Poincaré [3] and Logarithm Sobolev inequalities [6] are involved in such problems, where the energy $Ener_\mu(f)$ of a local integrable function f with $f \in L^2(\mathbb{R}^n, \mu)$ can be defined as:

$$Ener_\mu(f) := \mathbf{E}[|\nabla f|^2] \tag{2.1}$$

with ∇f the gradient of f and $\mathbf{E}$ the expectation of measure μ, i.e. $\mathbf{E}(f) = \int_{\mathbb{R}} f d\mu$, see for details [16].

Let X be a random variable with probability density function (pdf) f on $\mathbb{R}^p$. Recall that, in principle, a function $f : \mathbb{R}^p \mapsto \mathbb{R}$ is said to be in a Sobolev space $W^{1,\gamma}(\mathbb{R}^p), \gamma \geq 1$, [5], [29], if:

(i) $f \in L^\gamma(\mathbb{R}^p)$

(ii) its gradient $\nabla f \in L^\gamma(\mathbb{R}^p)$

For $f^{1/2} \in W^{1,2}(\mathbb{R}^p)$, Fisher's entropy type information of $f, J(X)$ say, is defined:

$$J(X) = 4 \int_{\mathbb{R}^p} |\nabla \sqrt{f}|^2 dx$$

see [6]. It is easy to see that:

$$\begin{aligned} J(X) &= \int_{\mathbb{R}^p} |\nabla \log f|^2 dx = \int_{\mathbb{R}^p} |\nabla f|^2 f^{-1} dx \\ &= \int_{\mathbb{R}^p} (\nabla f)(\nabla \log f) dx \end{aligned} \tag{2.2}$$

Considering (2.2), recall that the Shannon entropy is defined as:

$$H(X) = -\int_{\mathbb{R}^p} f \log f dx \tag{2.3}$$

while considering the family of densities of X parameterized by $\theta \in \Theta$, with Θ be a (compact, when limiting results are requested) subset of $\mathbb{R}^p$, the Fisher's (parametric) information matrix is:

$$I(\theta) := I_2(\theta) := \mathbf{E}_\theta |\nabla_\theta \log f_\theta(x)|^2 \tag{2.4}$$

The Vajda (parametric) information measure, [31], is defined as en extension of (2.4)

$$I_\gamma(\theta) := \mathbf{E}_\theta |\nabla_\theta \log f_\theta(X)|^\gamma, \gamma \geq 1 \tag{2.5}$$

Comparing $I_2(\theta)$ and the general $I_\gamma(\theta)$, see (2.4) and (2.5), it is easy for somebody to think to obtain, based on $J = J_2(X)$ in (2.2), a general form, $J_\gamma(X)$ say. Such a procedure was also considered in the sense that Rényi's entropy, [25], generalized Kullback - Leibler information, [18]. For a recent study on Rényi's divergence measure see [13].

Under this line of thought [14] worked and generalized $J(X)$ to $J_\gamma(X)$, as well as Shannon exponential entropy $N_\gamma(X)$ through Shannon entropy $H(X)$, see (2.3). Indeed:

It is easy to consider that the entropy-type Fisher's information measure, see (2.2) can be extended to:

$$\begin{aligned} J_\gamma(X) &= \int_{\mathbb{R}^p} |\nabla \log f|^\gamma dx \\ &= \int_{\mathbb{R}^p} |\nabla f|^\gamma f^{1-\gamma} dx \end{aligned} \tag{2.6}$$

with $\gamma \geq 1$ and f the pdf of a random variable X with $f^{1/\gamma} \in W^{1,\gamma}(\mathbb{R}^p)$. For $\gamma = 2$ we can verify that:

$$J_2(X) = J(X) \tag{2.7}$$

Moreover Shannon's exponentially entropy, $N_\gamma(X)$, can be extended and defined as:

$$N_\gamma(X) = const(\gamma, p) \exp\left\{\frac{\gamma}{p} H(X)\right\} \tag{2.8}$$

It is crucial that due to the generalized form of the entropy power, an extension of the Cramer-Rao inequality can be obtained, relating $J_\gamma(X)$ and $N_\gamma(X)$. This has been also obtained from the Heisenberg uncertainty inequality, see [4]. Indeed the following theorem holds, [14].

Theorem 2.1. *It holds for the introduced entropy-type measures in (2.6) and (2.8) that* $J_\gamma(X) N_\gamma(X) \geq p$.

Due to Theorem 2.1, the well-known Cramer-Rao inequality is obtained, when $\gamma = 2$, while an example for the exponential family is discussed in [14].

It has been pointed out, [3], that the Logarithm Sobolev Inequality (LSI) can be also interpreted as sharpening the uncertainty principle, related through Theorem 1 to $J_\gamma(X)$ and $N_\gamma(X)$. That is why LSI was adopted and a new distribution (3.2) emerged, as it is discussed in Section 3.

3. The γ-order Normal distribution

Due to the discussion in Section 2 we consider the [8] extension of the [7] work for $g \in W^{1,\gamma}(\mathbb{R}^p)$ and $1 < \gamma < p$ with $\|g\|_\gamma = 1$ of the form:

$$\int_{\mathbb{R}^p} |g|^\gamma \log |g| \leq \frac{p}{\gamma^2} \log\left(c_\gamma \int_{\mathbb{R}^p} |\nabla g|^\gamma dx\right) \tag{3.1}$$

where the optimal constant $c_\gamma = c(\gamma, p)$ equals:

$$c(p, \gamma) = \frac{\gamma}{p}\left(\frac{\gamma - 1}{e}\right)^{\gamma - 1} \pi^{-\gamma/2} \left(\frac{\Gamma(\frac{p}{2} + 1)}{\Gamma(p\frac{\gamma-1}{\gamma} + 1)}\right)^{\gamma/p}, \gamma \in \mathbb{R} \setminus [0, 1]$$

For the [7] LSI, externals are precisely Gaussians, [6], while for [8] as it was pointed out by [14] and presented by [15], [30] it is:

$$\phi_\gamma(x) = \phi_\gamma(x;\mu,\Sigma) = c(p,\gamma)[det\Sigma]^{-1/2}\exp\left\{-\frac{\gamma-1}{\gamma}Q^{\frac{1}{2}\frac{\gamma}{\gamma-1}}(x)\right\}, \tag{3.2}$$

with $x \in \mathbb{R}^p, Q(x) =< (x-\mu), \Sigma^{-1}(x-\mu) >$,

$c(p,\gamma) = \pi^{-\frac{p}{2}}\left(\frac{\gamma-1}{\gamma}\right)^{p\frac{\gamma-1}{\gamma}}\frac{\Gamma(\frac{p}{2}+1)}{\Gamma(p\frac{\gamma-1}{\gamma}+1)}, \gamma \in \mathbb{R}\setminus[0,1]$ and $< a, b >= ab^T$ the inner product in $\mathbb{R}^n$, for $a, b \in \mathbb{R}^n$.

Notice that with $\gamma = 2$ we obtain the classical multivariate Normal, while with $\Sigma = I\sigma^2, I = diag\{1\} \in \mathbb{R}^{p\times p}$ and $p = 1$ we obtain the standard normal distribution $\phi(x)$ with $\mu = 0$:

$$\begin{aligned}\phi_2(x;0,1) &= \frac{\Gamma(\frac{1}{2}+1)}{\Gamma(\frac{1}{2}+1)}\frac{(1/2)^{1/2}}{\sqrt{\pi}}\exp\left\{-\frac{1}{2}x^2\right\} \\ &= \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}x^2\right\} = \phi(x)\end{aligned} \tag{3.3}$$

The introduced γ-order Normal distribution $N(\mu,\Sigma;\gamma)$ is a Kotz-type distribution, as it was pointed out by [14]. In principle, heavy-tailed distributions are those probability distributions whose tails are not exponentially bounded, they have "heavier" tails than we usually assume, practically more that 0.05 as the standard Normal. The Normal Inverse Gaussian $NIG(\cdot)$ is a very nice attempt to take into consideration the tails of the distribution and tries to cover the "fat tails" problem, which appears mainly in Finance studies, see [1], [2], [9], [26]. As far as Environmental Economics concerned, for the uncertainty see [12] where (3.2) was used. Notice that the Brownian motion will be normally distributed at all points in time, while a Lévy process, which is Generalized Hyperbolic (GH) distribution, can be GH at one point and might fail to be GH at another point in time, [23]. Both the GH and the γ-order GN are closed to affine transformations, while the Generalized Laplace (GL) and the NIG obey to the "closed under convolution" principle.

Although the existent background of the γ-order GN, $GN \sim N(\mu,\sigma^2 I;\gamma)$ depends on LSI, from which it "emerged" then the extra shape parameter γ provides an easy generalization of the multivariate Normal. The involved "international constant" $\left(\frac{\gamma-1}{\gamma}\right)^{\frac{\gamma-1}{\gamma}}$, [29], plays an important role in our development and is very essential that it comes through LSI. When $\gamma \rightarrowtail 1$ the Uniform distribution is obtained and when $\gamma \rightarrowtail \infty$ the Laplace distribution is obtained. Therefore we believe that γ-GN covers more prons than cons and this distribution outperforms comparing the other two. Moreover, it is a generalization of the distribution directly connected to the Heat Equation. It is rather helpful to obtain results, for Fisher's entropy type information. So there is a well working set of applications based on γ-GN and everything is reduced to the classical Normal when $\gamma = 2$. It is true that from NIG we can also reach Normal distribution but the NIG depends on four parameters, $NIG(\tau,\alpha,\delta,\mu)$, where the parameters are τ : for tail heaviness, α : asymmetry, δ : scale parameter, μ : location parameter. So there are 4 parameters involved and a rather complicated probability density function (pdf) based on modified Bessel function, of second kind. So it is rather difficult to be completely clear to those who are not mathematicians. Notice that the NIG belongs to the GH type distribution, while the γ-GN is a Kotz type distribution. The log-Normal (LN) it seems easier to be handled, but there is no shape parameter - actually the shape changes due to the fact that the logarithm is applied.

The introduced γ-ordered generalized Normal offers the possibility to approximate heavy-tailed distributions. The cumulative distribution function $\Phi_\gamma(z)$, for $z = \frac{x-\mu}{\sigma}$, for the $\phi_\gamma(x;\mu,\sigma^2)$ as in (3.2) has been obtained due to the following Theorem, see [30].

Theorem 3.1. *Let X be a random variable from the univariate $N_\gamma(\mu,\sigma^2)$ with pdf $\phi_\gamma(x;\mu,\sigma^2)$*

and F_γ the cdf. Let Φ_γ be the cdf of the standardized $z = \frac{1}{\sigma}(x-\mu) \sim N_\gamma(0,1)$. Then:

$$\Phi_\gamma(z) = \frac{1}{2} + \frac{\sqrt{\pi}}{2\Gamma(\gamma_0+1)\Gamma(\gamma_1+1)} Erf_{\gamma_1}[\gamma_2 z], \tag{3.4}$$

with Erf being the usual error function and

$$\gamma_0 = \frac{\gamma-1}{\gamma}, \gamma_1 = \frac{\gamma}{\gamma-1}, \gamma_2 = \gamma_0^{\gamma_0}, z = \frac{x-\mu}{\sigma}, x \in \mathbb{R}.$$

Based on (3.4) a number of calculations have been proceeded and a part is presented here for different γ and p values, see Table 1. For a graph of a Bivariate 10-order generalized Normal $N(0,1;10)$ see [15] or [16], while for the Kullback-LeibLer (K-L) information of two p-variate density functions from $N_\gamma(\mu_1, I\sigma^2)$ and $N_\gamma(\mu_2, I\sigma^2)$ see [17].

	p		
γ	1	2	3
-2	0.6084	0.8100	0.8995
-1	0.5940	0.7737	0.8603
1*	1.0000	1.0000	1.000
2**	0.6827	0.9545	0.9973
5	0.6470	0.8953	0.9724
10	0.6390	0.8792	0.9614
$\pm\infty$***	0.6320	0.8666	0.9510

Table 1: Probability mass $\mathbf{P}(\|X\| \leq 1)$, where $X \sim N(0, \sigma^2 I_p; \gamma)$, for various p and γ. * Uniform ** Normal *** Laplace.

Considering (3.2). the (elliptical contoured) γ-order Normal is reduced to a spherical contoured when $\Sigma = \sigma^2 I_p$, while when only one variable is involved and $\mu = 0$ then:

$$\phi_\gamma(x; 0, \sigma^2) = \frac{\Gamma(\frac{1}{2}+1)}{\Gamma(\gamma+1)} \frac{\gamma_2}{\sqrt{\pi}\sigma} \exp\left\{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2\right\}, \tag{3.5}$$

The distribution described in (3.5) , $N_\gamma(0, \sigma^2)$ say, is fundamental for generalizing the Heat Equation as in (1.2). We introduce and prove the problem in the next section.

4. Generalizing the Heat Equation

Consider a Brownian motion $\{X(t); t \geq 0\}$ and assume that every increment is γ-order Normal distribution with mean 0 and variance $\sigma^2 t, \sigma$ is fixed, i.e. the definition of the Brownian motion is valid under the γ-order Normal distribution.

As usually $X(0) = 0$ and $X(t)$ is continuous at $t = 0$. We can assume without loss of generality that $\sigma = 1$, or that the Brownian motion is standard, i.e. $N_\gamma(0,t)$ is considered. That is, as we are working under $N_\gamma(0,t)$, considering (3.5) with $\sigma = \sqrt{t}$ the corresponding γ-order generalizing distribution function is:

$$\phi_\gamma(x; 0, t) = \frac{\lambda}{\sqrt{\pi t}} \exp\left\{-\gamma_0\left(\frac{x}{\sqrt{t}}\right)^{\gamma_1}\right\}, \tag{4.1}$$

with

$$\lambda = \frac{\Gamma(\frac{1}{2}+1)}{\Gamma(\gamma_0+1)}\gamma_2$$

$$\gamma_0 = \frac{\gamma - 1}{\gamma} > 0, \quad \gamma_1 = \frac{\gamma}{\gamma - 1}, \quad \gamma_2 = \left(\frac{\gamma - 1}{\gamma}\right)^{\frac{\gamma-1}{\gamma}} = \gamma_0^{\gamma_0}$$

Lemma 4.1. *Let $\phi_\gamma = \phi_\gamma(x;0,t)$ as in (4.1). Then it holds:*

$$\frac{\partial \phi_\gamma}{\partial t} = \phi_\gamma(x;0,t)A(x;t,\gamma), \tag{4.2}$$

where $A(x;t,\gamma)$ is a well defined function.

Proof of Lemma 4.1: From (4.1) differentiate with respect to t to get:

$$\frac{\partial \phi_\gamma}{\partial t} = \frac{\lambda}{\sqrt{\pi}}(t^{-1/2})' \exp\left\{-\gamma_0 \left(\frac{x}{\sqrt{t}}\right)^\gamma\right\} + \frac{\lambda}{\sqrt{\pi}} t^{-1/2} \left(\exp\left\{-\gamma_0\left(\frac{x}{\sqrt{t}}\right)^\gamma\right\}\right)'$$

We let:

$$Q(x,t) = -\gamma_0 \left(\frac{x}{\sqrt{t}}\right)^{\gamma_1}. \tag{4.3}$$

Then:

$$\begin{aligned}
\frac{\partial \phi_\gamma}{\partial t} &= -\frac{\lambda}{\sqrt{\pi}} t^{-3/2} \exp\{Q(x,t)\} \\
&\quad + \frac{\lambda}{\sqrt{\pi}} t^{-1/2} \exp\{Q(x,t)\} \cdot Q'(x,t) \\
&= \frac{\lambda}{\sqrt{\pi t}} \exp\{Q(x,t)\} \left(-\frac{1}{2}t^{-1} + \frac{1}{2}x^{\gamma_1} t^{-\frac{\gamma_1+2}{2}}\right) \\
&= \phi_\gamma(x;0,t)A(x;t,\gamma) \qquad (4.4)
\end{aligned}$$

□

Lemma 4.2. *Let $\phi_\gamma = \phi_\gamma(x;0,t)$ as in (4.1). Then the following are true:*

$$\frac{\partial \phi_\gamma}{\partial x} = \phi_\gamma(x;0,t)B_1(x;t,\gamma),$$

$$\frac{\partial^2 \phi_\gamma}{\partial x^2} = \phi_\gamma(x;0,t)B_2(x;t,\gamma),$$

with $B_1(x;t,\gamma), B_2(x;t,\gamma)$ well defined functions.

Proof of Lemma 4.2: Recall (4.3) and that $\gamma_0\gamma_1 = 1$. Differentiating with respect to x it is

$$\begin{aligned}
\frac{\partial \phi_\gamma}{\partial x} &= \frac{\lambda}{\sqrt{\pi t}} \exp\{Q(x,t)\}'_x \\
&= \frac{\lambda}{\sqrt{\pi t}} \exp\{Q(x,t)\}(Q(x,t))' \\
&= \phi_\gamma(x;0,t)\left(-\gamma_0\gamma_1 t^{-\frac{1}{2}\gamma_1} x^{\gamma_1 - 1}\right) \\
&= \phi_\gamma(x;0,t)\left(-t^{-\frac{1}{2}\gamma_1} x^{\gamma_1 - 1}\right) \\
&= \phi_\gamma(x;0,t)B_1(x;t,\gamma) \qquad (4.5)
\end{aligned}$$

Thus, from (4.5) we get:

$$\begin{aligned}
\frac{\partial^2 \phi_\gamma}{\partial x^2} &= \phi'_\gamma(x;0,t)B_1(x;t,\gamma) + \phi_\gamma(x;0,t)B'_1(x;t,\gamma) \\
&= \phi_\gamma(x;0,t)B_1^2(x;t,\gamma) + \phi_\gamma(x;0,t)\left(-t^{-\frac{1}{2}\gamma_1} \frac{1}{\gamma_1 - 1} x^{\gamma_1 - 2}\right) \\
&= \phi_\gamma(x;0,t)B_2(x;t,\gamma) \qquad (4.6)
\end{aligned}$$

□

Therefore we can state and prove the following Theorem, generalizing the Heat Equation (1.2) as in (4.4)

Theorem 4.1. *There exists a well defined function $K = K(x;t,\gamma)$ such that*

$$\frac{\partial^2 \phi_\gamma}{\partial x^2} = K \frac{\partial \phi_\gamma}{\partial t} \tag{4.7}$$

Proof of Theorem 4.1: From (4.4) it is, as $\gamma_1 = \frac{\gamma}{\gamma-1}$,

$$A(x;t,\gamma) = -\frac{1}{2}t^{-1} + \frac{1}{2}x^{\frac{\gamma}{\gamma-1}} / t^{\frac{3\gamma-1}{2(\gamma-1)}} \tag{4.8}$$

From (4.5) and (4.6) respectively we get

$$B_1(x;t,\gamma) = -t^{-\frac{\gamma}{2(\gamma-1)}} x^{\frac{1}{\gamma-1}} \tag{4.9}$$

$$\begin{aligned} B_2(x;t,\gamma) &= B_1^2(x;t,\gamma) + \left(-t^{-\frac{1}{2}\frac{\gamma}{\gamma-1}} \frac{1}{\gamma-1} x^{\frac{2-\gamma}{\gamma-1}}\right) && (4.10) \\ &= \left(-t^{-\frac{\gamma}{2(\gamma-1)}} x^{\frac{1}{\gamma-1}}\right)^2 + \left(-t^{-\frac{1}{2}\frac{\gamma}{\gamma-1}} \frac{1}{\gamma-1} x^{\frac{2-\gamma}{\gamma-1}}\right) \\ &= t^{-\frac{\gamma}{\gamma-1}} x^{\frac{2}{\gamma-1}} - \frac{1}{\gamma-1} t^{-\frac{1}{2}\frac{\gamma}{\gamma-1}} x^{\frac{2-\gamma}{\gamma-1}} && (4.11) \end{aligned}$$

Therefore:

$$\frac{\frac{\partial^2 \phi_\gamma}{\partial x^2}}{\frac{\partial \phi_\gamma}{\partial t}} = \frac{t^{-\frac{\gamma}{\gamma-1}} x^{\frac{2}{\gamma-1}} - \frac{1}{\gamma-1} t^{-\frac{1}{2}\frac{\gamma}{\gamma-1}} x^{\frac{2-\gamma}{\gamma-1}}}{\frac{1}{2}\left(-\frac{1}{t} + \frac{x^{\frac{\gamma}{\gamma-1}}}{t^{\frac{3\gamma-2}{2(\gamma-1)}}}\right)} := K(x;t,\gamma) := K, \tag{4.12}$$

i.e. $\frac{\partial^2 \phi_\gamma}{\partial x^2} = K \frac{\partial \phi_\gamma}{\partial t}$ and K is well defined as in (4.12). □

Corollary 4.1. *With $\gamma = 2$ the classical homogeneous heat equation is true, i.e. (1.2) holds.*

Proof of Corollary 4.1: From (4.12) with $\gamma = 2$ we obtain

$$K(x;t,2) = \frac{\frac{x^2}{t^2} - \frac{1}{t}}{\frac{1}{2}\left(-\frac{1}{t} + \frac{x^2}{t^2}\right)} = 2 \tag{4.13}$$

□

We can arrive at the same result through $\phi_2(x;0,t)$, verifying relation (1.2). Indeed:

$$\frac{\partial \phi_2}{\partial t} = \frac{1}{2}\phi_2(x;0,t)\left(-\frac{1}{t} + \frac{x^2}{t^2}\right)$$

$$\frac{\partial \phi_2}{\partial x} = \phi_2(x;0,t)\left(-\frac{x}{t}\right)$$

$$\frac{\partial^2 \phi_2}{\partial x^2} = \phi_2(x;0,t)\left(\frac{x^2}{t^2} - \frac{1}{t}\right) = 2\frac{\partial \phi_2}{\partial t}$$

Corollary 4.2. *With $t = 1$ it holds*

$$\frac{\partial \phi_\gamma}{\partial x} = \phi_\gamma(x)\left(-x^{\frac{1}{\gamma-1}}\right) \tag{4.14}$$

Proof of Corollary 4.2: From (4.5) and (4.9)

$$\begin{aligned}\frac{\partial \phi_\gamma}{\partial x} &= \phi_\gamma(x;0,1)B_1(x;1,\gamma)\\ &= \phi_\gamma(x;0,1)\left(-x^{\frac{1}{\gamma-1}}\right)\end{aligned}$$

□

Corollary 4.3. *From (4.14) with $\gamma = 2$ the well known relation:*

$$\phi_2'(x) = -x\phi_2(x) \tag{4.15}$$

holds.

Consider the generalized Heat equation (GHE) as in (4.7). In Table 2 and Figures 1 and 2 we present, for given values of γ, the corresponding γ-order GN, $\phi_\gamma(x;0,t)$ as in (4.1), with the value of λ as in (4.1). The calculations were performed through MATLAB and the corresponding $K = K_\gamma = K(x;t,\gamma)$ is also evaluated and presented.

γ	ϕ_γ	K_γ
1.5	$\frac{\lambda}{\sqrt{\pi t}}\exp\left\{-0.3\left(\frac{x}{\sqrt{t}}\right)^3\right\}$	$-\dfrac{\frac{1}{2}t^{3/2} - 0.5\left(\frac{x}{\sqrt{t}}\right)^2 xt}{t^{3/2}\left(-2\left(\frac{x}{\sqrt{t}}\right)+\left(\frac{x}{\sqrt{t}}\right)^4\right)}$
1.9	$\frac{\lambda}{\sqrt{\pi t}}\exp\left\{-0.4736842105\left(\frac{x}{\sqrt{t}}\right)^{2.1}\right\}$	$-\dfrac{\frac{1}{2}t^{3/2} - 0.5\left(\frac{x}{\sqrt{t}}\right)^{1.1} xt}{t^{3/2}\left(-1.1\left(\frac{x}{\sqrt{t}}\right)^{0.1}+\left(\frac{x}{\sqrt{t}}\right)^{2.2}\right)}$
2	$\frac{1}{\sqrt{2\pi t}}\exp\left\{-0.5\left(\frac{x}{\sqrt{t}}\right)^2\right\}$	2
2.1	$\frac{\lambda}{\sqrt{\pi t}}\exp\left\{-0.5\left(\frac{x}{\sqrt{t}}\right)^{1.9}\right\}$	$\dfrac{\frac{1}{2}t^{3/2} - 0.5\left(\frac{x}{\sqrt{t}}\right)^{0.9} xt}{t^{3/2}\left(0.9\left(\frac{x}{\sqrt{t}}\right)^{-0.09}-\left(\frac{x}{\sqrt{t}}\right)^{1.81}\right)}$
2.5	$\frac{\lambda}{\sqrt{\pi t}}\exp\left\{-0.6\left(\frac{x}{\sqrt{t}}\right)^{1.6}\right\}$	$\dfrac{\frac{1}{2}t^{3/2} - 0.5\left(\frac{x}{\sqrt{t}}\right)^{0.6} xt}{t^{3/2}\left(0.66\left(\frac{x}{\sqrt{t}}\right)^{-0.33}-\left(\frac{x}{\sqrt{t}}\right)^{1.33}\right)}$
3	$\frac{\lambda}{\sqrt{\pi t}}\exp\left\{-2/3\left(\frac{x}{\sqrt{t}}\right)^{3/2}\right\}$	$\dfrac{t^{3/2} - xt\sqrt{\frac{x}{\sqrt{t}}}}{t^{3/2} - 2xt\sqrt{\frac{x}{\sqrt{t}}}}$

Table 2: Evaluating $\phi_\gamma(x;0,t)$ as (4.1) with $\lambda = \frac{\Gamma(\frac{1}{2}+1)}{\Gamma(\gamma_0+1)}$ and K_γ for given values of γ, see (4.1) and (4.7) the GHE.

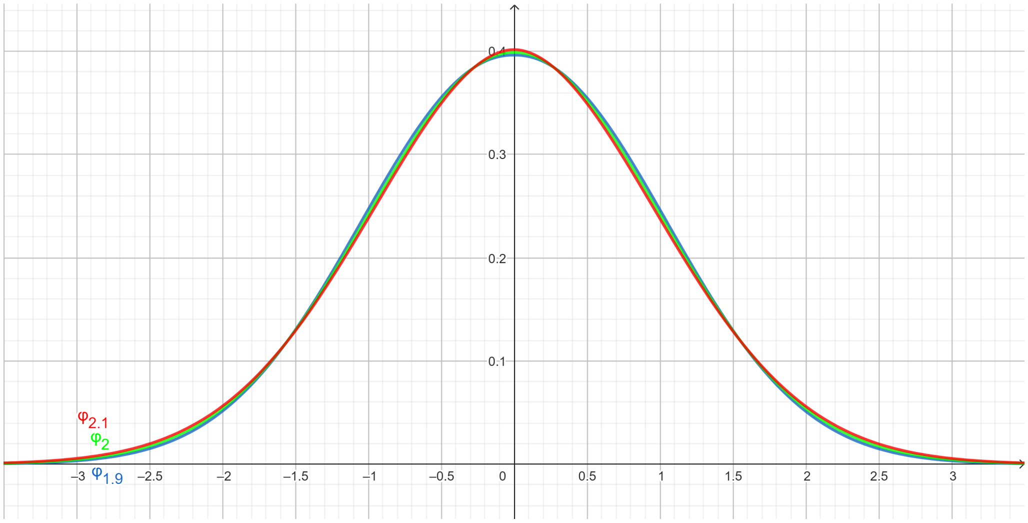

Figure 1: Plots of $\phi_\gamma(x; 0, 1)$ for values of γ close to 2. Namely $\gamma = 1.9, 2$ and 2.1

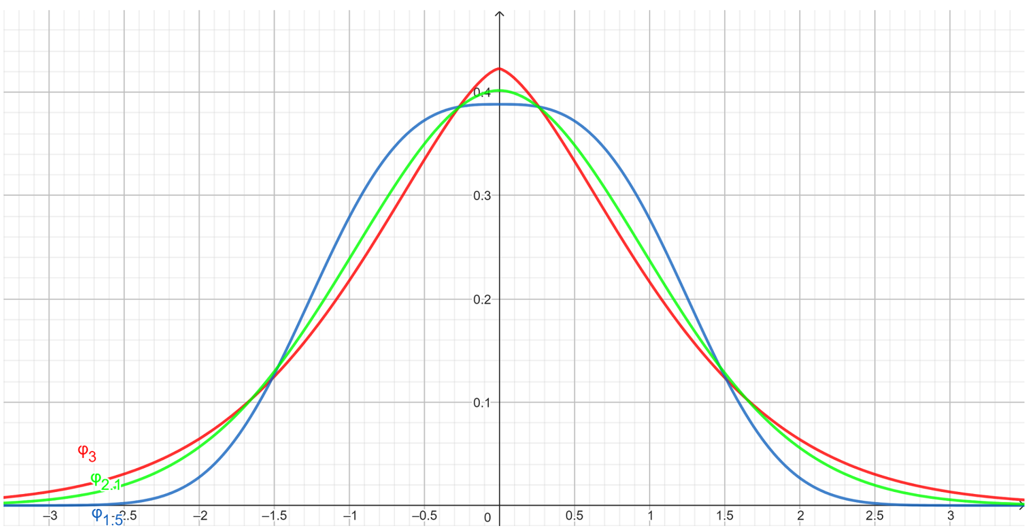

Figure 2: Plots of $\phi_\gamma(x; 0, 1)$ for values of γ. Namely $\gamma = 1.5, 2.1$ and 3.

Therefore not only a general form of the Heat Equation was provided due to Theorem 4.1, but also minor results can be generalized due to the γ-order Normal. Notice that when $t = 1$ (equivalently $\sigma^2 = 1$) the values of ϕ_γ and K_γ are simplified.

It is clear that for values of $\gamma = 1.9, 2, 2.1$ i.e close to 2, the corresponding graphs are close to the usual Normal, see Figure 1, but for values of $\gamma = 1.5, 2.1, 3$ i.e the corresponding graphs provide evidence for their "fat tails", see Figure 2.

5. Discussion

From a statistical point of view the Heat Equation has faced little attention. Most of the work is referring to the Brownian motion process as one of the two Lévy processes - the other one is the Poisson process. In Medical Physics, in their recent paper, [21], working on breast cancer and the involved mathematical equations, came across the Heat Equation, due to the fact that the thermal diffusivity, (α) in their notation, the time t and the penetration depth (δ) linked, in their notation, as $\delta = 3.65\sqrt{\alpha t}$ satisfy the transient conduction equation for the temperature $T = T(x,t)$ in one dimension

$$\frac{\partial^2 T(x,t)}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T(x,t)}{\partial t}$$

For a given value of the thermal diffusivity and the depth of tumor the cold stress time is evaluated. They also provided a review of the mathematical models concerning the subject. That is, the Heat Equation participated in problems facing the underground application rather through Mathematics, than Statistics, as the Gaussian is rather more familiar that any other distribution. But the Statistical background is solid and we only try to offer a more general Gaussian to face applications with "fat tails", and still obey to the GHE, as in (4.6).

From a physical point of view the appropriate name of the Heat Equation is diffusion equation with a source term. From an Analysis point of view the Heat Equation is known as a parabolic differential equation: briefly it describes the distribution of the heat flow (into/out) of a material in a given space over time. The proportional factor is the specific heat capacity of the material. The Heat Equation is a typical example of a continuity equation and it was related to the Gaussian. Schrodinger (1915) was investigating the Brownian motion and he came across, as we already mentioned, to the Normal Inverse Gaussian (NIG), see [27], [19]. The Brownian movement provides food for thought to continuous optimization models in Economics, [24] among others. Let $F_\gamma(x)$ to be the cdf of the rv $X \sim N(0,1;\gamma)$. For given different shape values for γ, the corresponding probability values have been evaluated, see also [12], in Table 3.

	$GN(0,1;\gamma)$	
γ	$F_\gamma(-3)$	$F_\gamma(2)$
2	0.0013	0.9772
3	0.0071	0.9598
10	0.0193	0.9396
-1/10	0.1656	0.8111

Table 3: Values of cdf of γ-order generalized Normal with $\mu = 0, \sigma = 1$.

Let $F_{NIG}(x)$ to be the cdf of a rv $X \sim NIG(\tau,\alpha,\delta,\mu) = NIG(\tau,0,1,0)$. For given values of tail heaviness taken from Table3 and keeping asymmetry parameter as well as location parameter equal to zero, while the scale parameter is one, we present in Table 4 the values of the cdf $F_{NIG}(x)$, with $x = -3, 2$ as for the $N(0,1;\gamma)$.

	$NIG(\tau,0,1,0)$	
τ	$F_{NIG}(-3)$	$F_{NIG}(2)$
0.0014	0.1018	0.9718
0.0193	0.0953	0.9701

Table 4: Values of cdf of $NIG(\tau,0,1,0)$ for different values of heavy tailness parameter τ.

Although the $NIG(\cdot)$ did not appear to our procedure, still it is clear that it provides "heavy tails" and this is evidence that the shape parameter γ in $N(0,1;\gamma)$ "adjusts" the value of tail as well as shape.

Now, with the γ-order Normal, we described a general family of distributions with a particular extra shape parameter γ. The shape parameter can describe "fat tails distributions", "close" to what is known as Gaussian or Normal, see Table 2. Consider that with $\sigma = \sqrt{t} = 1$ the values of $\phi(0,1;\gamma) = \phi_\gamma$ and $K_\gamma = K(x,1;\gamma)$ are simplified. In limited cases, as $\gamma \to 1$ the Uniform distribution can be obtained or the Laplace, when $\gamma \to \infty$, [30]. Notice that the "international constant" $\gamma_0^{\gamma_0}$ plays an important role to the described formulation. That certainly needs more investigation as the statistical generalization might offer chance for food of thought under Mathematical or Physical considerations. Moreover the generalization of the Normal provide generalized entropy type information measures, [17] and possible engineering implementations, [22]. Therefore we open a subject which can offer a number of different lines of thought to work in future. We shall try to continue the statistical line of generalization we are creating.

ACKNOWLEDGMENTS

I would like to thank Prof. N. Halidias, Univ. of Aegean, for useful discussions during the preparation of this paper. The comments of the referees are very much appreciated as eventually improved this paper.

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