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Bayesian Sampling Plan for Weibull Distribution with Type II Hybrid Censoring under Random Decision Function

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Abstract:

- This article studies the problem of conception of a Bayesian single variable sampling plan for Weibull distribution under type II hybrid censoring based on two-sided decision function with a linear random doubt zone. Based on an appropriate loss function, an explicit expression for the Bayes risk is determined. To find the optimal sampling plans, a simple algorithm based on the grid search method is described. Finally, simulation study is given to illustrate the proposed model.

Keywords:

- *Bayesian sampling; type II hybrid censoring; loss function; random decision function; grid search method.*

AMS Subject Classification:

- 62D05, 62F15, 62N05.

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1. INTRODUCTION

Quality control is one of the most important issues of the modern industry, to determine whether the quality of the products or process is satisfactory according to certain criteria established in advance. We distinguish two types of control, the control during production: which is the one carried out at different stages during the production process, and the reception control: which is the one carried out by the producer or the consumer during the inspection of a finished product, which also requires taking sampling plans. There have been several criteria to construct sampling plans. Criteria based on decision theory are the most efficient for quality control, in the sense that the sampling plan is determined by taking an optimal decision. Numerous study have investigated along with this approach, we refer to [9, 11, 19, 10].

Recently, a number of studies have investigated Bayesian sampling plans based on the lifetime censored data. Readers are referred to the sampling plan based on type II censored sample [12] and [5], sampling plan based on type I censored sample [13] and [18], interval censored sample [6]. The type I hybrid censored sample was initially introduced in [8]. In [7] the exact distribution of the maximum likelihood estimator (MLE) of the expected lifetime is provided where the lifetime of components follows exponential distribution under type I and type II hybrid censoring. Reference [14] have studied sampling plans under type I and type II hybrid censoring for quadratic loss function based on the results of [7]. Furthermore, a Bayesian sampling plan based on type I hybrid censored samples has been developed in [15] using a conventional one-sided decision function. Modified type II hybrid censoring has been provided by [20]. For exponential distribution under type I censoring and type I hybrid censoring a new shrinkage estimator for the expected lifetime has been studied in [17], which always exists even if no failure occurs at the termination time. In addition, Reference [17] provided that the construction of the Bayes decision function (as in [20], [15]), which is based on the posterior expectation, becomes more difficult if the loss function is not polynomial.

In some industrial process, the quality characteristics data are derived from a complex production process or from an uncertain environment. Much acceptance sampling plans have been proposed under this situation, [2, 3] have developed acceptance sampling plan for variable and attribute using the neutrosophic statistics. [4] discussed a Bayesian sampling plan under two-sided decision function based on linear random doubt zone.

In this work, we develop a Bayesian single variable sampling plan for Weibull distribution based on the modified type II hybrid censored sample under random decision function. However, we generalize the work of [4] into two valuable issues. The first issue, the Weibull distribution, which is frequently used in life testing due to flexibility in term of hazard function (see e.g. [1]), and with the commonly used of other distributions as special cases, such as the exponential and Rayleigh distributions. The second issue, the type II hybrid censoring which is a generalization of type II censoring. The type II hybrid censoring has the advantage that at least m failures or more can be observed at the censoring time, which leads to significant efficiency of the model. The rest of this paper is organized in the following way. In Section 2, we provide the proposed random decision function and all necessary assumptions. In Sections 3 and 4, we obtain an explicit expression for the Bayes risk using a polynomial and non polynomial loss respectively. A simple algorithm based on the grid search method to obtain an optimal sampling plan is provided in Section 5. In Section 6, we give numerical examples for the polynomial and non polynomial loss functions followed by some remarks. We finish by a conclusion in Section 7.

2. FORMULATION OF THE PROBLEM

Suppose that we have a batch of items prepared for inspection. The lifetime of each item is a random variable X which follows a Weibull distribution $W(\lambda, \mu)$:

$$f(x|\lambda,\mu) = \begin{cases} \lambda\mu x^{\mu-1}\exp(-\lambda x^{\mu}), & \text{for } x \geq 0, \\ 0, & \text{otherwise,} \end{cases}$$

with the shape parameter μ is known and the scale parameter λ is unknown. It is easy to show that X^{μ} follows an exponential distribution with expected lifetime $1/\lambda$. Further, We assume that λ has a prior distribution $\Gamma(\alpha, \beta)$ where α and β are known, with the pdf:

$$g(\lambda;\alpha,\beta) = \begin{cases} \lambda^{\alpha-1}\exp(-\beta\lambda)\beta^{\alpha}/\Gamma(\alpha), & \text{for } \lambda > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Given a random sample of size n, taken from a batch for life testing. Assume that the modified type II hybrid censoring is adopted. Let $\underline{X} = (X_{(1)}, X_{(2)}, ..., X_{(n)})$ be the order statistic of sample $(X_1, X_2, ..., X_n)$, the life test terminates at the random time $\tau_{n,m} = \min\{max(X_{(m)}, t), X_{(n)}\}$ with $m \leq n$. The likelihood function in this case is given by:

$$l(\underline{X}|\lambda) = \begin{cases} \dfrac{n!(\lambda\mu)^m \prod_{i=1}^{m} X_{(i)}^{\mu}}{(n-m)!} e^{-\lambda\left(\sum_{i=1}^{m} X_{(i)}^{\mu} + (n-m)X_{(m)}^{\mu}\right)} & \text{for } D = 0, 1, ..., m-1, \\ \dfrac{n!(\lambda\mu)^D \prod_{i=1}^{D} X_{(i)}^{\mu}}{(n-D)!} e^{-\lambda\left(\sum_{i=1}^{D} X_{(i)}^{\mu} + (n-D)t^{\mu}\right)} & \text{for } D = m, m+1, ..., n \end{cases}$$

where D represents the number of observed failures that occur before time t.
Then, the MLE of $\theta = 1/\lambda$ is given by:

$$\hat{\theta} = \begin{cases} \frac{\sum_{i=1}^{m} X_{(i)}^{\mu} + (n-m)X_{(m)}^{\mu}}{m}, & \text{for } D = 0, 1, ..., m-1, \\ \frac{\sum_{i=1}^{D} X_{(i)}^{\mu} + (n-D)t^{\mu}}{D}, & \text{for } D = m, m+1, ..., n, \end{cases} \tag{2.1}$$

According to [7], the exact distribution of the MLE of θ:

$$f_{\hat{\theta}}(y) = \sum_{d=0}^{n}\sum_{j=0}^{d}(-1)^j \binom{n}{d}\binom{d}{j} e^{-\lambda t^{\mu}(n-d+j)} g(y - a_{j,M}; M, \lambda M). \tag{2.2}$$

where $a_{j,M} = (n-d+j)t^{\mu}/M$, and $M = max\{d, m\}$.

Let C_s, C_t and C_r be positive constants and represent respectively the unit inspection cost, the cost per unit of time used for the test and the loss due to rejection of the batch. Let $a_0 + a_1\lambda + \cdots + a_k\lambda^k$ denote the loss of accepting the batch and be positive and increasing in λ. When the life test is interrupted, the unfailures items can be reused and therefore have the salvage value v_s, where $0 < v_s < C_s$, then the loss function is defined as follows:

$$L(\lambda,\delta(\underline{x})) = \begin{cases} nC_s - (n-D_{n,m})v_s + C_t\tau_{n,m} + \sum\limits_{i=0}^{k} a_i\lambda^i, & \text{for } \delta(\underline{x}) = d_0, \\ nC_s - (n-D_{n,m})v_s + C_t\tau_{n,m} + C_r, & \text{for } \delta(\underline{x}) = d_1, \end{cases} \tag{2.3}$$

where d_0 and d_1 represent the decisions of accepting and rejecting the batch respectively. The random variable $D_{n,m}$ denotes the number of failures that occur before the termination time $\tau_{n,m}$. $\delta(\underline{x})$ is the decision function which depends on the observation failures $\underline{x} = (x_{(1)}, x_{(2)}, ..., x_{(n)})$. We propose the following two-sided decision function:

$$\delta(\underline{x}) = \begin{cases} d_0, & \text{for } \hat{\theta} \geq T_0, \\ \begin{cases} d_1, & \text{with probability } p_\theta \\ d_0, & \text{with probability } 1-p_\theta \end{cases} & \text{for } T_1 \leq \hat{\theta} < T_0, \\ d_1, & \text{for } \hat{\theta} < T_1, \end{cases} \tag{2.4}$$

where $p_\theta = \frac{T_0-\hat{\theta}}{T_0-T_1}$, and $0 < T_1 < T_0$. Note that, the decision function in Equation (2.4) is described similarly as in [4].

3. COMPUTATION OF THE BAYES RISK

Based on the decision function $\delta(\underline{x})$, the Bayes risk can be computed as follows:

$$\begin{aligned} R(n,m,t,T_0,T_1) &= E\{E[L(\lambda,\delta(\underline{x}))]\} \\ &= E\left\{E\left[nC_s + C_t\tau_{n,m} - (n-D_{n,m})v_s + d_1C_r + (1-d_1)\sum_{i=0}^{k} a_i\lambda^i|\lambda\right]\right\} \\ &= n(C_s-v_s) + v_sE\{E[D_{n,m}|\lambda]\} + C_tE\{E[\tau_{n,m}|\lambda]\} + \sum_{i=0}^{k} a_i\gamma_i \\ &\quad + E\left\{E\left[d_1\sum_{i=0}^{k}\omega_i\lambda^i|\lambda\right]\right\} \\ &= n(C_s-v_s) + v_sE\{E[D_{n,m}|\lambda]\} + C_tE\{E[\tau_{n,m}|\lambda]\} + \sum_{i=0}^{k} a_i\gamma_i + r(n,m|d_1), \end{aligned}$$

here γ_i represents the i-th moment of λ, and

$$\omega_i = \begin{cases} C_r - a_0, & \text{for } i = 0, \\ -a_i & \text{for } i = 1, ..., k. \end{cases} \tag{3.1}$$

Such as

$$
\begin{aligned}
r(n,m|d_1) &= E\Big\{E\Big[\sum_{i=0}^{k}\omega_i\lambda^i d_1|\lambda\Big]\Big\} = E\Big\{\sum_{i=0}^{k}\omega_i\lambda^i E\Big[I_{\hat{\theta}<T_1} + p_\theta I_{T_1\le\hat{\theta}<T_0}|\lambda\Big]\Big\} \\
&= \sum_{i=0}^{k}\omega_i\int_0^\infty \tfrac{\beta^\alpha}{\Gamma(\alpha)}e^{-\beta\lambda}\lambda^{\alpha+i-1}\Bigg[\int_0^{T_1} f_{\hat{\theta}}(y)dy + \int_{T_1}^{T_0}\tfrac{T_0-y}{T_0-T_1}f_{\hat{\theta}}(y)dy\Bigg]d\lambda \\
&= \sum_{d=0}^{n}\sum_{j=0}^{d}\sum_{i=0}^{k}(-1)^j\omega_i\tbinom{n}{d}\tbinom{d}{j}\int_0^\infty\Bigg[\int_{a_{j,M}}^{T_1}\tfrac{\beta^\alpha M^M\left(y-a_{j,M}\right)^{M-1}}{\Gamma(\alpha)\Gamma(M)}e^{-(\beta+My)\lambda}\lambda^{\alpha+M+i-1}dy \\
&\quad + \int_{T_1}^{T_0}\tfrac{T_0-y}{T_0-T_1}\tfrac{\beta^\alpha M^M\left(y-a_{j,M}\right)^{M-1}}{\Gamma(\alpha)\Gamma(M)}e^{-(\beta+My)\lambda}\lambda^{\alpha+M+i-1}dy\Bigg]d\lambda \\
&= \sum_{d=0}^{n}\sum_{j=0}^{d}\sum_{i=0}^{k}\tfrac{\beta^\alpha M^M\Gamma(M+\alpha+i)}{\Gamma(\alpha)\Gamma(M)}\Bigg[\int_0^{T_1-a_{j,M}}\tfrac{y^{M-1}}{\left(\beta+Ma_{j,M}+My\right)^{\alpha+M+i}}dy \\
&\quad + \int_{T_1-a_{j,M}}^{T_0-a_{j,M}}\tfrac{T_0-y-a_{j,M}}{T_0-T_1}\tfrac{y^{M-1}}{\left(\beta+Ma_{j,M}+My\right)^{\alpha+M+i}}dy\Bigg],
\end{aligned}
$$

Using $z = \dfrac{My}{My+\beta+Ma_{j,M}}$ we obtain

$$
\begin{aligned}
&r(n,m|d_1) \\
&= \sum_{d=0}^{n}\sum_{j=0}^{d}\sum_{i=0}^{k}(-1)^j\omega_i\tbinom{n}{d}\tbinom{d}{j}\tfrac{\beta^\alpha\Gamma(M+\alpha+i)}{\Gamma(\alpha)\Gamma(M)\left(\beta+Ma_{j,M}\right)^{\alpha+i}}\Bigg[\int_0^{q_1} z^{M-1}(1-z)^{\alpha+i-1}dz \\
&+\tfrac{T_0-a_{j,M}}{T_0-T_1}\int_{q_1}^{q_0} z^{M-1}(1-z)^{\alpha+i-1}dz - \tfrac{\beta+Ma_{j,M}}{T_0-T_1}\int_{q_1}^{q_0} z^{M-1}(1-z)^{\alpha+i-1}dz\Bigg] \\
&= \sum_{d=0}^{n}\sum_{j=0}^{d}\sum_{i=0}^{k}\tfrac{(-1)^j\omega_i\binom{n}{d}\binom{d}{j}\beta^\alpha\Gamma(\alpha+i)}{\Gamma(\alpha)\left(\beta+Ma_{j,M}\right)^{\alpha+i}}\Big\{I_{q_1}(M,\alpha+i) + \tfrac{T_0-a_{j,M}}{T_0-T_1}[I_{q_0}(M,\alpha+i)- \\
&I_{q_1}(M,\alpha+i)] - \tfrac{\beta+Ma_{j,M}}{(\alpha+i-1)(T_0-T_1)}[I_{q_0}(M+1,\alpha+i-1) - I_{q_1}(M+1,\alpha+i-1)]\Big\},
\end{aligned}
$$

where $q_i = \frac{M\left(T_i-a_{j,M}\right)}{\beta+M\left(T_i-a_{j,M}\right)+Ma_{j,M}}$. $B_x(a,b)$ and $I_x(a,b)$ denote the incomplete Beta function and the cdf of Beta distribution respectively.

Hence, the Bayes risk $R(n,m,t,T_0,T_1)$ can be expressed as:

$$
\begin{aligned}
&R(n,m,t,T_0,T_1) \\
&= \sum_{d=0}^{n}\sum_{j=0}^{d}\sum_{i=0}^{k}\tfrac{(-1)^j\omega_i\binom{n}{d}\binom{d}{j}\beta^\alpha\Gamma(\alpha+i)}{\Gamma(\alpha)\left(\beta+Ma_{j,M}\right)^{\alpha+i}}\Big\{I_{q_1}(M,\alpha+i) + \tfrac{T_0-a_{j,M}}{T_0-T_1}[I_{q_0}(M,\alpha+i)- \\
&I_{q_1}(M,\alpha+i)] - \tfrac{\beta+Ma_{j,M}}{(\alpha+i-1)(T_0-T_1)}[I_{q_0}(M+1,\alpha+i-1) - I_{q_1}(M+1,\alpha+i-1)]\Big\} \\
&\quad + n(C_s-v_s) + v_s\sum_{d=0}^{n}\sum_{j=0}^{d}(-1)^{d-j}M\tbinom{n}{d}\tbinom{d}{j}\left(\tfrac{\beta}{\beta+(n-j)t^\mu}\right)^\alpha + \sum_{i=0}^{k}a_i\gamma_i + \tau^* C_t,
\end{aligned}
\tag{3.2}
$$

where, for $m < n$

$$\begin{aligned}\tau^* &= E\{E[\tau_{n,m}|\lambda]\}\\ &= m\binom{n}{m}\sum_{j=0}^{m-1}(-1)^{m-j-1}\tbinom{m-1}{j}\tfrac{\alpha\beta^{1/\mu}}{(n-j)^{1+1/\mu}}B_{1-q*}\Big(1+\tfrac{1}{\mu},\alpha-\tfrac{1}{\mu}\Big)\\ &\quad+\tfrac{tn!}{(m-1)!(n-m-1)!}\sum_{i=0}^{m-1}\sum_{j=0}^{n-m-1}\Big[(-1)^{n-i-j}\tbinom{m-1}{i}\tbinom{n-m-1}{j}\\ &\quad\times\tfrac{\beta^\alpha}{(m+j-i)(n-m-j)}\Big(\tfrac{1}{((n-m-j)t^\mu+\beta)^\alpha}-\tfrac{1}{((n-i)t^\mu+\beta)^\alpha}\Big)\Big]\\ &\quad+n\sum_{j=0}^{n-1}(-1)^{n-j-1}\tbinom{n-1}{j}\tfrac{\alpha\beta^{1/\mu}}{(n-j)^{1+1/\mu}}B_{q*}\Big(1+\tfrac{1}{\mu},\alpha-\tfrac{1}{\mu}\Big),\end{aligned}$$

and, for $m = n$

$$\tau^* = E\{E[\tau_{n,m}|\lambda]\} = n\alpha\beta^{1/\mu}B\Big(1+\tfrac{1}{\mu},\alpha-\tfrac{1}{\mu}\Big)\sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}\tfrac{1}{(j+1)^{1+1/\mu}},$$

with $q* = \frac{(n-j)t^\mu}{\beta+(n-j)t^\mu}$. The computation of $E\{E[D_{n,m}|\lambda]\}$ and $E\{E[\tau_{n,m}|\lambda]\}$ is provided in the appendix.

4. BAYES RISK FOR NON-POLYNOMIAL LOSS FUNCTION

In this section we provide an explicit expression for the Bayes risk under non-polynomial loss function, which can be written as:

$$L_{NP}(\lambda,\delta(\underline{x})) = \begin{cases} nC_s-(n-D_{n,m})v_s+C_t\tau_{n,m}+\exp(c\lambda)-c\lambda-1, & \text{for } \delta(\underline{x})=d_0,\\ nC_s-(n-D_{n,m})v_s+C_t\tau_{n,m}+C_r, & \text{for } \delta(\underline{x})=d_1,\end{cases} \tag{4.1}$$

where the loss of accepting the batch $\exp(c\lambda)-c\lambda-1$ is of the form LINEX loss (see e.g. [1, 16]). The value of c must be positive for ensuring that, the loss of accepting the batch is increasing in λ.

$$\begin{aligned}&R_{NP}(n,m,t,T_0,T_1) = E\{E[L_{NP}(\lambda,\delta(\underline{x}))]\}\\ &= E\{E[nC_s+C_t\tau_{n,m}-(n-D_{n,m})v_s+d_1C_r+(1-d_1)(\exp(c\lambda)-c\lambda-1)|\lambda]\}\\ &= n(C_s-v_s)+v_sE\{E[D_{n,m}|\lambda]\}+C_tE\{E[\tau_{n,m}|\lambda]\}+\Big(\tfrac{\beta}{\beta-c}\Big)^\alpha-c\tfrac{\alpha}{\beta}-1\\ &\quad+E\{E[d_1(C_r+1+c\lambda-\exp(c\lambda))|\lambda]\}\\ &= n(C_s-v_s)+v_sE\{E[D_{n,m}|\lambda]\}+C_tE\{E[\tau_{n,m}|\lambda]\}+\Big(\tfrac{\beta}{\beta-c}\Big)^\alpha-c\tfrac{\alpha}{\beta}-1+r'(n,m|d_1),\end{aligned}$$

with

$$\begin{aligned}r'(n,m|d_1) &= E\{E[d_1(C_r+1+c\lambda-\exp(c\lambda))|\lambda]\}\\ &= E\Big\{(C_r+1+c\lambda-\exp(c\lambda))E\Big[I_{\hat\theta<T_1}+p_\theta I_{T_1\le\hat\theta<T_0}|\lambda\Big]\Big\}\\ &= \sum_{i=0}^{1}\omega_i'\int_0^\infty\tfrac{\beta^\alpha}{\Gamma(\alpha)}e^{-\beta\lambda}\lambda^{\alpha+i-1}\left[\int_0^{T_1}f_{\hat\theta}(y)dy+\int_{T_1}^{T_0}\tfrac{T_0-y}{T_0-T_1}f_{\hat\theta}(y)dy\right]d\lambda\\ &\quad-\int_0^\infty\tfrac{\beta^\alpha}{\Gamma(\alpha)}e^{-(\beta-c)\lambda}\lambda^{\alpha-1}\left[\int_0^{T_1}f_{\hat\theta}(y)dy+\int_{T_1}^{T_0}\tfrac{T_0-y}{T_0-T_1}f_{\hat\theta}(y)dy\right]d\lambda,\end{aligned}$$

where $\omega_0' = C_r+1$, $\omega_1' = c$.

From the previous section, we have

$$\begin{aligned}
&\int_0^\infty \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-(\beta-c)\lambda}\lambda^{\alpha-1}\left[\int_0^{T_1} f_{\hat\theta}(y)dy + \int_{T_1}^{T_0}\frac{T_0-y}{T_0-T_1} f_{\hat\theta}(y)dy\right]d\lambda\\
&=\sum_{d=0}^{n}\sum_{j=0}^{d}(-1)^j\binom{n}{d}\binom{d}{j}\int_0^\infty\left[\int_{a_{j,M}}^{T_1}\frac{\beta^\alpha M^M\big(y-a_{j,M}\big)^{M-1}}{\Gamma(\alpha)\Gamma(M)}e^{-(\beta-c+My)\lambda}\lambda^{\alpha+M-1}dy\right.\\
&\quad\left.+\int_{T_1}^{T_0}\frac{T_0-y}{T_0-T_1}\frac{\beta^\alpha M^M\big(y-a_{j,M}\big)^{M-1}}{\Gamma(\alpha)\Gamma(M)}e^{-(\beta-c+My)\lambda}\lambda^{\alpha+M-1}dy\right]d\lambda\\
&=\sum_{d=0}^{n}\sum_{j=0}^{d}\frac{\beta^\alpha M^M\Gamma(M+\alpha)}{\Gamma(\alpha)\Gamma(M)}\left[\int_0^{T_1-a_{j,M}}\frac{y^{M-1}}{\big(\beta-c+Ma_{j,M}+My\big)^{\alpha+M}}dy\right.\\
&\quad\left.+\int_{T_1-a_{j,M}}^{T_0-a_{j,M}}\frac{T_0-y-a_{j,M}}{T_0-T_1}\frac{y^{M-1}}{\big(\beta-c+Ma_{j,M}+My\big)^{\alpha+M+i}}dy\right]\\
&=\sum_{d=0}^{n}\sum_{j=0}^{d}\frac{(-1)^j\binom{n}{d}\binom{d}{j}\beta^\alpha}{\big(\beta-c+Ma_{j,M}\big)^\alpha}\left\{I_{q_1'}(M,\alpha)+\frac{T_0-a_{j,M}}{T_0-T_1}\Big[I_{q_0'}(M,\alpha)-I_{q_1'}(M,\alpha)\Big]\right.\\
&\quad\left.-\frac{\beta-c+Ma_{j,M}}{(\alpha-1)(T_0-T_1)}\Big[I_{q_0'}(M+1,\alpha-1)-I_{q_1}(M+1,\alpha-1)\Big]\right\},
\end{aligned}$$

with $q_i' = \frac{M\big(T_i-a_{j,M}\big)}{\beta-c+M\big(T_i-a_{j,M}\big)+Ma_{j,M}}$.

Therefore, the Bayes risk expression under the loss function 4.1 is given by:

$$\begin{aligned}
(4.2)\qquad & R_{NP}(n,m,t,T_0,T_1)\\
&=\sum_{d=0}^{n}\sum_{j=0}^{d}\sum_{i=0}^{1}\frac{(-1)^j\omega_i'\binom{n}{d}\binom{d}{j}\beta^\alpha\Gamma(\alpha+i)}{\Gamma(\alpha)\big(\beta+Ma_{j,M}\big)^{\alpha+i}}\Big\{I_{q_1}(M,\alpha+i)+\frac{T_0-a_{j,M}}{T_0-T_1}[I_{q_0}(M,\alpha+i)-\\
&I_{q_1}(M,\alpha+i)]-\frac{\beta+Ma_{j,M}}{(\alpha+i-1)(T_0-T_1)}[I_{q_0}(M+1,\alpha+i-1)-I_{q_1}(M+1,\alpha+i-1)]\Big\}\\
&-\sum_{d=0}^{n}\sum_{j=0}^{d}\frac{(-1)^j\binom{n}{d}\binom{d}{j}\beta^\alpha}{\big(\beta-c+Ma_{j,M}\big)^\alpha}\left\{I_{q_1'}(M,\alpha)+\frac{T_0-a_{j,M}}{T_0-T_1}\Big[I_{q_0'}(M,\alpha)-I_{q_1'}(M,\alpha)\Big]\right.\\
&\left.-\frac{\beta-c+Ma_{j,M}}{(\alpha-1)(T_0-T_1)}\Big[I_{q_0'}(M+1,\alpha-1)-I_{q_1}(M+1,\alpha-1)\Big]\right\}+n(C_s-v_s)\\
&+v_s\sum_{d=0}^{n}\sum_{j=0}^{d}(-1)^{d-j}M\binom{n}{d}\binom{d}{j}\left(\frac{\beta}{\beta+(n-j)t^\mu}\right)^\alpha+\left(\frac{\beta}{\beta-c}\right)^\alpha-c\frac{\alpha}{\beta}-1+\tau^*C_t.
\end{aligned}$$

5. NUMERICAL APPROXIMATIONS

The expression of $R(n,m,t,T_0,T_1)$ and $R_{NP}(n,m,t,T_0,T_1)$ are quite complicated, so we cannot get the optimal sampling plan analytically. Using the grid search method we can obtain an optimal sampling plan numerically. As given in [17], we assume that T_0 has an upper bound since $0 < T_0 < T_0^*$, and for t as given in [13], we obtain a confidence interval $[t_L, t_U]$ such that $P(X > t_U) = \eta/2$ and $P(X < t_L) = \eta/2$ where:

$$P(X < t_L) = \int_0^\infty\int_0^{t_L}\frac{\beta^\alpha}{\Gamma(\alpha)}\lambda^{\alpha-1}e^{-\beta\lambda}\lambda x^{\mu-1}e^{-\lambda x^\mu}dxd\lambda = \eta/2,$$

and

$$P(X > t_U) = \int_0^\infty \int_{t_U}^\infty \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda} \lambda x^{\mu-1} e^{-\lambda x^\mu} dx d\lambda = \eta/2,$$

hence

$$t_L = \left\{\beta\left[\left(1-\tfrac{\eta}{2}\right)^{-1/\alpha} - 1\right]\right\}^{\frac{1}{\mu}}$$
$$t_U = \left\{\beta\left[\left(\tfrac{\eta}{2}\right)^{-1/\alpha} - 1\right]\right\}^{\frac{1}{\mu}}.$$

5.1. An upper bound for the optimal size sample

To obtain the optimal sampling plan, we provide an upper bound for the optimal sample size, and then the optimal sampling plan can be obtained in a finite number of search steps.

Theorem 5.1. *The optimal sample is bounded by:*

$$N = \min\left\{\left[\frac{C_r}{C_s - v_s}\right], \left[\frac{\sum_{i=0}^k a_i\gamma_i}{C_s - v_s}\right]\right\}, \tag{5.1}$$

where $[x]$ is the integer part of x.

Proof: Let $(0,0,0,0,0)$ and $(0,0,0,\infty,\infty)$ be the sampling plans that accepts and rejects the batch without taking sampling respectively. For (n',m',t',T_0',T_1') an optimal sampling plan, we have $R(n',m',t',T_0',T_1') \leq R(0,0,0,0,0) = \sum_{i=0}^k a_i\gamma_i$. and $R(n',m',t',T_0',T_1') \leq R(0,0,0,\infty,\infty) = C_r$.

As $n(C_s - v_s) \leq R(n',m',t',T_0',T_1')$, therefore

$$n(C_s - v_s) \leq \min\left\{C_r, \sum_{i=0}^k a_i\gamma_i\right\}$$
$$n \leq \min\left\{\left[\frac{C_r}{C_s - v_s}\right], \left[\frac{\sum_{i=0}^k a_i\gamma_i}{C_s - v_s}\right]\right\}.$$

Hence the result. □

Algorithm 5.1. To derive an optimal sampling plan (n',m',t',T_0',T_1') based on the minimization of the Bayes risk, a finite algorithm is described in the following steps:

a) Start with $(n,m,t) = (0,0,0)$, compute N from (5.1) and compute $R(0,0,0,T_0,T_1)$ $= \min\left\{R(0,0,0,\infty,\infty) = C_r, R(0,0,0,0,0) = \sum_{i=0}^k a_i\gamma_i\right\}$.

b) For fixed (n,m,t), compute the optimal $T'_{0,(n,m,t)}$ and $T'_{1,(n,m,t)}$ using grid search method, such that $R\left(n,m,t,T'_{0,(n,m,t)},T'_{1,(n,m,t)}\right) = \min_{0<T_1<T_0\leq T^*} R(n,m,t,T_1,T_0)$, with grid size 0.0125.

c) For fixed (n, m), compute the optimal $t'_{(n,m)}$ using grid search method, such that $R\Big(n, m, t'_{(n,m)}, T'_{0,(n,m,t)}, T'_{1,(n,m,t)}\Big) = \min_{t_L \le t \le t_U} R\Big(n, m, t, T'_{0,(n,m,t)}, T'_{1,(n,m,t)}\Big)$, with grid size $\frac{t_U - t_L}{100}$.

d) For $0 \le m \le n \le N$, choose (n', m', t', T'_0, T'_1) which corresponds to the smallest value of the Bayes risks $R\Big(n, m, t'_{(n,m)}, T'_{0,(n,m,t)}, T'_{1,(n,m,t)}\Big)$.

6. AN ILLUSTRATIVE EXAMPLE

To implement the Algorithm 5.1, we assume that the loss is a quadratic function with $(k = 2)$. Then, we assume that the loss function is a quintic polynomial. Using the upper bound of sample size and the grid search method various numerical examples are presented in Tables 1–4. In each table we indicate the optimal Bayesian sampling plans by $S_0 \equiv (n', m', t', T'_0, T'_1)$, and the corespondent Bayes risk by $R_0 \equiv R(n', m', t', T'_0, T'_1)$. Also, we denote the expected number of observation failures by $E[D_0]$, and the expected termination time by $E[\tau_0]$. During computation and in some cases the optimal sampling plan is achieved when T_1 close to T_0. So, to make a sense to the sampling plan (n, m, t, T_0, T_1) we assume that $T_0 - T_1 \ge 0.05$, $T^* = T^*_0 = 2$ and $\eta = 0.05$. As the true values of parameters and coefficients for the quadratic loss for which we made the calculations, we take $\mu = 2$, $\alpha = 2$, $\beta = 1$, $a_0 = a_1 = a_2 = 3$, $C_s = 0.5$, $v_s = 0.2$, $C_t = 2$, $C_r = 30$. For the previous standard values, the optimal sampling plan is $(5, 1, 0.3104, 0.7750, 0.2000)$, which means, we put 5 items for life testing, and when $t = 0.3104$ is less than the time of fifth failure $X_{(5)}$, the life test terminates after the maximum between the first failure and $t = 0.3104$, otherwise the life test terminates at $X_{(5)}$. We accept the batch if the estimator of the average lifetime $\hat{\theta}$ is greater than or equal 0.7750, and we reject it if $\hat{\theta}$ is less than 0.2000. For $\hat{\theta}$ is between 0.7750 and 0.2000, the batch is rejected and accepted with probability $p_{\hat{\theta}} = \Big(0.7750 - \hat{\theta}\Big)/(0.7750 - 0.2000)$ and $1 - p_{\hat{\theta}}$ respectively, the corresponding Bayes risk is $R_0 = 23.9637$.

In Table 1, we observe that for α fixed and β decreases while $\mu = 2$, $a_0 = a_1 = a_2 = 3$, $C_s = 0.5$, $v_s = 0.2$, $C_t = 2$ and $C_r = 30$, the Bayes risk R_0 increase. And for β fixed R_0 is increasing in α. On the other hand, we can see that the expected number of failure $E[D_0]$ is close to m' and the expected termination time $E[\tau_0]$ is always greater than t'. Furthermore, for each couple $(\alpha, \beta) = (1.5, 0.2)$,$(2.0, 0.4)$,$(2.5, 0.6)$,$(3.0, 0.8)$,$(3.5, 0.8)$,$(3.5, 1.0)$, the batch is rejected without any sample cost, and thus $R_0 = C_r = 30$. In Table 2, we can see that, the minimum Bayes risk R_0 significantly increases with the values of a_2, and the optimal sample size n' decreases for a_2 increasing. Furthermore, the optimal number of fixed failures m' is close to n' when a_2 increases. For $a_2 \le 2$ and the other parameters and coefficients are fixed, the sampling plan $S_0 = (0, 0, 0, 0, 0)$ with $R_0 = a_0 + a_1\alpha/\beta + a_2(\alpha^2 + \alpha)/\beta^2$ where the batch is accepted for no sampling case. And, for $a_2 \ge 15$ the optimal plan $S_0 = (0, 0, 0, \infty, \infty)$ with $R_0 = C_r = 30$, the batch is rejected without taking sampling. In Table 3, it is observed that $E[D_0] \ge m'$ and $E[\tau_0] \ge t'$, this indicates that the sampling plan S_0 takes more time to better observe the lifetime components, and can obtain more information about the expected lifetime of items. Also, the number of fixed failures brings closer to the optimal sample size when C_t closes to 0. On the other hand, for C_t increases the optimal sample size increases and the minimum Bayes risk increases. From Table 4, it can be seen that, R_0 is increasing in C_r.

And, for $C_r \leq 17.5$, the batch will be rejected with $R_0 = Cr$. For $Cr \geq 45$, the batch will be accepted with minimum Bayes risk $R_0 = 27$.

Table 1: Optimal sampling plans and Bayes risks for α and β vary.

α	β	n'	m'	t'	T_0'	T_1'	$E[D_0]$	$E[\tau_0]$	R_0
1.5	0.2	0	0	0.0000	∞	∞	0.0000	0.0000	30.0000
1.5	0.4	3	2	0.0825	0.6000	0.5500	2.0000	0.6113	28.3413
1.5	0.6	4	1	0.4172	0.1100	0.3000	1.5853	0.5178	25.4861
1.5	0.8	5	1	0.3413	0.8500	0.3250	1.3627	0.4815	22.3640
2.0	0.4	0	0	0.0000	∞	∞	0.0000	0.0000	30.0000
2.0	0.6	3	2	0.0874	0.6000	0.5500	2.0000	0.5881	28.8383
2.0	0.8	4	3	0.1009	0.4750	0.4250	3.0000	0.7894	26.5777
2.0	1.0	5	1	0.3104	0.7750	0.2000	1.2956	0.4212	23.9637
2.5	0.6	0	0	0.0000	∞	∞	0.0000	0.0000	30.0000
2.5	0.8	3	2	0.0902	0.6000	0.5500	2.0000	0.5764	29.2431
2.5	1.0	4	3	0.1009	0.4750	0.4250	3.0000	0.7492	27.3505
2.5	1.2	4	1	0.3577	0.8250	0.2000	1.3065	0.4528	24.9906
3.0	0.8	0	0	0.0000	∞	∞	0.0000	0.0000	30.0000
3.0	1.0	3	2	0.0921	0.6000	0.5500	2.0000	0.5694	29.5670
3.0	1.2	4	3	0.1008	0.4750	0.4250	3.0000	0.7252	27.9490
3.5	0.8	0	0	0.0000	∞	∞	0.0000	0.0000	30.0000
3.5	1.0	0	0	0.0000	∞	∞	0.0000	0.0000	30.0000
3.5	1.2	3	2	0.0933	0.5750	0.5250	2.0000	0.5648	29.8291

Table 2: Optimal sampling plans and Bayes risks for a_2 varies.

a_2	n'	m'	t	T_0'	T_1'	$E[D_0]$	$E[\tau_0]$	R_0
2.0	0	0	0.0000	0.0000	0.0000	0.0000	0.0000	21.0000
2.5	6	1	0.2884	0.7000	0.3000	1.3312	0.3873	22.8246
3.0	5	1	0.3104	0.7750	0.2000	1.2956	0.4212	23.9637
4.0	5	1	0.3762	1.0750	0.3500	1.5060	0.4607	25.4895
5.0	4	1	0.4421	1.2500	0.3250	1.5159	0.5278	26.0883
6.0	4	1	0.4860	1.5250	0.3750	1.6468	0.5575	27.4065
7.0	4	3	0.1129	0.7250	0.6750	3.0000	0.8827	28.0233
8.0	3	2	0.1129	0.8500	0.8000	2.0000	0.7592	28.5096
10.0	3	2	0.1129	1.0000	0.9500	2.0000	0.7592	29.1995
15.0	0	0	0.0000	∞	∞	0.0000	0.0000	30.0000

Table 3: Optimal sampling plans and Bayes risks for C_t varies.

C_t	n'	m'	t	T_0'	T_1'	$E[D_0]$	$E[\tau_0]$	R_0
1.0	3	3	0.2226	0.4000	0.3500	3.0000	1.1436	23.2092
1.5	4	3	0.1129	0.4000	0.3500	3.0000	0.8827	23.6896
2.0	5	1	0.3104	0.7750	0.2000	1.2956	0.4212	23.9637
2.5	5	1	0.3104	0.7750	0.2000	1.2956	0.4212	24.1743
3.0	6	1	0.2884	0.7500	0.2500	1.3312	0.3873	24.3711
4.0	6	1	0.2884	0.7500	0.2500	1.3312	0.3873	24.7584
5.0	6	1	0.2884	0.7500	0.2500	1.3312	0.3873	25.1458
6.0	6	1	0.2884	0.7500	0.2500	1.3312	0.3873	25.5331
8.0	7	1	0.2665	0.7500	0.2500	1.3437	0.3583	26.3014
10.0	0	0	0.0000	0.0000	0.0000	0.0000	0.0000	27.0000

Table 4: Optimal sampling plans and Bayes risks for C_r varies.

C_r	n'	m'	t	T_0'	T_1'	$E[D_0]$	$E[\tau_0]$	R_0
17.5	0	0	0.0000	∞	∞	0.0000	0.0000	17.5000
20.0	4	1	0.4640	1.4000	0.3250	1.5804	0.5425	19.2507
22.5	4	1	0.4421	1.2500	0.3250	1.5159	0.5278	20.6154
25.0	4	1	0.3982	1.0250	0.2500	1.3943	0.5000	21.8282
27.5	4	1	0.3982	0.9750	0.3000	1.3943	0.5000	22.9330
30.0	5	1	0.3104	0.7750	0.2000	1.2956	0.4212	23.9637
32.5	6	1	0.2884	0.7250	0.2750	1.3312	0.3873	24.7412
35.0	6	1	0.2884	0.7000	0.3000	1.3312	0.3873	25.4510
40.0	6	1	0.2884	0.6750	0.3250	1.3312	0.3873	26.1433
45.0	0	0	0.0000	0.0000	0.0000	0.0000	0.0000	27.0000

6.1. Numerical examples for higher degree polynomal and non polynomial loss

To simulate the Bayes risk performance and obtain the optimal sampling plan under non polynomial loss, a similar algorithm to the one in Section 5 is considered:

a) Start with $(n,m,t)=(0,0,0)$, compute N from (5.1) and compute $R_{NP}(0,0,0,T_0'',T_1'')=\min\Big\{R_{NP}(0,0,0,\infty,\infty)=C_r, R_{NP}(0,0,0,0,0)=\Big(\frac{\beta}{\beta-c}\Big)^{\alpha}-c\frac{\alpha}{\beta}-1\Big\}$.

b) For fixed (n,m,t), compute the optimal $T'_{0,(n,m,t)}$ and $T'_{1,(n,m,t)}$ using grid search method, such that $R\Big(n,m,t,T''_{0,(n,m,t)},T''_{1,(n,m,t)}\Big)=\min\limits_{0<T_1<T_0\le T^*}R(n,m,t,T_1,T_0)$, with grid size 0.0125.

c) For fixed (n,m), compute the optimal $t'_{(n,m)}$ using grid search method, such that $R\Big(n,m,t''_{(n,m)},T''_{0,(n,m,t)},T''_{1,(n,m,t)}\Big)=\min\limits_{t_L\le t\le t_U}R\Big(n,m,t,T''_{0,(n,m,t)},T''_{1,(n,m,t)}\Big)$, with grid size $\frac{t_U-t_L}{100}$.

d) For $0\le m\le n\le N$, choose $(n'',m'',t'',T_0'',T_1'')$ which corresponds to the smallest value of the Bayes risks $R\Big(n,m,t''_{(n,m)},T''_{0,(n,m,t)},T''_{1,(n,m,t)}\Big)$.

Table 5 provides some optimal sampling plans for the polynomial loss with order $k=5$. Under setting: $\mu=2$, $a_1=a_2=a_4=0$, $a_0=a_3=1$, $C_s=0.5$, $v_s=0.2$, $C_t=2$ and $C_r=30$, while α, β and a_5 vary. It appears from this table that the minimum Bayes risk R_0 increases quickly when a_5 increases while α and β fixed are fixed. On the other hand, the values of $E[\tau_0]$ are significant comparing with Table 2, in this case we may observe more than m' failures and this will result in an efficient life testing procedure.

In Table 6, various optimal sampling plans and their minimum Bayes risk are depicted for different values of α, β and c while $\mu=2$, $C_s=0.5$, $v_s=0.2$, $C_t=2$, $Cr=30$. Such that $S_{NP}(n'',m'',t'',T_0'',T_1'')\equiv S_{NP}$ and $R_{NP}(n'',m'',t'',T_0'',T_1'')\equiv R_{NP}$ denote optimal sampling plan and its minimum Bayes risk respectively. As shown in Table 6, the Bayes risk R_{NP} decreases when c is close to 0 for α and β fixed. When c is close to β, R_{NP} and $E[\tau_0]$ are large. There are some optimal sampling plans under no sampling case. For instance see $(\alpha,\beta,c)=(2,1,0.5),(2,1.5,0.7),(4,2,1)$, the optimal sampling plan $S_{NP}=(0,0,0,0,0)$ and the batch is accepted without any sample cost. When $(\alpha, \beta, c) = (5, 3, 2.5)$, $S_{NP} = (0, 0, 0, \infty, \infty)$ and the batch must be rejected without any sample cost.

Table 5: Optimal sampling plans and Bayes risks under polynomial loss with order 5, for α, β and a_5 vary.

α	β	a_5	n'	m'	t	T_0'	T_1'	$E[D_0]$	$E[\tau_0]$	R_0
2	1.0	1	6	5	0.1129	0.7625	0.7125	5.0000	1.0373	26.6566
2	1.0	2	6	5	0.1129	0.8875	0.8375	5.0000	1.0373	28.0317
2	1.0	3	5	4	0.1129	1.1250	1.0750	4.0000	0.9701	29.1941
2	1.5	1	7	5	0.1382	0.6625	0.6125	5.0000	1.1045	21.1053
2	1.5	2	7	5	0.1382	0.7875	0.7375	5.0000	1.1045	23.0980
2	1.5	3	7	5	0.1382	0.8750	0.8250	5.0000	1.1045	24.2304
3	1.5	1	6	5	0.1127	0.7750	0.7250	5.0000	0.9528	28.3606
3	1.5	2	5	4	0.1127	0.9875	0.9375	4.0000	0.8911	29.7964
3	1.5	3	0	0	0.0000	∞	∞	0.0000	0.0000	30.0000
3	2.0	1	7	6	0.1302	0.6625	0.6125	6.0000	1.1577	23.7820
3	2.0	2	7	6	0.1302	0.7875	0.7375	6.0000	1.1577	26.0688
3	2.0	2	6	5	0.1302	0.9125	0.8625	5.0000	1.1002	27.3013

Table 6: Optimal sampling plans and Bayes risks under non polynomial loss for α, β and c vary.

α	β	c	n''	m''	t''	T_0''	T_1''	$E[D_0]$	$E[\tau_0]$	R_{NP}
2	1.0	0.5	0	0	0.0000	0.0000	0.0000	0.0000	0.0000	02.0000
2	1.0	0.8	5	4	0.1129	0.2250	0.1750	4.0000	0.9701	12.5445
3	1.5	0.7	0	0	0.0000	0.0000	0.0000	0.0000	0.0000	04.1918
3	1.5	1.0	6	5	0.1127	0.2875	0.2375	5.0000	0.9528	15.8429
3	1.5	1.3	7	6	0.1127	0.4500	0.4000	6.0000	1.0026	21.8487
4	2.0	1.0	0	0	0.0000	0.0000	0.0000	0.0000	0.0000	13.0000
4	2.0	1.2	6	5	0.1127	0.3625	0.3125	5.0000	0.9168	19.9275
4	2.0	1.5	7	6	0.1127	0.5375	0.4875	6.0000	0.9647	25.4796
4	2.0	1.8	6	5	0.1127	0.7375	0.6875	5.0000	0.9168	28.9217
5	3.0	1.5	6	5	0.1234	0.4375	0.3875	5.0000	0.9825	20.9812
5	3.0	2.0	6	5	0.1234	0.7750	0.7250	5.0000	0.9825	28.5097
5	3.0	2.5	0	0	0.0000	∞	∞	0.0000	0.0000	30.0000

7. CONCLUSION

In [14], Bayesian sampling plans for exponential distribution based on type II hybrid censored samples under the quadratic loss have been discussed, since the time-consuming cost and the salvage value are not included in the loss function. However, Several single variables sampling plans have been improved in recent years, most improvements have been achieved by considering the one-sided decision function. Such that, these studies do not take into account that a doubt zone can be existed in the decision interval, e.g. this can be happened when the experimenter estimates that the minimum acceptable and the maximum rejectable surviving time are not equal. Nevertheless, there are still some interesting and relevant problems to be addressed in this situation. With this purpose, we have determined Bayesian sampling plans for Weibull distribution under type II hybrid censoring based on a two-sided decision function with a random doubt zone. We provided an explicit expression for the Bayes risk using a suitable polynomial loss, which includes the unit inspection cost, the time consuming-cost, the rejection cost, the salvage value, and the after-sales cost. Furthermore, we have expressed an explicit form for the Bayes risk under non polynomial loss with the LINEX form. It is noticed that, the Bayes risk under the polynomial loss (resp. non polynomial loss) is always quite complicated. So, we proposed an upper bound for the optimal size of the sample and a finite algorithm to simulate the risk function numerically based on the grid search method. Based on the results, it can be concluded that the Bayes risk based on the two-side decision function have robust behavior with considering the changes of the parameters and coefficients in the proposed sampling plan. However, in this paper we have considered Weibull distribution with known shape parameter. Further study of the issue is still required for completely Bayesian analysis to the two-parameter Weibull distribution. More research will be needed along with this issue for other censoring.

A. APPENDIX

A.1. Computation of $E\{E(D_{n,m}|\lambda)\}$

Let $F(x|\lambda,\mu)$ be the cdf of X. The probability function of $D_{n,m}$, such that $D_{n,m} = m, m+1, ..., n$ can be calculated as follows:
For $j = m+1, ..., n$

$$\begin{aligned} P(D_{n,m}=j) &= P(X_1 \le t, X_2 \le t, ..., X_j \le t, X_{j+1} > t, X_{j+2} > t, ..., X_n > t) \\ &= \binom{n}{j} F(t|\lambda,\mu)^j (1-F(t|\lambda,\mu))^{n-j} = \binom{n}{j}\left(1-e^{-\lambda t^\mu}\right)^j e^{-\lambda(n-j)t^\mu}, \\ P(D_{n,m}=m) &= 1-P(D_{n,m}>m) = 1-\sum_{d=m+1}^{n}\tbinom{n}{d}\left(1-e^{-\lambda t^\mu}\right)^d e^{-\lambda(n-d)t^\mu} \\ &= \sum_{d=0}^{m}\tbinom{n}{d}\left(1-e^{-\lambda t^\mu}\right)^d e^{-\lambda(n-d)t^\mu}, \end{aligned}$$

Then for $m \le n$

$$\begin{aligned} &E(D_{n,m}|\lambda) = \sum_{d=m}^{n} dP(D_{n,m}=d) \\ &= \sum_{d=m+1}^{n} d\tbinom{n}{d}\left(1-e^{-\lambda t^\mu}\right)^d e^{-\lambda(n-d)t^\mu} + m\sum_{d=0}^{m}\tbinom{n}{d}\left(1-e^{-\lambda t^\mu}\right)^d e^{-\lambda(n-d)t^\mu} \\ &= \sum_{d=m+1}^{n}\sum_{j=0}^{d}(-1)^{d-j} d\tbinom{n}{d}\tbinom{d}{j}e^{-\lambda(n-j)t^\mu} + m\sum_{d=0}^{m}\sum_{j=0}^{d}(-1)^{d-j}\tbinom{n}{d}\tbinom{d}{j}e^{-\lambda(n-j)t^\mu} \\ &= \sum_{d=0}^{n}\sum_{j=0}^{d}(-1)^{d-j} M\tbinom{n}{d}\tbinom{d}{j}e^{-\lambda(n-j)t^\mu}, \end{aligned}$$

it is easy to show that when $m = n$, $E(D_{n,m}|\lambda) = nP(D_{n,m} = n) = n$. Hence

$$\begin{aligned} E\{E(D_{n,m}|\lambda)\} &= \int_0^\infty E(D_{n,m}|\lambda)g(\lambda;\alpha,\beta)d\lambda \\ &= \sum_{d=0}^{n}\sum_{j=0}^{d}(-1)^{d-j}M\tbinom{n}{d}\tbinom{d}{j}\tfrac{\beta^\alpha}{\Gamma(\alpha)}\int_0^\infty e^{-\lambda(\beta+(n-j)t^\mu)}\lambda^{\alpha-1}d\lambda \\ &= \sum_{d=0}^{n}\sum_{j=0}^{d}(-1)^{d-j}M\tbinom{n}{d}\tbinom{d}{j}\left(\tfrac{\beta}{\beta+(n-j)t^\mu}\right)^\alpha. \end{aligned}$$

A.2. Computation of $E\{E(\tau_{n,m}|\lambda)\}$

The computation of $E\{E(\tau_{n,m}|\lambda)\}$ is similar as in [20]. Let I_A be the indicator function of a set A.

For $m < n$, when $X_{(m)} \geq t$, $\tau_{n,m} = X_{(m)}$, then

$$E\big(X_m I_{\{X_m \geq t\}}|\lambda\big) = \int_t^\infty y f_{X_{(m)}}(y) dy = m\binom{n}{m} \sum_{j=0}^{m-1} (-1)^{m-j-1} \binom{m-1}{j} \int_t^\infty e^{-\lambda(n-j)y^\mu} \lambda y^\mu dy$$

Therefore

$$E\big\{E\big(X_m I_{\{X_m \geq t\}}|\lambda\big)\big\} = m\binom{n}{m} \sum_{j=0}^{m-1} (-1)^{m-j-1} \binom{m-1}{j} \int_0^\infty \int_t^\infty \tfrac{\beta^\alpha}{\Gamma(\alpha)} e^{-\lambda(\beta+(n-j)y^\mu)} \lambda^\alpha y^\mu dy d\lambda = m\binom{n}{m} \sum_{j=0}^{m-1} (-1)^{m-j-1} \binom{m-1}{j} \int_t^\infty \alpha\beta^\alpha \tfrac{\mu y^\mu}{(\beta+(n-j)y^\mu)^{\alpha+1}} dy.$$

A simple transformation $z = (n-j)y^\mu/(\beta+(n-j)y^\mu)$ yields

$$E\big\{E\big[X_m I_{\{X_m \geq t\}}|\lambda\big]\big\} = m\binom{n}{m} \sum_{j=0}^{m-1} (-1)^{m-j-1} \binom{m-1}{j} \tfrac{\alpha\beta^{1/\mu}}{(n-j)^{1+1/\mu}} B_{1-q*}\Big(1+\tfrac{1}{\mu}, \alpha-\tfrac{1}{\mu}\Big).$$

For $X_m < t < X_n$, $\tau_{n,m} = t$, then

$$\begin{aligned} E\big[tI_{\{X_m<t<X_n\}}|\lambda\big] &= \int_0^t \int_t^\infty \tfrac{tn!(\lambda\mu)^2(xy)^{\mu-1}e^{-\lambda(x^\mu+y^\mu)}}{(m-1)!(n-m-1)!} \big(1-e^{-\lambda x^\mu}\big)^{m-1} \big(e^{-\lambda x^\mu} - e^{-\lambda y^\mu}\big)^{n-m-1} dy dx \\ &= \tfrac{tn!}{(m-1)!(n-m-1)!} \int_0^t \int_t^\infty \sum_{i=0}^{m-1} (-1)^{m-i-1} \tbinom{m-1}{i} \lambda\mu x^{\mu-1} e^{-\lambda(m-i)x^\mu} \\ &\quad \times \sum_{j=0}^{n-m-1} (-1)^{n-m-j-1} \tbinom{n-m-1}{j} \lambda\mu y^{\mu-1} e^{-\lambda(n-m-j)y^\mu} e^{-\lambda j x^\mu} \\ &= \tfrac{tn!}{(m-1)!(n-m-1)!} \sum_{i=0}^{m-1} \sum_{j=0}^{n-m-1} \Big[(-1)^{n-i-j} \tbinom{m-1}{i} \tbinom{n-m-1}{j} \int_0^t \lambda\mu x^{\mu-1} e^{-\lambda(m+j-i)x^\mu} dx \\ &\quad \times \int_t^\infty \lambda\mu y^{\mu-1} e^{-\lambda(n-m-j)y^\mu} dy\Big] \\ &= \tfrac{tn!}{(m-1)!(n-m-1)!} \sum_{i=0}^{m-1} \sum_{j=0}^{n-m-1} (-1)^{n-i-j} \tbinom{m-1}{i} \tbinom{n-m-1}{j} \tfrac{e^{-\lambda(n-m-j)t^\mu} - e^{-\lambda(n-i)t^\mu}}{(m+j-i)(n-m-j)}. \end{aligned}$$

Thus

$$\begin{aligned} E\big\{E\big[tI_{\{X_m<t<X_n\}}|\lambda\big]\big\} &= \int_0^\infty E\big[tI_{\{X_m<t<X_n\}}|\lambda\big] g(\lambda;\alpha,\beta) d\lambda \\ &= \tfrac{tn!}{(m-1)!(n-m-1)!} \sum_{i=0}^{m-1} \sum_{j=0}^{n-m-1} \Big[(-1)^{n-i-j} \tbinom{m-1}{i} \tbinom{n-m-1}{j} \\ &\quad \times \tfrac{\beta^\alpha}{(m+j-i)(n-m-j)} \Big(\tfrac{1}{((n-m-j)t^\mu+\beta)^\alpha} - \tfrac{1}{((n-i)t^\mu+\beta)^\alpha}\Big)\Big]. \end{aligned}$$

For $X_{(n)} \leq t$, $\tau_{n,m} = X_{(n)}$, then

$$\begin{aligned} E\big\{E\big[X_{(n)} I_{\{X_n \leq t\}}|\lambda\big]\big\} &= \int_0^\infty \int_0^t y f_{X_{(n)}}(y) g(\lambda;\alpha,\beta) dy d\lambda \\ &= n \sum_{j=0}^{n-1} (-1)^{n-j-1} \tbinom{n-1}{j} \tfrac{\alpha\beta^{1/\mu}}{(n-j)^{1+1/\mu}} B_{q*}\Big(1+\tfrac{1}{\mu}, \alpha-\tfrac{1}{\mu}\Big). \end{aligned}$$

Hence, for $m < n$

$$E\{E[\tau_{n,m}|\lambda]\} = m\binom{n}{m}\sum_{j=0}^{m-1}(-1)^{m-j-1}\binom{m-1}{j}\frac{\alpha\beta^{1/\mu}}{(n-j)^{1+1/\mu}}B_{1-q*}\Big(1+\tfrac{1}{\mu},\alpha-\tfrac{1}{\mu}\Big)$$
$$+\frac{tn!}{(m-1)!(n-m-1)!}\sum_{i=0}^{m-1}\sum_{j=0}^{n-m-1}\Big[(-1)^{n-i-j}\binom{m-1}{i}\binom{n-m-1}{j}$$
$$\times\frac{\beta^{\alpha}}{(m+j-i)(n-m-j)}\Big(\frac{1}{((n-m-j)t^{\mu}+\beta)^{\alpha}}-\frac{1}{((n-i)t^{\mu}+\beta)^{\alpha}}\Big)\Big]$$
$$+n\sum_{j=0}^{n-1}(-1)^{n-j-1}\binom{n-1}{j}\frac{\alpha\beta^{\frac{1}{\mu}}}{(n-j)^{1+1/\mu}}B_{q*}\Big(1+\tfrac{1}{\mu},\alpha-\tfrac{1}{\mu}\Big)$$

For $m = n$, $\tau_{n,m} = X_{(n)}$

$$E\{E[\tau_{n,m}|\lambda]\} = \int_0^{\infty}\int_0^{\infty} y f_{X_{(n)}}(y)g(\lambda;\alpha,\beta)dyd\lambda$$
$$= n\alpha\beta^{1/\mu}B\Big(1+\tfrac{1}{\mu},\alpha-\tfrac{1}{\mu}\Big)\sum_{j=0}^{n-1}(-1)^{j}\binom{n-1}{j}\frac{1}{(j+1)^{1+1/\mu}}.$$

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