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STATIONARY UNDERDISPERSED INAR(1) MODELS BASED ON THE BACKWARD APPROACH

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Abstract:

- Most of the stationary first-order autoregressive integer-valued (INAR(1)) models in the literature have been developed using the idea of binomial thinning. Two approaches have been adopted to establish the distributional properties of a stationary INAR(1) process: the forward approach and the backward approach. In the forward approach, the marginal distribution of the processs is specified and an appropriate distribution for the innovation sequence is sought. Whereas in the backward setting, the roles are reversed. The common distribution of the innovation sequence is specified and the marginal distribution of the process is studied. In this article we focus on the backward approach. Our motivation is mainly theoretical, in the context of statistical distribution theory. We establish a number of basic properties of a specific infinite convolution of distributions on $\mathbb{Z}_+$. We then proceed to interpret our results in the context of stationary INAR(1) models whose innovation has a finite mean. As an application, we present new distributional properties for some stationary INAR(1) processes that show underdispersion, including two new models with q-series innovation distributions.

Keywords:

- *Integer-valued time series, The Binomial thinning operator, q-series, Poissonian Binomial distribution, Heine distribution.*

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1. INTRODUCTION

The area of integer-valued time series has attracted a lot of interest in research and practice during the last 35 years. It started with the pioneering work of McKenzie ([18], [19], [20]) and Al-Osh and Alzaid [2], and Alzaid and Al-Osh [4]. Most of the existing models are based on the Binomial thinning operator of Steutel and van Harn [27]. Models under a variety of generalized thinning operators have been proposed by several authors. We refer to the review articles [21] and [25] for details and additional references.

The Binomial thinning (cf. [27]) of a $\mathbb{Z}_+$-valued random variable X denoted by $\alpha \odot X$, is defined as

$$\alpha \odot X = \sum_{i=1}^{X} Y_i, \tag{1.1}$$

where $\alpha \in (0,1)$ and $\{Y_i\}$ is a sequence of independent identically distributed (iid) Bernoulli(α) rv's independent of X. The operation $\odot$ incorporates the discrete nature of the variates and acts as the analogue of the standard multiplication used in the continuous time series models.

Assume that $0 < \alpha < 1$, and $(\varepsilon_t, t \geq 1)$ is an iid sequence of $\mathbb{Z}_+$-valued rv's. A sequence $(X_t, t \geq 0)$ of $\mathbb{Z}_+$-valued rv's is said to be an INAR (1) process if

$$X_t = \alpha \odot X_{t-1} + \varepsilon_t \qquad (t \geq 1), \tag{1.2}$$

such that the binomial thinning $\alpha \odot X_{t-1}$ in (1.2) is performed independently for each t. More precisely, we assume the existence of an array $(Y_{i,t},\ i \geq 1,\ t \geq 0)$ of iid Bernoulli(α) rv's, independent of $\{\varepsilon_t\}$, such that

$$\alpha \odot X_{t-1} = \sum_{i=1}^{X_{t-1}} Y_{i,t-1}.$$

In (1.2), $\{\varepsilon_t\}$ is referred to as the innovation sequence and α as the coefficient of the process $\{X_t\}$.

The main focus of this paper is on stationary INAR (1) models. The basic question of interest in this case is the choice of the marginal distribution of the process and that of its innovation.

Two approaches to this question prevail. One, which we will refer to as the forward approach, consists in selecting a specific marginal distribution for the process and then searching for the proper innovation distribution. The other approach, referred to as the backward approach, consists of the exact opposite: start out with the marginal distribution of the innovation sequence and then search for the proper marginal distribution of the process.

Both approaches have been widely used in the literature. For the forward approach, we refer to the review articles cited above and references therein. For models based on the backward approach, we cite a number of fairly recent articles: [13], [22], [28], [23], [24], [7], and [16].

In the current work, we adopt the backward approach to develop INAR (1) models driven by (1.2) and whose innovation has finite mean. Our motivation is mainly theoretical, in the context of statistical distribution theory. We establish a number of basic properties of a specific infinite convolution of distributions on $\mathbb{Z}_+$. These results are then used to obtain most of the needed properties of the marginal and the conditional distributions of a stationary INAR (1) model. That is the object of Section 2. As an application, we present new distributional properties for some stationary INAR (1) models that show underdispersion, including two new INAR (1) models with q-series innovation distributions. More specifically, in Sections 3-7 we study in details the models whose innovation follow the Bernoulli distribution, the Binomial distribution, a q-series called the Poissonian Binomial distribution, the logarithmic distribution, and the Heine distribution, another q-series, respectively. We note that the INAR (1) models with Bernoulli, binomial and logarithmic innovations have been discussed in [7]. Our results provide additional properties for these processes. We also note the backward approach has been used by the authors in a related article (see [3]) that develops INAR (1) models with compound Poisson innovations.

We will use throughout the rest of this paper the notation $\overline{a} = 1 - a$, $a \in (0,1)$.

We designate by $\mu_r^{(u)}(\kappa_r^{(u)})$ and $\mu_{[r]}^{(u)}(\kappa_{[r]}^{(u)})$ the r-th moment (cumulant) and the r-th factorial moment (factorial cumulant) of the pmf $\{u_r\}$, respectively. We will make use of the formulas (see [12], Sections 1.2.7 and 1.2.8):

$$\mu_r^{(u)} = \sum_{j=1}^{r} S(r,j)\mu_{[j]}^{(u)} \quad \text{and} \quad \kappa_r^{(u)} = \sum_{j=1}^{r} S(r,j)\kappa_{[j]}^{(u)}, \tag{1.3}$$

where $S(r,j)$ are the Stirling numbers of the second kind defined as $S(0,0)=1, S(0,k)=S(r,0)=0$ and

$$S(r,j) = \frac{1}{j!}\sum_{k=0}^{j}(-1)^{j-k}\binom{j}{k}k^r. \tag{1.4}$$

2. Basic results on the backward approach

Our goal in this section is to establish several properties of a specific infinite convolution of distributions on $\mathbb{Z}_+$. We then proceed to interpret our results in the context of stationary INAR(1) models whose innovation has finite mean.

Theorem 2.1. *Let $\Psi(z)$ be the pgf of a pmf $\{f_r\}$. Assume $\Psi'(1) < \infty$, i.e., $\{f_r\}$ has finite mean. Then, the function*

$$\varphi(z) = \prod_{i=0}^{\infty}\Psi(1-\alpha^i+\alpha^i z) \tag{2.1}$$

is a pgf. Moreover, the convergence of the infinite product is uniform over the interval $[0,1]$ and $\varphi(z)$ satisfies

$$\varphi(z) = \varphi(1-\alpha+\alpha z)\Psi(z), \qquad z \in [0,1]. \tag{2.2}$$

Proof: First, we recall some basic results on pgf's (we refer to [10]). For $k \geq 0$, let $q_k = \sum_{i=k+1}^{\infty} f_i$ be the sequence of the tail probabilities corresponding to $\{f_k\}$ and let

$$Q(z) = \sum_{k=0}^{\infty} q_k z^k,$$

be the generating function of $\{q_k\}$. We have $1-\Psi(z) = (1-z)Q(z)$, $z \in [0,1]$, and $Q(1) = \sum_{k=0}^{\infty} q_k = \sum_{k=0}^{\infty} k f_k = \Psi'(1) < \infty$. Define $h_i(z) = 1-\Psi(1-\alpha^i+\alpha^i z)$. It follows that $h_i(z) = \alpha^i(1-z)Q(1-\alpha^i+\alpha^i z)$. Noting that Q is increasing over $[0,1], 0 \leq 1-z \leq 1$, and $Q(1)$ is finite, we conclude that $0 \leq h_i(z) \leq Q(1)\alpha^i$ and $\sum_{i=n+1}^{\infty} h_i(z) \leq Q(1)\sum_{i=n+1}^{\infty}\alpha^i$. This implies $\sum_{i=n+1}^{\infty} h_i(z)$ converges uniformly to 0 over the interval $[0,1]$. For every $n \geq 0$, define

$$\varphi_{n+1}(z) = \prod_{i=0}^{n}\Psi(1-\alpha^i+\alpha^i z), \tag{2.3}$$

which can be rewritten as $\varphi_{n+1}(z) = \prod\limits_{i=0}^{n}(1-h_i(z))$. It follows by Theorem 1, p. 381, in [17], that the sequence $\{\varphi_{n+1}(z)\}$ converges uniformly over the interval $[0,1]$ to

$$\varphi(z) = \prod_{i=0}^{\infty}(1-h_i(z)) = \prod_{i=0}^{\infty}\Psi(1-\alpha^i+\alpha^i z).$$

Next, we show that $\lim_{z\uparrow 1}\varphi(z) = 1$. Define $r_n(z) = \prod_{i=n+1}^{\infty}\Psi(1-\alpha^i+\alpha^i z)$ and let $\delta > 0$ be arbitrary. By the uniform convergence of $\{\varphi_{n+1}(z)\}$ to $\varphi(z)$, there exists a positive integer $N(\delta)$ such that for any

$n > N(\delta)$, $\sup_{z\in[0,1]} |r_n(z) - 1| < \delta$. Note that $\varphi_{n+1}(\cdot)$ of (2.3) satisfies $\varphi_{n+1}(1) = 1$ and $\varphi_{n+1}(z) \le 1$. Since

$$|\varphi(z) - 1| = |\varphi_{n+1}(z)(r_n(z) - 1) + \varphi_{n+1}(z) - 1|,$$

it follows that for any $n > N(\delta)$, $|\varphi(z) - 1| \le \delta + |\varphi_{n+1}(z) - 1|$, which in turn implies

$$\limsup_{z\uparrow 1} |\varphi(z) - 1| = \limsup_{z\uparrow 1} (1 - \varphi(z)) \le \delta + \liminf_{z\uparrow 1} (1 - \varphi_{n+1}(z)) \le \delta.$$

Since $\varphi(z)$ is the limit of the sequence of pgf's $\{\varphi_{n+1}(z)\}$, we conclude that $\varphi(z)$ is a pgf by the Continuity Theorem. Equation (2.2) is easily shown to hold. □

Theorem 2.2. *Let $\{f_r\}$ be a pmf with finite mean and with pgf $\Psi(z)$. Let $\{p_r\}$ be the pmf with pgf $\varphi(z)$ of (2.1) and let $\{f_r^{(i)}\}$ be the pmf with pgf $\Psi(1 - \alpha^i + \alpha^i z)$, $i \ge 0$. The following assertions are true.*

1. *$f_r^{(0)} = f_r$ and*

$$f_r^{(i)} = \begin{cases} f_0 + \sum_{n=1}^{\infty} (1 - \alpha^i)^n f_n, & \text{if } r = 0 \\ \alpha^{ir} \sum_{n=r}^{\infty} \binom{n}{r} f_n (1 - \alpha^i)^{n-r}, & \text{if } r \ge 1. \end{cases} \tag{2.4}$$

2.

$$p_r = \lim_{k\to\infty} \left(f^{(0)} * f^{(1)} * \cdots * f^{(k-1)} \right)_r, \tag{2.5}$$

where $f^{(0)} * f^{(1)} * \cdots * f^{(k-1)}$ designates the k-factor convolution of the pmf's $\{f_r^{(0)}\}, \{f_r^{(1)}\}, \cdots, \{f_r^{(k-1)}\}$.

3. *Assume the factorial cumulant generating function (fcgf) $\ln \Psi(1 + t) = \sum_{r=1}^{\infty} \kappa_{[r]}^{(f)} \frac{t^r}{r!}$ of the pmf $\{f_r\}$ exists for $|t| < \rho_0$ for some $\rho_0 > 0$. Then, for every $r \ge 1$, $\kappa_{[r]}^{(p)}$ and $\kappa_r^{(p)}$ are finite and are given by (cf. (1.3)–(1.4))*

$$\kappa_{[r]}^{(p)} = \frac{\kappa_{[r]}^{(f)}}{1 - \alpha^r} \quad \text{and} \quad \kappa_r^{(p)} = \sum_{j=1}^{r} S(r, j) \frac{\kappa_{[j]}^{(f)}}{1 - \alpha^j}. \tag{2.6}$$

4. *If $\{f_r\}$ has a finite second cumulant, then the mean $\mu^{(p)}$, the variance $(\sigma^{(p)})^2$ and the dispersion index of $I^{(p)}$ of $\{p_r\}$ are obtained in terms of their $\{f_r\}$ counterparts, $\mu^{(f)}$, $(\sigma^{(f)})^2$ and $I^{(f)}$ as follows:*

$$\mu^{(p)} = \frac{\mu^{(f)}}{1 - \alpha}, \quad (\sigma^{(p)})^2 = \frac{(\sigma^{(f)})^2 + \alpha\mu^{(f)}}{1 - \alpha^2} \quad \text{and} \quad I^{(p)} = 1 + \frac{I^{(f)} - 1}{1 + \alpha}. \tag{2.7}$$

Proof: The proof of (2.4) is straightforward. Since $\varphi(z) = \lim_{k\to\infty} \prod_{i=0}^{k-1} \varphi_k(z)$, with $\varphi_k(z)$ of (2.3), we obtain (2.5) by the Continuity Theorem and (2.4). Since the fcgf $\ln \Psi(1 + t)$ of the pmf $\{f_r\}$ exists for $|t| < \rho_0$, we have $\ln \Psi(1 + \alpha^i t) = \sum_{r=1}^{\infty} \alpha^{ir} \kappa_{[r]}^{(f)} \frac{t^r}{r!}$. It follows by (2.1) that $\ln \varphi(1 + t) = \sum_{i=0}^{\infty} \ln \Psi(1 + \alpha^i t)$. One can show by a standard argument that the series $\ln \varphi(1+t)$ converges uniformly in the interval $|t| \le \rho$ for every $0 < \rho < \rho_0$. Therefore, by Weierstrass Theorem, p. 430 in [17], we have

$$\ln \varphi(1 + t) = \sum_{r=1}^{\infty} \sum_{i=0}^{\infty} \alpha^{ir} \kappa_{[r]}^{(f)} \frac{t^r}{r!} = \sum_{r=1}^{\infty} \frac{\kappa_{[r]}^{(f)}}{1 - \alpha^r} \frac{t^r}{r!} \quad (|t| < \rho_0),$$

proving the first part of (2.6). The second part of (2.6) is deduced from (1.3)-(1.4). The formulas in (2.7) follow from (2.2). □

Remark 2.1. We make a number of useful remarks.

1. Equations (2.6) and (2.7) are known (see [28]). We note that as a function of α the dispersion index $I^{(p)}$ is increasing and concave down if the innovation distribution is underdispersed.

2. As noted in [28], it is easily seen from (2.7), that $\{p_r\}$ of (2.5) is underdispersed (i.e., $(\sigma^{(p)})^2 < \mu^{(p)}$) if and only if $\{f_r\}$ is underdispersed.

3. There are no simple formulas linking the r-th moment $\mu_r^{(p)}$ and the r-th factorial moment $\mu_{[r]}^{(p)}$ of $\{p_r\}$ to their $\{f_r\}$ counterparts. However, if either $\kappa_{[r]}^{(p)}$ or $\kappa_r^{(p)}$ can be calculated for every $r \geq 1$, then one can compute $\mu_r^{(p)}$ and $\mu_{[r]}^{(p)}$ recursively using standard formulas that link moments and cumulants (see [12], Sections 1.2.7 and 1.2.8, and [26]).

Next, we interpret Theorem 2.1 and Theorem 2.2 in the context of INAR (1) modeling.

If the INAR (1) process $\{X_t\}$ of (1.2) is stationary, then its marginal pgf $\varphi_X(z)$ and the common pgf $\Psi(z)$ of the innovation sequence $\{\varepsilon_t\}$ must satisfy the functional equation (2.2) with $\varphi(z) = \varphi_X(z)$.

The backward approach we have adopted in this paper translates as follows: one chooses a pgf $\Psi(\cdot)$ and solve for $\varphi_X(\cdot)$ that satisfies (2.2). It can be shown that in this case $\varphi_X(z) = \lim_{n\to\infty} \varphi_n(z)$, with $\varphi_n(z)$ of (2.3), provided that the limit exists and is a pgf.

The backward approach leads to the following existence theorem for a stationary INAR (1) process.

Theorem 2.3. *Let $\alpha \in (0,1)$. Any pgf $\Psi(z)$ such that $\Psi'(1) < \infty$ gives rise to a stationary INAR (1) process $\{X_t\}$ defined on some probability space $(\Omega, \mathcal{F}, P)$ and driven by equation (1.2). Its marginal pgf is*

$$\varphi_X(z) = \prod_{i=0}^{\infty} \Psi(1 - \alpha^i + \alpha^i z). \tag{2.8}$$

Proof: Since $\Psi'(z) < 1$, by Theorem 2.1 $\varphi_X(z)$ is a pgf that satisfies equation (2.2). By Proposition 2.1 in [8], there exists a stationary INAR (1) process $\{X_t\}$ on some probability space $(\Omega, \mathcal{F}, P)$ such that its marginal distribution and that of its innovation sequence $\{\varepsilon_t\}$ have respective pgf's $\varphi_X(z)$ and $\Psi(z)$. $\square$

The following additional results (we refer to [2] and [20]) are needed in the sequel. An INAR (1) model driven by (1.2) is necessarily a homogeneous Markov chain with the 1-step transition probabilities,

$$P(X_t = k | X_{t-1} = l) = \sum_{j=0}^{\min(l,k)} \binom{l}{j} \alpha^j (1-\alpha)^{l-j} P(\varepsilon = k - j). \tag{2.9}$$

The k-step-ahead version of (1.2) for $k \geq 1$ is given by

$$X_{t+k} \overset{d}{=} \alpha^k \circ X_t + \sum_{j=1}^{k} \alpha^{j-1} \circ \varepsilon_{t+k-j+1} \tag{2.10}$$

and the k-step autocorrelation of $\{X_t\}$ is

$$Corr(X_t, X_t + k) = \alpha^k. \tag{2.11}$$

It follows from (2.10) that the conditional pgf of X_{t+k} given X_t satisfies

$$\varphi_{X_{t+k}|X_t}(z) = \left(1 - \alpha^k + \alpha^k z\right)^{X_t} \times \prod_{i=0}^{k-1} \Psi(1 - \alpha^i + \alpha^i z). \tag{2.12}$$

Therefore, given $X_t = n$, the distribution of X_{t+k} is the convolution of a Binomial(n, α^k) distribution and the pmf $\left\{\left(f^{(0)} * f^{(1)} * \cdots * f^{(k-1)}\right)_r\right\}$ of Theorem 2.2.

Remark 2.2. It is a well known fact that the INAR (1) process (1.2) is a branching process with a Benoulli(α) offspring distribution and an immigration sequence of iid random variables with common pgf $\Psi(z)$. It follows by Theorem in [11] that Theorem 2.3 holds under the weaker condition $\sum_{k=0}^{\infty} q_k(k+1)^{-1} < \infty$, where $\{q_k\}$ is the sequence of tail probabilities of $\Psi(z)$. For a more general result, we refer to Theorem 2, p. 264, in [5].

3. Stationary INAR (1) models with Bernoulli innovations

In this section and subsequent ones, we describe the properties of the marginal and conditional distributions of stationary INAR (1) processes with specific innovation sequences. We obtain useful representations of the marginal pgf's of these models as well as formulas for moments and cumulants of their marginal distributions.

We start out with the case of Bernoulli innovations. First, we recall a definition.

Let $q, c \in (0,1)$ and $m \geq 1$. Kemp [14] (see also [12], p. 467) introduced and studied the Poissonian Binomial (m, q, c) distribution as the distribution of a finite convolution of Bernoulli(cq^i) distributions, $i = 0, 1, 2, \cdots, m-1$ with pgf

$$\Psi(z) = \prod_{i=0}^{m-1} (1 - cq^i(1-z)) \tag{3.1}$$

and pmf

$$q_r(m,q,c) = \sum_{k=r}^{m} (-1)^{k-r} \binom{k}{r} c^k q^{\binom{k}{2}} \prod_{l=0}^{k-1} \frac{1-q^{m-l}}{1-q^{l+1}} \quad , r = 0, 1, \cdots, m. \tag{3.2}$$

We will expand more on this distribution in Section 5.

The main result of this section follows next. Its proof is long and is deferred to the appendix

Theorem 3.1. *Let $\{X_t\}$ be the stationary INAR (1) process driven by (1.2) and with a Bernoulli(p) innovation sequence for some $p \in (0,1)$. Then,*

1. *the marginal pmf $\{p_r\}$ of $\{X_t\}$ is the weak limit of Poissonian Binomial(n, α, p) (see (3.1) and (3.2)) as $n \to \infty$ and is given by*

$$p_r = \lim_{n\to\infty} q_r(n, \alpha, p) = \sum_{k=r}^{\infty} (-1)^{k-r} \binom{k}{r} \frac{p^k \alpha^{\binom{k}{2}}}{\prod_{l=1}^{k}(1-\alpha^l)}, r \geq 0; \tag{3.3}$$

2. *the tail probabilities $P(X_t \geq r) = \sum_{j=r}^{\infty} p_j$ of X_t are obtained by the formula*

$$P(X_t \geq r) = \sum_{k=r}^{\infty} (-1)^{k-r} \binom{k-1}{r-1} \frac{p^k \alpha^{\binom{k}{2}}}{\prod_{l=1}^{k}(1-\alpha^l)}, \quad r \geq 1; \tag{3.4}$$

3. *the marginal pgf $\varphi_X(z)$ of $\{X_t\}$ admits two useful representations:*

$$\varphi_X(z) = 1 + \sum_{n=1}^{\infty} \frac{p^n (z-1)^n \alpha^{\binom{n}{2}}}{\prod_{l=1}^{n}(1-\alpha^l)} \tag{3.5}$$

and

$$\varphi_X(z) = \exp\left\{ - \sum_{n=1}^{\infty} \frac{p^n}{n(1-\alpha^n)} (1-z)^n \right\}. \tag{3.6}$$

Additional properties of $\{X_t\}$ are given next.

By (2.9), the 1-step transition probability is given by

$$P(X_t = k|X_{t-1} = l) = \begin{cases} 0, & k > l+1 \\ p\alpha^{k-1}, & k = l+1 \\ \alpha^{k-1}\overline{\alpha}^{l-k}\left\{p\binom{l}{k-1}\overline{\alpha} + \overline{p}\binom{l}{k}\alpha\right\}, & k \leq l \end{cases}. \tag{3.7}$$

By (2.12), the conditional pgf of X_{t+k} given X_t satisfies

$$\varphi_{X_{t+k}|X_t}(z) = \left(1 - \alpha^k + \alpha^k z\right)^{X_t} \times \prod_{i=0}^{k-1}(1 - p\alpha^i(1-z)).$$

Therefore, given $X_t = n$, the distribution of X_{t+k} is the convolution of a Binomial(n, α^k) distribution and the Poissonian Binomial (k, α, p) distribution of (3.2).

Next, we derive the factorial moments $(\mu_{[r]}^{(p)}, r \geq 1)$ of X_t. Using the version (3.5) of $\varphi_X(z)$, we deduce that

$$\varphi_X(1+t) = 1 + \sum_{r=1}^{\infty} \frac{r! p^r \alpha^{\binom{r}{2}}}{\prod_{i=1}^{r}(1-\alpha^i)} \cdot \frac{t^r}{r!}.$$

Since the series converges everywhere, the factorial moments (the coefficients of $t^r/r!$) and therefore the moments of X_t (by (1.3)-(1.4)) of all orders are finite and are given by

$$\mu_{[r]}^{(p)} = \frac{r! p^r \alpha^{\binom{r}{2}}}{\prod_{i=1}^{r}(1-\alpha^i)} \quad \text{and} \quad \mu_r^{(p)} = \sum_{j=1}^{r} S(r,j) \frac{j! p^j \alpha^{\binom{j}{2}}}{\prod_{i=1}^{j}(1-\alpha^i)} \qquad (r \geq 1). \tag{3.8}$$

By (2.7), the mean, the variance and the index of dispersion of X_t are

$$\mu_X = \frac{p}{1-\alpha}, \quad \sigma_X^2 = \frac{p(1-p) + \alpha p}{1-\alpha^2} \quad \text{and} \quad I_X = 1 - \frac{p}{1+\alpha}.$$

As expected, the marginal distribution of $\{X_t\}$ is underdispersed. We note that I_X is decreasing and linear affine in p and increasing and concave down in α (see Figure 1).

By (3.6), the fcgf of X_t is given by

$$\ln \varphi_X(1+t) = \sum_{r=1}^{\infty} \frac{(-1)^{r+1}(r-1)! p^r}{(1-\alpha^r)} \cdot \frac{t^r}{r!}.$$

Since the series above converges everywhere, the factorial cumulants and the cumulants of X_t of all orders are finite and given by (applying (1.3)-(1.4)):

$$\kappa_{[r]}^{(p)} = (-1)^{r+1} \frac{(r-1)! p^r}{(1-\alpha^r)} \qquad \text{and} \quad \kappa_r^{(p)} = \sum_{j=0}^{r} S(r,j)(-1)^{j+1} \frac{(j-1)! p^j}{(1-\alpha^j)} \quad (r \geq 1). \tag{3.9}$$

Remark 3.1. We note that if the innovation sequence $\{\varepsilon_t\}$ has the Power-Law distribution of the first kind $(PL_1(\lambda, p))$, i.e., $\varepsilon_t \sim Pois(\lambda) * Bernoulli(p)$, $0 < p < 1$, then its marginal distribution will result from the convolution of a Poisson$(\frac{\lambda}{1-\alpha})$ and the pmf $\{p_r\}$ of (3.3) in Theorem 3.1. The $PL_1(\lambda, p)$ law was discussed in Section 2.3 of [28]. Additional distributional properties of this law such as moments and cumulants, can be obtained from Theorem 3.1 and subsequent results in this section.

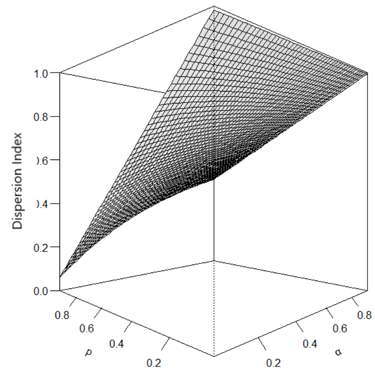

Figure 1: Variance-mean ratio of the marginal distribution of an INAR(1) process with a Bernoulli innovation.

4. Stationary INAR (1) models with Binomal innovations

The treatment is essentially similar to the Bernoulli case ($m = 1$). We summarize the main results with minimal justifications for the most part.

Theorem 4.1. *Let $\{X_t\}$ be the stationary INAR (1) process driven by (1.2) and with a Binomial(m, p) innovation sequence for some positive integer m and some $p \in (0, 1)$. Then*

1. *the marginal pmf $\{p_r\}$ of $\{X_t\}$ is the m-fold convolution of the marginal distribution (3.3) of the INAR (1) process with a Bernoulli(p) innovation.*

2. *the marginal pgf $\varphi_X(z)$ of $\{X_t\}$ admits two representations:*

$$\varphi_X(z) = \left[1 + \sum_{n=1}^{\infty} \frac{p^n (z-1)^n \alpha^{\binom{n}{2}}}{\prod_{l=1}^{n}(1-\alpha^l)}\right]^m \tag{4.1}$$

and

$$\varphi_X(z) = \exp\Big\{ - m \sum_{n=1}^{\infty} \frac{p^n}{n(1-\alpha^n)}(1-z)^n \Big\}. \tag{4.2}$$

Proof: Straightforward. We omit the details. □

We proceed to give additional properties of $\{X_t\}$.

By (2.9), the 1-step transition probability of $\{X_t\}$ is

$$P(X_t = k | X_{t-1} = l) = p^k \overline{p}^{m-k} \overline{\alpha}^l \sum_{j=\max(k-m,0)}^{\min(l,k)} \binom{l}{j}\binom{m}{k-j}\Big(\frac{\alpha\overline{p}}{p\overline{\alpha}}\Big)^j, \quad k \leq l+m. \tag{4.3}$$

By (2.12), the conditional pgf of X_{t+k} given X_t satisfies

$$\varphi_{X_{t+k}|X_t}(z) = \left(1-\alpha^k+\alpha^k z\right)^{X_t} \times \left[\prod_{i=0}^{k-1}(1-p\alpha^i(1-z))\right]^m.$$

Therefore, the conditional distribution of X_{t+k} given $X_t = n$ is the convolution of a Binomial(n,α^k) distribution and the m-fold convolution of the Poissonian Binomial (k,α,p) distribution of (3.2).

Since the power series expansion of $\varphi_X(1+t)$ for $\varphi_X(z)$ of (4.1) is not easily computable, we proceed to derive simpler recurrence formulas for the factorial moments $(\mu_{[r]}^{(p)}, r \geq 1)$ of X_t by using instead the representation (4.2).

Let

$$\phi(z) = \sum_{n=1}^{\infty} b_n(1-z)^n, \quad b_n = \frac{p^n}{n(1-\alpha^n)}. \tag{4.4}$$

The series (4.4) converges uniformly over the interval $(0,1)$ due to the fact that for every $n \geq 1$ and $z \in (0,1)$, $b_n(1-z)^n \leq b_n$, and that $\sum_{n=1}^{\infty} b_n$ converges. It follows that $\phi'(z)$ and subsequent higher order derivatives exist and converge uniformly over $(0,1)$ (see [17]). The r-th derivative of $\phi(z)$ admits the representation

$$\phi^{(r)}(z) = (-1)^r \sum_{n=r}^{\infty} \frac{p^n}{1-\alpha^n}\frac{(n-1)!}{(n-r)!}(1-z)^{n-r} \qquad (r \geq 1). \tag{4.5}$$

Uniform convergence allows for the interchange of limit (as $z \uparrow 1$) and summation in (4.5). Hence,

$$\phi^{(r)}(1) = (-1)^r \frac{(r-1)!p^r}{1-\alpha^r} \qquad (r \geq 1). \tag{4.6}$$

Since $\ln\varphi_X(z) = -m\phi(z)$, it follows that $\varphi_X'(z) = -m\varphi_X(z)\phi'(z)$. An induction argument shows that the r^{th} derivative, $\varphi_X^{(r)}(z)$, of $\varphi_X(z)$ can be obtained by the following forward recursion (with $\varphi_X^{(0)}(z) = \varphi_X(z)$ and $\binom{0}{0} = 1$):

$$\varphi_X^{(r)}(z) = -m\sum_{j=0}^{r-1}\binom{r-1}{j}\varphi_X^{(j)}(z)\phi^{(r-j)}(z). \tag{4.7}$$

Therefore, the factorial moments $\mu_{[r]}^{(p)} = \varphi_X^{(r)}(1)$, $r \geq 1$, are finite and satisfy the recurrence relation (with $\mu_{[0]}^{(p)} = 1$),

$$\mu_{[r]}^{(p)} = -m\sum_{j=0}^{r-1}\binom{r-1}{j}\mu_{[j]}^{(p)}\phi^{(r-j)}(1) \qquad (r \geq 1). \tag{4.8}$$

By (2.7), the mean, the variance and the index of dispersion of X_t are

$$\mu_X = \frac{mp}{1-\alpha}, \quad \sigma_X^2 = \frac{mp(1+\alpha-p)}{1-\alpha^2} \quad \text{and } I_X = 1-\frac{p}{1+\alpha},$$

implying the marginal of $\{X_t\}$ is underdispersed.

We note that the dispersion indexes for INAR (1) processes with Bernoulli and binomial innovations are identical. However, as pointed out by the referee, the additional parameter m of the model with Binomial innovations gives further flexibilty for the parameterization of the $INAR(1)$ model. For example, we may estimate α using the sample autocorrelation function of order one, $ACF(1)$ (cf. (2.11)), and $\lambda = mp$ using the sample mean $\widehat{\lambda}$. Thus, the remaining degree of freedom, p in $\left\{\widehat{\lambda}, \frac{\widehat{\lambda}}{2}, \frac{\widehat{\lambda}}{3}, \cdots\right\}$ can be used to adjust the dispersion index.

The moments $(\mu_r^{(p)}, r \geq 1)$ of X_t are finite and can be obtained from their factorial counterparts via (4.8) and equations (1.3)-(1.4).

Finally, and similarly to the Bernoulli case, the factorial cumulants and the cumulants of X_t are obtained via the pgf representation (4.2) and equations (1.3)-(1.4):

$$\kappa_{[r]}^{(p)} = \frac{m(-1)^{r+1}(r-1)!p^r}{1-\alpha^r} \quad \text{and} \quad \kappa_r^{(p)} = m\sum_{j=0}^{r} S(r,j)(-1)^{j+1}\frac{(j-1)!p^j}{1-\alpha^j} \quad (r \geq 1). \tag{4.9}$$

5. Stationary INAR (1) models with Poissonian Binomial innovations

In this section, we develop a stationary INAR (1) process with a Poissonian Binomial innovation sequence with pgf and pmf given respectively in (3.1) and (3.2), for some positive integer m and some real numbers $q, c \in (0,1)$. This distribution belongs to the family of discrete q-series distributions with finite range. It results from the convolution of m independent Bernoulli(p_j) distributions , $j = 1, 2, \cdots, m$, where the p_j's vary according to the geometric progression $p_j = cq^{j-1}$. For more on q-distributions, we refer to the monograph [9].

We recall for further reference that a Poissonian Binomial(m, q, c) is underdispersed with mean, variance, and dispersion index given by (see [14])

$$\mu_\varepsilon = \frac{(1-q^m)c}{1-q}, \quad \sigma_\varepsilon^2 = \frac{(1-q^m)c}{1-q} - \frac{(1-q^{2m})c^2}{1-q^2}, \quad \text{and} \quad I_\varepsilon = 1 - \frac{(1+q^m)c}{1+q}. \tag{5.1}$$

Theorem 5.1. *Let $\{X_t\}$ be the stationary INAR (1) process driven by (1.2) and with a Poissonian Binomial(m, q, c) innovation sequence for some positive integer m and some real numbers $q, c \in (0,1)$.*

1. *The marginal pgf $\varphi_X(z)$ of $\{X_t\}$ admits the following representations:*

$$\varphi_X(z) = \prod_{j=0}^{m-1}\Big[1 + \sum_{n=1}^{\infty}\frac{(cq^j)^n(z-1)^n\alpha^{\binom{n}{2}}}{\prod_{l=1}^{n}(1-\alpha^l)}\Big]. \tag{5.2}$$

and

$$\varphi_X(z) = \exp\left\{-\sum_{n=1}^{\infty}\frac{1-q^{mn}}{1-q^n}\frac{c^n}{n(1-\alpha^n)}(1-z)^n\right\}. \tag{5.3}$$

2. *The marginal pmf $\{p_r\}$ of $\{X_t\}$ is the convolution of the pmf's $(\{p_r^{(j)}\}, 0 \leq j \leq m-1)$,*

$$p_r = (p^{(0)} * p^{(1)} * \cdots * p^{(m-1)})_r \quad (r \geq 0), \tag{5.4}$$

where

$$p_r^{(j)} = \sum_{k=r}^{\infty}(-1)^{k-r}\binom{k}{r}\frac{(cq^j)^k\alpha^{\binom{k}{2}}}{\prod_{l=1}^{k}(1-\alpha^l)}, \quad r \geq 0. \tag{5.5}$$

Proof: Let $\Psi(z)$ be the pgf of the Poissonian Binomial (m, q, c) distribution as given in (3.1). Then,

$$\Psi(1-\alpha^i+\alpha^i z)=\prod_{j=0}^{m-1}(1+c\alpha^i q^j(z-1)),$$

which is the pgf of a Poissonian Binomial$(m, q, c\alpha^i)$. By Theorem 2.1, the marginal pgf $\varphi_X(z)$ is

$$\varphi_X(z)=\prod_{j=0}^{m-1}\prod_{i=0}^{\infty}(1+c\alpha^i q^j(z-1)).$$

Noting that $\prod_{i=0}^{\infty}(1+c\alpha^i q^j(z-1))$ is the marginal pgf of a stationary INAR (1) process with Bernoulli(cq^j) innovations, representations (5.2) and (5.3) follow from (3.5) and (3.6), respectively. By Theorem 3.1 and (3.3), for each $j \geq 0$, the pmf with pgf

$$\varphi_j(z)=1+\sum_{n=1}^{\infty}\frac{(cq^j)^n(z-1)^n\alpha^{\binom{n}{2}}}{\prod_{l=1}^{n}(1-\alpha^l)}$$

is $\{p_r^{(j)}\}$ of (5.5). Therefore, part 3 and (5.5) follow from (5.2). $\square$

Some additional properties of $\{X_t\}$ are presented next.

By (2.9), the 1-step transition probability of $\{X_t\}$ is given by

$$P(X_t=k|X_{t-1}=l)=\sum_{j=\max(k-m,0)}^{\min(l,k)}\binom{l}{j}\alpha^j(1-\alpha)^{l-j}q_{k-j}(m,q,c), \quad k \leq l+m. \tag{5.6}$$

By (2.12), the conditional pgf of X_{t+k} given X_t satisfies

$$\varphi_{X_{t+k}|X_t}(z)=\left(1-\alpha^k+\alpha^k z\right)^{X_t}\times\prod_{j=0}^{m-1}\Big[\prod_{i=0}^{k-1}(1-(cq^j)\alpha^i(1-z))\Big].$$

Therefore, the conditional distribution of X_{t+k} given $X_t=n$ is the convolution of a Binomial(n, α^k) distribution and the Poissonian Binomial (k, α, cq^j) distributions, $j=0, 1, \cdots, m-1$.

By (5.2), the power series expansion of $\varphi_X(1+t)$ exists but is not easily computable. We proceed as in the Binomial case (Section 4) to derive the factorial moments $(\mu_{[r]}^{(p)}, r \geq 1)$ of X_t via the representation (5.3) of $\varphi_X(z)$ and a recurrence relation.

Let

$$\phi_1(z)=-\ln\varphi_X(z)=\sum_{n=1}^{\infty}\frac{1-q^{mn}}{1-q^n}\frac{c^n}{n(1-\alpha^n)}(1-z)^n. \tag{5.7}$$

The argument we used to derive (4.5)–(4.8) in Section 4 carries over almost verbatim. We state the main steps without further explanations. The r-th derivative of $\phi_1(z)$ admits the representation

$$\phi_1^{(r)}(z)=(-1)^r\sum_{n=r}^{\infty}\frac{(1-q^{mn})c^n}{(1-q^n)(1-\alpha^n)}\frac{(n-1)!}{(n-r)!}(1-z)^{n-r} \qquad r \geq 1. \tag{5.8}$$

Hence,

$$\phi_1^{(r)}(1) = (-1)^r \frac{(1-q^{mr})c^r}{(1-q^r)(1-\alpha^r)}(r-1)! \qquad (r \geq 1). \tag{5.9}$$

Since $\ln \varphi_X(z) = -\phi_{(}z)$, the r^{th} derivative, $\varphi_X^{(r)}(z)$, of $\varphi_X(z)$ can be obtained by the following forward recursion (with $\varphi_X^{(0)}(z) = \varphi_X(z)$ and $\binom{0}{0} = 1$):

$$\varphi_X^{(r)}(z) = -\sum_{j=0}^{r-1} \binom{r-1}{j} \varphi_X^{(j)}(z)\phi_1^{(r-j)}(z). \tag{5.10}$$

The factorial moments $\mu_{[r]}^{(p)} = \varphi_X^{(r)}(1)$, $r \geq 1$, are finite and satisfy the recurrence relation (with $\mu_{[0]}^{(p)} = 1$),

$$\mu_{[r]}^{(p)} = -\sum_{j=0}^{r-1} \binom{r-1}{j} \mu_{[j]}^{(p)} \phi_1^{(r-j)}(1) \qquad (r \geq 1). \tag{5.11}$$

The moments of X_t, $\mu_r^{(p)} = E(X_t^r)$, $r \geq 1$, are finite and can be obtained from their factorial counterparts via equations (1.3)-(1.4).

By (2.7) and (5.1), the marginal distribution of $\{X_t\}$ is underdispersed with mean, variance and dispersion index given by

$$\mu_X = \frac{(1-q^m)c}{(1-\alpha)(1-q)}, \quad \sigma_X^2 = \frac{(1-q^m)c}{(1-\alpha)(1-q)} - \frac{(1-q^{2m})c^2}{(1-\alpha^2)(1-q^2)}$$

and

$$I_X = 1 - \frac{(1+q^m)c}{(1+\alpha)(1+q)}.$$

We note that I_X is increasing in α and m ($m \geq 2$) and decreasing in c. Moreover, it is concave down in q with concavity becoming more pronounced as c increases (see Figure 2).

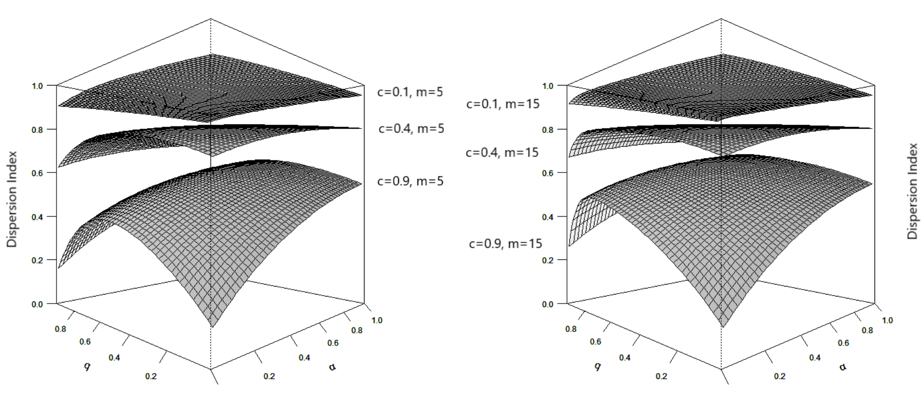

Figure 2: Variance-mean ratio of the marginal distribtution of an INAR(1) process with a Poissonian Binomial innovation.

Similarly to the Bernoulli and the Binomial cases (Sections 3 and 4, respectively), the factorial cumulants and the cumulants of X_t are obtained in straightforward fashion from the pgf representation (5.3) and equations (1.3)-(1.4):

$$\kappa_{[r]}^{(p)} = \frac{(-1)^{r+1}(r-1)!(1-q^{mr})c^r}{(1-q^r)(1-\alpha^r)} \qquad (r \geq 1) \tag{5.12}$$

and

$$\kappa_r^{(p)} = \sum_{j=0}^{r} S(r,j) \frac{(-1)^{j+1}(j-1)!(1-q^{mj})c^j}{(1-q^j)(1-\alpha^j)}, \qquad r \geq 1. \tag{5.13}$$

6. Stationary INAR (1) models with logarithmic innovations

We revisit in this section the INAR (1) process with a logarithmic(p) distribution introduced by [7]. Most of the discussion will focus on the underdispersion version of the process.

We start out by recalling a few facts about the logarithmic distribution (see [12]). The pmf of the logarithmic(p) distribution is given by $f_r = \frac{p^r}{-r \ln \overline{p}}, r \geq 1$, where $p \in (0,1)$. Its pgf, mean, variance and dispersion index are given respectively by

$$\Psi(z) = \frac{\ln(1-pz)}{\ln \overline{p}}, \tag{6.1}$$

$$\mu^{(f)} = -\frac{p}{\overline{p}\ln \overline{p}}, \quad (\sigma^{(f)})^2 = -\frac{p(p+\ln \overline{p})}{(\overline{p}\ln \overline{p})^2} \quad \text{and} \quad I^{(f)} = \frac{p+\ln \overline{p}}{\overline{p}\ln \overline{p}}. \tag{6.2}$$

Note that the logarithmic distribution is underdispersed if $p < 1 - 1/e$, equidispersed if $p = 1 - 1/e$ and overdispersed if $p > 1 - 1/e$.

The factorial moments and moments of $\{f_r\}$ are respectively (cf. (1.3)-(1.4))

$$\mu_{[r]}^{(f)} = -\frac{p^r(r-1)!}{(1-p)^r \ln \overline{p}} \quad \text{and} \quad \mu_r^{(f)} = -\frac{1}{\ln \overline{p}} \sum_{j=1}^{r} S(r,j) \frac{p^j(j-1)!}{(1-p)^j} \qquad (r \geq 1). \tag{6.3}$$

We will also refer to the logarithmic-with–zeros(c,p) distribution, with $c, p \in (0,1)$, that arises as a two-mixture of the Dirac measure δ_0 sitting at 0 and the logarithmic(p) distribution with respective mixing probabilities c and $1-c$ (see [12], Sections 7.1 and 8.2). Its pgf is $P(z) = c+(1-c)\ln[(1-pz)/\ln \overline{p}]$.

Lemma 6.1. *Let $p \in (0,1)$ and let $\{f_r\}$ be the pmf of a logarithmic(p) distribution with pgf $\Psi(z)$ of (6.1). Then for every $i \geq 0$, the pmf, $\{f_r^{(i)}\}$ of (2.4) with pgf $\Psi(1-\alpha^i+\alpha^i z)$ is a logarithmic-with-zeros(b_i, q_i) distribution with $b_i = 1 - \frac{\ln \overline{q}_i}{\ln \overline{p}}$ and $q_i = \frac{p\alpha^i}{1-p(1-\alpha^i)}$ ($q_0 = p, b_0 = 0$), i.e.,*

$$f_r^{(i)} = b_i \delta_0(\{r\}) + (1-b_i)\frac{q_i^r}{-r \ln \overline{q}_i}. \tag{6.4}$$

Noting $f_r^{(0)} = f_r$, the k-factor convolution of the pmf's $\{f_r^{(0)}\}, \{f_r^{(1)}\}, \cdots, \{f_r^{(k-1)}\}, k \geq 2$, is a finite mixture of convolutions of logarithmic distributions, namely,

$$\left(f^{(0)} * f^{(1)} * \cdots * f^{(k-1)}\right)_r = C_{I,0}\, g_r^{(0)} + \sum_{l=1}^{k-1} \sum_{\mathbf{j} \in \mathbf{J}_l} C_{\mathbf{j},l} \left(g^{(0)} * g^{(j_1)} * g^{(j_2)} \cdots * g^{(j_l)}\right)_r, \tag{6.5}$$

where $\{g_r^{(j)}\}$ is the pmf of the logarithmic(q_j), $I = \{1, 2, \cdots, k-1\}$, $\mathbf{J}_l$ is the collection of ordered l-tuples $\mathbf{j} = (j_1, j_2, \cdots, j_l)$, $1 \le j_1 < j_2 < \cdots < j_l \le k-1$ and $\mathbf{j}_u = \{j_1, j_2, \cdots, j_l\}$ is the corresponding unordered l-tuple. The mixing probabilities are

$$C_{I,0} = \prod_{j=1}^{k-1} b_j \quad \text{and} \quad C_{\mathbf{j},l} = \Big(\prod_{j \in I \setminus \mathbf{j}_u} b_j\Big)\Big(\prod_{h=1}^{l}(1 - b_{j_h})\Big). \tag{6.6}$$

Proof: If $i = 0$, (6.4) is true since $\{f_r^{(0)}\} = \{f_r\}$. Assume $i \ge 1$. By (2.4),

$$f_0^{(i)} = \frac{-1}{\ln \overline{p}} \sum_{n=1}^{\infty} \frac{\big(p(1-\alpha^i)\big)}{n} = \frac{\ln(1 - p(1-\alpha^i))}{\ln \overline{p}},$$

and for $r \ge 1$,

$$f_r^{(i)} = -\frac{(p\alpha^i)^r}{r \ln \overline{p}} \sum_{n=r}^{\infty} \binom{n-1}{r-1} \big(p(1-\alpha^i)\big)^{n-r}.$$

Using the power series expansion $(1-t)^{-r-1} = \sum_{n=r}^{\infty} \binom{n}{r} t^{n-r}$, with $t = p(1-\alpha^i)$, it follows that

$$f_r^{(i)} = -\frac{1}{r \ln \overline{p}} \left[\frac{p\alpha^i}{1 - p(1-\alpha^i)}\right]^r \qquad (r \ge 1).$$

Setting $q_i = \frac{p\alpha^i}{1-p(1-\alpha^i)}$, it is easily verified that $f_r^{(i)}$ satisfies (6.4). The second part of the Lemma and equations (6.5) and (6.6) are proved by a tedious but straightforward induction argument. The details are omitted. $\square$

Theorem 6.1. *Let $\{X_t\}$ be the stationary INAR(1) process driven by (1.2) and with a logarithmic(p) innovation sequence for some $p \in (0,1)$. Then,*

(i) *the marginal distribution $\{p_r\}$ of $\{X_t\}$ is the weak limit, as $k \to \infty$, of the sequence of pmf's $\big(f^{(0)} * f^{(1)} * \cdots * f^{(k-1)}, k \ge 1\big)$, where $f^{(0)} * f^{(1)} * \cdots * f^{(k-1)}$ is described by equations (6.5) and (6.6).*

(ii) *the marginal pgf $\varphi_X(z)$ of $\{X_t\}$ admits the representation*

$$\varphi_X(z) = \prod_{i=0}^{\infty} \Big[1 - \frac{1}{\ln \overline{p}} \cdot \ln \frac{\overline{q}_i}{1 - q_i z}\Big] \qquad (0 < z \le 1), \tag{6.7}$$

with $p_0 = \varphi_X(0) = 0$.

Proof: Part (i) is a direct consequence of (2.5) and Lemma 6.1. For part (ii), (6.7) follows from (2.8) and the fact that when $z = 0$ the first factor in (6.7) is equal to 0 . $\square$

Next, we provide additional properties of $\{X_t\}$, some of which appeared in [7].

By (2.9), the 1-step transition probability of $\{X_t\}$ is given by

$$P(X_t = k | X_{t-1} = l) = -\frac{p^k}{\ln \overline{p}} \sum_{j=0}^{\min(l,k-1)} \binom{l}{j} \frac{(\alpha/p)^j \, \overline{\alpha}^{l-j}}{k-j}, \quad k, l \ge 1.$$

By (2.12), the conditional pgf of X_{t+k} given X_t satisfies

$$\varphi_{X_{t+k}|X_t}(z) = \Big(1 - \alpha^k + \alpha^k z\Big)^{X_t} \times \prod_{i=0}^{k-1} \Big[1 - \frac{1}{\ln \overline{p}} \cdot \ln \frac{\overline{q}_i}{1 - q_i z}\Big].$$

Therefore, given $X_t = n$, the distribution of X_{t+k} is the convolution of a Binomial(n, α^k) distribution and the finite mixture of convolutions of logarithmic distributions described by (6.5) and (6.6).

By (2.7) and (6.3), the mean μ_X, the variance σ_X^2 and the dispersion index I_X of the marginal distribution of $\{X_t\}$ are given by

$$\mu_X = -\frac{p}{\overline{p}(1-\alpha)\ln\overline{p}}, \quad \sigma_X^2 = -\frac{p(p+\ln\overline{p}+\alpha(\overline{p}\ln\overline{p})^2)}{(\overline{p}\ln\overline{p})^2(1-\alpha^2)} \quad \text{and} \quad I_X = 1+\frac{p(1+\ln\overline{p})}{\overline{p}(1+\alpha)\ln\overline{p}}. \tag{6.8}$$

Note that the distribution of X_t is underdispersed if and only if $p < 1 - 1/e$. The graph of I_X restricted to that range is given in Figure 3 below. I_X is increasing and concave down in α and decreasing and concave up in p.

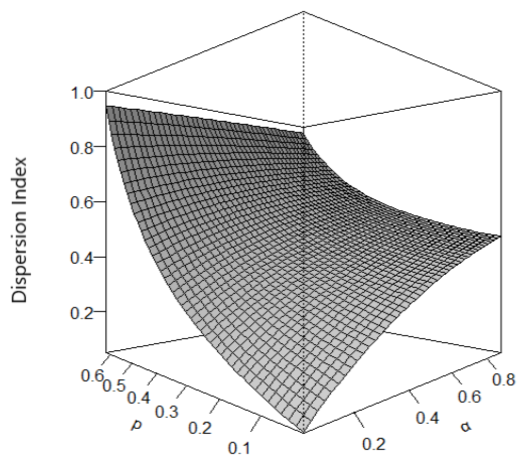

Figure 3: Variance-mean ratio of the marginal distribution of an INAR(1) process with an underdispersed logarithmic innovation.

Unlike the previously encountered models, the representations of the functions $\varphi_X(z)$ and $\ln\varphi_X(z)$ of the distribution of X_t are too complex to lead to manageable formulas for moments and cumulants of X_t. Instead, we proceed as in [28], Section 4.2, and use a number of recurrence formulas that will compute these quantities in the following order:

1. Compute the r-th cumulant $\kappa_r^{(f)}$ of ε_t using the formula, derived in [26],

$$\kappa_r^{(f)} = \mu_r^{(f)} - \sum_{i=1}^{r-1} \binom{r-1}{i} \kappa_{r-i}^{(f)} \mu_i^{(f)}, \tag{6.9}$$

along with (6.3) (recall $\kappa_1^{(f)} = \mu^{(f)}$ and $\kappa_2^{(f)} = (\sigma^{(f)})^2$, cf. (6.2)).

2. Compute the r-th factorial cumulant $\kappa_{[r]}^{(f)}$ of ε_t using the formula (see [12], Sections 1.2.7 and 1.2.8)

$$\kappa_{[r]}^{(f)} = \sum_{j=0}^{r} s(r,j)\kappa_j^{(f)}, \tag{6.10}$$

where $s(r,j)$ is the Stirling number of the first kind satisfying the recurrence relation

$$s(r+1,j) = s(r,j-1) - ns(r,j), \tag{6.11}$$

with $s(n,0)=0$ and $s(1,1)=1$.

3. Compute the r-th factorial cumulant, $\kappa_{[r]}^{(p)}$, and cumulant, $\kappa_r^{(p)}$, of X_t using (2.6).

4. Use the formulas in [12] to compute the moments $\{\mu_r\}$ of X_t, eq. (1.252), p. 54, and its factorial moments $\{\mu_{[r]}\}$, eq. (1.244), p. 53.

Remark 6.1. Since both the innovation and the marginal distribution of the INAR (1) process in this section have support in $\mathbb{N}^* = \{1,2,3,\cdots\}$, it may be more suitable to adopt a measure of dispersion different from the one relative to the Poisson distribution. We briefly discuss two approaches and submit the relevant graphs as an illustration. For a random variable Y with support in $\mathbb{N}^*$ such that $0 < P(Y=1) < 1$, we suggest using dispersion indexes of the zero-shifted distribution of Y, that is the distribution of $Y-1$. We consider two such indexes: the standard one, we denote by I_{Y-1} (as in previous sections) and the one introduced in [1], we denote by $I_{Y-1}^{(geo)}$, relative to the zero-shifted geometric distribution (with the usual interpretation of under/equi/over-dispersion). The formulas are as follows:

$$I_{Y-1} = \frac{Var(Y)}{E(Y)-1} \quad \text{and} \quad I_{Y-1}^{(geo)} = \frac{I_Y}{E(Y)-1}. \tag{6.12}$$

It is easily seen that for any $p \in (0,1)$, the zero-shifted version of the logarithmic(p) distribution is overdispersed relative to both the Poisson and the zero-shifted geometric distributions. Applying the formulas (6.12), along with (6.8), to the zero-shifted version of the marginal distribution of the stationary INAR (1) with a logarithmic innovation, we obtain the following graphs for the two indexes:

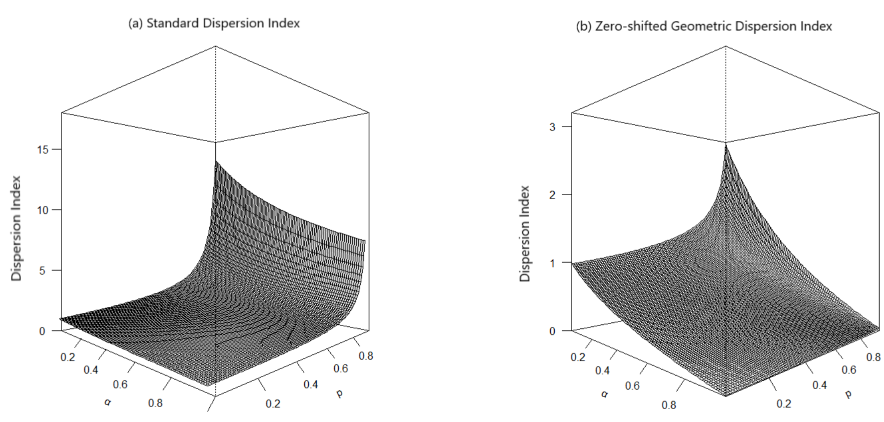

Figure 4: Two variance-mean ratios of the zero-shifted marginal distribution of an INAR(1) process with a Logarithmic innovation.

Both graphs show increase in p and decrease in α. Note also that for every $\alpha \in (0,1)$ and $p \in (0,1)$, I_{X-1} exhibits all three dispersion states, whereas $I_{X-1}^{(geo)}$ shows underdispersion for every $\alpha \geq 0.58$ and $p \in (0,1)$.

7. Stationary INAR (1) processes with Heine innovations

A distribution on $\mathbb{Z}_+$ is said to have the Heine distribution (Heine(λ, q)) with parameters $\lambda > 0$ and $q \in (0,1)$ if its pgf and pmf are respectively given by

$$\Psi(z) = \prod_{j=0}^{\infty} \big(1 - \beta_j + \beta_j z\big) \tag{7.1}$$

and

$$f_0 = \prod_{j=0}^{\infty} (1 - \lambda q^j)^{-1} \quad \text{and} \quad f_r = \frac{\lambda^r q^{r(r-1)/2}}{\prod_{l=1}^{r}(1-q^l)} f_0 \quad (r \geq 1), \tag{7.2}$$

where $\beta_j = \frac{\lambda q^j}{1+\lambda q^j}$ for $j \geq 0$.

The Heine distribution was introduced by Benkherouf and Bather [6]. It is a q-series distribution with infinite support. Many of its properties were studied in [15] . More details can be found in these references and in [12], Section 10.8.2. The Heine distribution is underdispersed and its mean and variance are

$$\mu = \sum_{r=0}^{\infty} \frac{\lambda q^r}{1+\lambda q^r} \quad \text{and} \quad \sigma^2 = \sum_{r=0}^{\infty} \frac{\lambda q^r}{(1+\lambda q^r)^2}. \tag{7.3}$$

We recall a few facts about double infinite products. Let $\{a_{mn}\}$ be a double sequence. The double infinite product $\prod_{i=0}^{\infty}\prod_{j=0}^{\infty}(1+a_{ij})$ is defined as the limit of the double sequence $P_{mn} = \prod_{i=0}^{m}\prod_{j=0}^{n}(1+a_{ij})$ as $m, n \to \infty$. If $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}|a_{ij}| < \infty$, then the double infinite product $\prod_{i=0}^{\infty}\prod_{j=0}^{\infty}(1+a_{ij})$ converges. Moreover, if $\prod_{i=0}^{\infty}\prod_{j=0}^{\infty}(1+a_{ij})$ and the iterated infinite products

$$\prod_{i=0}^{\infty}\Big[\prod_{j=0}^{\infty}(1+a_{ij})\Big] \quad \text{and} \quad \prod_{j=0}^{\infty}\Big[\prod_{i=0}^{\infty}(1+a_{ij})\Big]$$

are all convergent, then they necessarily have the same value.

The main result of this section is given next. Its proof is deferred to the appendix.

Theorem 7.1. *Let $\{X_t\}$ be the stationary INAR (1) process driven by (1.2) and with a Heine(λ, q) innovation sequence for some $\lambda > 0$ and $0 < q < 1$. Then the marginal pgf $\varphi_X(z)$ of $\{X_t\}$ admits the following representations:*

1. $$\varphi_X(z) = \prod_{j=0}^{\infty}\Big[1 + \sum_{n=1}^{\infty} \frac{\beta_j^n (z-1)^n \alpha^{\binom{n}{2}}}{\prod_{l=1}^{n}(1-\alpha^l)}\Big], \tag{7.4}$$

   where β_j is as in (7.1).

2. $$\varphi_X(z) = \exp\left\{-\sum_{n=1}^{\infty} \frac{B_n}{n(1-\alpha^n)}(1-z)^n\right\} \tag{7.5}$$

   with $B_n = \sum_{j=0}^{\infty} \beta_j^n$, $n \geq 1$.

3. *The marginal pmf $\{p_r\}$ of $\{X_t\}$ is*

$$p_r = \lim_{k\to\infty}(q^{(0)} * q^{(1)} * \cdots * q^{(k-1)})_r \quad (r \geq 0), \tag{7.6}$$

where

$$q_r^{(j)} = \sum_{k=r}^{\infty}(-1)^{k-r}\binom{k}{r}\frac{\beta_j^k \alpha^{\binom{k}{2}}}{\prod_{l=1}^{k}(1-\alpha^l)}, \quad r \geq 0. \tag{7.7}$$

Next, we discuss additional properties of $\{X_t\}$.

The 1-step transition probability can be obtained from (2.9). Given there are no notable simplifications of the formulas, we omit the details and refer to the following discussion by setting $k = 1$.

By (2.12), the conditional distribution of X_{t+k} given $X_t = n$ results from the convolution of $k+1$ distributions, namely a Binomial(n, α^k) distribution and the distributions $(\{g_r^{(i)}\}, 0 \leq i \leq k-1)$ defined as follows:

$$g_r^{(i)} = \alpha^{ir}\sum_{l=0}^{\infty}\binom{r+l}{r}(1-\alpha)^l f_{r+l},$$

where $\{f_r\}$ is the pmf of the Heine(λ, q) distribution (7.2).

The factorial cumulants and the cumulants of X_t are obtained in straightforward fashion from the pgf representation (7.5) and equations (1.3)-(1.4):

$$\kappa_{[r]}^{(p)} = (-1)^{r+1}\frac{(r-1)!B_r}{1-\alpha^r} \quad \text{and} \quad \kappa_r^{(p)} = \sum_{j=0}^{r}S(r,j)(-1)^{j+1}\frac{(j-1)!B_j}{(1-\alpha^j)} \qquad (r \geq 1). \tag{7.8}$$

By (2.7) and (7.3), the mean, the variance and the index of dispersion of X_t are given by

$$\mu_X = \frac{1}{1-\alpha}\sum_{r=0}^{\infty}\frac{\lambda q^r}{1+\lambda q^r}, \quad \sigma_X^2 = \frac{1}{1-\alpha^2}\sum_{r=0}^{\infty}\frac{\lambda q^r}{1+\lambda q^r}([1+\lambda q^r]^{-1}+\alpha)$$

and

$$I_X = \sum_{r=0}^{\infty}\frac{\lambda q^r}{1+\lambda q^r}([1+\lambda q^r]^{-1}+\alpha)\Big/\Big[(1+\alpha)\sum_{r=0}^{\infty}\frac{\lambda q^r}{1+\lambda q^r}\Big].$$

Since the Heine distribution is underdispersed, the INAR (1) process with a Heine innovation is underdispersed. We note that I_X is increasing in α and q and decreasing in λ (see Figure 5).

Similarly to the way (6.9)–(6.10) were derived, the moments and factorial moments of X_t can be computed using the formulas

$$\mu_r^{(p)} = \sum_{j=0}^{r-1}\binom{r-1}{j}\kappa_{r-j}^{(p)}\mu_j \quad \text{and} \quad \mu_{[r]}^{(p)} = \sum_{j=0}^{r}s(r,j)\mu_j^{(p)}, \tag{7.9}$$

with initial conditions $\mu_0^{(p)} = 1$ and $\mu_1^{(p)} = \kappa_1^{(p)}$, and where $\{s(r,j)\}$ are the Stirling numbers of the first kind of (6.11).

In turn, the factorial moments $\mu_{[r]}^{(p)}, r \geq 1$, of X_t can be obtained via the formula (see [12], Section 1.2.7):

$$\mu_{[r]}^{(p)} = \sum_{j=0}^{r}s(r,j)\mu_j^{(p)}, \tag{7.10}$$

where $\{s(r,j)\}$ are the Stirling numbers of the first kind of (6.11).

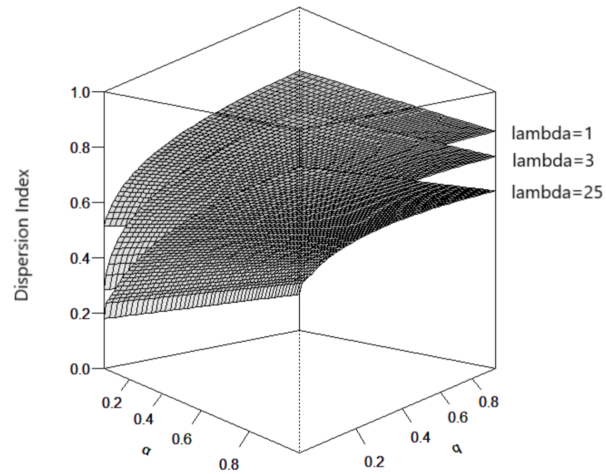

Figure 5: Variance-mean ratio of the marginal distribution of an INAR(1) process with a Heine innovation.

8. Conclusion

In this article we formalized a theoretical approach to study the distributional properties of a stationary INAR (1) process based on binomial thinning when the innovation distribution is known. We established a number of basic properties of a specific infinite convolution of distributions on $\mathbb{Z}_+$and interpreted our results in the context of stationary INAR(1) models whose innovation has a finite mean. As an application, we presented new distributional properties for some stationary INAR (1) models that show underdispersion, including two new INAR (1) models with q-series innovation distributions. Simulations and statistical analysis for some of these models will be the object of the authors future work. Another direction of research would be to extend the results in this paper by using more general thinning operators.

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Appendix

This section is devoted to the proof of Theorem 3.1 and Theorem 7.1. We first extablish a Lemma.

Lemma 8.1. *Assume $n \geq 2$ and $a_i \in (0,1)$ for $i = 0, 1, 2, \cdots, n-1$. Then,*

1.

$$\prod_{i=0}^{n-1}(1-a_i) = 1 + \sum_{k=1}^{n}(-1)^k\Big[\sum_{0\leq j_1<j_2<\cdots<j_k\leq n-1}\prod_{l=1}^{k}a_{j_l}\Big]. \tag{8.1}$$

2.

$$\sum_{0\leq j_1<j_2<\cdots<j_k\leq n-1}\alpha^{j_1}\alpha^{j_2}\cdots\alpha^{j_k} = \alpha^{\binom{k}{2}}\prod_{l=0}^{k-1}\frac{1-\alpha^{n-l}}{1-\alpha^{l+1}} \tag{8.2}$$

for every $k \in \{1.\cdots, n\}$.

Proof: (1) follows by a straightforward induction.

(2) We also proceed by induction. The result is trivially true for $n = 2$ (forces $k = 1$). Assume the assertion is true up to n. Equation (8.2) holds for $n+1$ and $k = n+1$, as in this case

$$\sum_{0\leq j_1<j_2<\cdots<j_{n+1}\leq n}\alpha^{j_1}\alpha^{j_2}\cdots\alpha^{j_k} = \alpha^{\sum_{k=0}^{n}k} = \alpha^{\binom{n+1}{2}} = \alpha^{\binom{n+1}{2}}\prod_{l=0}^{n}\frac{1-\alpha^{n+1-l}}{1-\alpha^{l+1}}.$$

Assume now $k \in \{1, 2, \cdots, n\}$. Setting $J = (j_1, j_2, \cdots, j_k) \in \mathbb{N}^k$, it is clear that

$$\{J \in \mathbb{N}^k : 0 \leq j_1 < j_2 < \cdots < j_k \leq n\} = A \cup B,$$

where $A = \{J \in \mathbb{N}^k : 0 \leq j_1 < j_2 < \cdots < j_k \leq n-1\}$ and $B = \{J \in \mathbb{N}^k : 0 \leq j_1 < j_2 < \cdots < j_{k-1} \leq n-1, j_k = n\}$. Therefore,

$$\sum_{0 \leq j_1 < j_2 < \cdots < j_k \leq n} \prod_{l=1}^{k} \alpha^{j_l} = \sum_{J \in A} \prod_{l=1}^{k} \alpha^{j_l} + \sum_{J \in B} \alpha^n \prod_{l=1}^{k-1} \alpha^{j_l}.$$

Using the induction hypothesis, it follows that

$$\sum_{J \in A} \prod_{l=1}^{k} \alpha^{j_l} = \alpha^{\binom{k}{2}} \prod_{l=0}^{k-1} \frac{1-\alpha^{n-l}}{1-\alpha^{l+1}}$$

and

$$\sum_{J \in B} \alpha^n \prod_{l=1}^{k-1} \alpha^{j_l} = \alpha^n \alpha^{\binom{k-1}{2}} \prod_{l=0}^{k-2} \frac{1-\alpha^{n-l}}{1-\alpha^{l+1}}$$

which implies

$$\sum_{0 \leq j_1 < j_2 < \cdots < j_k \leq n} \prod_{l=1}^{k} \alpha^{j_l} = \frac{\prod_{l=0}^{k-2}(1-\alpha^{n-l})\left[(1-\alpha^{n-k+1})\alpha^{\binom{k}{2}} + (1-\alpha^k)\alpha^n \alpha^{\binom{k-1}{2}}\right]}{\prod_{l=0}^{k-1}(1-\alpha^{l+1})}.$$

Now, noting that $\binom{k}{2} = \binom{k-1}{2} + k - 1$, it is easily seen that

$$(1-\alpha^{n-k+1})\alpha^{\binom{k}{2}} + (1-\alpha^k)\alpha^n \alpha^{\binom{k-1}{2}} = \alpha^{\binom{k}{2}}(1-\alpha^{n+1}).$$

Therefore, (8.2) holds for $n+1$. □

Proof of Theorem 3.1:

Let $\{X_t\}$ be the stationary INAR (1) process with a Bernoulli(p) innovation sequence. By Theorem 2.3 and (2.8), its marginal pgf is

$$\varphi_X(z) = \prod_{i=0}^{\infty}(1 - p\alpha^i(1-z)). \tag{8.3}$$

Since $\varphi_X(z) = \lim_{n\to\infty} \prod_{i=0}^{n-1}(1-p\alpha^i(1-z))$, we conclude by the continuity theorem that the marginal pmf $\{p_r\}$ of $\{X_t\}$ is the weak limit of a sequence of Poissonian Binomial distributions of (3.1) and (3.2), with $m = n$, $q = \alpha$ and $c = p$. Let $r \geq 0$. We define a purely atomic measure, we denote $meas_r$, on $\mathbb{N}_r = \{r, r+1, r+2, \cdots\}$ and its power set $\mathcal{P}(\mathbb{N}_r)$ as follows:

$$meas_r(\{k\}) = \frac{p^k \alpha^{\binom{k}{2}}}{\prod_{l=1}^{k}(1-\alpha^l)}, \quad (k \geq r), \tag{8.4}$$

with $meas_0(\{0\}) = 1$. It is clear that $\sum_{k=r}^{\infty} meas_r(\{k\}) < \infty$. Therefore, $meas_r$ is a finite measure. Define now the sequence of functions $\{f_n(\cdot)\}$ on $\mathbb{N}_r$ by

$$f_n(k) = \begin{cases} (-1)^{k-r}\binom{k}{r} \prod_{l=0}^{k-1}(1-\alpha^{n-l}) & \text{if } k = r, r+1, \cdots, n \\ 0 & \text{if } k > n. \end{cases}$$

Define $h(k) = \binom{k}{r}$ on $\mathbb{N}_r$. It is clear that $|f_n(k)| \leq h(k)$ (recall $\alpha \in (0,1)$) and that $\sum_{k=r}^{\infty} h(k) meas_r(\{k\}) < \infty$ by the ratio test). Moreover, for every $k \in \mathbb{N}_r$,

$$f(k) = \lim_{n\to\infty} f_n(k) = (-1)^{k-r}\binom{k}{r}.$$

Rewriting $p_r^{(n)}$ in terms of the discrete integral of $f_n(k)$ on the measure space $(\mathbb{N}_r, \mathcal{P}(\mathbb{N}_r), meas_r)$ and calling on the dominated convergence theorem, we have

$$p_r = \lim_{n\to\infty} \int_{\mathbb{N}_r} f_n(k)\, meas_r(dk) = \int_{\mathbb{N}_r} f(k)\, meas_r(dk),$$

which is precisely (3.3) and thus part 1 of the Theorem is established. To show part 2, note that

$$P(X_t \geq r) = \sum_{j=r}^{\infty}\sum_{k=j}^{\infty}(-1)^{k-j}\binom{k}{j}\frac{p^k\alpha^{\binom{k}{2}}}{\prod_{l=1}^{k}(1-\alpha^l)}.$$

Since the double series above converges absolutely, interchanging summations is allowed, leading to

$$P(X_t \geq r) = \sum_{k=r}^{\infty}\left(\sum_{j=r}^{k}(-1)^{k-j}\binom{k}{j}\right)\frac{p^k\alpha^{\binom{k}{2}}}{\prod_{l=1}^{k}(1-\alpha^l)}.$$

We have by induction on k that $\sum_{j=r}^{k}(-1)^{k-j}\binom{k}{j} = (-1)^{k-r}\binom{k-1}{r-1}$, thus establishing (3.4).

For part 3, we note first that $\varphi_X(z)$ of (8.3) can be rewritten as

$$\varphi_X(z) = \exp\left\{\sum_{i=0}^{\infty}\ln(1-p\alpha^i(1-z))\right\}. \tag{8.5}$$

The representation (3.6) of $\varphi_X(z)$ follows by way of the power series expansion of

$$-\ln(1-x) = \sum_{n=1}^{\infty} x^n/n, \quad 0 \leq x < 1$$

applied to $x = p\alpha^i(1-z)$ in (8.5).

To prove (3.5), we first note that by letting $a_i = p\alpha^i(1-z)$ in (8.1) and using (8.2), we obtain the following expression for $\varphi_n(z)$ of (2.3):

$$\varphi_{n-1}(z) = 1 + \sum_{k=1}^{n} p^k(z-1)^k\alpha^{\binom{k}{2}}\prod_{l=0}^{k-1}\frac{1-\alpha^{n-l}}{1-\alpha^{l+1}} \tag{8.6}$$

and therefore,

$$\varphi_X(z) = \lim_{n\to\infty}\Big[1 + \sum_{k=1}^{n}(z-1)^k\prod_{l=0}^{k-1}(1-\alpha^{n-l})\frac{p^k\alpha^{\binom{k}{2}}}{\prod_{l=1}^{k}(1-\alpha^l)}\Big]. \tag{8.7}$$

We proceed as in the proof of (3.3). We define a sequence of functions $g_n(k)$ on the finite measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), meas_0)$, where $meas_0$ is defined in (8.4):

$$g_n(k) = \begin{cases} 1 & \text{if } k = 0 \\ (z-1)^k\prod_{l=0}^{k-1}(1-\alpha^{n-l}) & \text{if } 1 \leq k \leq n \\ 0 & \text{if } k > n. \end{cases}$$

It is easily seen that $|g_n(k)| \leq 1$ (recall $\alpha \in (0,1)$ and $z \in [0,1]$) and that

$$g(k) = \lim_{n\to\infty} g_n(k) = \begin{cases} 1 & \text{if } k = 0 \\ (z-1)^k & \text{if } k \geq 1. \end{cases}$$

Rewriting (8.7) in terms of the discrete integral on the measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), meas_0)$ and calling on the Dominated Convergence Theorem, we have

$$\varphi_X(z) = \lim_{n\to\infty}\int_{\mathbb{N}} g_n(k)\, meas_0(dk) = \int_{\mathbb{N}} g(k)\, meas_0(dk),$$

which is precisely (3.5).

Proof of Theorem 7.1:

First, we note that $0 < \beta_j < 1$ for any $j \geq 0$. Moreover, for any $n \geq 1$,

$$B_n = \sum_{j=0}^{\infty} \frac{\lambda^n (q^n)^j}{(1+\lambda q^j)^n} \leq \frac{\lambda^n}{1-q^n} < \infty. \tag{8.8}$$

The pgf $\Psi(z)$ of the innovation sequence of $\{X_t\}$ (cf. (7.1)) yields

$$\Psi(1-\alpha^i+\alpha^i z) = \prod_{j=0}^{\infty} \big(1-\beta_j \alpha^i (1-z)\big) \qquad (i \geq 0).$$

It follows by Theorem 2.3 and (2.8) that

$$\varphi_X(z) = \prod_{i=0}^{\infty} \Big[\prod_{j=0}^{\infty} \big(1-\beta_j \alpha^i (1-z)\big)\Big]. \tag{8.9}$$

Clearly, the right hand side of (8.9) converges. A straightforward argument shows that the double infinite product $\prod_{i=0}^{\infty}\prod_{j=0}^{\infty} \big(1-\beta_j\alpha^i(1-z)\big)$ converges. In order to be able to interchange the order of the infinite products in (8.9), it remains to show that the iterated infinite product

$$\prod_{j=0}^{\infty} \Big[\prod_{i=0}^{\infty} \big(1-\beta_j \alpha^i (1-z)\big)\Big] = \prod_{j=0}^{\infty} P_j(z)$$

converges, where for each $j \geq 0$,

$$P_j(z) = \prod_{i=0}^{\infty} \big(1-\beta_j \alpha^i (1-z)\big).$$

Note that for each $j \geq 0$, $P_j(\cdot)$ has the form of the pgf of the marginal of an INAR(1) process with a Bernoulli(β_j) innovation (see (8.3)). Therefore, by Theorem 3.1 and (3.5),

$$P_j(z) = 1 + \sum_{n=1}^{\infty} \frac{\beta_j^n (z-1)^n \alpha^{\binom{n}{2}}}{\prod_{l=1}^{n}(1-\alpha^l)}. \tag{8.10}$$

For $j \geq 0$, denote

$$\zeta_j(z) = \sum_{n=1}^{\infty} \frac{\beta_j^n (z-1)^n \alpha^{\binom{n}{2}}}{\prod_{l=1}^{n}(1-\alpha^l)}.$$

By (8.8) and $0 \leq z \leq 1$, we have

$$\sum_{j=0}^{\infty} |\zeta_j(z)| \leq \sum_{n=1}^{\infty}\sum_{j=0}^{\infty} \frac{\beta_j^n \alpha^{\binom{n}{2}}}{\prod_{l=1}^{n}(1-\alpha^l)} = \sum_{n=1}^{\infty} \frac{B_n \alpha^{\binom{n}{2}}}{\prod_{l=1}^{n}(1-\alpha^l)} \leq \sum_{n=1}^{\infty} a_n,$$

where

$$a_n = \frac{\lambda^n}{1-q^n} \frac{\alpha^{\binom{n}{2}}}{\prod_{l=1}^{n}(1-\alpha^l)}.$$

Since $\alpha, q \in (0,1)$, we have

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\lambda(1-q^n)\alpha^n}{(1-q^{n+1})(1-\alpha^{n+1})} = 0.$$

By the ratio test, $\sum_{j=0}^{\infty} |\zeta_j(z)|$ converges uniformly over $z \in [0,1]$. This in turn implies (see [17]) that $\prod_{j=0}^{\infty} P_j(z)$ converges uniformly over $z \in [0,1]$. The representation (7.4) then follows by interchanging the order of the infinite products in (8.9) and by using (8.10). We now prove (7.5). By the first part of the proof, we have

$$\varphi_X(z) = \prod_{j=0}^{\infty}\Big[\prod_{i=0}^{\infty}\big(1-\beta_j\alpha^i(1-z)\big)\Big].$$

Applying the representation (3.6) to $\prod_{i=0}^{\infty}\big(1-\beta_j\alpha^i(1-z)\big)$ with $p=\beta_j$, we have

$$\varphi_X(z) = \exp\left\{-\sum_{j=0}^{\infty}\Big[\sum_{n=1}^{\infty}\frac{\beta_j^n}{n(1-\alpha^n)}(1-z)^n\Big]\right\}. \tag{8.11}$$

This implies that the double series in (8.11) is convergent. Since its terms are nonnegative (as $0 \le z \le 1$), the order of summation can be interchanged (by Cauchy's criterion for double series). This establishes the representation (7.5). By Theorem 3.1 and (3.3), $P_j(z)$ of (8.10) is the pgf of the pmf $\{q_r^{(j)}\}$ of (7.7). Therefore, part 3 and (7.6) follow from the representation (7.4), Theorem 2.2 and (2.5).