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Pitman's Measure of Closeness for Weighted Random Variables

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Abstract:

- Statistical inference based on weighted random variables is developed in the sense of Pitman's measure of closeness. Some general formulas are presented to compute the Pitman closeness of two weighted random variables to the parameter of interest. A new general weighted model is also introduced and some properties are investigated. Also, the concept of Pitman's measure of closeness is used for measuring the nearness of some weighted random variables with respect to each other. The results are illustrated using some real data sets. Eventually, some conclusions are stated.

Keywords:

- *weighted distribution; exponential distribution; ordered data; population quantile; skew distributions.*

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1. INTRODUCTION

There are situations in which the observed data would not be modeled by a proper stochastic model. In these cases, some general models leading to weighted distributions may be used. The concept of weighted distributions has been introduced by Fisher [17] and Rao [35] in connection with modeling statistical data in situation that the standard distribution was not appropriate for their purposes. A formal definition of a weighted distribution for random variable X with density (mass) function $f(x)$ and non-negative weight $w(x)$ is obtained by $g(x) = \frac{w(x)f(x)}{E(w(X))}$, where $E(w(X)) > 0$. The corresponding weighted random variable is denoted by X_w in this paper. For more details about the statistical applications of weighted distributions, we refer the readers to Patil and Rao [31, 32] and Rao [38]. From Gupta and Keating [20] to Bartoszewicz [12] and the references therein, the idea of the weighted distributions is developed to various applications and properties. Saghir *et al.* [44] carried out a brief review of weighted distributions, and investigated the implications of the differing weight models as well as characterizations of these distributions based on a simple relationship between two truncated moments. In recent years, this concept has been applied in many areas of statistics such as biomedical, human study, wildlife populations, electronic, etc. For example, Jiang [21] used the weight function distribution to analyze measurement errors of an electromagnetic flowmeter. It is of great importance to note that the models of ordered data are special cases of weighted distributions, for example the i-th order statistic in sample of size n, denoted by $X_{i:n}$ $1 \le i \le n$, from a population with probability density function (pdf) $f(\cdot)$ and cumulative distribution function (cdf) $F(\cdot)$ is a weighted random variable with weight function $w(x) = (F(x))^{i-1}(\bar{F}(x))^{n-i}$. The n-th upper k-record and the n-th lower k-record are also weighted random variables with weight functions $w(x) = \left[-\log \bar{F}(x)\right]^{n-1}(\bar{F}(x))^{k-1}$ and $w(x) = [-\log F(x)]^{n-1}F(x)^{k-1}$, respectively. For more details and literature on order statistics and record values, see for example, Arnold *et al.* [6, 7] and David and Nagaraja [14]. The skew distributions are also another kind of weighted distributions. Recently, Gómez-Déniz *et al.* [19] investigated the properties and applications of a new family of skew distributions. Azzalini [8] has done an overview on the progeny of the skew-normal family.

It is known that the theory of estimation is a fundamental discipline dealing with the specification of probabilistic model in terms of observed data. There are many measures to evaluate the performance of an estimator like mean squared error (MSE), mean absolute deviation, etc. The Pitman's measure of closeness (PMC) introduced by Pitman [33] is another criterion which has been used by several authors to compare the performance of the estimators. Let us first recall the formal definition:

Definition 1.1. Let $\hat{\theta}_1$ and $\hat{\theta}_2$ be two estimators of a common parameter θ. The Pitman's measure of closeness of $\hat{\theta}_1$ relative to $\hat{\theta}_2$ is denoted by $\pi(\hat{\theta}_1, \hat{\theta}_2|\theta)$ and defined as

$$\pi(\hat{\theta}_1, \hat{\theta}_2|\theta) = \Pr(|\hat{\theta}_1 - \theta| < |\hat{\theta}_2 - \theta|), \qquad \forall\ \theta \in \Theta, \tag{1.1}$$

where Θ is the parameter space. If $\Pr(\hat{\theta}_1 = \hat{\theta}_2) = 0$ and $\pi(\hat{\theta}_1, \hat{\theta}_2|\theta) \ge 1/2$ for all $\theta \in \Theta$, with strict inequality holding for at least one θ, then $\hat{\theta}_1$ is said to be a Pitman closer estimator than $\hat{\theta}_2$ with respect to θ.

It is obvious that $\pi(\hat{\theta}_1, \hat{\theta}_2|\theta)$ determines the relative frequency that the estimator $\hat{\theta}_1$ is closer than $\hat{\theta}_2$ to the true but unknown value of the parameter θ. Note that the absolute difference in (1.1) may be replaced by any other loss function, in this case the associated probability is called generalized PMC. Several developments in estimation theory via PMC arguments have been written by Efron [16]. Also, Rao [36, 37] concluded noticeable comments about the necessity of inference based on PMC. Keating [23] derived various aspects of PMC. Keating and Mason [24] provided many practical examples in which the PMC may be more useful than MSE. Sen [45] presented the condition on variance for an estimator being closer than another. Ghosh and Sen [18] have shown under certain conditions a median unbiased estimator of parameter is the Pitman closest within a certain class of estimators. Some characterization results in statistical inference and decision theory with examples have been explained by Lee [27] in view of the PMC for location and scale families. Nayak [30] used the PMC to find best estimators of a location or scale parameter. Many relevant references can be found for example in the monograph by Keating *et al.* [25]. A useful discussion to the question "Is Pitman closeness a reasonable criterion?" with comments and rejoinder about this measure can be found in Robert *et al.* [43]. Kourouklis [26] used the PMC to get an improved estimation.

It is of great importance to note that two various statistics in estimating a common parameter would be considered as two weighted random variables. This motivated us to study the PMC in view of the weighted distributions. Toward this end, the PMC of weighted random variables with respect to any unknown parameter is first investigated. A new general weighted model is also introduced and some properties are investigated. Then, the concept of PMC is used to measure the nearness of some weighted random variables with respect to each other.

The rest of the paper is organized as follows: In Section 2, some general results are given to compute the PMC of the weighted random variables to the parameter of interest. In Section 3, a new general weighted model is introduced and some numerical results and conclusions are presented. A real data set is used in this section to illustrate the proposed procedure. In Section 4, the PMC of two weighted random variables with respect to another one is investigated. Moreover, the results are applied to another real data set. Finally, some conclusions are presented in Section 5.

2. GENERAL RESULTS

Let X be a random variable with pdf $f(\cdot)$ and cdf $F(\cdot)$ which is defined on finite interval $[a, b]$. Assume X_{w_1} and X_{w_2} are two independent weighted random variables of X with weight functions $w_1(\cdot)$ and $w_2(\cdot)$, respectively. Again, it is pointed out that these random variables may be considered as two various estimators for a common parameter θ. Based on (1.1), the PMC of X_{w_1} and X_{w_2} with respect to θ is given by

$$\pi(X_{w_1}, X_{w_2}|\theta) = P(|X_{w_1} - \theta| < |X_{w_2} - \theta|). \tag{2.1}$$

A general expression for the probability in (2.1) can be written as follows:

$$\begin{aligned}\pi(X_{w_1}, X_{w_2}|\theta) &= P(X_{w_1} < X_{w_2}, X_{w_1}+X_{w_2} > 2\theta) + P(X_{w_1} > X_{w_2}, X_{w_1}+X_{w_2} < 2\theta)\\ &= \int_{\max\{a,2\theta-b\}}^{\theta}\int_{2\theta-x}^{b} g_1(x)g_2(y)dydx + \int_{\theta}^{b}\int_{x}^{b} g_1(x)g_2(y)dydx\\ &\quad + \int_{a}^{\theta}\int_{a}^{x} g_1(x)g_2(y)dydx + \int_{\theta}^{\min\{b,2\theta-a\}}\int_{a}^{2\theta-x} g_1(x)g_2(y)dydx\\ &= \int_{\max\{a,2\theta-b\}}^{\theta} \bar{G}_2(2\theta-x)g_1(x)dx + \int_{\theta}^{b}\bar{G}_2(x)g_1(x)dx\\ &\quad + \int_{a}^{\theta} G_2(x)g_1(x)dx + \int_{\theta}^{\min\{b,2\theta-a\}} G_2(2\theta-x)g_1(x)dx, \end{aligned} \tag{2.2}$$

where for $i=1,2$, the functions $g_i(x)$ and $G_i(x)$ are respectively the pdf and cdf of X_{w_i}. It can be shown that $\bar{G}_i(x) = 1 - G_i(x) = \frac{B_i(x)}{E(w_i(X))}$, where

$$B_i(x) = \bar{F}(x)E(w_i(X)|X>x) = \int_x^{\infty} w_i(t)f(t)dt.$$

Therefore, we have

$$\begin{aligned}\pi(X_{w_1}, X_{w_2}|\theta) &= \int_{\max\{a,2\theta-b\}}^{\theta} \frac{B_2(2\theta-x)}{E(w_2(X)}\frac{w_1(x)f(x)}{E(w_1(X))}dx + \int_{\theta}^{b}\frac{B_2(x)}{E(w_2(X))}\frac{w_1(x)f(x)}{E(w_1(X))}dx\\ &\quad + \int_a^{\theta}\left(1-\frac{B_2(x)}{E(w_2(X))}\right)\frac{w_1(x)f(x)}{E(w_1(X))}dx\\ &\quad + \int_{\theta}^{\min\{b,2\theta-a\}}\left(1-\frac{B_2(2\theta-x)}{E(w_2(X))}\right)\frac{w_1(x)f(x)}{E(w_1(X))}dx.\end{aligned} \tag{2.3}$$

In special case of $w_i(x) = \varphi_i\big(F(x)\big)$ $(i=1,2)$, by transforming $u = F(x)$, we get

$$\begin{aligned}\pi(X_{w_1}, X_{w_2}|\theta) &= \frac{1}{\gamma_1\gamma_2}\left\{\int_{F(\max\{a,2\theta-b\})}^{F(\theta)} B_2(2\theta - F^{-1}(u))\varphi_1(u)du + \int_{F(\theta)}^{1} B_2(F^{-1}(u))\varphi_1(u)du\right\}\\ &\quad -\frac{1}{\gamma_1\gamma_2}\left\{\int_0^{F(\theta)} B_2(F^{-1}(u))\varphi_1(u)du\right.\\ &\qquad\left. + \int_{F(\theta)}^{F(\min\{b,2\theta-a\})} B_2(2\theta - F^{-1}(u))\varphi_1(u)du\right\}\\ &\quad + \frac{1}{\gamma_1}\left\{\int_0^{F(\theta)}\varphi_1(u)du + \int_{F(\theta)}^{F(\min\{b,2\theta-a\})}\varphi_1(u)du\right\},\end{aligned} \tag{2.4}$$

where $\gamma_i = E[\varphi_i(U)]$ $(i=1,2)$, such that U is a Uniform(0, 1) random variable. Table 1 shows some special cases of $\varphi_i(u)$ $(i=1,2)$, which have been previously studied by several authors.

The definition of the weighted random variables may be extended to a random sample of size n. Let $X_1, X_2, ..., X_n$ be independent and identically distributed (iid) random variables with cdf $F(\cdot)$ and pdf $f(\cdot)$, then a weighted version of this sample can be defined by $(X_1^*, X_2^*, ..., X_n^*)$, which have the joint pdf

$$h_{X_1^*,X_2^*,...,X_n^*}(x_1^*, x_2^*, ...x_n^*) = \frac{w(x_1^*, x_2^*, ...x_n^*)}{E[w(X_1^*, X_2^*, ..., X_n^*)]}\prod_{i=1}^{n} f(x_i^*). \tag{2.5}$$

Table 1: Some special cases of $\varphi_i(u)$ related to PMC.

References	$\varphi_1(u)$	$\varphi_2(u)$	Descriptions
Ahmadi and Raqab [4]	$u^{i-1}(1-u)^{m-i+1}$	$u^{j-1}(1-u)^{n-j+1}$	Order statistics in two samples
Raqab and Ahmadi [39]	$[-\log(1-u)]^i$	$[-\log(1-u)]^j$	Record values from two sequences
Volterman *et al.* [47]	$\sum_{l=1}^{i} a_l^R(i)(1-u)^{\gamma_l^R-1}$	$\sum_{l=1}^{j} a_l^S(j)(1-u)^{\gamma_l^S-1}$	Two-sample progressive type-II censoring
Ahmadi and Mohtashami Borzadaran [5]	$[-\log(1-u)]^m$	$u^{i-1}(1-u)^{n-i+1}$	Record values and order statistics

Notice that the joint pdf in (2.5) is a weighted version of $\prod_{i=1}^n f(x_i^*)$. Now, for the special case of $n=2$, this joint distribution can be expressed as

$$h_{X_1^*,X_2^*}(x,y) = \frac{w(x,y)}{E[w(X_1^*,X_2^*)]} f(x)f(y), \tag{2.6}$$

where include the joint distribution of two order statistics in a finite random sample or two k-records in a sequence of iid random variables. By assuming $w(x,y)=\varphi(u,v)$, where $u=F(x)$ and $v=F(y)$, the PMC of X_1^* and X_2^* with respect to parameter θ can be determined as follows:

$$\begin{aligned}
\pi(X_1^*,X_2^*|\theta) &= \frac{1}{A}\Bigg\{\int_{F(\max(a,2\theta-b))}^{F(\theta)} \int_{F(2\theta-F^{-1}(u))}^{1} \varphi(u,v)dvdu \\
&\quad + \int_{F(\theta)}^{1}\int_{u}^{1} \varphi(u,v)dvdu + \int_{0}^{F(\theta)}\int_{0}^{u} \varphi(u,v)dvdu \\
&\quad + \int_{F(\theta)}^{F(\min(b,2\theta-a))}\int_{0}^{F(2\theta-F^{-1}(u))} \varphi(u,v)dvdu\Bigg\},
\end{aligned} \tag{2.7}$$

where $A=E[\varphi(F(X_1^*),F(X_2^*))]$. Some special cases of the weighted model in (2.6) have been previously studied in view of PMC by some authors which are summarized in Table 2.

Table 2: Some special cases of $\varphi(u,v)$ related to PMC.

References	$\varphi(u,v)$	Descriptions
Balakrishnan *et al.* [11]	$u^{m-1}(v-u)^{i-m-1}(1-v)^{n-i}$	Sample median and the i-th order statistic
Balakrishnan *et al.* [9]	$u^{i-1}(v-u)^{j-i-1}(1-v)^{n-i}$	Two order statistics in one sample
Ahmadi and Balakrishnan [1]	$\frac{[-\log(1-u)]^i}{1-u}\Big[-\log(\frac{1-v}{1-u})\Big]^{j-i-1}$	Two upper records in one sequence
Ahmadi and Balakrishnan [3]	$[-\log(u+1-v)]^{i-1}$	The i-th lower and upper records in one sequence
Ahmadi and Balakrishnan [2]	$[-\log(1-u)]^i\Big[-\log(\frac{1-v}{1-u})\Big]^{j-i-1}\frac{(1-v)^{k-1}}{1-u}$	Two k-records in one sequence

Remark 2.1. It is worthwhile to note that not only for the special cases presented in Tables 1 and 2, but also the PMC of the general versions of weighted random variables with respect to the parameter of interest can be derived by the use of (2.4) and (2.7).

3. A GENERAL WEIGHTED MODEL

In Section 2, a weighted random variable whose weight function was a function of the baseline cdf was considered in some special cases. Here, we introduce a more general model contains all previous ones. Let X be a continuous random variable with pdf $f(\cdot)$ and cdf $F(\cdot)$. For positive real constants α, β, γ and δ, one may consider a general weighted random variable with the following pdf

$$g_F(x;\lambda) = \frac{1}{M(\lambda)}[F(x)]^{\alpha-1}[\bar{F}(x)]^{\gamma-1}[-\log F(x)]^{\beta-1}[-\log \bar{F}(x)]^{\delta-1}f(x), \tag{3.1}$$

where $\lambda = (\alpha, \gamma, \beta, \delta)$ and $M(\lambda)$ is the normalizing constant which is given by

$$\begin{aligned} M(\lambda) &= M(\alpha,\gamma,\beta,\delta) \\ &= \int_0^1 u^{\alpha-1}(1-u)^{\gamma-1}(-\log u)^{\beta-1}[-\log(1-u)]^{\delta-1}du. \end{aligned} \tag{3.2}$$

The model defined in (3.1) includes many famous families of distributions, such as distribution of order statistics (see David and Nagaraja, [14]), distributions of upper and lower k-records (see, Arnold *et al.*, [6]), Jones model (see Jones, [22]) and proportional (reversed) hazard rate model. This model can be also considered as the pdf of a weighted k-record statistic and also weighted order statistics. In the next two subsection we consider two famous member of the proposed model.

3.1. Results based on records

Let us recall the sequences of upper k-record times, $T_{n,k}$, and upper k-record values, $R_{n,k}$, which are defined as follows: $T_{1,k} = k$ with probability one, $R_{1,k} = X_{1:k}$ and for $n \geq 2$

$$T_{n,k} = \min\{j : j > T_{n-1,k},\ X_j > X_{T_{n-1,k}-k+1:T_{n-1,k}}\},$$

and the n-th upper k-record value is defined by $R_{n,k} = X_{T_{n,k}-k+1:T_{n,k}}$, for $n \geq 1$. For $k = 1$, the ordinary records are recovered. Then, the pdf of $R_{n,k}$ is given by

$$f_{n,k}(x) = \frac{k^n}{\Gamma(n)}\big[-\log \bar{F}(x)\big]^{n-1}(\bar{F}(x))^{k-1}f(x),$$

where $\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}dx$ stands for the complete gamma function; see Arnold *et al.* [6] for more details. It is obvious that by taking $\gamma = k$ and $\delta = n$, the pdf in (3.1) can be interpreted as the pdf of a weighted the n-th upper k-record statistic, which is denoted by $R^w_{n,k}$. Similarly, the weighted lower k-records may be defined. The first question arises here is that whether k-records or the weighted version ones are closer to the parameter of interest. By using (2.4) and performing some algebraic calculations, it can be shown that the PMC of $R^w_{n,k}$ and $R_{n,k}$ with respect to θ is

$$\begin{aligned}\pi_{\alpha,\beta}(R^w_{n,k}, R_{n,k}|\theta) \;=\; & \frac{1}{M(\alpha,\beta,k,n)} \sum_{j=\beta-1}^{\infty} \sum_{r=0}^{\alpha-1} \binom{\alpha-1}{r} (-1)^r C_j(\beta-1) \\ & \times \Bigg\{ \frac{\Gamma\big(n, -(r+j+k)\log \bar{F}(\max(a, 2\theta-b))\big)}{(r+j+k)^n} \\ & + \sum_{i=n}^{\infty} \frac{k^i}{i!} \Bigg(\frac{\Gamma(n) - 2\Gamma\big(n, -(r+j+2k)\log \bar{F}(\theta)\big)}{(n+j+2k)^{n+i}} \\ & + \int_{-\log \bar{F}(\theta)}^{-\log \bar{F}(\min(b,2\theta-a))} \psi(y;i,j,r)dy \\ & - \int_{-\log \bar{F}(\max(a,2\theta-b))}^{-\log \bar{F}(\theta)} \psi(y;i,j,r)dy \Bigg) \Bigg\}, \end{aligned} \tag{3.3}$$

where $\Gamma(n,a) = \int_a^\infty x^{n-1}e^{-x}dx$ stands for the incomplete gamma function, the function $M(\alpha,\beta,k,n)$ is as defined in (3.2), $C_j(n)$ is the coefficient of w^j in the expansion of $(\sum_{i=1}^\infty \frac{w^i}{i})^n$, and

$$\psi(y;i,j,r) \;=\; \big(-\log \bar{F}(2\theta - F^{-1}(1-e^{-y}))\big)^i \big(\bar{F}(2\theta - F^{-1}(1-e^{-y}))\big)^k \, y^{n-1} e^{-(r+j+k)y}.$$

Remark 3.1. It is clear that for other different versions of weight functions, we can express the PMC via similar arguments in (3.3), for which they can be interpreted as applications and specifications of several forms.

In the rest of this section, the population quantile is considered as the parameter of interest θ. We recall that the population quantile ξ_p of order p $(0 < p < 1)$ of the cdf $F(\cdot)$ is defined by $\xi_p = \inf\{x : F(x) \geq p\}$. Balakrishnan *et al.* [11, 10] determined the closest order statistic in a random sample of size n to a specific population quantile and specially studied the PMC of sample median to population median. Ahmadi and Balakrishnan [1] examined the PMC of record statistics to the population quantiles of location-scale family of distributions. Moreover, Ahmadi and Balakrishnan [2] investigated the PMC of k-records and Ahmadi and Balakrishnan [3] obtained the PMC of current records for location-scale families. Similar work was done by Razmkhah and Ahmadi [41] regarding the current k-records. The PMC of upper (lower) records in two independent sequences of iid continuous random variables to population quantiles was studied by Raqab and Ahmadi [39]. Similarly, the PMC of order statistics in a two-sample problem was investigated by Ahmadi and Raqab [4]. Moreover, a comparison study for order statistics and records was performed with some remarks by Ahmadi and Mohtashami Borzadaran [5]. Davies [15] studied some PMC results for type-I hybrid censored data from exponential distribution. Morabbi and Razmkhah [29] used the PMC to get the quantile estimation based on modified ranked set sampling schemes. The weighted versions of the aforementioned papers, for example, weighted order statistics and weighted k-records may be of great interest which are discussed in this paper.

From (3.3), it is seen that the PMC is not distribution-free. So, the baseline distribution has to be specified. Therefore, for as an example, the standard exponential distribution has been considered and the PMC investigated for the population quantiles. With this in mind, let $\{X_i, i \geq 1\}$ be a sequence of iid random variables from standard exponential distribution.

The PMC of $R^w_{n,k}$ and $R_{n,k}$ with respect to the p-th quantile of this distribution, can be simplified as follows:

$$\begin{aligned}\pi_{\alpha,\beta}(R^w_{n,k}, R_{n,k}|\xi_p) \;=\; & \frac{1}{M(\alpha,\beta,k,n)} \sum_{j=\beta-1}^{\infty} \sum_{r=0}^{\alpha-1} \binom{\alpha-1}{r} (-1)^r C_j(\beta-1) \\ & \times \Bigg\{ \frac{\Gamma(n)}{(r+j+k)^n} + \sum_{i=n}^{\infty} \frac{k^i}{i!} \Bigg(\frac{\Gamma(n) - 2\Gamma(n, -(r+j+2k)\log q)}{(r+j+2k)^{n+i}} \\ & + q^{2k} \int_{-\log q}^{-\log q^2} h(y;i,j,r)dy - q^{2k} \int_0^{-\log q} h(y;i,j,r)dy \Bigg) \Bigg\}, \end{aligned} \tag{3.4}$$

where

$$h(y;i,j,r) = \left(-\log(q^2 e^y)\right)^i y^{n-1} e^{-(r+j)y}.$$

Using (3.4), the numerical values of $\pi_{\alpha,\beta}(R^w_{n,k}, R_{n,k}|\xi_p)$ have been computed for $n = 3, 4$, $k = 3, 4$ and some selected values of α, β and p. The results are presented in Table 3.

Table 3: Values of $\pi_{\alpha,\beta}(R^{w,U}_{n,k}, R^U_{n,k}|\xi_p)$ for standard exponential distribution.

n	k	β	α	p								
				0.10	0.25	0.5	0.60	0.75	0.80	0.90	0.95	0.99
3	3	1	2	0.4036	0.4044	0.4439	0.4899	**0.5654**	**0.5819**	**0.5955**	**0.5964**	**0.5964**
			3	0.3310	0.3315	0.3805	0.4540	**0.5989**	**0.6347**	**0.6667**	**0.6689**	**0.6690**
			4	0.2754	0.2757	0.3213	0.4077	**0.6099**	**0.6664**	**0.7204**	**0.7244**	**0.7246**
			5	0.2321	0.2322	0.2702	0.3591	**0.6056**	**0.6827**	**0.7613**	**0.7676**	**0.7679**
		2	1	**0.6789**	**0.6745**	**0.5730**	0.4839	0.3626	0.3397	0.3222	0.3212	0.3211
			2	**0.5748**	**0.5759**	**0.5727**	**0.5437**	0.4675	0.4463	0.4267	0.4253	0.4252
			3	0.4888	0.4902	**0.5363**	**0.5613**	**0.5467**	**0.5319**	**0.5130**	**0.5113**	**0.5112**
			4	0.4184	0.4194	0.4838	**0.5497**	**0.6030**	**0.5985**	**0.5838**	**0.5816**	**0.5815**
			5	0.3609	0.3614	0.4273	**0.5201**	**0.6400**	**0.6491**	**0.6414**	**0.6392**	**0.6391**
4	3	1	2	0.4291	0.4292	0.4371	0.4560	**0.5166**	**0.5392**	**0.5674**	**0.5706**	**0.5709**
			3	0.3720	0.3720	0.3812	0.4088	**0.5149**	**0.5596**	**0.6200**	**0.6275**	**0.6280**
			4	0.3254	0.3254	0.3335	0.3633	**0.5012**	**0.5663**	**0.6609**	**0.6737**	**0.6746**
			5	0.2869	0.2870	0.2933	0.3219	0.4800	**0.5631**	**0.6928**	**0.7116**	**0.7130**
		2	1	**0.6884**	**0.6881**	**0.6554**	**0.5937**	0.4321	0.3790	0.3183	0.3120	0.3116
			2	**0.6123**	**0.6123**	**0.6099**	**0.5914**	0.4988	0.4559	0.3959	0.3883	0.3877
			3	**0.5461**	**0.5462**	**0.5566**	**0.5678**	**0.5440**	**0.5163**	**0.4632**	**0.4546**	**0.4539**
			4	0.4890	0.4890	**0.5032**	**0.5325**	**0.5711**	**0.5618**	**0.5211**	**0.5118**	**0.5110**
			5	0.4397	0.4397	0.4535	0.4921	**0.5835**	**0.5943**	**0.5708**	**0.5613**	**0.5603**
	4	1	2	0.4167	0.4169	0.4456	0.4889	**0.5612**	**0.5745**	**0.5830**	**0.5833**	**0.5833**
			3	0.3498	0.3500	0.3871	0.4577	**0.5992**	**0.6291**	**0.6495**	**0.6501**	**0.6502**
			4	0.2959	0.2960	0.3318	0.4166	**0.6189**	**0.6674**	**0.7028**	**0.7040**	**0.7041**
			5	0.2522	0.2523	0.2830	0.3722	**0.6246**	**0.6927**	**0.7456**	**0.7477**	**0.7477**
		2	1	**0.6601**	**0.6589**	**0.5828**	0.4942	0.3715	0.3518	0.3402	0.3399	0.3399
			2	**0.5715**	**0.5718**	**0.5667**	**0.5370**	0.4599	0.4417	0.4289	0.4285	0.4285
			3	0.4949	0.4953	**0.5269**	**0.5485**	**0.5310**	**0.5179**	**0.5056**	**0.5051**	**0.5050**
			4	0.4297	0.4299	0.4763	**0.5371**	**0.5855**	**0.5809**	**0.5710**	**0.5703**	**0.5703**
			5	0.3743	0.3744	0.4232	**0.5103**	**0.6250**	**0.6317**	**0.6264**	**0.6257**	**0.6257**
		3	1	**0.7803**	**0.7753**	**0.5810**	0.4165	0.2483	0.2289	0.2198	0.2197	0.2197
			2	**0.6988**	**0.6983**	**0.6137**	0.4973	0.3368	0.3136	0.3015	0.3012	0.3012
			3	**0.6226**	**0.6231**	**0.6117**	**0.5510**	0.4180	0.3926	0.3778	0.3774	0.3774
			4	**0.5537**	**0.5542**	**0.5856**	**0.5789**	0.4892	0.4639	0.4468	0.4463	0.4463

The bold numbers in Table 3 are referring to the fact that the weighted k-records are Pitman closer than the usual k-records to the specific population quantile. From this table, it is intuitively observed that for given α and β, there exist real values α_0 and $p_0 \in (0,1)$ such that for $\alpha \leq \alpha_0$ and $p \leq p_0$, $\pi(R_{n,k}^{U,w}, R_{n,k}^{U}|\xi_p) > 0.5$. Similarly for $\alpha > \alpha_0$, there exist a real value $p_* \in (0,1)$, so that for $p \geq p_*$, $\pi(R_{n,k}^{U,w}, R_{n,k}^{U}|\xi_p) > 0.5$.

3.2. Results based on order statistics

We recall that the pdf of the r-th order statistic from an iid sample of size n from cdf $F(\cdot)$ and pdf $f(\cdot)$ is given by

$$f_{r:n}(x) = k\binom{n}{r} F^{r-1}(x)\bar{F}^{n-r}(x)f(x). \tag{3.5}$$

Let us denote the weighted order statistic by $X_{r:n}^w$. Here, the PMC of $X_{r:n}^w$ and $X_{r:n}$ with respect to the p-th quantile ξ_p is studied when the baseline distribution is standard exponential distribution. From (2.4) and (3.5), we find

$$\begin{aligned}
\pi_{\beta,\delta}(X_{r:n}^w, X_{r:n}|\xi_p) \;=\; & \frac{1}{M(r,n-r+1,\beta,\delta)}\Bigg\{\sum_{j=0}^{r-1}\sum_{k=1}^{\infty}(-1)^j\binom{r-1}{j}C_k(\beta-1) \\
& \times\Bigg(\frac{\Gamma(\delta)}{(j+n-r+k+1)} - \frac{1}{M(r,n-r+1,1,1)} \\
& \times\sum_{i=0}^{n-r}\sum_{l=0}^{r+i}\Bigg(\frac{(-1)^{i+l}\binom{n-r}{i}\binom{r+i}{l}(1-p)^{2l}}{r+i} \\
& \times\frac{\Gamma(\delta)-\Gamma\big(\delta,-(j+n-r+k-l+1)\ln(1-p)\big)}{(j+n-r+k-l+1)^{\delta}}\Bigg)\Bigg) \\
& +\frac{1}{M(r,n-r+1,1,1)}\sum_{i=0}^{n-r}\sum_{j=0}^{2r+i-1}\sum_{k=1}^{\infty}\Bigg(\frac{(-1)^{i+j}\binom{n-r}{i}\binom{2r+i-1}{j}}{r+i} \\
& \times C_k(\beta-1)\frac{\Gamma(\delta)-2\Gamma\big(\delta,-(j+n-r+k+1)\ln(1-p)\big)}{(j+n-r+k+1)^{\delta}}\Bigg)\Bigg\}.
\end{aligned} \tag{3.6}$$

For extreme order statistics $X_{1:n}$ and $X_{n:n}$ we have

$$\begin{aligned}
\pi_{\beta,\delta}(X_{1:n}^w, X_{1:n}|\xi_p) \;=\; & \frac{1}{M(1,n,\beta,\delta)}\sum_{k=1}^{\infty}C_k(\beta-1)\Bigg\{\frac{\Gamma(\delta)}{(n+k)} \\
& +n\sum_{i=0}^{n-1}\sum_{j=0}^{i+1}\frac{(-1)^{i+j}\binom{n-1}{i}\binom{i+1}{j}}{i+1} \\
& \times\Bigg(\frac{\Gamma(\delta)-2\Gamma\big(\delta,-(n+k+j)\ln(1-p)\big)}{(n+k+j)^{\delta}} \\
& -\frac{\Gamma(\delta)-\Gamma\big(\delta,-(n+k-j)\ln(1-p)\big)}{(n+k-j)^{\delta}}(1-p)^{2j}\Bigg)\Bigg\}
\end{aligned}$$

and

$$\begin{aligned}\pi_{\beta,\delta}(X^w_{n:n}, X_{n:n}|\xi_p) &= \frac{1}{M(n,1,\beta,\delta)} \sum_{k=1}^{\infty} C_k(\beta-1) \Bigg\{ \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} \frac{\Gamma(\delta)}{(j+k+1)} \\ &+ \sum_{j=0}^{2n-1} (-1)^j \binom{2n-1}{j} \frac{\Gamma(\delta) - 2\Gamma\big(\delta, -(j+k+1)\ln(1-p)\big)}{(j+k+1)^\delta} \\ &- n \sum_{i=0}^{n-r} \sum_{j=0}^{n-1} \sum_{l=0}^{n} \Bigg(\frac{(-1)^{j+l}\binom{n-1}{j}\binom{n}{l}(1-p)^{2l}}{n+i} \\ &\times \frac{\Gamma(\delta) - \Gamma\big(\delta, -(j+k-l+1)\ln(1-p)\big)}{(j+k-l+1)^\delta} \Bigg) \Bigg\},\end{aligned}$$

respectively. Using (3.6), the numerical values of $\pi_{\beta,\delta}(X^w_{r:n}, X_{r:n}|\xi_p)$ have been computed for some selected values of n, r, β, δ and p. The results are presented in Table 4. The bold numbers in this table are referring to the fact that the weighted order statistics are Pitman closer than the usual order statistics to the specific population quantile.

Table 4: Values of $\pi_{\beta,\delta}(X^w_{r:n}, X_{r:n}|\xi_p)$ for standard exponential distribution.

n	r	β	δ	p				
				0.10	0.25	0.5	0.75	0.90
5	1	2	3	0.2326	**0.5067**	**0.7858**	**0.8018**	**0.8019**
		3	2	**0.5243**	**0.5585**	**0.5299**	**0.5276**	**0.5275**
	3	2	3	0.3628	0.3658	0.4716	**0.6292**	**0.6372**
		3	2	**0.6371**	**0.6360**	**0.5349**	0.3758	0.3629
	4	2	3	0.4301	0.4302	0.4447	**0.5515**	**0.5751**
		3	2	**0.7064**	**0.7063**	**0.6731**	0.4180	0.3005
8	1	2	3	0.2361	**0.6905**	**0.8162**	**0.8169**	**0.8169**
		3	2	**0.5307**	**0.5699**	**0.5636**	**0.5635**	**0.5635**
	7	2	3	0.4833	0.4834	0.4839	**0.5180**	**0.5341**
		3	2	**0.7338**	**0.7337**	**0.7313**	**0.5615**	0.2974
10	2	2	3	0.2741	**0.5501**	**0.7414**	**0.7429**	**0.7434**
		3	2	**0.5131**	**0.5259**	**0.5170**	**0.5024**	0.5000
	5	2	3	0.3758	0.3777	**0.5190**	**0.6237**	**0.6241**
		3	2	**0.5966**	**0.5958**	0.4858	0.4038	0.4033
	7	2	3	0.4216	0.4307	0.4415	**0.5573**	**0.5693**
		3	2	**0.6758**	**0.6547**	**0.6320**	0.3869	0.3454

From Table 4 we observe that, if $\beta \leq \delta$ (or $\beta > \delta$), there exist $p_0 \in (0,1)$ such that for $p \geq p_0$ (or $p \leq p_0$), we get $\pi_{\beta,\delta}(X^w_{n:n}, X_{n:n}|\xi_p) \geq 0.5$.

4. PITMAN CLOSENESS AND WEIGHTED RANDOM VARIABLES

Note that in the field of PMC, there is no necessity to restrict our attention to parameters and their estimators. There are situations in which the problem of closeness of some random variables with respect to each others may be of great importance. For instance, if Z is a random variable and X and Y are two other random variables, then for an appropriate measure of distance $d(\cdot,\cdot)$, we may define $GPN(X,Y,Z) = P(d(X,Z) < d(Y,Z))$ to determine the closer random variable between X and Y to Z; see for example, Mendes and Merkle [28]. Here, we focus on the PMC by using the distance $d(X,Z) = |X - Z|$.

Now, consider the situation in which the cdf of a distribution varies in some steps when the time proceeds. Let us show the corresponding random variable to the baseline (initial) distribution by X. Then, the transformed distribution at the i-th temporal step can be characterized by a weighted random variable, such as X_{w_i} $(i \geq 1)$ with pdf $g_i(\cdot)$. So, the PMC of any two transformed distribution to the baseline distribution is discussed in what follows.

Let X_{w_1}, X_{w_2} and X_{w_3} be independent weighted random variables which are considered to describe the transformations on a baseline distribution by pdf $f(\cdot)$ and cdf $F(\cdot)$. Therefore, the pdf of X_{w_i} is given by

$$g_i(x) = \frac{w_i(x)f(x)}{E(w_i(X))}, \quad E(w_i(X)) > 0, \quad i = 1,2,3. \tag{4.1}$$

Here, we generally focus our attention to compute the probability of closeness of X_{w_1} and X_{w_2} with respect to X_{w_3}. Note that in the special case of $w_3(x) = 1$, in fact the closeness of two transformed distributions with respect to the baseline distribution is studied. Using (1.1), we get

$$\begin{aligned}
\pi(X_{w_1}, X_{w_2}|X_{w_3}) &= P(|X_{w_1} - X_{w_3}| < |X_{w_2} - X_{w_3}|) \\
&= P(-|X_{w_2} - X_{w_3}| < X_{w_1} - X_{w_3} < |X_{w_2} - X_{w_3}|) \\
&= P(X_{w_2} < X_{w_1} < 2X_{w_3} - X_{w_2}, X_{w_2} < X_{w_3}) \\
&\quad + P(2X_{w_3} - X_{w_2} < X_{w_1} < X_{w_2}, X_{w_2} > X_{w_3}).
\end{aligned}$$

Assuming the random variables are defined on $[a,b]$, we have

$$\begin{aligned}
\pi(X_{w_1}, X_{w_2}|X_{w_3}) &= \int_a^b \int_a^z \int_y^{2z-y} g_1(x)g_2(y)g_3(z)dxdydz \\
&\quad + \int_a^b \int_z^b \int_{2z-y}^{y} g_1(x)g_2(y)g_3(z)dxdydz \\
&= \frac{1}{\Lambda}\Big\{ \int_a^b \int_a^z A_1(y)\bar{F}(y)w_2(y)f(y)w_3(z)f(z)dydz \\
&\quad - \int_a^b \int_a^z A_1(2z-y)\bar{F}(2z-y)w_2(y)f(y)w_3(z)f(z)dydz \\
&\quad + \int_a^b \int_z^b A_1(2z-y)\bar{F}(2z-y)w_2(y)f(y)w_3(z)f(z)dydz \\
&\quad - \int_a^b \int_z^b A_1(y)\bar{F}(y)w_2(y)f(y)w_3(z)f(z)dydz \Big\},
\end{aligned} \tag{4.2}$$

where

$$\Lambda = E(w_1(X))E(w_2(X))E(w_3(X))$$

and

$$A_1(x) = E(w_1(X)|X > x) = \frac{1}{\bar{F}(x)} \int_x^b w_1(t)f(t)dt.$$

Let us now suppose that $w_i(x) = \varphi_i(F(x))$, for $i = 1, 2, 3$, then by using transformations $u = F(y)$ and $v = F(z)$, the PMC in (4.2) can be rewritten as follows:

$$\pi(X_{w_1}, X_{w_2}|X_{w_3}) = \frac{1}{\Lambda}\left\{ \int_0^1\int_0^v [C_1(u,v) - C_2(u,v)]dudv - \int_0^1\int_v^1 [C_1(u,v) - C_2(u,v)]dudv \right\},$$

where

$$C_1(u,v) = A_1(F^{-1}(u))(1-u)\varphi_2(u)\varphi_3(v)$$

and

$$C_2(u,v) = A_1(2F^{-1}(v) - F^{-1}(u))\bar{F}(2F^{-1}(v) - F^{-1}(u))\varphi_2(u)\varphi_3(v).$$

By selecting various choices of $w_i(x)$, specially those of presented in the previous sections, the above probability of closeness can be derived via tedious calculations.

It is worth to mention that formula (4.2) can also be used to assess the relationship among different statistics. Suppose we have three independent sample from the same distribution. It is known that in the context of nonparametric $X_{[np]:n}$ used as an estimator for ξ_p. Denote the the sample mean of a standard exponential population by $\bar{X}$, and consider the sample quantiles of two independent standard exponential populations by $X_{r:n}$ and $X_{s:n}$, respectively. Then by using (4.2) we have

$$\begin{aligned}\pi(X_{r:n}, X_{s:n}|\overline{X}) = {} & n^n rs\binom{n}{r}\binom{n}{s}\sum_{i=1}^{r-1}\sum_{j=1}^{s-1}(-1)^{i+j}\binom{r-1}{i}\binom{s-1}{j} \\ & \times\left\{\frac{(n-r+i+j)^{-1}}{(2n-r-s+i+j+2)}\left(\frac{1}{n^n} - \frac{1}{(3n-r-s+i+j+3)^n}\right.\right. \\ & \left. - \frac{1}{(3n-2r+2i+2)^n} + \frac{1}{(4n-3r-s+3i+j+4)^n}\right) \\ & \left. + \frac{(3n-r-s+i+j+2)^{-n}}{(r-s-i+j)(2n-r-s+i+j+2)}\right\}.\end{aligned}$$

5. APPLICATIONS

To illustrate the performance of the proposed procedure in Sections 4 and 5, we use two real data sets in the following examples.

Example 1 (*Telephone calls*)**.** Table 5 contains the data concerning the times (in minutes) between 48 consecutive telephone calls to a company's switchboard which is presented by Castillo *et al.* [13]. They assumed that the data come from the exponential distribution.

Table 5: Times (in minutes) between 48 consecutive calls.

1.34	0.14	0.33	1.68	1.86	1.31	0.83	0.33	2.20	0.62	3.20	1.38
0.96	0.28	0.44	0.59	0.25	0.51	1.61	1.85	0.47	0.41	1.46	0.09
2.18	0.07	0.02	0.64	0.28	0.68	1.07	3.25	0.59	2.39	0.27	0.34
2.18	0.41	1.08	0.57	0.35	0.69	0.25	0.57	1.90	0.56	0.09	0.28

By using the definition of k-records, as presented in subsection 3.1, from this data set, the second upper 2-records (values of $R_{2,2}$) have been extracted which are 0.33, 1.86, 1.38, 0.44, 0.51, 1.85, 0.07, 0.68, 2.39, 0.34, 0.57 and 0.57. Note that these data are indeed the second largest observations in the partial samples. Moreover, after observing each data point, the procedure of collecting the next second upper 2-record has been restarted. Moreover, the initial sample maxima to attain the second upper 2-records are observed as

$$1.34,\ \ 2.2,\ \ 3.2,\ \ 0.96,\ \ 0.59,\ \ 2.18,\ \ 0.64,\ \ 1.07,\ \ 3.25,\ \ 2.18,\ \ 1.08,\ \ 0.69.$$

The null hypothesis that the above data are coming from a weighted distribution with pdf (3.1) and the parameters $\lambda = (\alpha, \beta, 2, 2)$ (distribution of the second upper 2-record), is checked by using the Kolmogorov–Smirnov (K-S) distances between the empirical distribution functions and the fitted distribution function. The observed MLEs of α and β are 3.5 and 0.8, respectively. Also, the observed value of K-S statistic is 0.2158 and the associated p-value is 0.6311. So, based on these observations, the weighted distribution is adequate for the data regarding the sample maxima to attain the second upper 2-records. That is, one may accept that these data are some observed values of $R^w_{2,2}$.

Using (3.4), the values of $\pi_{3.5,0.8}(R^w_{2,2}, R_{2,2}|\xi_p)$ have been numerically obtained and presented in Table 6 for some choices of p. It is observed that for upper quantiles ($p \geq 0.75$), $R^w_{2,2}$ is Pitman closer to ξ_p than $R_{2,2}$.

Table 6: Values of $\pi_{3.5,0.8}(R^w_{2,2}, R_{2,2}|\xi_p)$ for standard exponential distribution.

p	0.10	0.25	0.5	0.60	0.75	0.80	0.90	0.95	0.99
$\pi_{3.5,0.8}(R^w_{2,2}, R_{2,2}\|\xi_p)$	0.2378	0.2389	0.2935	0.3727	0.5616	0.6341	0.7366	0.7581	0.7621

Example 2 (*Air conditioning system*)**.** In this example we use the data set which consist of the intervals between failures (in hours) of the air conditioning system in three Boeing 720 jet aircrafts. The data are reported in Table 7. See Proschan [34] for a detailed description of the data set. He tested and accepted the hypothesis that the successive failure times were iid exponential for each plane, but with different failure rates.

Table 7: Intervals between failures of the air conditioning system in three Boeing 720 jet aircraft.

Plane 7909	90,10,60,186,61,49,14,24,56,20,79,84,44,59,29,118,25,156,310,76,26,44,23,62,130,208,70,101,208
Plane 7912	23,261,87,7,120,14,62,47,225,71,246,21,42,20,5,12,120,11,3,14,71,11,14,11,16,90,1,16,52,95
Plane 7913	97,51,11,4,141,18,142,68,77,80,1,16,106,206,82,54,31,216,46,111,39,63,18,191,18,163,24

Let us denote the intervals between failures of the air conditioning system of Planes 7913, 7912 and 7909 by X, X_{w_1} and X_{w_2}, respectively. As well as assumed by Proschan [34], the observed values of X come from exponential distribution. Moreover, it can be deduced that X_{w_1} and X_{w_2} are two different weighted versions of X. More precisely, according to the general weighted pdf presented in (3.1) with baseline exponential distribution, the null hypotheses that the associated data for Planes 7912 and 7909 are coming from

$$g_F(x; 1, 1.75, 1, 1) = 1.75[\bar{F}(x)]^{0.75} f(x)$$

and

$$g_F(x; 1, 1, 1, 1.15) = \frac{1}{\Gamma(1.15)}[-\log \bar{F}(x)]^{0.15} f(x),$$

respectively, are checked by using the K-S distances. The observed p-values are 0.6455 and 0.9123, respectively. So, based on these observations, the mentioned weighted versions of the exponential distribution are adequate for the data. By using (4.2) and doing some algebraic calculations and numerical computations, we get $\pi(X_{w_1}, X_{w_2}|X) = 0.0452$. That is, the intervals between failures of the air conditioning system of Plane 7909 are Pitman closer than those of Plane 7912 with respect to Plane 7913.

6. CONCLUSION

In this paper, the weighted random variables were considered and their closeness to a common parameter was investigated in the sense of PMC. Some general results were derived and a new general weighted model was introduced which subsumes most of the previous works as special cases. Some numerical results and conclusions were presented for exponential distribution in details. It was seen that the weighted upper k-records are Pitman closer than the usual ones to certain population quantiles. To illustrate the proposed procedure, a real data set was used. Furthermore, the concept of PMC was applied for measuring the nearness of some weighted random variables with respect to each other. This procedure was also explained via application to a real data set.

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