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Trend Resistant General Efficiency Balanced Block Designs for Two Disjoint Sets of Treatments

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Abstract:

- General Efficiency Balance (GEB) is an important property of designs. Variance balance and efficiency balance are special cases of GEB. Here, GEB block designs for comparing treatments belonging to two disjoint sets in the presence of systematic trend have been discussed. Methods of constructing Trend Resistant General Efficiency Balanced Bipartite Block Designs (TR-GEBBPB) designs have been presented. The block designs so obtained are trend resistant, general efficiency balanced and are more efficient for estimating the contrasts pertaining to two treatments from two different sets.

Keywords:

- *block design; balanced bipartite; general efficiency; trend.*

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1. INTRODUCTION

Balancing is an essential and desirable statistical property of block design. The concept of balance has been used in several senses in the literature, viz., variance balance, efficiency balance, pairwise balance, general efficiency balance etc. The concept of general efficiency balance was given by [9]. When an incomplete block design is compared against any other design i.e. either a completely randomized design (CRD) or randomized complete block design (RCBD) both having same number of treatments, but not necessarily the same number of replications such that the ratio of variances of the estimates of any treatment contrast for two designs is constant, then such an incomplete block design has been called GEB design.

Definition 1.1. A connected block design is called a General Efficiency Balanced (GEB) design, if for some $\theta, s_1, s_2, ..., s_v (> 0)$, the information matrix ($\mathbf{C}$) can be expressed as

$$\mathbf{C} = \theta\left[\mathbf{S} - \frac{1}{g}\mathbf{ss}'\right], \tag{1.1}$$

where $\theta = \{n - \text{trace}(\mathbf{NK}^{-1}\mathbf{N}')\}/(g - \frac{1}{g}\mathbf{s}'\mathbf{s})$, $\mathbf{S} = \text{diag}(s_1, s_2, ..., s_v)$, $\mathbf{s} = (s_1, s_2, ..., s_v)'$ and $\mathbf{s}'\mathbf{1} = g$. $\mathbf{N}$ is the $v \times b$ incidence matrix of treatments vs blocks, $\mathbf{K}$ is the diagonal matrix of block sizes and n is the total number of observations.

Several series of variance balanced and efficiency balanced designs as subclasses of GEB designs through the technique of reinforcement were constructed by [9]. It was pointed out that a variance balanced design or an efficiency balanced design cannot be constructed in (v+2) or more treatments through reinforcement. A method of constructing GEB designs through method of reinforcement of a Balanced Incomplete Block (BIB) design was given by [24]. They found that if one new treatment is added to each block of the BIB design, then the resultant design will be a GEB design with (v+1) treatments. Different aspects of efficiency-balanced designs have been studied in [30] and [34].

Definition 1.2. A connected block design with $v^* = v + 1$ treatments, b^* blocks, block sizes k, replication numbers $\mathbf{r} = (r\mathbf{1}_v' \;\; r_0)'$ and $\mathbf{C}$ of the form

$$\mathbf{C} = \begin{bmatrix} (a+b)\mathbf{I}_v - b\mathbf{1}_v\mathbf{1}_v' & -c\mathbf{1}_v \\ -c\mathbf{1}_v' & d \end{bmatrix} \tag{1.2}$$

is a GEB design with $\mathbf{s} = [b\mathbf{1}_v' \;\; c]'$ and $g = vb+c$, where a, b, c and d are positive integers satisfying $a - b(v - 1) = c$, $cv = d$, r is the replication of v treatments and r_0 is the replication of $(v + 1)$-th treatment.

The above $\mathbf{C}$-matrix is identical to the structure of the $\mathbf{C}$-matrix of a **Balanced Test Treatment Incomplete Block (BTIB)** design given by [2]. This equivalence shows that GEB designs are identical to BTIB designs and can be useful in making test treatments control comparisons. In [32], it has been shown that all the BTIB designs are also GEB designs and vice-versa for a single control case. However for many controls, this result does not hold good.

Example 1.1. The block contents of a GEB design (see in [32]) with parameters $v^* = 7$ (= 6+1), $b^* = 11$, $r = 7$, $r_0 = 2$ and $k = 4$ are (1, 2, 3, 4), (5, 6, 1, 2), (3, 4, 5, 6), (5, 2, 1, 4), (3, 6, 5, 2), (1, 6, 3, 2), (5, 4, 1, 6), (3, 2, 5, 4), (1, 4, 3, 6), (1, 3, 5, 7) and (2, 4, 6, 7). The information matrix for this design is

$$\mathbf{C} = \frac{1}{4}\begin{bmatrix} 25\mathbf{I}_6 - 4\mathbf{1}_6\mathbf{1}_6 & -\mathbf{1}_6' \\ -\mathbf{1}_6' & 6 \end{bmatrix}.$$

It is seen that this is a GEB design with $\mathbf{s} = \frac{1}{4}[4\mathbf{1}_6' \;\; 1]'$ and $g = 6.25$. Some methods of constructing GEB design with equal and unequal block sizes were given by [10] along with a catalogue of GEB designs with efficiencies.

Example 1.2. The block contents of a GEB design (see in [10]) with parameters $v^* = 7$ (= 6+1), $b^* = 11$, $r = 4$, $r_0 = 9$ and $k = 3$ are (1, 2, 7), (3, 4, 7), (5, 6, 7), (1, 6, 7), (3, 2, 7) (5, 4, 7), (1, 4, 7), (3, 6, 7), (5, 2, 7), (1, 3, 5) and (2, 4, 6). The information matrix for the given design is

$$\mathbf{C} = \frac{1}{3}\begin{bmatrix} 8\mathbf{I}_6 - 4\mathbf{1}_6\mathbf{1}_6' & -\mathbf{1}_6 \\ -\mathbf{1}_6' & 18 \end{bmatrix}.$$

This is a GEB design with $\mathbf{s} = \frac{1}{3}[4\mathbf{1}_6' \;\; 1]'$ and $g = 8.33$.

A definition of GEB design for the case when there are treatments belonging to two disjoint sets is given below.

Definition 1.3. Consider a design d with $v = v_1 + v_2$ treatments (where v_1 is the number of treatments belonging to 1-st set and v_2 is the number of treatments belonging to 2-nd set; v_1, $v_2 \geq 2$) having a $\mathbf{C}$ matrix of the form

$$\mathbf{C} = \begin{bmatrix} (f_1 - f_2)\mathbf{I}_{v_1} + f_2\mathbf{1}_{v_1}\mathbf{1}_{v_1}' & -f_3\mathbf{1}_{v_1}\mathbf{1}_{v_2}' \\ -f_3\mathbf{1}_{v_2}\mathbf{1}_{v_1}' & (f_4 - f_5)\mathbf{I}_{v_2} + f_5\mathbf{1}_{v_2}\mathbf{1}_{v_2}' \end{bmatrix}, \tag{1.3}$$

where f_1, f_2, f_3, f_4, $f_5 > 0$ and $f_1 = f_2v_1 + f_3v_2$, and $f_4 = f_3v_1 + f_5v_2$. The design d is said to be a GEB design if and only if $f_2f_5 = f_3^2$. It can be shown that the $\mathbf{C}$-matrix of a GEB design given in (1.3) can be expressed in the form of $\mathbf{C}$-matrix of (1.1) with

$$\mathbf{s} = [f_2\mathbf{1}_{v_1}' \;\; f_3\mathbf{1}_{v_2}']', \quad \mathbf{S} = \begin{bmatrix} f_2\mathbf{I}_{v_1} & 0 \\ 0 & f_3\mathbf{I}_{v_2} \end{bmatrix}, \quad g = f_2v_1 + f_3v_2 \quad \text{and} \quad \theta = \frac{f_1}{f_2}.$$

The above $\mathbf{C}$-matrix in (1.3) is identical to the structure of the $\mathbf{C}$-matrix of the balanced block design obtained for comparing two disjoint sets of treatments called Balanced Bipartite Block (BBPB) design. The interest here is to estimate the contrasts of the type $(\tau_i - \tau_j)$ with as high precision as possible, where τ_i and τ_j belong to 1-st and 2-nd set of treatments, respectively.

Example 1.3. The block contents of a GEB design (see in [18]) with parameters $v_1 = 8$, $v_2 = 2$, $b = 18$, $r_1 = 5$, $r_2 = 16$ and $k = 4$ are (1, 2, 9, 10), (3,4, 9, 10), (5, 6, 9, 10), (7, 8, 9, 10), (1, 4, 9, 10), (3, 2, 9, 10), (5, 8, 9, 10), (7, 6, 9, 10), (6, 1, 9, 10), (8, 3, 9, 10), (2, 5, 9, 10), (4, 7, 9, 10), (8, 1, 9, 10), (6, 3, 9, 10), (4, 5, 9, 10), (2, 7, 9, 10), (1, 3, 5, 7) and (2, 4, 6, 8). The information matrix of this design is

$$\mathbf{C} = \frac{1}{4}\begin{bmatrix} 16\mathbf{I}_8 - \mathbf{1}_8\mathbf{1}_8' & -4\mathbf{1}_8\mathbf{1}_2' \\ -4\mathbf{1}_2\mathbf{1}_8' & 64\mathbf{I}_2 - 16\mathbf{1}_2\mathbf{1}_2' \end{bmatrix}.$$

Here $f_3^2 = f_2 f_5$ and the design is a GEB design with $\mathbf{s} = \frac{1}{4}[\mathbf{1}_8' \;\; 4\mathbf{1}_2']$, g=3 and $\theta = 4$.

An overview of block designs for comparing test treatments with control treatments was given in [11]. A method of constructing GEB block designs with unequal block sizes for comparing two disjoint sets of treatments, with each set consisting of two or more treatments, has been developed by [22]. Optimal first order circular block designs with fewer blocks considering the correlated observations for an even number of treatments have been constructed in [31]. They developed GEB circular block designs with correlated observations for an even number of treatments. A-optimal/efficient designs for making the comparison between treatments that belongs to two disjoint sets with equal and unequal blocks were obtained by different authors (see for details [25, 21, 19, 20, 23, 15, 12]). Some methods of construction of BBPB designs using incidence matrices of BIB designs and two-associate-class partially balanced incomplete block group divisible designs were discussed in [33]. In another case of block design setup, experiments may be carried out using plots occurring in long, narrow rows wherein spatial fertility trends may occur. In such situations, the response may also depend on the spatial position of the experimental unit within a block. One way to overcome such situations is the suitable arrangement of treatments over plots within a block such that the arranged design is capable of completely eliminating the effects of defined components of a common trend. Such designs have been called Trend Free Block (TFB) designs (see in [6]). These designs are constructed so that treatment effects and trend effects are orthogonal. A necessary and sufficient condition for a block design to be linear trend free was obtained in [35], and the concepts and properties of Nearly TFB designs with linear and quadratic trends over plots within blocks were highlighted in [36]. A lot of literature is also available which deals with different aspects of block designs incorporating trend effects (see, for instance, [4, 5, 3, 13, 14, 16, 17, 26]). An algorithm to construct a series of exact optimum designs resistant to linear and quadratic time trends has been developed by [1]. An integer programming approach for the construction of trend-free split-plot designs was developed by [7].

This article deals with Trend Resistant General Efficiency Balanced Bipartite Block (TR-GEBBPB) designs when there are two disjoint sets of treatments (one set may be tests and other may be controls). Series of TR-GEBBPB designs for comparing a treatment from set 1 to a treatment from set 2, with more precision have been developed. The interest here is to estimate the contrasts pertaining to test treatments vs. control treatments with higher precision in the presence of trend.

2. GEBBPB DESIGNS IN THE PRESENCE OF TREND

Consider the following model in block design set-up for v treatments ($v = v_1 + v_2$; v_1 treatments in first set and v_2 treatments in second set) and b blocks of size k each incorporating trend component (within-block trend effects are represented by orthogonal polynomials of p-th degree, $p \leq k$):

$$\mathbf{Y} = \mu\mathbf{1} + \mathbf{\Delta}'\boldsymbol{\tau} + \mathbf{D}'\boldsymbol{\beta} + \mathbf{Z}\boldsymbol{\rho} + \mathbf{e}, \tag{2.1}$$

where $\mathbf{Y}$ is a $n \times 1$ vector of observations, μ is general mean, $\mathbf{1}$ is a $n \times 1$ vector of unity, $\mathbf{\Delta}'$ is a $n \times (v_1 + v_2)$ matrix of observations versus treatments, τ is a $(v_1 + v_2) \times 1$ vector of treatment effects, $\mathbf{D}'$ is a $n \times b$ incidence matrix of observations versus blocks, $\boldsymbol{\beta}'$ is a $b\times1$ vector of block effects, $\mathbf{Z}\boldsymbol{\rho}$ represents the trend effects. The matrix $\mathbf{Z}$, of order $n\times p$, is the matrix of coefficients given by $\mathbf{Z} = \mathbf{1}_b \otimes \mathbf{F}$ where $\mathbf{F}$ is a $k \times p$ matrix with columns representing the (normalized) orthogonal polynomials and $\mathbf{e}$ is a $n\times1$ vector of errors with $\mathrm{E}(\mathbf{e}) = 0$ and $\mathrm{V}(\mathbf{e}) = \sigma^2\ \mathbf{I}_n$. Further, $\mathbf{1}'\ \mathbf{F} = 0$, $\mathbf{F}'\ \mathbf{F} = \mathbf{I}_p$.

Let $\mathbf{N}$ be a $(v_1 + v_2) \times b$ incidence matrix, which is partitioned as

$$\mathbf{\Delta}\mathbf{D}' = \mathbf{N} = \begin{pmatrix} \mathbf{N}_1 \\ \mathbf{N}_2 \end{pmatrix},$$

where $\mathbf{N}_1$ is a $v_1 \times b$ incidence matrix pertaining to v_1 treatments and $\mathbf{N}_2$ is a $v_2 \times b$ incidence matrix pertaining to v_2 treatments. The model (2.1) can be written as

$$\mathbf{Y} = \mathbf{X}_1\boldsymbol{\theta}_1 + \mathbf{X}_2\boldsymbol{\theta}_2 + \mathbf{e}, \tag{2.2}$$

where $\mathbf{X}_1 = [\mathbf{\Delta}'] = [\mathbf{\Delta}_1'\ \ \mathbf{\Delta}_2']$, $\mathbf{X}_2 = [1\ \ \mathbf{D}'\ \ \mathbf{Z}]$, $\boldsymbol{\theta}_1 = \boldsymbol{\tau}$ and $\boldsymbol{\theta}_1 = [\mu\ \ \boldsymbol{\beta}'\ \ \boldsymbol{\rho}']'$.

$\mathbf{X}_1$ is the matrix of effects of interest and $\mathbf{X}_2$ is the matrix of nuisance effects. The joint information matrix for estimating different effects is obtained as:

$$\mathbf{C} = \begin{bmatrix} r_1\mathbf{I}_{v_1} - \frac{1}{k}\mathbf{N}_1\mathbf{N}_1' - \frac{1}{b}\mathbf{\Delta}_1\mathbf{Z}\mathbf{Z}'\mathbf{\Delta}_1' & -\frac{1}{k}\mathbf{N}_1\mathbf{N}_2' - \frac{1}{b}\mathbf{\Delta}_1\mathbf{Z}\mathbf{Z}'\mathbf{\Delta}_2' \\ -\frac{1}{k}\mathbf{N}_2\mathbf{N}_1' - \frac{1}{b}\mathbf{\Delta}_2\mathbf{Z}\mathbf{Z}'\mathbf{\Delta}_1' & r_2\mathbf{I}_{v_2} - \frac{1}{k}\mathbf{N}_2\mathbf{N}_2' - \frac{1}{b}\mathbf{\Delta}_2\mathbf{Z}\mathbf{Z}'\mathbf{\Delta}_2' \end{bmatrix},$$

where r_1 and r_2 are the replications of the first and second set of treatments, respectively.

Definition 2.1. A bipartite block design is said to be balanced with respect to set 1 vs set 2 if each treatment from a set appears together with every other treatment of the same set a constant number of times (say, λ_{ii}^*, i = 1,2) and each treatment from a set appears together with every other treatment of a different set a constant number of times (say, λ_{12}^*).

Definition 2.2. A bipartite block design is said to be general efficiency balanced i.e. GEBBPB if its information matrix ($\mathbf{C}$) is of the form (1.3).

Definition 2.3. A GEBBPB design is said to be Trend Resistant (TR-GEBBPB) design if the adjusted treatment sum of squares of block model with trend is same as adjusted treatment sum of squares of block model without trend.

3. METHODS OF CONSTRUCTING TR-GEBBPB DESIGNS

3.1. Method 1

Consider a Semi-regular (SR) group divisible design with parameters $v_1 = mn$ $(m < n)$, b_1, r_1, k_1, λ_{11} and λ_{12}. Consider the $(m,\ n)$ group divisible association scheme in m blocks each of size n each with $v_1 = mn$, $b_2 = m$, $r_2 = 1$, $k_2 = n$, $\lambda_{21} = 1$ and $\lambda_{22} = 0$. Augment $(k_2 - k_1) = v_2$ number of treatments to the SR design and juxtapose both the design and the association scheme. Fold-over the whole plan and the resultant design is a TR-GEBBPB design with parameters $v_1 = mn$, v_2, $b = 2(b_1 + b_2)$, $\mathbf{r}' = [2(r_1 + r_2)\mathbf{1}'_{v_1}\ \ 2b_1\mathbf{1}'_{v_2}], k = k_2, \lambda^*_{11} = 2\lambda_{12}, \lambda^*_{12} = 2r_1$ and $\lambda^*_{22} = 2b_1$. The information matrix for this design is given by

$$\mathbf{C} = \frac{2}{k}\begin{bmatrix} r_1^2\mathbf{I}_{v_1} - \mathbf{1}_{v_1}\mathbf{1}'_{v_1} & -r_1\mathbf{1}_{v_1}\mathbf{1}'_{v_2} \\ -r_1\mathbf{1}_{v_2}\mathbf{1}'_{v_1} & r_1^2k_2\mathbf{I}_{v_2} - r_1^2\mathbf{1}_{v_2}\mathbf{1}'_{v_2} \end{bmatrix}.$$

Example 3.1.1. Consider a SR group divisible design (SR 9 in [8]) with parameters $v_1 = 8$, $b_1 = 16$, $r_1 = 4$, $k_1 = 2$, $m = 2$, $n = 4$, $\lambda_{11} = 0$ and $\lambda_{12} = 1$. The (2, 4) group divisible association scheme with two blocks each of size four is (1, 3, 5, 7) and (2, 4, 6, 8) with $v_1 = 8$, $b_2 = 2$, $r_2 = 1$, $k_2 = 4$, $\lambda_{21} = 1$ and $\lambda_{22} = 0$. Following above procedure a TR-GEBBPB design with parameters $v_1 = 8$, $v_2 = 2$, $b = 36$, $k = 4$, $\mathbf{r}' = [10\mathbf{1}'_8\ \ 32\mathbf{1}'_2], \lambda^*_{11} = 2, \lambda^*_{12} = 8$ and $\lambda^*_{22} = 32$ is obtained with block contents as (1, 2, 9, 10), (3,4, 9, 10), (5, 6, 9, 10), (7, 8, 9, 10), (6, 1, 9, 10), (8, 3, 9, 10), (2, 5, 9, 10), (4, 7, 9, 10), (1, 4, 9, 10), (3, 2, 9, 10), (5, 8, 6, 8), (7, 6, 9, 10), (8,1, 9, 10), (6, 3, 9, 10), (4, 5, 9, 10), (2, 7, 9, 10), (1, 3, 5, 7), (2, 4, 6, 8), (10, 9, 2, 1), (10, 9, 2, 1), (10, 9, 4, 3), (10, 9, 6, 5), (10, 9, 8, 7), (10, 9, 1, 6), (10, 9, 3, 8), (10, 9, 5, 2), (10, 9, 7, 4), (10, 9, 4, 1), (10, 9, 2, 3), (10, 9, 8, 5), (10, 9, 6, 7), (10, 9, 1, 8), (10, 9, 3, 6), (10, 9, 5, 4), (10, 9, 7, 2), (7, 5, 3, 1) and (8, 6, 4, 2). Here, for the given design, the normalized orthogonal polynomial of degree 1 is given as

$$\mathbf{F} = \begin{bmatrix} \frac{-3}{\sqrt{20}} & \frac{-1}{\sqrt{20}} & \frac{1}{\sqrt{20}} & \frac{3}{\sqrt{20}} \end{bmatrix}' = \begin{bmatrix} -0.67 & -0.22 & 0.22 & 0.67 \end{bmatrix}'.$$

The information matrix for this design is given as

$$\mathbf{C} = \frac{1}{2}\begin{bmatrix} 16\mathbf{I}_8 - \mathbf{1}_8\mathbf{1}'_8 & -4\mathbf{1}_8\mathbf{1}'_2 \\ -4\mathbf{1}_2\mathbf{1}'_8 & 64\mathbf{I}_2 - 16\mathbf{1}_2\mathbf{1}'_2 \end{bmatrix}.$$

It can be seen that here $f_2 f_5 = f_3^2$. Variance of any estimated elementary contrast among the treatments belonging to the first set is $V_{11} = 0.2500\ \sigma^2$ and the variance of any estimated elementary contrast between the treatments belonging to the first and second set is $V_{12} = 0.1562\ \sigma^2$.

Example 3.1.2. Consider a SR group divisible design (SR 11 in [8]) with parameters $v_1 = 10$, $b_1 = 25$, $r_1 = 5$, $k_1 = 2$, $m = 2$, $n = 5$, $\lambda_{11} = 0$ and $\lambda_{12} = 1$. The (2, 5) group divisible association scheme with $v_1 = 10$, $b_2 = 2$, $r_2 = 1$, $k_2 = 5$, $\lambda_{21} = 1$ and $\lambda_{22} = 0$ is as follows:

$$\begin{matrix} 1 & 3 & 5 & 7 & 9 \\ 2 & 4 & 6 & 8 & 10 \end{matrix}$$

The block contents of the TR-GEBBPB design obtained with parameters $v_1 = 10$, $v_2 = 3$, $b = 54$, $k = 5$, $\mathbf{r}' = \left[\, 12\mathbf{1}'_{10} \;\; 50\mathbf{1}'_{2} \,\right]$, $\lambda^*_{11} = 2$, $\lambda^*_{12} = 10$ and $\lambda^*_{22} = 50$, are (1, 2, 11, , 12, 13), (3, 10, 11, 12, 13), (5, 8, 11, 12, 13), (7, 6, 11, 12, 13), (9, 4, 11, 12, 13), (1, 8, 11, 12, 13), (3, 6, 11, 12, 13), (5, 4, 11, 12, 13), (7, 2, 11, 12, 13), (9, 10, 11, 12, 13), (1, 4, 11, 12, 13), (3, 2, 11, 12, 13), (5, 10, 11, 12, 13), (7, 8, 11, 12, 13), (9, 6, 11, 12, 13), (1, 10, 11, 12, 13), (3, 8, 11, 12, 13), (5, 6, 11, 12, 13), (7, 4, 11, 12, 13), (9, 2, 11, 12, 13), (1, 6, 11, 12, 13), (3, 4, 11, 12, 13), (5, 2, 11, 12, 13), (7, 10, 11, 12, 13), (9, 8, 11, 12, 13), (1, 3, 5, 7, 9), (2, 4, 6, 8, 10), (13, 12, 11, 2, 1), (13, 12, 11, 10, 3), (13, 12, 11, 8, 5), (13, 12, 11, 6, 7), (13, 12, 11, 4, 9), (13, 12, 11, 8, 1), (13, 12, 11, 6, 3), (13, 12, 11, 4, 5), (13, 12, 11, 2, 7), (13, 12, 11, 10, 9), (13, 12, 11, 4,1), (13, 12, 11, 2, 3), (13, 12, 11, 10, 5), (13, 12, 11, 8, 7), (13, 12, 11, 6, 9), (13, 12, 11, 10, 1), (13, 12, 11, 8, 3), (13, 12, 11, 6, 5), (13, 12, 11, 4, 7), (13, 12, 11, 2, 9), (13, 12, 11, 6, 1), (13, 12, 11, 4, 3), (13, 12, 11, 2, 5), (13, 12, 11, 10, 7), (13, 12, 11, 8, 9), (9, 7, 5, 3, 1) and (10, 8, 6, 4, 2).

The normalized orthogonal polynomial of degree 1 for the design is

$$\mathbf{F} = \left[\frac{-2}{\sqrt{10}} \;\; \frac{-1}{\sqrt{10}} \;\; 0 \;\; \frac{1}{\sqrt{10}} \;\; \frac{2}{\sqrt{10}}\right]' = \left[-0.63 \;\; -0.32 \;\; 0 \;\; 0.32 \;\; 0.63\right]'.$$

The information matrix obtained for this design

$$\mathbf{C} = \frac{2}{5}\begin{bmatrix} 25\mathbf{I}_{10} - \mathbf{1}_{10}\mathbf{1}'_{10} & -5\mathbf{1}_{10}\mathbf{1}'_{3} \\ -5\mathbf{1}_{3}\mathbf{1}'_{10} & 125\mathbf{I}_{3} - 25\mathbf{1}_{3}\mathbf{1}'_{3} \end{bmatrix}.$$

The variance of any estimated elementary contrast among the treatments belonging to the first set is $V_{11} = 0.200\sigma^2$ and the variance of any estimated elementary contrast between the treatments belonging to the first and second set is $V_{12} = 0.120\sigma^2$.

3.2. Method 2

Consider a BIB design with parameters v^*, b^*, r^*, k^* and λ^*. From each block of this design, develop $(k^* - 1)$ more blocks by rotating the treatments clockwise resulting into b^*k^* blocks. Substitute the last u (u = 2, 3,...,v^*−2) set of treatments of the design with the last treatment of the second set, the second last set of treatments with the second last treatment of the second set, likewise v^*−3 number of treatments can be replaced by p number of treatment of the second set. The resulting design is a TR-GEBBPB with parameters $v_1 = (v^* - pu)$, $v_2 = p$, $b = k^*b^*$, $\mathbf{r}' = [k^*r^*\mathbf{1}'_{v_1} \;\; 2k^*r^*\mathbf{1}'_{v_2}]$, $k = k^*$, $\lambda^*_{11} = k^*\lambda^*$, $\lambda^*_{12} = 2k^*\lambda^*$ and $\lambda^*_{22} = 5k^*\lambda^*$.

The information matrix for this design is

$$\mathbf{C} = \lambda^*\begin{bmatrix} v^*\mathbf{I}_{v_1} - \mathbf{1}_{v_1}\mathbf{1}'_{v_1} & -u\mathbf{1}_{v_1}\mathbf{1}'_{v_2} \\ -u\mathbf{1}_{v_2}\mathbf{1}'_{v_1} & uv^*\mathbf{I}_{v_2} - u^2\mathbf{1}_{v_2}\mathbf{1}'_{v_2} \end{bmatrix},$$

with $V_{11} = \dfrac{2}{v^*\lambda^*}\sigma^2$ and $V_{12} = \dfrac{(u+1)}{uv^*\lambda^*}\sigma^2$.

Example 3.2.1. Let $v^* = 9$,$b^* = 12$, $r^* = 4$, $k^* = 3$ and $\lambda^* = 1$ be the parameters of a BIB design with blocks as (1, 2, 3), (4, 5, 6), (7, 8, 9), (1, 4, 7), (2, 5, 8), (3, 6, 9), (1, 6, 8), (2, 4, 9), (3, 5, 7), (1, 5, 9), (2, 6, 7) and (3, 4, 8). From each block of this design, develop two more blocks by rotating the treatments clockwise resulting into 36 blocks. Let $p = 2$ and $u = 3$, substitute the last two treatments of the design with the last treatment of the second set, second last two treatments with second last treatment of the second set, i.e., substitute treatments (8, 9) by treatment number 5 and treatments (6, 7) by treatment number 4. The resulting design is TR-GEBBPB with parameters $v_1 = 3$, $v_2 = 2$, $b = 36$, $\mathbf{r}' = [12\mathbf{1}_3' \;\; 36\mathbf{1}_2']$, $k = 3$, $\lambda_{11}^* = 3$, $\lambda_{12}^* = 6$ and $\lambda_{22}^* = 15$. The blocks of the design are: (1, 2, 3), (2, 3, 1), (3, 1, 2), (4, 5, 4), (5, 4, 4), (4, 4, 5), (4, 5, 5), (5, 5, 4), (5, 4, 5), (1, 4, 4), (4, 4, 1), (4, 1, 4), (2, 5, 5), (5, 5, 2), (5, 2, 5), (3, 4, 5), (4, 5, 3), (5, 3, 4), (1, 4, 5), (4, 5, 1), (5, 1, 4), (2, 4, 5), (4, 5, 2), (5, 2, 4), (3, 5, 4), (5, 4, 3), (4, 3, 5), (1, 5, 5), (5, 5, 1), (5, 1, 5), (2, 4, 4), (4, 4, 2), (4, 2, 4), (3, 4, 5), (4, 5, 3) and (5, 3, 4).

For the above design, the normalized orthogonal polynomial of degree 1 is given as

$$\mathbf{F} = \begin{bmatrix} \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}' = \begin{bmatrix} -0.71 & 0 & 0.71 \end{bmatrix}'.$$

The information matrix for this design is given as

$$\mathbf{C} = \begin{bmatrix} 9\mathbf{I}_3 - \mathbf{1}_3\mathbf{1}_3' & -3\mathbf{1}_3\mathbf{1}_2' \\ -3\mathbf{1}_2\mathbf{1}_3' & 27\mathbf{I}_2 - 9\mathbf{1}_2\mathbf{1}_2' \end{bmatrix},$$

with $V_{11} = 0.2222\sigma^2$ and $V_{12} = 0.1481\sigma^2$.

3.3. Method 3

Consider a BIB design with parameters $v^* = sm+1$ (prime or prime power), $b^* = sv^*$, $r^* = sm$, $k^* = m$ and $\lambda^* = m-1$ obtained by developing following initial block(s) modulo v:

$$x^w, x^{w+s}, x^{w+2s}, ..., x^{(m-1)s}, \qquad \text{for} \quad w = 0, 1, ..., s-1,$$

where x is the primitive element of GF (v^*). Substitute the last u set of treatments of the design with the last treatment of the second set, second last u set of treatment with second last treatment of the second set, likewise, v^*-3 number of treatments can be replaced by p number of treatment of the second set. The resulting design is a TR-GEBBPB design with parameters $v_1 = (v^* - pu)$, $v_2 = p$, $b = sv^*$, $r_1 = sm$, $r_2 = usm$, $k = m$, $\lambda_{11}^* = \lambda^*$, $\lambda_{12}^* = 2\lambda^*$ and $\lambda_{22}^* = 4\lambda^*$.

The joint information matrix for this design is given as

$$\mathbf{C} = \frac{(k-1)}{k} \begin{bmatrix} v^*\mathbf{I}_{v_1} - \mathbf{1}_{v_1}\mathbf{1}_{v_1}' & -u\mathbf{1}_{v_1}\mathbf{1}_{v_2}' \\ -u\mathbf{1}_{v_2}\mathbf{1}_{v_1}' & u(v^*\mathbf{I}_{v_2} - u\mathbf{1}_{v_2}\mathbf{1}_{v_2}') \end{bmatrix},$$

with $V_{11} = \dfrac{uk}{v^*(k-1)}\sigma^2$ and $V_{12} = \dfrac{k(u+1)}{uv^*(k-1)}\sigma^2$.

Example 3.3.1. The blocks of a TR-GEBBPB design with parameters $v_1 = 3$, $v_2 = 2$, $b = 7$, $r_1 = 6$, $r_2 = 12$, $k = 6$, $\lambda_{11}^* = 5$, $\lambda_{12}^* = 10$ and $\lambda_{22}^* = 20$ obtained from BIB design of parameters $v^* = 7$ ($s = 1$, $m = 6$), $b^* = 7$, $r^* = 6$, $k^* = 6$ and $\lambda^* = 6$ by taking $p = 2$ and $u = 2$ are given as: (1, 3, 2, 5, 4, 4), (2, 4, 3, 5, 4, 5), (3, 4, 4, 1, 5, 5), (4, 5, 4, 2, 5, 1), (4, 5, 5, 3, 1, 2), (5, 1, 5, 4, 2, 3) and (5, 2, 1, 4, 3, 4).

The normalized orthogonal polynomial of degree 1 for the above design is

$$\mathbf{F} = \begin{bmatrix} \frac{-5}{\sqrt{70}} & \frac{-3}{\sqrt{70}} & \frac{-1}{\sqrt{70}} & \frac{1}{\sqrt{70}} & \frac{3}{\sqrt{70}} & \frac{5}{\sqrt{70}} \end{bmatrix}' = \begin{bmatrix} -0.60 & -0.36 & -0.12 & 0.12 & 0.36 & 0.60 \end{bmatrix}'.$$

The information matrix for the above design is

$$\mathbf{C} = \frac{5}{6}\begin{bmatrix} 7\mathbf{I}_3 - \mathbf{1}_3\mathbf{1}_3' & -2\mathbf{1}_3\mathbf{1}_2' \\ -2\mathbf{1}_2\mathbf{1}_3' & 14\mathbf{I}_2 - 4\mathbf{1}_2\mathbf{1}_2' \end{bmatrix},$$

with $V_{11} = 0.3428\sigma^2$ and $V_{12} = 0.2571\sigma^2$.

4. DISCUSSION

This article attempts to study general efficiency balanced block designs for comparing treatments belonging to two disjoint sets in the presence of systematic trend. The advantage of the block designs, named TR-GEBBPB, obtained here is that these are robust against the presence of trend effects. Besides, these designs are general efficiency balanced and are more efficient for estimating the contrasts pertaining to two treatments from two different sets. As the designs are completely trend resistant, the analysis of the data generated from these designs can be carried out in the usual manner as if no trend effect is present in the model. A possible extension of the present study is to develop some methods to obtain smaller designs under the present experimental situation, for which an algorithmic approach can be an alternative. Attempts can also be made to obtain designs for comparing treatments belonging to two disjoint sets in the presence of trend under unequal block structure. The effects of repeated blocks (see for instance [28], [27], [29]) in TR-GEBBPB designs obtained through BIB designs can also be explored in selecting optimal designs for testing block effects.

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