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EXTREMES FOR SOLUTIONS TO STOCHASTIC DIFFERENCE EQUATIONS WITH REGULARLY VARYING TAILS

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Abstract:

- The main purpose of this paper is to look at the extremal properties of

$$X_k = \sum_{j=1}^{\infty} \left(\prod_{s=1}^{j-1} A_{k-s} \right) B_{k-j} , \qquad k \in \mathbb{Z} ,$$

where $(A_k, B_k)_{k\in\mathbb{Z}}$ is a periodic sequence of independent $\mathbb{R}^2_+$-valued random pairs. The so-called complete convergence theorem we prove enable us to give in detail the weak limiting behavior of various functional of the underlying process including the asymptotic distribution of upper and lower order statistics. In particular, we investigate the limiting distribution of the maximum and its corresponding extremal index. An application to a particular class of bilinear processes is included. These results generalize the ones obtained for the stationary case.

Key-Words:

- *periodic stochastic difference equations; extremal index; point processes.*

AMS Subject Classification:

- 62-02; 60G70.

1. INTRODUCTION

A general approach to look at the extremal properties of non-linear processes is through the analysis of stochastic difference equations (SDEs hereafter) of the form

$$\mathbf{X}_k = \mathbf{A}_k \mathbf{X}_{k-1} + \mathbf{B}_k \ , \qquad k \in \mathbb{Z} \ , \tag{1.1}$$

where $(\mathbf{A}_k)$ are $d{\times}d$ random matrices with possibly negative entries, $(\mathbf{B}_k)$ are $[0,\infty)^d$-valued random (column) vectors such that $(\mathbf{A}_k, \mathbf{B}_k)$ are independent and identically distributed (i.i.d.), and independent of the random column vector $\mathbf{X}_0 \in [0,\infty)^d$. The literature of SDEs is vast mainly for i.i.d. and stationary ergodic sequences $(\mathbf{A}_k, \mathbf{B}_k)$. The existence of a solution to (1.1) has been addressed by Kesten [23], Vervaat [37], and Goldie [18]; for more general results see also Brandt *et al.* [8], Bougerol and Picard [7], and Babillot *et al.* [2]. SDEs play a central role in fields such as finance, economics, insurance mathematics and biology. Examples can be found in Dufresne [12], Embrechts *et al.* [13], Baxendale and Khasminskii [6], Stărică [36], Mikosch [28], and Konstantinides and Mikosch [24]. The interest in these equations is generally justified by the fact that many non-linear processes, including (G)ARCH, threshold, and bilinear processes can be embedded in SDEs.

Extremal properties of the solution of one-dimensional SDEs were first studied by de Haan *et al*. [11] and then by Perfekt [31]. de Haan *et al*. [11] proved the convergence of the point processes of exceedances to a compound Poisson process. As an application, these authors obtained the extremal behavior of the ARCH(1) process. Perfekt [31] extended de Haan *et al*.'s results to Markov processes including SDEs as special cases with possibly negative A_k and B_k. More recently, Scotto [35] derived the extremal behavior of stationary solutions of SDEs where $(A_k, B_k)_{k\in\mathbb{Z}}$ are i.i.d. $\mathbb{R}^2_+$-valued random pairs, the distribution of B_1 being heavy-tailed and the distribution of A_1 having relatively lighter tails compared to the one of B_1 (cf. Grincevičius, [20] and Grey, [19]).

The primary objective of this paper is to derive the extremal properties of one-dimensional SDEs when $(X_k)_{k\in\mathbb{Z}}$ forms a periodic sequence, i.e., when there exists an integer $M \geq 1$ such that for every choice of integers $i_1, ..., i_n$, $(X_{i_1}, ..., X_{i_n})$ and $(X_{i_1+M}, ..., X_{i_n+M})$ are identically distributed. We will refer to such a sequence as an M-periodic sequence if M is the smallest integer as above. Note that if $M=1$ then $(X_k)_{k\in\mathbb{Z}}$ is a stationary sequence. The study of the extremal properties of non-stationary (periodic) stochastic processes plays a central role when modelling environmental time series, because of its wide applicability to the analysis of phenomena such as extreme concentration of air pollution, floods, wind storms, and extreme temperatures. Extreme value theory of non-stationary processes has been discussed under certain conditions. Horowitz [21] considered the model $\log(Y_k) = g(k) + X_k$, for daily ozone maxima $(Y_k)_{k\in\mathbb{Z}}$, where $g(k)$ is a deterministic function and $(X_k)_{k\in\mathbb{Z}}$ is a normal stationary autoregressive process. Ballerini and McCormick [4] discussed limit theory for non-stationary random sequences of the form $Y_k = g(k) + h(k)X_k$, where $(X_k)_{k\in\mathbb{Z}}$ is a stationary random sequence, satisfying some mixing conditions, and $h(k)$ is a positive, periodic function with integer period $p > 1$ as the variance function. The authors derived the limiting distribution of the maximum term based on the assumption that the distribution of X_k belongs to the domain of attraction of an extreme value distribution. The results were applied in a rainfall study; see also Ballerini and Waylen [5] and Ballerini [3]. Niu [29] introduced a class of nonlinear additive time series models for daily maxima of ozone concentrations in which both mean levels and variances are nonlinear functions of relevant meteorological variables. As an alternative approach to analyze tropospheric ozone data Niu [30] focus on estimating probabilities of monthly maximum ozone observations exceeding some specific levels, calculating the mean rate of exceedances of daily maximum ozone over the national standard level 120 ppb (parts per billion). For further examples see Coles [9].

Extreme value theory for periodic sequences was first considered by Alpuim [1] who showed that under Leadbetter's D condition (Leadbetter *et al.*, [26]) the only possible limit laws for the normalized maxima of the periodic sequence are the three extreme value distributions. Extensions for randomly indexed periodic sequences under long range dependence conditions were established by Ferreira [16]. Further results can be found in Ferreira [15] who studied the extremal behavior of periodic sequences under local mixing conditions. Generalizations under weaker local mixing conditions have been considered by Ferreira and Martins [17]. More recently, Martins and Ferreira [27] derived the expression of the extremal index (and hence the limiting distribution of the maximum) of a periodic moving average sequence driven by heavy-tailed innovations.

The rest of the paper is organized as follows: Section 2 deals with the tail behavior of X_r, $r = 1, ..., M$. Section 3 is devoted to a detailed point process analysis of asymptotic properties of the periodic sequence $(X_k)_{k\in\mathbb{Z}}$. In particular we deduce the maximum limiting distribution and the extremal index. Finally, in Section 4 the results are applied to a particular class of bilinear processes.

2. TAIL BEHAVIOR

Let $(A_k, B_k)_{k\in\mathbb{Z}}$ be a one-dimensional M-periodic sequence of independent $\mathbb{R}^2_+$-valued random pairs, such that $\bar{F}_{B_r}(x) = P(B_r > x)$, $r = 1, ..., M$, are regularly varying with tail index $-\alpha$, for some $\alpha > 0$, i.e.,

$$\bar{F}_{B_r}(x) = x^{-\alpha} L_r(x) \;, \qquad r = 1, ..., M \;, \tag{2.1}$$

for some slowly varying functions $L_r \colon \mathbb{R}_+ \to \mathbb{R}_+$ $(r = 1, ..., M)$ at infinity.

We further assume that the tails are equivalent in the sense that

$$\lim_{x\to\infty} \frac{\bar{F}_{B_l}(x)}{\bar{F}_{B_k}(x)} = \gamma_{l,k} \ , \qquad (0<\gamma_{l,k}<\infty) \quad l,k \in \mathbb{Z} \ . \tag{2.2}$$

Note that $\gamma_{l,k} = \gamma_{l+M,k}$ and $\gamma_{l,k} = \gamma_{l,k+M}$. In addition, we assume that for $r = 1, ..., M$

$$EA_r^\alpha < 1 \ , \qquad \text{and} \qquad EA_r^{\alpha+\delta} < \infty, \quad \text{for some } \ \delta > 0 \ . \tag{2.3}$$

Note that no further assumptions are needed since the central role in determining the tail behavior of X_r is played by the distributions F_{B_r}. Furthermore, we assume that X_k admits the representation

$$X_k = \sum_{j=1}^{\infty} \left(\prod_{s=1}^{j-1} A_{k-s} \right) B_{k-j} \ , \tag{2.4}$$

where we use the convention $\prod_{s=1}^{0} = 1$. This series representation is possible a.s. by virtue of the assumptions on A_k and B_k. Clearly, $(X_k)_{k\in\mathbb{Z}}$ forms an M-periodic sequence and satisfies the SDEs

$$X_k = A_k X_{k-1} + B_k \ .$$

We start with the analysis of the tail behavior of X_r, $r = 1, ..., M$. In doing so, the following alternative representation of X_r is very useful.

Proposition 2.1. *For the process defined in* (2.4), *it holds that for* $r = 1, ..., M$

$$X_r = \sum_{i=0}^{M-1} X_r^{(i)} \ ,$$

with

$$X_r^{(i)} = \sum_{j=1}^{\infty} \left(\prod_{s=1}^{M(j-1)+i} A_{r-s} \right) B_{r-(j-1)M-i-1} \ .$$

We now begin with a series of results designed to understand the tail behavior of $X_r^{(i)}$ as well as sums of these variables. The tail behavior of $X_r^{(i)}$ will be derived in two stages: first we obtain the tail behavior of the approximation $X_{r,m}^{(i)}$, with $m = KM$ $(K \geq 1)$, defined as

$$X_{r,m}^{(i)} = \sum_{j=1}^{m} W_{r,i}^{(j)} \ ,$$

with

$$W_{r,i}^{(j)} = \left(\prod_{s=1}^{M(j-1)+i} A_{r-s} \right) B_{r-(j-1)M-i-1} \ ;$$

then the results are extended so that the number of summands can be infinite.

Lemma 2.1. *Let $(A_k, B_k)_{k\in\mathbb{Z}}$ be an M-periodic sequence of independent $\mathbb{R}^2_+$-valued random pairs satisfying (2.1), (2.2), and (2.3). For a fixed value $0 \le i \le M-1$ and $1 \le j \le m$, we have as $x \to \infty$*

$$(2.5)\quad P\big(W^{(j)}_{r,i} > x\big) \sim \gamma_{r+M-i-1,r} \left(\prod_{s=1}^{i} E(A^\alpha_{r-s})^j\right)\left(\prod_{s=i+1}^{M} E(A^\alpha_{r-s})^{j-1}\right) P\big(B_r > x\big)\,.$$

Furthermore, for all fixed values $1 \le j_1 < j_2 \le m$ and $0 \le i \le M-1$, as $x \to \infty$

$$(2.6)\qquad \frac{P\Big(W^{(j_1)}_{r,i} > x,\, W^{(j_2)}_{r,i} > x\Big)}{P\big(B_r > x\big)} \to 0\;, \qquad r = 1, ..., M\,.$$

Proof: The first statement follows as an application of Breiman's result (cf. Davis and Resnick [10], p. 1197). Let $C_{s,j_h} = \prod_{s=1}^{M(j_h-1)+i} A_{r-s}$, for $h = 1, 2$. In proving (2.6) observe that

$$\begin{aligned}
&P\Big(W^{(j_1)}_{r,i} > x,\, W^{(j_2)}_{r,i} > x\Big) = \\
&\quad = P\Big(C_{s,j_1} B_{r-(j_1-1)M-i-1} > x,\; C_{s,j_1} C_{s,j_2} B_{r-(j_2-1)M-i-1} > x\Big) \\
&\quad \le P\Big(C_{s,j_1} \le \epsilon,\; C_{s,j_1} B_{r-(j_1-1)M-i-1} > x\Big) \\
&\qquad + P\Big(C_{s,j_1} > \epsilon,\; C_{s,j_1} B_{r-(j_1-1)M-i-1} > x,\; C_{s,j_1} C_{s,j_2} B_{r-(j_2-1)M-i-1} > x\Big) \\
&\quad \le P\Big(C_{s,j_1} 1_{[C_{s,j_1} \le \epsilon]}\, B_{r-(j_1-1)M-i-1} > x\Big) \\
&\qquad + P\Big(B_{r-(j_1-1)M-i-1} > \frac{x}{\epsilon},\; C_{s,j_2} B_{r-(j_2-1)M-i-1} > \frac{x}{\epsilon}\Big)\,.
\end{aligned}$$

Now, by Breiman's result

$$\begin{aligned}
\limsup_{x\to\infty} \frac{P\Big(C_{s,j_1} 1_{[C_{s,j_1}\le\epsilon]}\, B_{r-(j_1-1)M-i-1} > x\Big)}{P\big(B_r > x\big)} &= \gamma_{r+M-i-1,r}\, E\Big(C_{s,j_1} 1_{[C_{s,j_1}\le\epsilon]}\Big)^\alpha \\
&\to 0\;,
\end{aligned}$$

as $\epsilon \to 0$. Moreover,

$$\begin{aligned}
&\frac{P\Big(B_{r-(j_1-1)M-i-1} > \frac{x}{\epsilon},\; C_{s,j_2} B_{r-(j_2-1)M-i-1} > \frac{x}{\epsilon}\Big)}{P\big(B_r > x\big)} \sim \\
&\qquad\qquad \sim \epsilon^{2\alpha}\, \gamma_{r+M-i-1,r}\, E(C_{s,j_2})^\alpha\, P\Big(B_{r-(j_2-1)M-i-1} > x\Big)\,,
\end{aligned}$$

as $x \to \infty$. Note that as $\epsilon \to 0$, the right-hand side converges to 0. This completes the proof. $\square$

Lemma 2.2. *Let $(A_k, B_k)_{k\in\mathbb{Z}}$ be an M-periodic sequence of independent $\mathbb{R}^2_+$-valued random pairs satisfying* (2.1), (2.2), *and* (2.3). *For a fixed value of* $0 \le i \le M-1$

$$\lim_{x\to\infty} \frac{P\big(X^{(i)}_{r,m} > x\big)}{P\big(B_r > x\big)} = \left(\prod_{s=1}^{i} EA^{\alpha}_{r-s}\right) \frac{1-\Big(\prod_{s=1}^{M} EA^{\alpha}_{r-s}\Big)^m}{1-\prod_{s=1}^{M} EA^{\alpha}_{r-s}}\, \gamma_{r+M-i-1,r}\ , \tag{2.7}$$

for $r = 1, ..., M$. *Moreover, as* $m \to \infty$

$$\lim_{x\to\infty} \frac{P\big(X^{(i)}_{r} > x\big)}{P\big(B_r > x\big)} = \frac{\prod_{s=1}^{i} EA^{\alpha}_{r-s}}{1-\prod_{s=1}^{M} EA^{\alpha}_{r-s}}\, \gamma_{r+M-i-1,r}\ . \tag{2.8}$$

Proof: The first statement follows as an application of Lemma 2.1 in Davis and Resnick [10] and Lemma 2.1. The proof is complete upon showing that by letting $m \to \infty$ we obtain (2.8). First note that the first statement implies that

$$\begin{aligned}\liminf_{x\to\infty} \frac{P\big(X^{(i)}_{r} > x\big)}{P\big(B_r > x\big)} &\ge \liminf_{x\to\infty} \frac{P\big(X^{(i)}_{r,m} > x\big)}{P\big(B_r > x\big)}\\ &= \left(\prod_{s=1}^{i} EA^{\alpha}_{r-s}\right) \frac{1-\Big(\prod_{s=1}^{M} EA^{\alpha}_{r-s}\Big)^m}{1-\prod_{s=1}^{M} EA^{\alpha}_{r-s}}\, \gamma_{r+M-i-1,r}\ .\end{aligned}$$

Hence, as $m \to \infty$

$$\liminf_{x\to\infty} \frac{P\big(X^{(i)}_{r} > x\big)}{P\big(B_r > x\big)} \ge \frac{\prod_{s=1}^{i} EA^{\alpha}_{r-s}}{1-\prod_{s=1}^{M} EA^{\alpha}_{r-s}}\, \gamma_{r+M-i-1,r}\ .$$

The arguments needed to get the upper bound follow closely the arguments outlined in Resnick ([33], p. 228): decompose the event $\big[X^{(i)}_r > x\big]$ according to whether $\big[\max_{j\in\mathbb{N}} W^{(j)}_{r,i} > x\big]$ or $\big[\max_{j\in\mathbb{N}} W^{(j)}_{r,i} \le x\big]$

$$\begin{aligned}P\big(X^{(i)}_r > x\big) &= P\Big(X^{(i)}_r > x,\ \max_{j\in\mathbb{N}} W^{(j)}_{r,i} > x\Big) + P\Big(X^{(i)}_r > x,\ \max_{j\in\mathbb{N}} W^{(j)}_{r,i} \le x\Big)\\ &\le P\left(\bigcup_{j\in\mathbb{N}} W^{(j)}_{r,i} > x\right) + P\left(\sum_{j=1}^{\infty} W^{(j)}_{r,i}\, 1_{\{W^{(j)}_{r,i}\le x\}} > x,\ \max_{j\in\mathbb{N}} W^{(j)}_{r,i} \le x\right)\\ &\le \sum_{j=1}^{\infty} P\big(W^{(j)}_{r,i} > x\big) + P\left(\sum_{j=1}^{\infty} W^{(j)}_{r,i}\, 1_{\{W^{(j)}_{r,i}\le x\}} > x\right).\end{aligned}$$

By Markov's inequality

$$\begin{aligned}\frac{P\big(X^{(i)}_r > x\big)}{P\big(B_r > x\big)} &\le \frac{\sum_{j=1}^{\infty} P\big(W^{(j)}_{r,i} > x\big)}{P\big(B_r > x\big)} + \frac{\sum_{j=1}^{\infty} EW^{(j)}_{r,i}\, 1_{\{W^{(j)}_{r,i}\le x\}}}{x\, P\big(B_r > x\big)}\\ &= I(x) + J(x)\ .\end{aligned}$$

To handle $I(x)$, note that by Kamarata's Theorem quoted in Resnick ([33], p. 17), the result in (2.5) along with condition (2.3) and dominated convergence, lead us to obtain

$$\lim_{x\to\infty} I(x) \;=\; \frac{\prod_{s=1}^{i} EA_{r-s}^{\alpha}}{1-\prod_{s=1}^{M} EA_{r-s}^{\alpha}}\; \gamma_{r+M-i-1,r}\;.$$

For $J(x)$ let us start by considering the case $0<\alpha<1$. By Lemma 2.1, the distribution tail of $W_{r,i}^{(j)}$ is regularly varying with index $-\alpha$. Now

$$\frac{EW_{r,i}^{(j)}\,1_{\{W_{r,i}^{(j)}\le x\}}}{x\,P\big(B_r>x\big)} \;=\; \frac{EW_{r,i}^{(j)}\,1_{\{W_{r,i}^{(j)}\le x\}}}{x\,P\big(W_{r,i}^{(j)}>x\big)}\;\frac{P\big(W_{r,i}^{(j)}>x\big)}{P\big(B_r>x\big)}\;.$$

From an integration by parts along with the result in (2.5), and Kamarata's Theorem

$$\frac{EW_{r,i}^{(j)}\,1_{\{W_{r,i}^{(j)}\le x\}}}{x\,P\big(W_{r,i}^{(j)}>x\big)} \;\to\; \alpha(1-\alpha)^{-1}\;, \qquad x\to\infty\;. \tag{2.9}$$

Since $P(B_r>x)$ is regularly varying with index $-\alpha$ we can use its Kamarata representation and (2.5) to obtain that for sufficiently large x and some constant $K>0$

$$\frac{P\big(W_{r,i}^{(j)}>x\big)}{P\big(B_r>x\big)} \;\le\; K\,\gamma_{r+M-i-1,r}\left(\prod_{s=1}^{i} E(A_{r-s}^{\alpha})^{j}\right)\left(\prod_{s=i+1}^{M} E(A_{r-s}^{\alpha})^{j-1}\right)\,, \tag{2.10}$$

for $r=1,...,M$. Combining (2.9) and (2.10), we conclude, for sufficiently large x

$$\frac{EW_{r,i}^{(j)}\,1_{\{W_{r,i}^{(j)}\le x\}}}{x\,P\big(B_r>x\big)} \;\le\; K_1\,\gamma_{r+M-i-1,r}\left(\prod_{s=1}^{i} E(A_{r-s}^{\alpha})^{j}\right)\left(\prod_{s=i+1}^{M} E(A_{r-s}^{\alpha})^{j-1}\right)\,,$$

for some constant $K_1>0$. This bound is summable providing, by dominated convergence

$$\begin{aligned}\limsup_{x\to\infty} J(x) \;&\le\; K_1\sum_{j=1}^{\infty}\gamma_{r+M-i-1,r}\left(\prod_{s=1}^{i} E(A_{r-s}^{\alpha})^{j}\right)\left(\prod_{s=i+1}^{M} E(A_{r-s}^{\alpha})^{j-1}\right)\\ &=\; K_1\,\frac{\prod_{s=1}^{i} EA_{r-s}^{\alpha}}{1-\prod_{s=1}^{M} EA_{r-s}^{\alpha}}\;\gamma_{r+M-i-1,r}\end{aligned}$$

and hence

$$\limsup_{x\to\infty}\frac{P\big(X_r^{(i)}>x\big)}{P\big(B_r>x\big)} \;\le\; (K_1+1)\,\frac{\prod_{s=1}^{i} EA_{r-s}^{\alpha}}{1-\prod_{s=1}^{M} EA_{r-s}^{\alpha}}\;\gamma_{r+M-i-1,r}\;. \tag{2.11}$$

If $\alpha \geq 1$, we proceed as follows:

pick $\beta \in (\alpha, \alpha\delta^{-1})$ and consider $A^{(i)} = \sum_{j=1}^{\infty} \prod_{s=1}^{M(j-1)+i} A_{r-s}$, $(i=1, ..., M)$ and $P_j^{(i)} = \big(\prod_{s=1}^{M(j-1)+i} A_{r-s}\big) \{A^{(i)}\}^{-1}$, $(i=1, ..., M,\ j \in \mathbb{N})$.

By Jensen's inequality

$$\begin{aligned}
\big(X_r^{(i)}\big)^\beta &= \big\{A^{(i)}\big\}^\beta \left(\sum_{j=1}^{\infty} P_j^{(i)} B_{r-(j-1)M-i-1}\right)^\beta \\
&\leq \big\{A^{(i)}\big\}^\beta \sum_{j=1}^{\infty} P_j^{(i)} B^\beta_{r-(j-1)M-i-1} \\
&= \big\{A^{(i)}\big\}^{\beta-1} \sum_{j=1}^{\infty} \left(\prod_{s=1}^{M(j-1)+i} A_{r-s}\right) B^\beta_{r-(j-1)M-i-1} \,,
\end{aligned}$$

providing

$$\frac{P\big(X_r^{(i)} > x\big)}{P\big(B_r > x\big)} \leq \frac{P\Big(\big\{A^{(i)}\big\}^{\beta-1} \sum_{j=1}^{\infty}\Big(\prod_{s=1}^{M(j-1)+i} A_{r-s}\Big) B^\beta_{r-(j-1)M-i-1} > x^\beta\Big)}{P\big(B_r^\beta > x^\beta\big)} \,.$$

Using the fact that $P(B_r^\beta > x) \in RV_{-\alpha\beta^{-1}}$ with $\delta < \alpha\beta^{-1}$, for $r = 1, ..., M$ and $i = 0, ..., M-1$ it follows that

$$\begin{aligned}
\limsup_{x\to\infty} \frac{P\big(X_r^{(i)} > x\big)}{P\big(B_r > x\big)} &\leq (K_1+1) \sum_{j=1}^{\infty} \left(\prod_{s=1}^{i} E(A^\alpha_{r-s})^j\right)\left(\prod_{s=i+1}^{M} E(A^\alpha_{r-s})^{j-1}\right) \\
&\qquad \times \big\{EA^{(i)}\big\}^{\alpha(1-\beta^{-1})} \gamma_{r+M-i-1,r} < \infty \,.
\end{aligned} \tag{2.12}$$

On the other hand, for any $\epsilon > 0$

$$\frac{P\big(X_r^{(i)} > x\big)}{P\big(B_r > x\big)} \leq \frac{P\Big(\sum_{j=1}^{m} W_{r,i}^{(j)} > (1-\epsilon)\,x\Big)}{P\big(B_r > x\big)} + \frac{P\Big(\sum_{j=m+1}^{\infty} W_{r,i}^{(j)} > \epsilon\, x\Big)}{P\big(B_r > x\big)} \,,$$

and for (2.7) and (2.11)

$$\begin{aligned}
\limsup_{x\to\infty} \frac{P\big(X_r^{(i)} > x\big)}{P\big(B_r > x\big)} &\leq (1-\epsilon)^{-\alpha} \left(\prod_{s=1}^{i} EA^\alpha_{r-s}\right) \frac{1 - \Big(\prod_{s=1}^{M} EA^\alpha_{r-s}\Big)^m}{1 - \prod_{s=1}^{M} EA^\alpha_{r-s}} \gamma_{r+M-i-1,r} \\
&\quad + K_1\, \epsilon^{-\alpha} \\
&\quad \times \sum_{j=m+1}^{\infty} \left(\prod_{s=1}^{i} E(A^\alpha_{r-s})^j\right)\left(\prod_{s=i+1}^{M} E(A^\alpha_{r-s})^{j-1}\right) \gamma_{r+M-i-1,r} \,,
\end{aligned}$$

for the case $0 \leq \alpha \leq 1$ with a similar bound for the second piece provided by (2.12) when $\alpha \geq 1$. Let $m \to \infty$ and then send $\epsilon \to 0$ to obtain

$$\limsup_{x\to\infty} \frac{P\big(X_r^{(i)} > x\big)}{P\big(B_r > x\big)} \leq \frac{\prod_{s=1}^{i} EA^\alpha_{r-s}}{1 - \prod_{s=1}^{M} EA^\alpha_{r-s}} \gamma_{r+M-i-1,r}$$

and this combined with the liminf statement concludes the proof. $\square$

Combining Lemmas 2.1 and 2.2 yields the following result.

Theorem 2.1. *Let $(X_k)_{k\in\mathbb{Z}}$ be the M-periodic sequence defined in* (2.4). *Let $(A_k, B_k)_{k\in\mathbb{Z}}$ be an M-periodic sequence of independent $\mathbb{R}^2_+$-valued random pairs satisfying* (2.1), (2.2), *and* (2.3). *For $r=1,...,M$*

$$\lim_{x\to\infty} \frac{P\big(X_r > x\big)}{P\big(B_r > x\big)} = \frac{1}{1-\prod_{s=1}^{M} EA^{\alpha}_{r-s}} \sum_{i=0}^{M-1} \gamma_{r+M-i-1,r} \left(\prod_{s=1}^{i} EA^{\alpha}_{r-s}\right) . \tag{2.13}$$

Proof: Note that by Lemma 2.1 in Davis and Resnick [10] it is sufficient to show that for $0 \le i_1 < i_2 \le M-1$, as $x \to \infty$

$$\frac{P\Big(X_r^{(i_1)} > x,\ X_r^{(i_2)} > x\Big)}{P\big(B_r > x\big)} \sim 0 \ , \qquad r=1,...,M\,. \tag{2.14}$$

Now an argument similar to the one in the proof of Lemma 2.1 shows that (2.14) holds. □

3. POINT PROCESS APPROACH

In this section we investigate the limit behavior of a sequence of point processes based on the periodic sequence $(X_k)_{k\in\mathbb{Z}}$. Since our results are based on point process theory, we briefly discuss some notation and background about point processes; for further details see Kallenberg [22] and Resnick [33]. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space and E a state space where points reside and assume that E is Euclidian. Let $\mathcal{E}$ be the σ-algebra on E generated by open sets of E. For $x \in E$, define $\epsilon_x(\cdot)$ on $\mathcal{E}$ as the simple point measure with unit mass at x. Let $\{x_j\}$ be a countable collection of points on E. A point measure N on $\mathcal{E}$ is defined to be

$$N(\cdot) = \sum_{j=1}^{\infty} \epsilon_{x_j}(\cdot) \ ,$$

which is a non-negative integer valued Radon measure on compact subsets of E. Let $M_p(E)$ be the class of such Radon measures on $\mathcal{E}$ and $\mathcal{M}_p(E)$ the smallest σ-algebra, making maps $N \to N(A^*)$ measurable, where $N \in M_p(E)$ and $A^* \in \mathcal{E}$. $\mathcal{M}_p(E)$ can be made into a complete separable metric space, hence we assume that it is a metric space with vague metric d. A point process on E is a measurable map from $(\Omega, \mathcal{F})$ to $\big(M_p(E), \mathcal{M}_p(E)\big)$. Let $C^+_K(E)$ be the set of all continuous, non-negative functions on the state space E with compact support. If $N_n \in M_p(E)$ then N_n converges vaguely to N ($N_n \Rightarrow N$) if $N_n(f)$ converges to $N(f)$ for every

$f \in C_K^+(E)$, where $N(f) = \int f\, dN$. A Poisson process on $(E, \mathcal{E})$ with mean measure μ is a point process N such that, for every $A^* \in \mathcal{E}$, $N(A^*)$ is a Poisson random variable with mean measure $\mu(A^*)$. If $A_1^*, ..., A_m^*$ are mutually independent sets then $N(A_1^*), ..., N(A_m^*)$ are independent random variables. We call N a Poisson random measure with mean measure μ or PRM(μ) for short.

In this section, we investigate the limiting behavior of a sequence of point processes $(N_n)_{n\in\mathbb{N}}$ defined as

$$N_n = \sum_{k=1}^{\infty} \epsilon_{\{k/n,\, a_n^{-1} X_k\}} \ ,$$

based on $(a_n^{-1} X_k)_{k\in\mathbb{Z}}$ with the sequence of norming constants $(a_n)_{n\in\mathbb{N}}$ satisfying

$$\lim_{n\to\infty} n P\big(B_r > a_n x\big) = \tau_r \ , \qquad (\tau_r > 0), \quad r = 1, ..., M \, .$$

Note that such a sequence exists by the assumption of regular variation of each $\bar{F}_{B_r}$, $(r = 1, ..., M)$, and implies that

$$n P\big(X_r > a_n x\big) \ \to \ \tau_r \left\{ \frac{1}{1 - \prod_{s=1}^{M} E A_{r-s}^{\alpha}} \ \sum_{i=0}^{M-1} \left(\prod_{s=1}^{i} E A_{r-s}^{\alpha} \right) \gamma_{r+M-i-1,r} \right\} ,$$

as $n \to \infty$. It is important to point out the fact that $\tau_r = \tau_h \gamma_{r,l}$, for $r, l \in \{1, ..., M\}$. Hence, without lost of generality it will be assumed that $\tau_r = \tau_1 \gamma_{r,1}$ with $\tau_1 = x^{-\alpha}$.

The main result of this section is formalized through the following theorem, which discusses the weak convergence of the sequence of point processes $(N_n)_{n\in\mathbb{N}}$ to a function of PRM. For simplicity of notation we define $E_h = (0,\infty) \times [-\infty,\infty]^h \backslash \{\mathbf{0}\}$, with $h \in \mathbb{N}$.

Theorem 3.1. *Let $(X_k)_{k\in\mathbb{Z}}$ be an M-periodic sequence defined as in* (2.4) *where $(A_k, B_k)_{k\in\mathbb{Z}}$ is an M-periodic sequence of independent $\mathbb{R}_+^2$-valued random pairs satisfying* (2.1), (2.2), *and* (2.3). *Then, as $n \to \infty$*

$$N_n = \sum_{k=1}^{\infty} \epsilon_{\left\{\frac{k}{n},\, a_n^{-1} X_k\right\}} \ \Rightarrow \ N = \sum_{r=1}^{M} \sum_{i=0}^{M-1} \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \epsilon_{\left\{T_{k,r}^{(i)},\ J_{k,r}^{(i)} U_{k,1,r} \cdots U_{k,M(j-1)+i-1,r}\right\}} \ ,$$

in the space $M_p(E_1)$, where $\sum_{k=1}^{\infty} \epsilon_{\left\{T_{k,r}^{(i)},\, J_{k,r}^{(i)}\right\}}$ are PRM $(dt \times d\nu_{r,i})$ with

$$\nu_{r,i} = \frac{1}{M} \ \gamma_{r,1}\, \gamma_{r+M-i-1,r} \ \mu(dx) \ ,$$

where $\mu(dx) = \alpha\, x^{-\alpha-1} 1_{(0,\infty]}(x)\, dx$ and $(U_{k,1,r}, ..., U_{k,M,r})$ having the same distribution as $(A_1, ..., A_M)$.

Proof: First note that

$$\sum_{k=1}^{\infty} \epsilon_{\left\{\frac{k}{n},\, a_n^{-1} X_k\right\}} = \sum_{r=1}^{M} \sum_{k=1}^{\infty} \epsilon_{\left\{\frac{(k-1)M+r}{n},\, a_n^{-1} X_{(k-1)M+r}\right\}} .$$

As an application of Proposition 3.2 in Feigin *et al.* [14], for fixed values of $r = 1..., M$ and $i = 0, ..., M-1$, it follows that

$$\begin{aligned}
&\sum_{k=1}^{\infty} \epsilon_{\left\{\frac{(k-1)M+r}{n},\ a_n^{-1}\left(B_{k-(j-1)M-i-1)},\, j=1,...,m\right),\ A_{k-s},\ s=1,...,M(j-1)+i\right\}} \Rightarrow \\
&\qquad \Rightarrow \sum_{k=1}^{\infty} \epsilon_{\left\{T_{k,r}^{(i)},\, J_{k,r}^{(i)} \mathbf{e}_1,\, \infty,\, U_{k,1,r},\, ...,\, U_{kM(j-1)+i,r}\right\}} \\
&\qquad\quad + \sum_{k=1}^{\infty} \epsilon_{\left\{T_{k,r}^{(i)},\, J_{k}^{(i)} \mathbf{e}_2,\, U_{k,1,r},\, \infty,\, ...,\, U_{k,M(j-1)+i,r}\right\}} \\
&\qquad\quad \vdots \\
&\qquad\quad + \sum_{k=1}^{\infty} \epsilon_{\left\{T_{k,r}^{(i)},\, J_{k,r}^{(i)} \mathbf{e}_m,\, U_{k,1,r},\, ...,\, U_{k,M(j-1)+i,r},\, \infty\right\}} ,
\end{aligned}$$

in $M_p\big(E_m \times (0,\infty)^{M(j-1)+i}\big)$, where $\mathbf{e}_s$ is the unit vector in $\mathbb{R}^m$ with 1 in the s-th component and the rest zero. By the lines of reasoning given in Resnick and Van den Berg ([34], Theorem 4.1) it follows that, for a fixed value of $r = 1..., M$ and $i = 0, ..., M-1$

$$\sum_{k=1}^{\infty} \epsilon_{\left\{\frac{(k-1)M+r}{n},\, a_n^{-1} X_{(k-1)M+r}^{(i)}\right\}} \Rightarrow \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \epsilon_{\left\{T_{k,r}^{(i)},\, J_{k,r}^{(i)} U_{k,1,r} \cdots U_{k,M(j-1)+i,r}\right\}} ,$$

in $M_p(E_1)$. Next we have to show that the point processes

$$N_n^{(1)} = \sum_{k=1}^{\infty} \epsilon_{\left\{\frac{(k-1)M+r}{n},\, a_n^{-1}\left(X_{(k-1)M+r}^{(0)},...,X_{(k-1)M+r}^{(M-1)}\right)\right\}}$$

and

$$N_n^{(2)} = \sum_{i=0}^{M-1} \sum_{k=1}^{\infty} \epsilon_{\left\{\frac{(k-1)M+r}{n},\, a_n^{-1} X_{(k-1)M+r}^{(i)} \mathbf{v}_i\right\}} ,$$

where $\mathbf{v}_s$ is the unit vector in $\mathbb{R}^M$ with 1 in the s-th component and the rest zero, differ negligibly, as $n \to \infty$. In doing so we must prove that

$$d\big(N_n^{(1)}, N_n^{(2)}\big) \to 0 , \tag{3.1}$$

in probability, where d is the vague metric on the space of point measures in which $N_n^{(1)}$ and $N_n^{(2)}$ live. Here $N_n^{(2)}$ concentrates all its points on the axes $\mathbf{v}_s$, and (3.1) is expressing the fact that, for each k, at most one of the M components $X_k^{(i)}$

is non-negligible as compared to a_n. From the definition of vague convergence, (3.1) follows if

$$N_n^{(1)}(f) - N_n^{(2)}(f) \;\to\; 0 \;, \tag{3.2}$$

in probability for each $f \in C_K^+(E_M)$, the space of continuous non-negative functions with compact support on E_M. To prove (3.2), suppose that f is such a function. Because of compactness, the support of f is contained in the set

$$[0,\xi] \times \Big\{\mathbf{x}\colon\; \mathbf{x} \in [0,\infty]^M \backslash \{\mathbf{0}\},\; \max_{0\le i\le M-1} x_i > \delta\Big\} \;,$$

for some $\xi > 0$ and $\delta > 0$. Note therefore that f vanishes in $[0,\xi] \times [0,\delta]^M$. For an arbitrary $y \in (0,\delta)$ we define S_y as

$$S_y \Big\{\mathbf{x}\colon\; \mathbf{x} \in [0,\infty]^M \backslash \{\mathbf{0}\},\; \text{at most one component } x_i > y\Big\} \;,$$

and

$$N_n^{(h)}(f) \;= \int_{[0,\xi]\times S_y} f\,dN_n^{(h)} + \int_{[0,\xi]\times S_y^c} f\,dN_n^{(h)} \;, \qquad h = 1,2 \;.$$

Note that

$$E\bigg(\int_{[0,\xi]\times S_y^c} f\,dN_n^{(1)}\bigg) \;\le$$

$$\begin{aligned}
&\le\; \big(\sup f\big)\, E\Big(N_n^{(1)}\big([0,\xi]\times S_y^c\big)\Big) \\
&\le\; \big(\sup f\big) \left[\frac{n}{M}\right] \xi\, P\Big[2 \text{ or more } X_{M(k-1)+r}^{(0)}, ..., X_{M(k-1)+r}^{(M-1)} > a_n\, y\Big] \\
&\le\; \big(\sup f\big) \left[\frac{n}{M}\right] \xi \binom{M}{2} P\Big(X_{M(k-1)+r}^{(i_1)} > a_n\, y,\; X_{M(k-1)+r}^{(i_2)} > a_n\, y\Big) \;\to\; 0, \quad n \to \infty \;,
\end{aligned}$$

which follows by (2.14). Furthermore, it is also true that

$$\int_{[0,\xi]\times S_y^c} f\,dN_n^{(2)} \;=\; 0 \;.$$

Thus, in proving (3.2) it is enough to show that

$$\int_{[0,\xi]\times S_y} f\,dN_n^{(1)} \;-\; \int_{[0,\xi]\times S_y} f\,dN_n^{(2)} \;\to\; 0 \;,$$

in probability. This last statement follows by the same arguments used in the proof of Proposition 4.26 in Resnick [33]. We skip the details. Consider now the map $T\colon (M_p(E_1))^M \to M_p(E_M)$ such that, for a fixed value of $r = 1..., M$

$$\begin{aligned}
&T\left(\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \epsilon_{\left\{T_{k,r}^{(i)},\, J_{k,r}^{(i)} U_{k,1,r} \cdots U_{k,M(j-1)+i,r}\right\}},\quad i = 0, ..., M-1\right) = \\
&\qquad\qquad = \sum_{i=0}^{M-1}\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \epsilon_{\left\{T_{k,r}^{(i)},\, J_{k,r}^{(i)} U_{k,1,r} \cdots U_{k,M(j-1)+i,r}\, \mathbf{v}_i\right\}} \;.
\end{aligned}$$

Note that this map is continuous and hence by the continuous mapping theorem

$$
\begin{aligned}
&T\left(\sum_{k=1}^{\infty} \epsilon_{\left\{\frac{(k-1)M+r}{n},\, a_n^{-1} X^{(i)}_{(k-1)M+r}\right\}},\quad i=0,...,M-1\right) = \\
&\qquad = \sum_{i=0}^{M-1} \sum_{k=1}^{\infty} \epsilon_{\left\{\frac{(k-1)M+r}{n},\, a_n^{-1} X^{(i)}_{(k-1)M+r}\mathbf{v}_i\right\}} \implies \\
&\implies T\left(\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \epsilon_{\left\{T^{(i)}_{k,r},\, J^{(i)}_{k,r} U_{k,1,r}\cdots U_{k,M(j-1)+i,r}\right\}},\quad i=0,...,M-1\right) = \\
&\qquad = \sum_{i=0}^{M-1} \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \epsilon_{\left\{T^{(i)}_{k,r},\, J^{(i)}_{k,r} U_{k,1,r}\cdots U_{k,M(j-1)+i,r}\mathbf{v}_i\right\}}\ ,
\end{aligned}
\tag{3.3}
$$

in $M_p(E_M)$. Finally the map $T\colon M_p(E_M) \to M_p(E_1)$ defined by

$$
T\left(\sum_{k=1}^{\infty} \epsilon_{\left\{\frac{(k-1)M+r}{n},\, a_n^{-1}\left(X^{(0)}_{(k-1)M+r},...,X^{(M-1)}_{(k-1)M+r}\right)\right\}}\right) = \sum_{k=1}^{\infty} \epsilon_{\left\{\frac{(k-1)M+r}{n},\, a_n^{-1} X_{(k-1)M+r}\right\}}\ ,
$$

is almost surely continuous with respect to the distribution of (3.3). Hence applying the continuous mapping theorem we obtain

$$
\begin{aligned}
&T\left(\sum_{k=1}^{\infty} \epsilon_{\left\{\frac{(k-1)M+r}{n},\, a_n^{-1}\left(X^{(0)}_{(k-1)M+r},...,X^{(M-1)}_{(k-1)M+r}\right)\right\}}\right) = \\
&\qquad = \sum_{k=1}^{\infty} \epsilon_{\left\{\frac{(k-1)M+r}{n},\, a_n^{-1} X_{(k-1)M+r}\right\}} \implies \\
&\implies T\left(\sum_{i=0}^{M-1} \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \epsilon_{\left\{T^{(i)}_{k,r},\, J^{(i)}_{k,r} U_{k,1,r}\cdots U_{k,M(j-1)+i,r}\mathbf{v}_i\right\}}\right) = \\
&\qquad = \sum_{i=0}^{M-1} \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \epsilon_{\left\{T^{(i)}_{k,r},\, J^{(i)}_{k,r} U_{k,1,r}\cdots U_{k,M(j-1)+i,r}\right\}}\ ,
\end{aligned}
$$

providing

$$
\sum_{k=1}^{\infty} \epsilon_{\left\{\frac{k}{n},\, a_n^{-1} X_k\right\}} \implies \sum_{r=1}^{M} \sum_{i=0}^{M-1} \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \epsilon_{\left\{T^{(i)}_{k,r},\, J^{(i)}_{k,r} U_{k,1,r}\cdots U_{k,M(j-1)+i,r}\right\}}\ . \qquad \square
$$

The distribution of $M_n = \max_{1\le k\le n}(X_k)$ and its corresponding extremal index can now be obtained.

Corollary 3.1. *Under the conditions of the above theorem,*

1. *as* $n \to \infty$

$$P\big(M_n \le a_n x\big) \;\to\; \exp\left\{-\frac{1}{M}\, E W^\alpha x^{-\alpha}\right\} , \tag{3.4}$$

with

$$W \;=\; \bigvee_{r=1}^{M} \gamma_{r,1} \bigvee_{i=0}^{M-1} \left(\gamma_{r+M-i-1,r} \bigvee_{j=1}^{\infty} \big\{U_{1,1,r} \cdots U_{1,M(j-1)+i,r}\big\}\right) ;$$

2. *the periodic sequence* $(X_k)_{k\in\mathbb{Z}}$ *has extremal index*

$$\theta \;=\; \frac{\left\{1 - \prod_{s=1}^{M} E A_{r-s}^{\alpha}\right\} E W^\alpha}{\sum_{r=1}^{M} \gamma_{r,1} \sum_{i=0}^{M-1} \left(\prod_{s=1}^{i} E A_{r-s}^{\alpha}\right) \gamma_{r+M-i-1,r}} \;.$$

Proof:

$$P\big(M_n \le a_n x\big) \;=\; P\left(\sum_{k=1}^{\infty} \epsilon_{\left\{\frac{k}{n},\, a_n^{-1} X_k\right\}}\big((0,1]\times(x,\infty]\big) = 0\right) \;\Longrightarrow$$

$$\Longrightarrow\; P\left(\sum_{r=1}^{M} \sum_{i=0}^{M-1} \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \epsilon_{\left\{T_{k,r}^{(i)},\, J_{k,r}^{(i)} U_{k,1,r} \cdots U_{k,M(j-1)+i,r}\right\}}\big((0,1]\times(x,\infty]\big) = 0\right) .$$

The event

$$\left\{\sum_{r=1}^{M} \sum_{i=0}^{M-1} \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \epsilon_{\left\{J_{k,r}^{(i)} U_{k,1,r} \cdots U_{k,M(j-1)+i,r}\right\}}(x,\infty] \;=\; 0\right\} ,$$

is equivalent to the event that none of the points

$$\left\{J_{k,r}^{(i)} U_{k,1,r} \cdots U_{k,M(j-1)+i,r}\,,\;\; r=1,...,M,\;\; i=0,...,M-1,\;\; k,j\in\mathbb{N}\right\} ,$$

exceeds x. The latter can be expressed as the set

$$\bigcap_{r=1}^{M} \bigcap_{i=0}^{M-1} \bigcap_{k=1}^{\infty} \left\{J_{k,r}^{(i)} V_{k,r}^{(i)} \le x\right\} , \tag{3.5}$$

where

$$V_{k,r}^{(i)} \;=\; \bigvee_{j=1}^{\infty} \big\{U_{k,1,r} \cdots U_{k,M(j-1)+i,r}\big\} .$$

For a fixed value of $r = 1...,M$ and $i = 0,...,M-1$ it follows that $\{J_{k,r}^{(i)} V_{k,r}^{(i)}\}_{k\in\mathbb{N}}$ are the points of a PRM on $(0,\infty]$ with mean measure

$$(1/M)\; \gamma_{r,1}\; \gamma_{r+M-i-1,r}\; E\big(V_{1,r}^{(i)}\big)^\alpha\, x^{-\alpha} ,$$

(cf. Resnick, [32]). Since the set in (3.5) can be expressed as

$$\left\{J_{k,r}^{(i)} W \leq x,\;\; r=1,...,M,\; i=0,...,M-1,\; k \in \mathbb{N}\right\} ,$$

with

$$W = \bigvee_{r=1}^{M} \bigvee_{i=0}^{M-1} V_{1,r}^{(i)} ,$$

the set $\left\{J_{k,r}^{(i)} W,\;\; r=1,...,M,\; i=0,...,M-1,\; k \in \mathbb{N}\right\}$ contains the points of a PRM on $(0,\infty)$ with mean measure $EW^{\alpha} x^{-\alpha}$ and the result follows.

Finally, we concentrate on the examination of the extremal index. Define $(\hat{X}_k)_{k\in\mathbb{Z}}$ as the *associated independent* M-periodic sequence of $(X_k)_{k\in\mathbb{Z}}$, i.e., $\hat{X}_1, \hat{X}_2 ...$, are independent random variables being the tail distribution of $\hat{X}_r$ as in (2.13), for $r=1,...,M$. Further we define $M_n^{\hat{X}} = \max_k(\hat{X}_k)$. From Theorem 2.1 and classical extreme value theory we obtain that

$$\begin{aligned} & P\big(M_n^{\hat{X}} \leq a_n x\big) \;\to \\ & \to\; \exp\left\{-\frac{1}{M}\sum_{r=1}^{M} \tau_r \left(\frac{1}{1-\prod_{s=1}^{M} EA_{r-s}^{\alpha}} \sum_{i=0}^{M-1}\left(\prod_{s=1}^{i} EA_{r-s}^{\alpha}\right) \gamma_{r+M-i-1,r}\right)\right\} . \end{aligned} \tag{3.6}$$

By comparing (3.4) with (3.6) the expression of the extremal index is obtained; see Leadbetter *et al.* [26]. □

4. EXAMPLES

Consider that X_k is given in the form

$$X_k = \sum_{j=1}^{\infty} \left(\prod_{s=1}^{j-1} b\, Z_{k-s}\right) b\, Z_{k-j}^2 , \qquad k \in \mathbb{Z} ,$$

with $b > 0$ a positive constant. Note that the process $(Y_k)_{k\in\mathbb{Z}}$ defined as

$$Y_k = X_k + Z_k ,$$

satisfies the bilinear recursion

$$Y_k = b\, Y_{k-1} Z_{k-1} + Z_k , \qquad k \in \mathbb{Z} .$$

The reason in considering the tail behavior of X_k rather that Y_k itself is due to the fact that the contribution of the term Z_k on the extremal behavior of Y_k is negligible.

For deriving probabilistic and extremal properties of this process we will make extensive use of the fact that X_k can be embedded in the form (2.4) if $(A_k, B_k) = (bZ_k, bZ_k^2)$. We further assume that

$$\bar{F}_r(x) = P\big(Z_r^2 > x\big) = x^{-\alpha/2} L_r(x) , \qquad r = 1, ..., M ,$$

and that

$$b^{\alpha/2} E Z_r^{\alpha/2} < 1 , \qquad r = 1, ..., M .$$

It follows by Lemma 2.2 and the fact that

$$P\big(B_r > x\big) = P\big(b Z_r^2 > x\big) = b^{\alpha/2} x^{-\alpha/2} L_r(x) ,$$

for $r = 1, ..., M$ and $i = 0, ..., M-1$

$$\lim_{x\to\infty} \frac{P\big(X_r^{(i)} > x\big)}{P\big(Z_r^2 > x\big)} = \frac{b^{(i+1)\alpha/2} \prod_{s=1}^{i} E Z_{r-s}^{\alpha/2}}{1 - b^{M\alpha/2} \prod_{s=1}^{M} E Z_{r-s}^{\alpha/2}} \gamma_{r+M-i-1,r} .$$

Furthermore, by Theorem 2.1, for $r = 1, ..., M$

$$\lim_{x\to\infty} \frac{P\big(X_r > x\big)}{P\big(Z_r^2 > x\big)} = \frac{b^{\alpha/2}}{1 - b^{M\alpha/2} \prod_{s=1}^{M} E Z_{r-s}^{\alpha/2}} \sum_{i=0}^{M-1} \left(\prod_{s=1}^{i} E Z_{r-s}^{\alpha/2} \right) b^{\alpha i/2} \gamma_{r+M-i-1,r} .$$

The expression of the extremal index can be calculated from Corollary 3.1, providing

$$\theta = \frac{\Big(1 - b^{M\alpha/2} \prod_{s=1}^{M} E Z_{r-s}^{\alpha/2}\Big) E W^{\alpha/2}}{b^{\alpha/2} \sum_{r=1}^{M} \gamma_{r,1} \sum_{i=0}^{M-1} \Big(\prod_{s=1}^{i} E Z_{r-s}^{\alpha/2}\Big) \gamma_{r+M-i-1,r}\, b^{\alpha i/2}} .$$

Extensions for bivariate bilinear models can be easily obtained from the previous results; see Kumar [25] for details.

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