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Control Monitoring Schemes for Percentiles of Generalized Exponential Distribution with Hybrid Censoring

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Abstract:

- In this article, a parametric bootstrap "control monitoring scheme" equivalently known as "control chart", is proposed for process monitoring of percentiles of the generalized exponential distribution for type-I hybrid censored data assuming in-control parameters to be unknown. Monte Carlo simulations are carried out to evaluate the in-control and out-of-control performance of the proposed scheme in terms of average run lengths. Conventional Shewhart-type scheme is also proposed under the same set-up asymptotically and compared with bootstrap scheme using a skewed data set. Finally, an application of the proposed scheme is shown from clinical practice.

Keywords:

- *Average run length, Control monitoring scheme, False alarm rate, Generalized exponential distribution, Hybrid censoring, Parametric bootstrap.*

AMS Subject Classification:

- 62P30 (Primary), 62F40 (Secondary), 62N01 (Secondary)

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1. INTRODUCTION

Researchers have often shown interest in developing control charts, also known as control monitoring scheme, for monitoring percentiles of an underlying distribution in reliability studies. Padgett and Spurrier [25] and Nichols and Padgett [24] argued in favor of monitoring lower percentile of strength distribution over average to confirm the quality of carbon fiber strength to be in control. Chowdhury *et al.* [4] emphasized on monitoring the upper (lower) percentile of the proportion of non-conforming (conforming) units, as an upward (downward) shift in the upper (lower) percentile indicates a deterioration in quality. All the available control schemes for monitoring distribution quantiles (see for example, Padgett and Spurrier [25], Nichols and Padgett [24], Erto and Pallotta [7], Lio and Park [18, 19], Lio *et al.*[20], Erto *et al.* [8], Chiang *et al.* [2, 3], Leiva *et al.*[17] and Ma *et al.*[23]) used parametric bootstrap method with different distributional assumptions under classical or/and Bayesian set-up. Additionally, Chiang *et al.* [3] used model selection approaches to choose between competing underlying distributions. Due to the non-availability of closed form expressions of the sampling distribution of the percentiles, computational methods such as parametric bootstrap is used to obtain the control limits. For more discussion on the bootstrap technique and its advantages, one can refer to Efron and Tibshirani [5], Liu and Tang [21], Jones and Woodall [15] and Seppala *et al.* [26].
The results obtained from the aforementioned papers are useful and valuable, and can be applied to complete data setting only. In practice, reliability data are skewed and censored. Recently Vining *et al.* [27] emphasized on using censored data in reliability studies as customers expected products and processes to perform with high quality over the entire expected lifetime of the product/process. Most of the available schemes for censored data monitor mean of a process. Few papers are found in the literature for monitoring percentiles of a process using censored data. Haghighi *et al.* [12] proposed control charts for the quantiles of the Weibull distribution for type-II censored data, based on the distribution of a pivotal quantity conditioned on ancillary statistics. Wang *et al.* [28] proposed EWMA and CUSUM charts for monitoring the lower Weibull percentiles under complete data and type-II censoring using the same approach as used in Haghighi *et al.* [12]. Encouraged by these findings, in this paper, a control monitoring scheme is proposed based on bootstrap method using hybrid censoring which generalizes control monitoring schemes under type-I and type-II censoring. In type-I censoring scheme, the experiment is aborted after a pre-decided time $T = x_0$; whereas in type-II censoring, the termination is subject to failure of a pre-fixed number of items r. The hybrid censoring scheme which is popularly known in the literature as type-I hybrid censoring scheme was initially introduced by Epstein [6] and can be considered as a mixture of type-I and type-II censoring schemes. It can be described briefly as follows: Suppose n identical units are put on an experiment. Now if $X_{1:n}, ..., X_{n:n}$ are the ordered lifetimes of the units, then the experiment is aborted either when a pre-chosen number $r(< n)$ out of n items has failed or when a pre-determined time x_0 has elapsed. Hence the life test

can be terminated at a random time $X^* = min\{X_{r:n}, T\}$. One of the following two types of observations can be witnessed under type-I hybrid censoring scheme.
Case I: $\{X_{1:n} < ... < X_{r:n}\}$ if $X_{r:n} < x_0$.
Case II: $\{X_{1:n} < ... < X_{d:n} < T\}$ if $d + 1 \leq r < n$ and $x_0 \leq X_{r:n}$.

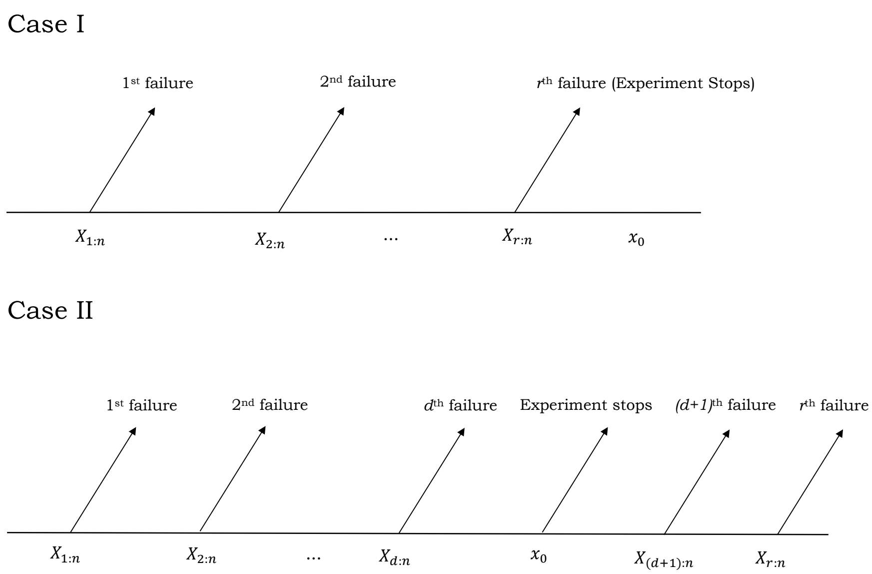

Figure 1: Schematic illustration of type-I hybrid censoring scheme

In reliability studies, two-parameter Weibull is the most popular distribution to the practitioners. Gupta and Kundu [10] proposed the two parameter generalized exponential (GE) distribution as an alternative to the Weibull and studied its properties extensively. The scale and shape parameters of the GE distribution bring quite a bit of flexibility in the distribution to analyze any positive real data. Both the Weibull and GE distributions have increasing or decreasing failure rates depending on the shape parameter. Many authors pointed out (see for example, Bain [1]) that since the hazard function of a GE distribution is bounded above or bounded below as opposed to Webull which is unbounded, the GE may be more appropriate as a population model when the items in the population are in a regular maintenance environment. The hazard rate may increase initially, but after some times the system reaches a stable condition because of maintenance. Therefore, if it is known that the data are from a regular maintenance environment, it may make more sense to fit the GE distribution

over the Weibull. As opposed to Weibull distribution, GE represents a parallel system of independent and identically distributed exponential components. GE has likelihood ratio ordering on the shape parameter indicating the possibility of constructing a uniformly most powerful test for testing a one-sided hypothesis on the shape parameter keeping the scale parameter known. The Weibull distribution doesn't enjoy any such ordering properties and hence no such uniformly most powerful test exists for Weibull. One of the disadvantages of Weibull can be pointed out that the asymptotic convergence to normality for the distribution of the maximum likelihood estimators is very slow (Bain [1]). Therefore most of the asymptotic inferences may not be very accurate unless the sample size is very large. For a detailed comparison between Weibull and GE, one can refer to Gupta and Kundu [11]. Motivated by these findings, GE is chosen as the underlying distribution to develop a bootstrap control monitoring scheme for hybrid censored data.
The rest of the paper is organized as follows. Section 2 provides the statistical background of the paper. The proposed bootstrap and Shewhart-type control monitoring schemes for GE percentiles with hybrid censored data are introduced in Section 3. Section 4 is devoted to the practical implementation of the schemes including tabulation of the control limits and average run length (ARL). Simulation results of both in-control (IC) and out-of-control (OOC) performance of the bootstrap scheme are presented in Section 4. The effectiveness of the proposed scheme is evaluated in Section 5 using a skewed data set. Bootstrap control monitoring scheme for type-I and type-II censored data are also obtained in Section 5 as a special case and are compared with bootstrap chart under hybrid censoring scheme. An application of the proposed scheme is shown from clinical practice in Section 6. Section 7 concludes the paper.

2. STATISTICAL FRAMEWORK

2.1. MAXIMUM LIKELIHOOD ESTIMATORS

Let X be a random variable following two parameter GE distribution with the shape parameter $\theta > 0$ and scale parameter $\lambda > 0$. Then probability density function (pdf) and cumulative distribution function (cdf) of X are given by

$$f(x|\theta,\lambda) = \theta\lambda e^{-\lambda x}(1-e^{-\lambda x})^{\theta-1}, \tag{2.1}$$

and

$$F(x|\theta,\lambda) = \left(1-e^{-\lambda x}\right)^{\theta}. \tag{2.2}$$

Let ξ_p be the $100p^{th}$ percentile of the GE distribution and is obtained as

$$\xi_p = -\frac{1}{\lambda}\ln\left(1-p^{\frac{1}{\theta}}\right). \tag{2.3}$$

Now, let $x_{i_1}, x_{i_2}, ..., x_{i_n}$ be i^{th} in-control (IC) random subgroup of size n ($i = 1, 2, ..., k$) drawn from phase I process following GE distribution as in (2.1). On the basis of the observed data and ignoring the additive constant, the log-likelihood function under hybrid censoring (for Case I and II as introduced in Section 1) is given by

$$\begin{aligned} L(\theta, \lambda|data) &= d\ln\theta + d\ln\lambda - \lambda\sum_{i=1}^{d} x_{i:n} + (\theta-1)\sum_{i=1}^{d}\ln(1-e^{-\lambda x_{i:n}}) \\ &+ (n-d)\ln\left(1-(1-e^{-\lambda c})^{\theta}\right). \end{aligned} \tag{2.4}$$

Note that for Case I, $d = r$ and $c = x_{r:n}$, and for Case II, $0 \le d \le r-1$ and $c = x_0$. Also it can be shown that for $\lambda \to 0$, and for any fixed θ, maximum likelihood estimators (MLE) of θ and λ do not exist when $d = 0$. Assuming d to be positive, the MLEs $\hat{\theta}$ and $\hat{\lambda}$ are obtained by maximizing the log-likelihood function (2.4), and subsequently solving the non-linear equations

$$\frac{\partial L}{\partial \theta} = 0,\ \frac{\partial L}{\partial \lambda} = 0.$$

As closed-form solutions of these two equations are not available, EM algorithm is used to obtain the MLEs. Let the observed data and the censored data be denoted by $\mathbf{X} = (x_{1:n}, \ldots, x_{d:n})$ and $\mathbf{Y} = (y_1, \ldots, y_{n-d})$ respectively. Here for given d, $\mathbf{Y}$ is not observable and hence can be thought of as missing data. The combination of $\mathbf{Z} = (\mathbf{X}, \mathbf{Y})$ forms the complete data set. Ignoring the additive constant, the log-likelihood function of the uncensored data set, denoted by $L_c(\theta, \lambda|Z)$ is give by

$$\begin{aligned} L_c(\theta, \lambda|data) &= n\ln\theta + n\ln\lambda - \lambda\left(\sum_{i=1}^{d} x_{i:n} + \sum_{i=1}^{n-d} y_i\right) \\ &+ (\theta-1)\left(\sum_{i=1}^{d}\ln\left(1-e^{-\lambda x_{i:n}}\right) + \sum_{i=1}^{n-d}\ln\left(1-e^{-\lambda y_i}\right)\right). \end{aligned} \tag{2.5}$$

Now for 'E'-step of the EM algorithm, one needs to compute the pseudo log-likelihood function as $L_s(\theta, \lambda|data) = E(L_c(Z; \theta, \lambda)|X)$, obtained as

$$\begin{aligned} L_s(\theta, \lambda|data) &= n\ln\theta + n\ln\lambda - \lambda\sum_{i=1}^{d} x_{i:n} + (\theta-1)\sum_{i=1}^{d}\ln\left(1-e^{-\lambda x_{i:n}}\right) \\ &- \lambda\sum_{i=1}^{n-d} E\left[Y_i|Y_i > c\right] + (\theta-1)\sum_{i=1}^{n-d} E\left[\ln(1-e^{-\lambda Y_i})|Y_i > c\right]. \end{aligned} \tag{2.6}$$

Now the 'M'-step involves maximization of the pseudo log-likelihood function given in (2.6). Therefore, if at the k-th stage the estimate of (θ, λ) is (θ^k, λ^k), then $(\theta^{(k+1)}, \lambda^{(k+1)})$ can be obtained by maximizing

$$\begin{aligned} g(\theta, \lambda) &= n\ln\theta + n\ln\lambda - \lambda\sum_{i=1}^{d} x_{i:n} + (\theta-1)\sum_{i=1}^{d}\ln\left(1-e^{-\lambda x_{i:n}}\right) \\ &- \lambda(n-d)A\left(c, \theta^{(k)}, \lambda^{(k)}\right) + (\theta-1)(n-d)B\left(c, \theta^{(k)}, \lambda^{(k)}\right), \end{aligned} \tag{2.7}$$

where

$$A(c,\theta,\lambda) = -\frac{\theta}{\lambda\left(1-F(c,\theta,\lambda)\right)}u(\lambda c,\theta),$$
$$B(c,\theta,\lambda) = \frac{1}{\theta\left(1-F(c,\theta,\lambda)\right)}\left[\left(1-e^{-c\lambda}\right)^{\theta}\left(1-\theta\ln\left(1-e^{-c\lambda}\right)\right)-1\right].$$

The maximization of (2.7) can be performed by using similar technique as of Gupta and Kundu [11]. First, $\lambda^{(k+1)}$ can be obtained by solving a fixed point type equation $h(\lambda)=\lambda$, where the function $h(\lambda)$ is defined as

$$h(\lambda) = \left[\frac{1}{n}\sum_{i=1}^{d} x_{i:n} + \frac{n-d}{n}A - \frac{1}{n}\left(\hat{\theta}(\lambda)-1\right)\sum_{i=1}^{d}\frac{x_{i:n}e^{-\lambda x_{i:n}}}{1-e^{-\lambda x_{i:n}}}\right]^{-1},$$

with $A = A\left(c,\theta^{(k)},\lambda^{(k)}\right)$, $B = B\left(c,\theta^{(k)},\lambda^{(k)}\right)$ and $\hat{\theta}(\lambda) = -\frac{n}{\sum_{i=1}^{d}\ln\left(1-e^{-\lambda x_{i:n}}\right)+(n-d)B}$. Once $\lambda^{(k+1)}$ is obtained, $\theta^{(k+1)}$ is obtained by solving the equation $\theta^{(k+1)} = \hat{\theta}\left(\lambda^{(k+1)}\right)$. For more detail on the estimation of GE parameters under hybrid censoring (see Kundu and Pradhan [16]).
The MLE of the $100p^{th}$ percentiles, denoted by $\hat{\xi_p}$, is also obtained as

$$\hat{\xi_p} = -\frac{1}{\hat{\lambda}}\ln\left(1-p^{\frac{1}{\hat{\theta}}}\right). \qquad (2.8)$$

2.2. ASYMPTOTIC PROPERTIES

An outline of the Fisher information matrix and asymptotic properties of the estimators are discussed here. For more detail, one may refer to Gupta and Kundu [11] and Kundu and Pradhan [16]. Using the missing value principle of Louise [22], it can be written that

(2.9) Observed information = Complete information - Missing information,

and can be expressed as

$$I_X(\Theta) = I_W(\Theta) - I_{W|X}(\Theta), \qquad (2.10)$$

with $\Theta = (\theta,\lambda)$; X = the observed vector; W = the complete data; $I_W(\Theta)$ = the complete information; $I_{W|X}(\Theta)$ = the missing information. The complete information $I_W(\Theta)$ is given by

$$I_W(\Theta) = -E\left[\frac{\partial^2 L_c(W;\Theta)}{\partial\Theta^2}\right],$$

with the Fisher information matrix of the censored observations being written as

$$I_{W|X}(\Theta) = -(n-d)E_{Z|X}\left[\frac{\partial^2 \ln f_Z(z|X,\Theta)}{\partial\Theta^2}\right].$$

The asymptotic variance covariance matrix of $\hat{\Theta}$, the MLE of Θ, can be obtained by inverting $I_X\left(\hat{\Theta}\right)$. The elements of the matrix $I_X(\Theta)$ for the complete data set can be obtained in Kundu and Pradhan [16].
Let $\hat{\xi}_p\left(\hat{\Theta}_n\right)$ be the value of ξ_p at $\Theta = \hat{\Theta}_n$, obtained from (2.3) and calculated on the basis of n observations. Then as in Chiang *et al.* [2], it can be shown that $\hat{\xi}_p\left(\hat{\Theta}_n\right)$ follows asymptotic normal distribution with mean $\xi_p(\Theta)$ and variance $\frac{1}{n}\nabla\xi_p^T(\mathbf{\Theta})\,\mathbf{I_Y}^{-1}(\mathbf{\Theta})\,\nabla\xi_p(\mathbf{\Theta})$, where $\nabla\xi_p(\Theta)$ is the gradient of $\xi_p(\Theta)$ with respect to Θ. In practice, $\mathbf{I_Y}(\mathbf{\Theta})$ is replaced by the observed Fisher Information matrix $\hat{\mathbf{I}}_Y\left(\hat{\mathbf{\Theta}}_n\right)$, obtained by substituting the unknown parameters θ and λ by their respective MLEs.

3. CONSTRUCTION OF PROPOSED CONTROL MONITORING SCHEMES

3.1. CHARTING PROCEDURE FOR THE BOOTSTRAP HYBRID-CENSORED CONTROL (BHCC) MONITORING SCHEME

Here, the bootstrap hybrid-censored control (BHCC) monitoring scheme for GE percentiles is developed using the following charting procedure.

Step-1: Collect and establish k reference samples $X_m = (x_{i1}, x_{i2}, \ldots, x_{im})$ of size m each from an IC process (Phase I process) following GE cdf $F(x|\theta, \lambda)$ as in (2.2).

Step-2: Obtain the MLEs of θ and λ from Step-1 under hybrid censoring following the procedure described in Section 2 and estimate the cdf as $F(x|\hat{\theta}, \hat{\lambda})$.

Step-3: Generate a bootstrap sample of size m, $x_1^*, x_2^*, \ldots, x_m^*$, from $F(x|\hat{\theta}, \hat{\lambda})$ as obtained in Step-2.

Step-4: Obtain the MLEs of θ and λ under hybrid censoring using the bootstrap sample obtained in Step-3, and denote these as θ^* and λ^*.

Step-5: Using (2.3) and (2.8), compute the bootstrap estimate of the $100p^{th}$ percentile as

$$\hat{\xi}_p^* = -\frac{1}{\lambda^*}\ln\left(1 - p^{\frac{1}{\theta^*}}\right). \tag{3.1}$$

Step-6: Repeat Steps 3-5 large number of times (B) to obtain bootstrap estimates of $\hat{\xi}_p^*$, denoted by $\hat{\xi}_{1p}^*, \hat{\xi}_{2p}^*, \ldots, \hat{\xi}_{Bp}^*$.

Step-7: Using B bootstrap estimates as obtained in Step 6, find the $\frac{\nu}{2}^{th}$ and $(1-\frac{\nu}{2})^{th}$ empirical percentiles as the lower control limit (LCL) and upper control limit (UCL) respectively to construct a two-sided BHCC chart, where ν is the false alarm rate (FAR) defined as the probability that an observation is considered as out of control (OOC) when the process is actually IC. Here, empirical sample percentiles are obtained following a method proposed by Hyndman and Fan [14].

Step-8: Sequentially observe the j^{th} phase II (test) sample $Y_{j:m} = (Y_{j1}, Y_{j2}, \ldots, Y_{jm})$ of size $m, j = 1, 2, \ldots$.

Step-9: Sequentially obtain $\hat{\xi}_{jp}$ using (3.1) after obtaining MLEs of the parameters under hybrid censoring scheme using the j^{th} test sample as described in Step-5.

Step-10: Plot $\hat{\xi}_{jp}$ against LCL and UCL as obtained in Step-7 of the Phase I process.

Step-11: If $\hat{\xi}_{jp}$ falls in between the LCL and UCL, then the process is assumed to be in-control, otherwise, an OOC signal is activated.

3.2. CHARTING PROCEDURE FOR THE SHEWART-TYPE HYBRID-CENSORED CONTROL (SHCC) MONITORING SCHEME

Shewhart-type control monitoring scheme for the percentiles of GE distribution, named as SHCC scheme is derived in this section following the asymptotic properties of the MLEs obtained in Section 2.2. The steps for designing the SHCC scheme for $100p^{th}$ percentile of proportion, $\xi_p(\Theta)$, are described as follows:
In phase I, samples are drawn from in-control process following GE distribution in k independent random subgroups of size m each with $n = m \times k$ being the total sample size.

Step-1: As described in Section 2.1, the MLEs $\hat{\Theta}_n = \left(\hat{\theta}_n, \hat{\lambda}_n\right)$ are computed on the basis of n in-control sample values of Phase I. Then the asymptotic standard error of $\hat{\xi}_{p,m}\left(\hat{\Theta}_m\right)$ is computed as

$$SE_{\xi_{p,m}} = \sqrt{\frac{1}{m}\nabla\xi_p^T\left(\hat{\mathbf{\Theta}}_n\right)\mathbf{I}_Y^{-1}\left(\hat{\mathbf{\Theta}}_n\right)\nabla\xi_p\left(\hat{\mathbf{\Theta}}_n\right)}, \tag{3.2}$$

where $\nabla\xi_p\left(\hat{\mathbf{\Theta}}_n\right)$ is the gradient of $\xi_p(\mathbf{\Theta})$ at $\mathbf{\Theta} = \hat{\mathbf{\Theta}}_n$. $\mathbf{I}_Y^{-1}\left(\hat{\mathbf{\Theta}}_n\right)$ is calculated following the procedure as described in Section 2.2.

Step-2: The MLEs $\hat{\mathbf{\Theta}}_m^j$ of Θ and $\xi_p^j\left(\hat{\mathbf{\Theta}}_m^j\right)$ are calculated based on j^{th} $(j = 1, 2, \ldots, k)$ IC samples of size m each. The sample mean of $\xi_p^j\left(\hat{\mathbf{\Theta}}_m^j\right)$s is calculated as

$$\bar{\xi}_p\left(\hat{\mathbf{\Theta}}_m\right) = \frac{1}{k}\sum_{i=1}^{k}\xi_p^j\left(\hat{\mathbf{\Theta}}_m^j\right).$$

Step-3: The Shewhart-type control monitoring scheme has the center line $CL_{SH} = \bar{\xi}_p\left(\hat{\boldsymbol{\Theta}}_m\right)$. If ν is the false alarm rate (FAR), then for $0 < \nu < 1$, the upper and lower control limits of the SHCC scheme are found to be

$$UCL_{SH} = \bar{\xi}_p\left(\hat{\boldsymbol{\Theta}}_m\right) + z_{(1-\nu/2)} SE_{\xi_{p,m}},$$

and

$$LCL_{SH} = \bar{\xi}_p\left(\hat{\boldsymbol{\Theta}}_m\right) - z_{(1-\nu/2)} SE_{\xi_{p,m}},$$

respectively, where $z_{(1-\nu/2)}$ is the $(1-\nu/2)^{th}$ quantile of standard normal distribution.

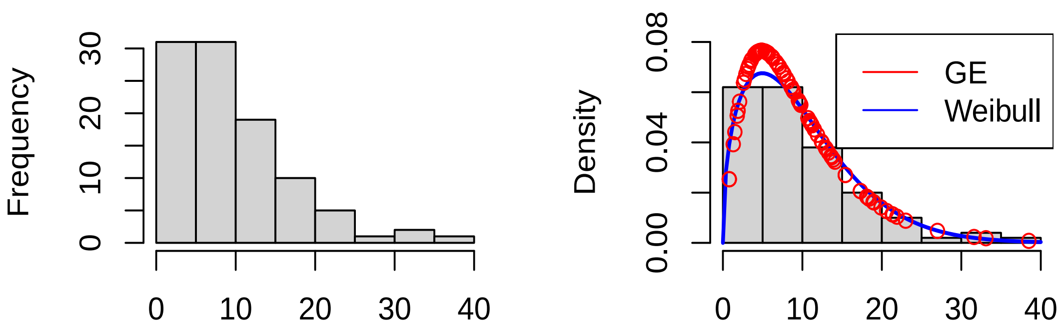

Figure 2: Histogram and density plot of waiting times of 100 patients

4. SIMULATION STUDY

In this section, the IC and OOC performances of the proposed BHCC monitoring scheme are evaluated through a comprehensive simulation study. Numerical computations in R (version 4.0.2) based on Monte-Carlo simulations are used to determine the average UCL and LCL. The MLEs of the parameters θ and λ are obtained for the pair $(\theta = 5.5, \lambda = 0.05)$. The control limits are obtained based on $B = 5,000$ bootstrap samples. The simulations are carried out with different bootstrap sample sizes m with $k = 20$ subgroups, different percentiles $(p = 0.1, 0.5, 0.9)$, different levels of FAR $(\nu = 0.1, 0.005, 0.0027, 0.002, 0.001)$ and the following censoring schemes: Scheme $1:\ m = 25,\ r = 15,\ x_0 = 55$; Scheme $2:\ m = 25,\ r = 20,\ x_0 = 55$; Scheme $3:\ m = 40,\ r = 30,\ x_0 = 55$; Scheme $4:\ m = 40,\ r = 35,\ x_0 = 55$; Scheme $5:\ m = 25,\ r = 15,\ x_0 = 70$; Scheme $6:\ m = 25,\ r = 20,\ x_0 = 70$; Scheme $7:\ m = 40,\ r = 30,\ x_0 = 70$; and Scheme

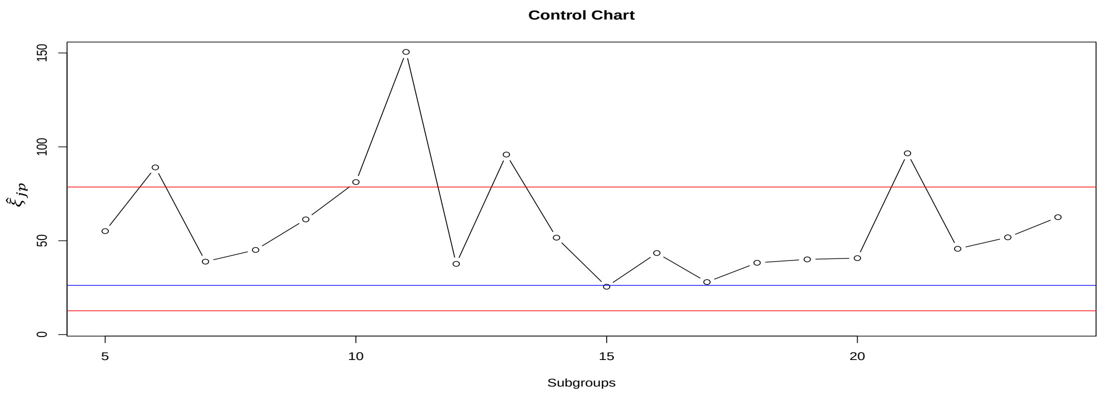

Figure 3: BHCC monitoring scheme for 90^{th} percentile of the waiting time data with $\Delta\theta = 0.15$, $\Delta\lambda = -0.15$, $UCL = 78.597$, $CL = 26.255$, $LCL = 12.702$

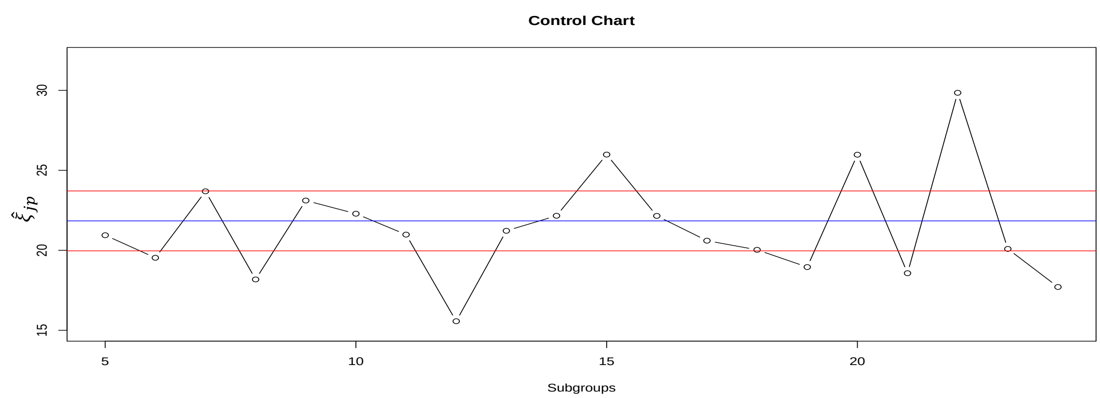

Figure 4: SHCC monitoring scheme for 90^{th} percentile of the waiting time data with $\Delta\theta = 0.15$, $\Delta\lambda = -0.15$, $UCL_{SH} = 23.711$, $CL_{SH} = 21.839$, $LCL_{SH} = 19.966$; Example 1

8 : $m = 40$, $r = 35$, $x_0 = 70$. The performance of the scheme is assessed by run length, defined as the number of cases required to observe the first OOC signal. For each simulation, the run length is obtained, followed by obtaining the average run length (ARL) and the standard deviation of run length (SDRL) by using $5,000$ simulation runs.

4.1. IC MONITORING SCHEME PERFORMANCE

The estimated IC control limits of the BHCC scheme are displayed in Table 1 along with the respective ARL and SDRL as the scheme performance measures, denoted by ARL_0 and $SDRL_0$ respectively. It is easy to show that

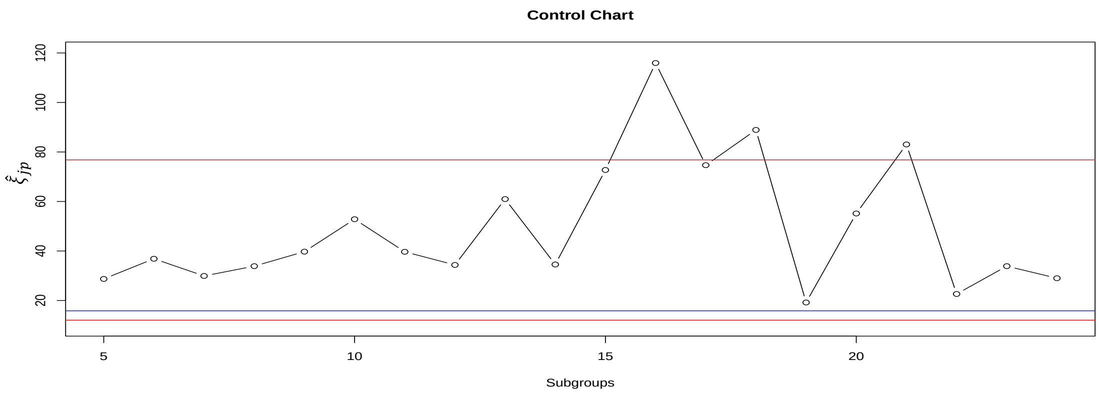

Figure 5: BT^{I}CC monitoring scheme for 90^{th} percentile of the waiting time data with $\Delta\theta = 0.15$, $\Delta\lambda = -0.15$, $UCL = 76.812$, $CL = 15.833$, $LCL = 12.066$

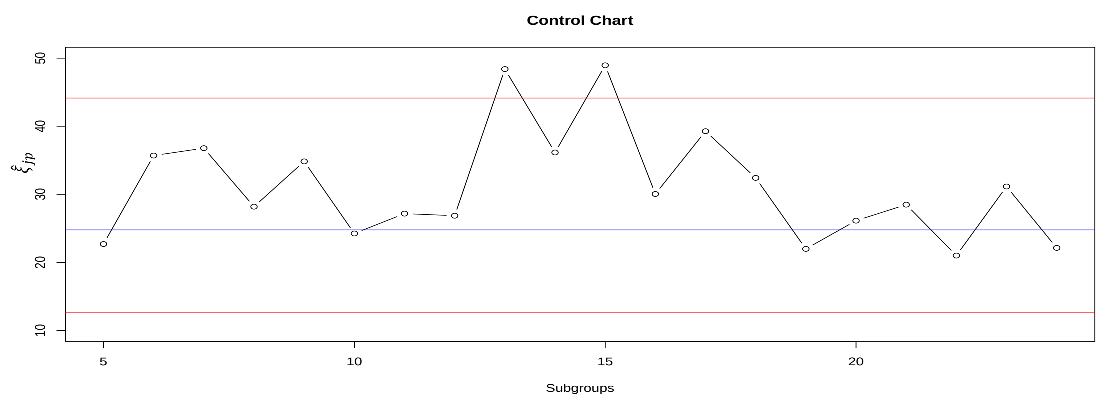

Figure 6: BT^{II}CC monitoring scheme for 90^{th} percentile of the waiting time data with $\Delta\theta = 0.15$, $\Delta\lambda = -0.15$, $UCL = 44.144$, $CL = 24.770$, $LCL = 12.599$

the reciprocal of FAR is same as the nominal (theoretical) ARL, viz. for $\nu = 0.1, 0.005, 0.0027, 0.002$ and 0.001, the nominal ARL should be equal to $10, 200, 370$, 500 and 1000 respectively. In general, the smaller ARLs indicate narrower control limits, while ARLs larger than 370 specifies wider limits that the bootstrap control schemes give fewer false signals. The simulated values of ARL_0 in Table 1 of the supplementary article are found to be closer to the theoretical results implying that the BHCC monitoring scheme for percentiles perform well with skewed data. As the bootstrap sample size (m) increases, the estimated control limits get closer together. Moreover, for fixed m, the control limits become farther apart as the percentile (p) increases. Also, $SDRL_0$ is found to be closer to the ARL_0, satisfying the theoretical result of the geometric distribution used as the run length model.

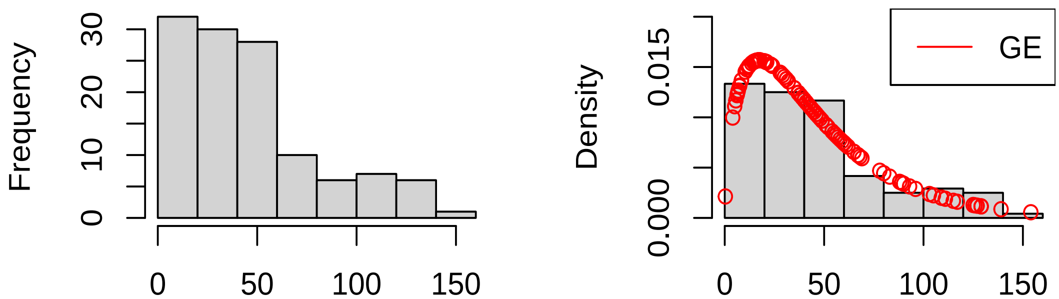

Figure 7: Histogram and density plot of survival times of 120 patients with breast cancer

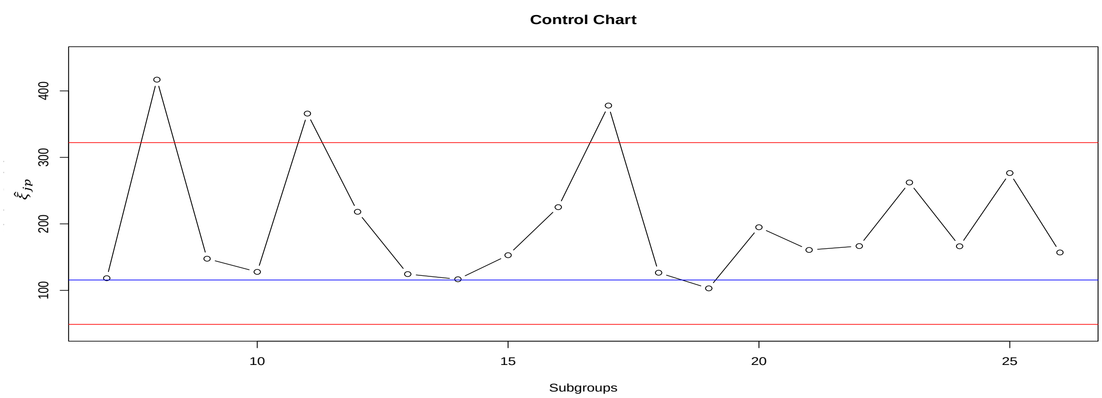

Figure 8: BHCC monitoring scheme for 90^{th} percentile of the survival time data with $\Delta\theta = 0.15$,$\Delta\lambda = -0.15$, $UCL = 322.248$, $CL = 115.577$, $LCL = 48.912$

4.2. OOC MONITORING SCHEME PERFORMANCE

The OOC performance of the BHCC monitoring scheme is investigated by measuring impact of changes in the IC parameter estimates on ARL. In other words, the phase II sample is considered taken from $GE(\theta + \Delta\theta, \lambda + \Delta\lambda)$, while the IC sample comes from $GE(\theta, \lambda)$. The effects of shifts ($\Delta\theta$ and/or $\Delta\lambda$) in the parameters of the GE distribution on ARL of the percentile scheme is examined and exhibited in Table 2 of the supplementary article. In general, the simulation results reveal that for fixed m, r, and x_0, the OOC ARL values (denoted by

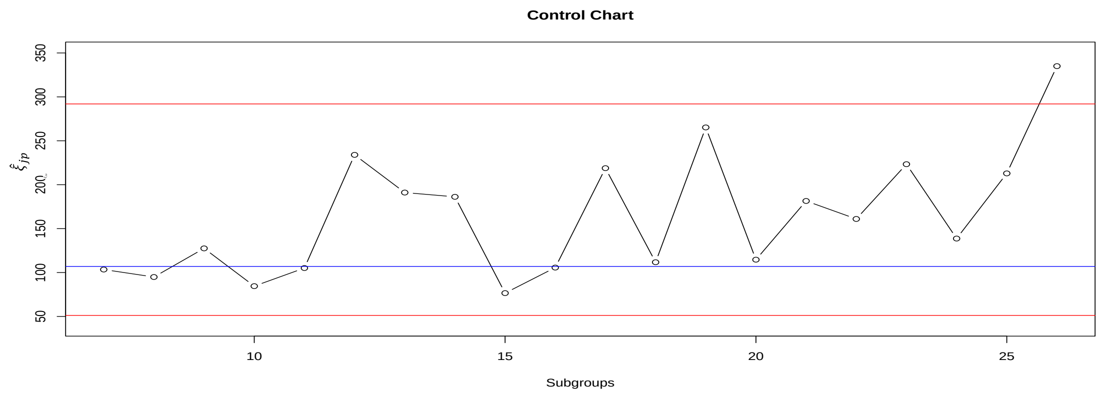

Figure 9: BT^ICC monitoring scheme for 90^{th} percentile of the survival time data with $\Delta\theta = 0.15$, $\Delta\lambda = -0.15$, $UCL = 291.965$, $CL = 106.950$, $LCL = 51.203$

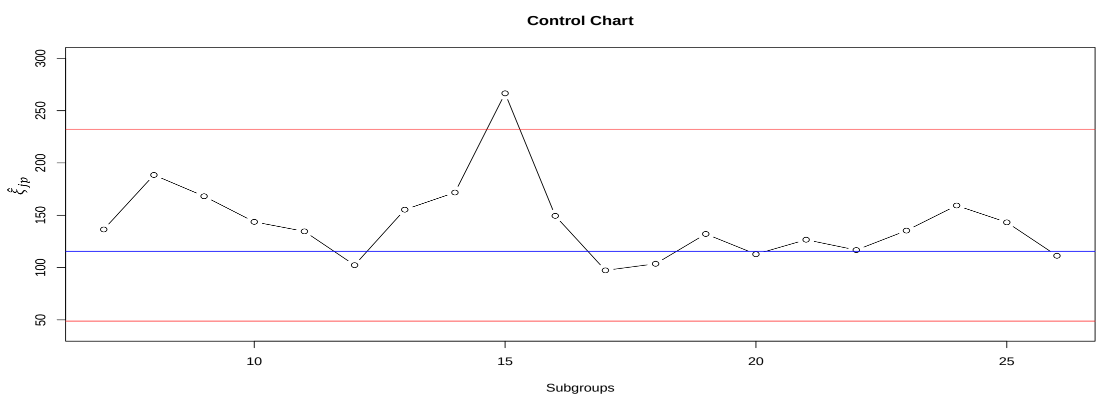

Figure 10: BT^{II}CC monitoring scheme for 90^{th} percentile of the survival time data with $\Delta\theta = 0.15$, $\Delta\lambda = -0.15$, $UCL = 232.296$, $CL = 115.565$, $LCL = 48.846$

ARL_1) for the percentiles decrease sharply with both downward and upward small, medium and large shifts in the parameters indicating the effectiveness and usefulness of the scheme. However, the speed of detection varies depending on the type of shifts, the parameters, and the percentile being considered. Except for minor sampling fluctuations, in general, the monitoring scheme detects OOC signal in percentiles faster for downward shifts than the upward shifts (refer Table 2 and Figure 11). In particular, when θ is IC, the ARLs around 50^{th} percentile are smaller than the other percentiles for both upward and downward shifts in λ as is evident from Table 2 and Figure 11(a). For example, for a 4% decrease (increase) in λ when θ is IC ($\Delta\theta = 0$), there is about 27.8% (21%) reduction in the ARL of the 50^{th} percentile. On the other hand, when λ is IC, the ARLs for the lower percentiles (around 10^{th} percentile) is found to be smaller than the other percentiles for downward (upward) shift in θ (refer Table 2 and Figure 11(b)). For example, there is about 44.8% (13.8%) reduction in the ARL of the 10^{th} percentile for a 6% decrease (increase) in θ when λ is IC. From Table 2 and Figure 11(c) it is also clear that for 10% deviation in θ the ARLs around 50^{th} percentile are smaller than the other percentiles for both upward and downward shifts in λ. Again, from Figure 11(d) it can also be observed that, for 10% deviation in λ the ARLs around 50^{th} percentile are smaller than the other percentiles for both upward and downward shifts in θ.

5. ILLUSTRATIVE EXAMPLE WITH COMPARISONS

In this section, the BHCC and SHCC monitoring schemes are illustrated by a numerical example which records the waiting times (in minutes) of 100 customers before getting their services (see Ghitany *et al.* [9]). The BHCC scheme is then compared with bootstrap scheme with Type I amd Type II censoring. Various summary measures of the data set can be found below:

Min	5%	10%	25%	50%	75%	90%	95%	Max
0.800	1.895	2.880	4.675	8.100	13.000	19.090	21.955	38.500

First, the Weibull and GE distributions are compared for fitting the data set. For Weibull model, the MLEs of the shape and scale parameters are found to be 1.458 and 10.954 respectively with Kolmogorov-Smirnov test (K-S) statistic value $D = 0.0577$, and p-value, $p = 0.8927$. For GE model, the MLEs are obtained as $\widehat{\theta} = 2.183$ and $\widehat{\lambda} = 0.159$ with $D = 0.0402$ and $p = 0.9970$. The histogram of the data and two fitted densities are provided in Figure 2. The fit results confirm that the GE distribution provides a better fit than Weibull in this case. Moreover, logarithm of the ratio of maximized likelihood (RML), defined as $T = \log L = l_{GE}(\hat{\theta}, \hat{\lambda}) - l_{WE}(sh\hat{a}pe, sc\hat{a}le) = -317.0884 - (-318.7261) = 1.6377 > 0$ indicates to choose GE distribution over Weibull.

In order to achieve service excellence, the bank may find extreme percentiles of the waiting times worth investigating over the average waiting time. An upward shift in the upper percentile of the waiting times indicates deterioration in the service quality and requires monitoring. In view of this objective, the BHCC chart is constructed for monitoring 90^{th} percentile of the waiting times. The complete data is censored either at the waiting time of the first 60% of the total number of customers ($r = 60$) or at the waiting time of 10 minutes ($x_0 = 10$), whichever occurs earlier. The censoring time $x_0 = 10$ is chosen near to 60^{th} percentile. The complete set of 100 observations is considered as four ($k = 4$) reference samples of size $m = 25$ each. The MLEs of θ and λ under the stated hybrid censoring scheme are obtained as $\widehat{\theta} = 1.760$ and $\widehat{\lambda} = 0.127$ respectively. Using these MLEs, $B = 5,000$ bootstrap samples of size $m = 25$ each are drawn with $r = 15$ (60% of the subgroup size) and $x_0 = 10$. Following the steps $4-7$ in subsection 3.1, and using

$\nu = 0.0027$ as FAR, the control limits of the BHCC scheme for the 90^{th} percentile are obtained as $UCL = 78.597$, $CL = 26.255$ and $LCL = 12.702$, while the same for the SHCC scheme are computed as $UCL_{SH} = 23.711$, $LCL_{SH} = 19.966$ and $CL_{SH} = 21.839$. It is observed that both schemes provide asymmetric control limits from the respective CL, while the SHCC scheme has narrower interval than the BHCC scheme. Twenty subgroups of size $m = 25$ each are generated from the OOC process under similar hybrid censoring plan having shape parameters $\theta = 2.024$ and $\lambda = 0.108$ (15% increase in θ and 15% decrease in λ).
The OOC performance of the BHCC and SHCC schemes for the 90^{th} percentile are presented in Figure 3 and Figure 4 respectively. The BHCC scheme is able to produce OOC signals quite efficiently with five 90^{th} percentile points falling above the UCL with the first OOC signal being obtained at test sample 2 indicating effectiveness of the scheme in terms of quick detection as well. On the other hand, nine OOC signals are produced by the SHCC scheme with test sample 2 producing the first OOC signal. It is to be noted here that the SHCC scheme grossly underestimates the IC ARL (computation of IC ARL for SHCC scheme is not shown for brevity) due to the narrow band of the control limits which may eventually produce false OOC signals.
Next, bootstrap monitoring scheme is used for type-I (denoted as BT^{I}CC) and type-II (denoted as BT^{II}CC) censored data coming from the GE distribution and their performance is compared with the BHCC monitoring scheme with the same data set and the procedure as used before. The control monitoring schemes for type-I and type-II censored data can be derived as a special case of hybrid censored data for $r = n$ and $T = x_{n:n}$ respectively. The MLEs of θ and λ under type-I censoring with $x_0 = 10$ are obtained as $\widehat{\theta} = 1.803$ and $\widehat{\lambda} = 0.131$ respectively, while the same under type-II censoring with $r = 60$ are found to be $\widehat{\theta} = 1.766$ and $\widehat{\lambda} = 0.127$ respectively. While the control limits of BT^{I}CC scheme for the 90^{th} percentile are obtained as $UCL = 76.812, CL = 15.833$ and $LCL = 12.066$, the same for the BT^{II}CC scheme are calculated as $UCL = 44.144, CL = 24.770$, and $LCL = 12.599$. Both the schemes provide asymmetric control limits with the BT^{II}CC scheme having narrower interval than the BT^{I}CC scheme. After the first four IC subgroups, twenty subgroups of size $m = 25$ each are generated from the OOC process having $\theta = 2.073$ and $\lambda = 0.111$ (15% increase in θ and 15% decrease in λ). Figure 5 and Figure 6 provide the OOC performance of the control monitoring schemes for the 90^{th} percentile. Figure 5 shows that the type-I censored scheme is able to generate three OOC points falling above the UCL with the first being produced at test sample 12. The type-II censored scheme as is shown in Figure 6 produces two OOC signals just above the UCL with test sample 9 providing the first signal. It is evident from the data analysis that the hybrid censored control monitoring scheme performs better than type-I and type-II censored control monitoring schemes in terms of both frequency and speed of detection of OOC signals.

6. APPLICATION

This section provides an application of the BHCC monitoring scheme in clinical practice. The scheme is used to monitor the top percentile of the survival times of 120 patients (see Hamedani [13]) with breast cancer obtained from a large hospital in a period from 1929 to 1938. The histogram of the data set as shown in Figure 7 and the summary measures below suggest the skewed nature of the data set.

Min	5%	10%	25%	50%	75%	90%	95%	Max
0.3	6.585	10.110	17.800	40.000	60.000	105.400	125.050	154.0

The MLEs of θ and λ for the complete data set coming from the GE distribution are found to be $\widehat{\theta} = 1.649$ and $\widehat{\lambda} = 0.029$ respectively. The fitted density is provided in Figure 7. The one sample K-S statistic and corresponding p-value are found to be 0.0717 and 0.5681 respectively. The fit results recommend GE distribution to model the survival time data and subsequent development of BHCC scheme. The complete sample data is split into six subgroups of size 20 each. Under hybrid censoring with $r = 72$ and $x_0 = 60$, the estimates of the GE parameters are obtained as $\widehat{\theta} = 1.415$ and $\widehat{\lambda} = 0.024$. Using $5,000$ bootstrap samples of size $m = 20$ each with $r = 12$ and $x_0 = 60$, the control limits of the BHCC monitoring scheme for the 90^{th} percentile are evaluated as $UCL = 322.248, CL = 115.577, LCL = 48.912$. Next, twenty phase II samples of size $m = 20$ each are generated from the process under similar hybrid censoring plan with $\Delta\theta = 0.15$ and $\Delta\lambda = -0.15$ to develop the BHCC monitoring scheme for the 90^{th} percentile as presented in Figure 8. The scheme has been able to detect OOC signals at 2^{nd}, 5^{th} and 11^{th} samples. For the same data set, the control limits for the BT^{I}CC monitoring scheme for 90^{th} percentile with $x_0 = 60$ are found to be $UCL = 291.965, CL = 106.950, LCL = 51.203$. Figure 9 shows that this scheme has been able to detect only one OOC signal at the 20^{th} sample. On the other hand, BT^{II}CC monitoring scheme for 90^{th} percentile with $r = 12$ is presented in Figure 10 with $UCL = 232.296, CL = 115.565, LCL = 48.846$. Figure 10 shows that this scheme also detects only one OOC signal at the 9^{th} sample. The frequency and speed of detection of OOC signals further justify the use of BHCC monitoring scheme over BT^{I}CC and BT^{II}CC monitoring schemes for the percentiles of survival time in a healthcare set-up.

7. CONCLUDING REMARKS

In this work, hybrid censoring is employed to develop control monitoring schemes for percentiles of GE distribution using bootstrap and asymptotic methods. Bootstrap monitoring schemes for type-I and type-II censored data are also developed under similar set-up as a special case of hybrid censoring plan. An extensive simulation study is conducted to evaluate the IC and OOC performance of the schemes. The hybrid censored schemes are found to be effective in the detection of OOC signals in terms of both magnitude and speed as demonstrated by a real data set. One application from healthcare is also provided to establish the effectiveness of the schemes. In this sense, the present work is the first attempt to apply a new censoring scheme in the process control and generalizes available control monitoring schemes for the GE data. As a scope for future research, hybrid censored schemes may be proposed under Bayesian set-up measuring uncertainty in the parameter(s). One can also think of using progressive censoring scheme to construct such control mechanism. For highly reliable products, accelerated life testing scheme may be employed under various censoring plans for the same purpose.

DATA AVAILABILITY STATEMENT

The data sets used in this manuscript are available in Ghitany *et al.* [9] and Hamedani [13].

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Table 1: Control limits, ARL_0 and $SDRL_0$ for $\theta = 5.5, \lambda = 0.05$

$m = 25,\ x_0 = 55,\ r = 15$						
ν	θ_1	θ_2	LCL	UCL	ARL_0	$SDRL_0$
$p = 0.1$						
0.1	2.812	0.030	11.830	23.037	8.865	9.269
0.005	2.876	00.314	8.968	27.711	199.281	202.163
0.0027	2.866	0.031	8.518	28.718	373.620	382.851
0.002	2.848	0.031	8.269	29.288	510.406	511.501
0.001	2.881	0.031	7.893	30.523	1012.984	1019.954
$p = 0.5$						
0.1	2.909	0.032	42.012	86.471	9.654	9.153
0.005	2.915	0.031	34.638	128.473	200.125	197.813
0.0027	2.869	0.031	33.919	144.963	371.171	373.865
0.002	2.932	0.032	33.336	147.457	516.495	518.838
0.001	2.895	0.0313	32.330	163.980	946.275	942.008
$p = 0.9$						
0.1	2.858	0.031	95.772	248.542	9.240	8.821
0.005	2.853	0.031	76.570	381.303	199.350	195.763
0.0027	2.837	0.031	74.073	421.948	366.788	369.880
0.002	2.882	0.032	73.003	439.777	548.506	539.569
0.001	2.877	0.031	70.660	491.325	935.637	917.314
$m = 25,\ x_0 = 55,\ r = 20$						
$p = 0.1$						
0.1	3.705	0.038	13.612	24.275	9.593	8.978
0.005	3.743	0.0387	10.616	28.571	197.046	202.377
0.0027	3.955	0.040	10.543	29.553	371.029	378.865
0.002	3.857	0.397	10.096	29.733	499.323	497.201
0.001	3.870	0.039	9.715	30.573	963.830	962.839
$p = 0.5$						
0.1	3.758	0.039	38.684	75.995	9.586	8.993
0.005	3.843	0.039	32.782	107.914	199.891	199.839
0.0027	3.834	0.034	31.638	112.757	372.903	371.694
0.002	3.671	0.038	31.317	120.484	498.172	501.858
0.001	3.917	0.039	30.986	130.329	1012.002	1025.377
$p = 0.9$						
0.1	3.835	0.039	73.389	187.241	8.782	9.306
0.005	4.068	0.041	59.618	291.500	187.599	189.073
0.0027	3.922	0.040	58.869	329.939	344.867	344.185
0.002	3.887	0.039	58.189	486.579	485.553	485.330
0.001	3.833	0.039	56.391	376.658	1067.507	1052.423

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Table 1 – *Continued from previous page*

ν	θ	λ	LCL	UCL	ARL_0	$SDRL_0$
$m = 40,\ x_0 = 55,\ r = 30$						
$p = 0.1$						
0.1	3.830	0.039	14.549	22.933	9.072	9.741
0.005	3.837	0.039	12.097	26.294	194.992	197.399
0.0027	3.894	0.039	11.780	26.922	371.586	375.765
0.002	3.786	0.039	11.434	27.066	501.07	514.918
0.001	4.095	0.041	11.528	27.985	983.974	972.453
$p = 0.5$						
0.1	2.503	0.0278	44.272	84.018	9.229	9.564
0.005	2.494	0.027	37.495	115.207	193.264	194.567
0.0027	2.512	0.027	36.734	123.208	362.996	370.67
0.002	2.503	0.027	36.005	124.74	503.296	488.532
0.001	2.531	0.028	35.167	132.979	988.48	971.968
$p = 0.9$						
0.1	3.900	0.039	79.890	170.495	8.999	9.708
0.005	3.997	0.041	67.896	232.891	217.252	218.698
0.0027	3.952	0.040	66.733	252.688	358.866	358.729
0.002	3.829	0.039	66.681	267.657	483.185	478.586
0.001	3.784	0.039	65.095	284.924	1055.368	1045.77
$m = 40,\ x_0 = 55,\ r = 35$						
$p = 0.1$						
0.1	3.982	0.040	14.788	23.134	9.198	9.786
0.005	3.736	0.038	11.898	26.137	197.942	200.359
0.0027	3.932	0.040	11.784	26.912	377.912	380.658
0.002	3.917	0.040	11.632	27.214	493.621	499.410
0.001	3.814	0.039	11.072	27.667	998.736	993.715
$p = 0.5$						
0.1	3.878	0.039	40.204	67.263	9.132	9.362
0.005	3.794	0.039	35.513	89.176	198.183	198.474
0.0027	3.898	0.039	34.708	92.318	359.332	356.474
0.002	3.825	0.039	34.375	95.111	496.614	493.678
0.001	3.842	0.039	33.474	98.635	974.028	967.235
$p = 0.9$						
0.1	3.852	0.039	80.054	171.3	9.423	8.836
0.005	4.008	0.041	66.133	232.144	210.664	207.081
0.0027	3.924	0.040	65.055	252.767	370.094	370.317
0.002	3.845	0.039	64.789	264.826	501.823	497.035
0.001	3.823	0.039	63.645	287.609	1060.917	1072.800
$m = 25,\ x_0 = 70,\ r = 15$						
$p = 0.1$						
0.1	2.85	0.031	11.932	24.484	9.075	9.822
0.005	2.869	0.031	8.954	29.986	194.054	189.316
0.0027	2.855	0.031	8.459	30.878	356.592	354.291
0.002	2.870	0.031	8.359	31.497	498.327	498.327
0.001	2.883	0.032	7.855	32.205	1006.289	983.451
$p = 0.5$						
0.1	2.832	0.031	41.750	70.441	9.128	9.419
0.005	2.914	0.031	35.082	96.917	199.993	204.249
0.0027	2.864	0.031	33.606	101.857	373.372	378.762
0.002	2.838	0.031	33.179	106.356	506.805	526.741
0.001	2.876	0.031	32.025	111.392	976.043	962.025
$p = 0.9$						
0.1	2.851	0.031	96.561	182.511	8.788	9.209
0.005	2.821	0.031	77.351	270.358	194.856	200.642
0.0027	2.843	0.031	74.817	286.522	400.526	400.402
0.002	2.581	0.031	73.916	302.909	458.016	457.852
0.001	2.841	0.031	70.663	319.773	1025.175	1004.552
$m = 25,\ x_0 = 70,\ r = 20$						
$p = 0.1$						
0.1	4.748	0.0465	15.550	27.219	9.091	9.500
0.005	4.792	0.046	12.467	32.308	198.842	197.303
0.0027	4.891	0.046	12.337	33.719	378.966	382.839
0.002	4.968	0.047	12.034	33.675	490.950	489.029
0.001	4.738	0.045	11.221	34.438	984.621	984.048
$p = 0.5$						
0.1	4.811	0.046	37.059	55.666	8.974	9.612
0.005	4.804	0.046	32.221	68.410	199.735	202.615
0.0027	4.783	0.045	31.681	72.234	366.767	362.857
0.002	4.929	0.046	31.323	72.598	508.148	511.174
0.001	4.759	0.046	30.461	76.763	982.873	994.352
$p = 0.9$						
0.1	4.913	0.046	67.604	117.085	9.308	8.976
0.005	4.774	0.046	57.159	161.266	202.340	200.612
0.0027	4.817	0.046	56.277	176.398	363.095	356.985
0.002	4.789	0.046	55.146	178.642	518.166	529.807
0.001	4.823	0.046	53.239	187.111	1013.308	1055.570
$m = 40,\ x_0 = 70,\ r = 30$						
$p = 0.1$						
0.1	4.490	0.044	15.967	25.259	9.017	9.654
0.005	4.410	0.043	13.298	29.299	195.595	196.360
0.0027	4.491	0.044	13.071	30.153	358.369	351.506
0.002	4.563	0.044	12.992	30.353	494.728	497.457
0.001	4.450	0.043	12.346	31.002	969.354	992.311

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Table 1 – *Continued from previous page*

ν	θ	λ	LCL	UCL	ARL_0	$SDRL_0$
$p = 0.5$						
0.1	4.354	0.043	39.241	54.206	8.975	9.327
0.005	4.363	0.043	35.250	64.082	194.275	195.113
0.0027	4.348	0.434	34.324	65.286	361.189	345.430
0.002	4.531	0.044	34.147	65.598	499.626	504.093
0.001	4.403	0.043	33.746	69.507	967.592	974.619
$p = 0.9$						
0.1	4.466	0.043	75.657	115.239	9.128	9.807
0.005	4.431	0.044	65.931	146.569	197.499	195.845
0.0027	4.387	0.044	64.441	153.629	374.160	366.423
0.002	4.416	0.043	64.231	159.318	490.342	487.078
0.001	4.332	0.042	63.081	170.417	998.072	1004.605
$m = 40,\ x_0 = 70,\ r = 35$						
$p = 0.1$						
0.1	5.099	0.048	16.987	31.039	9.009	9.445
0.005	5.211	0.048	14.599	30.020	200.179	198.202
0.0027	5.157	0.048	13.956	30.332	378.016	379.920
0.002	5.253	0.048	13.965	30.812	483.591	494.707
0.001	5.178	0.048	13.372	31.252	969.366	978.296
$p = 0.5$						
0.1	5.207	0.048	37.905	51.624	9.263	9.805
0.005	5.129	0.047	34.250	59.701	198.445	201.138
0.0027	5.077	0.048	33.582	61.249	371.043	369.694
0.002	4.905	0.046	33.499	63.380	504.436	500.516
0.001	5.154	0.048	32.929	64.138	981.467	995.158
$p = 0.9$						
0.1	5.208	0.048	68.413	104.205	8.935	9.597
0.005	5.209	0.048	61.156	132.089	196.665	197.578
0.0027	5.072	0.047	60.212	139.842	362.635	361.404
0.002	5.092	0.048	58.875	138.727	498.193	493.579
0.001	5.113	0.048	58.059	146.944	993.288	1002.445

Table 2: OOC performance for $m = 25, x_0 = 55, r = 15, \theta = 5.5, \lambda = 0.05$

$\Delta\lambda$	$p = 0.1$	$p = 0.5$	$p = 0.9$
$\Delta\theta = -0.3$			
	ARL(SDRL)	ARL(SDRL)	ARL(SDRL)
-0.2	8.071 (8.594)	16.451 (16.023)	15.109 (14.112)
-0.1	9.415 (10.004)	16.300 (17.133)	17.019 (17.138)
-0.08	9.975(10.444)	18.050(19.192)	16.078(17.290)
-0.06	10.134(9.291)	19.007(18.010)	18.653(18.365)
-0.04	10.950(8.723)	19.946(18.721)	21.894(21.022)
-0.02	11.639(9.904)	21.681(20.257)	32.333(32.958)
0	10.541 (9.251)	20.041 (21.078)	36.721 (36.002)
0.02	5.334(5.781)	6.766(7.240)	20.751(21.542)
0.04	4.723(5.139)	5.463(5.996)	17.341(17.914)
0.06	4.293(4.852)	4.620(5.132)	14.310(14.473)
0.08	3.882(4.308)	3.931(4.366)	12.156(12.797)
0.1	3.570 (4.023)	3.364 (3.846)	9.961 (10.623)
0.2	2.183 (2.702)	1.453 (1.927)	4.538 (5.118)
0.3	1.387 (1.822)	0.756 (1.182)	2.333 (2.825)
$\Delta\theta = -0.2$			
-0.3	35.950 (36.976)	23.445 (23.949)	24.664 (25.280)
-0.1	61.271 (60.222)	127.080 (127.506)	157.182 (158.833)
-0.08	54.435 (55.689)	94.709 (193.538)	207.804 (206.153)
-0.06	46.315 (46.097)	74.016 (73.762)	164.721 (168.005)
-0.04	33.551 (34.483)	54.724 (55.509)	127.318 (124.859)
-0.02	27.908 (28.031)	41.884 (41.318)	96.785 (98.045)
0	25.954 (26.304)	31.898 (32.191)	75.108 (77.694)
0.02	20.762 (21.244)	25.563 (26.411)	59.844 (60.967)
0.04	18.362 (17.736)	20.345 (20.977)	46.644 (46.517)
0.06	15.654 (16.147)	15.624 (16.179)	36.663 (37.373)
0.08	13.655 (14.395)	12.501 (13.140)	29.571 (30.682)
0.1	11.972 (12.555)	10.264 (10.795)	23.464 (24.721)
0.2	6.861 (7.329)	3.990 (4.371)	9.508 (9.862)
0.3	4.073 (4.489)	1.880 (2.343)	4.361 (4.842)
$\Delta\theta = -0.1$			
-0.3	12.388 (12.896)	8.710 (9.099)	9.430 (9.973)
-0.2	63.085 (64.036)	45.780 (46.784)	49.016 (50.760)
-0.08	198.434 (197.024)	262.096 (266.725)	314.749 (317.814)
-0.06	178.580 (173.978)	263.839 (264.534)	340.554 (341.785)
-0.04	155.577 (156.147)	228.006 (226.347)	311.412 (311.228)
-0.02	135.275 (136.893)	181.486 (179.450)	266.279 (269.387)
0	123.055 (124.551)	140.192 (142.782)	215.486 (214.190)
0.02	91.102 (90.584)	102.635 (100.961)	164.303 (161.264)
0.04	77.828 (78.596)	77.164 (77.172)	125.107 (125.513)
0.06	64.256 (65.345)	59.231 (58.065)	96.188 (96.402)

Continued on next page

Table 2 – *Continued from previous page*

$\Delta\lambda$	$p = 0.1$	$p = 0.5$	0.9
0.08	54.182 (55.813)	45.851 (45.705)	75.810 (76.306)
0.1	51.479 (52.428)	35.402 (35.681)	59.754 (59.929)
0.2	24.233 (24.276)	11.190 (11.619)	20.097 (20.594)
0.3	12.371 (12.949)	4.371 (4.957)	8.461 (9.041)
	$\Delta\theta = -0.08$		
-0.3	11.886(12.673)	7.775(8.267)	7.899(8.255)
-0.2	61.563 (61.407)	37.695 (37.675)	39.176 (38.924)
-0.1	215.642 (209.788)	205.235 (202.819)	226.998 (230.446)
-0.06	226.573 (227.238)	279.552 (279.899)	306.655 (307.643)
-0.04	205.435 (205.503)	270.143 (268.302)	315.048 (318.793)
-0.02	174.515 (171.644)	236.296 (231.238)	287.769 (287.914)
0	150.818 (152.212)	180.879 (181.724)	211.515 (210.278)
0.02	129.346 (129.425)	133.565 (132.856)	198.244 (202.245)
0.04	106.854 (106.836)	102.946 (103.456)	153.449 (152.253)
0.06	88.418 (89.227)	78.821 (77.653)	121.086 (121.408)
0.08	75.401 (75.139)	57.324 (57.387)	91.256 (92.481)
0.1	63.326 (64.821)	44.105 (44.193)	72.867 (73.186)
0.2	28.112 (28.448)	13.641 (14.115)	23.975 (24.209)
0.3	14.353(14.892)	5.161(5.548)	9.592(10.032)
	$\Delta\theta = -0.06$		
-0.3	9.893(10.463)	6.448(7.056)	6.586(6.998)
-0.2	48.117 (49.646)	30.908 (31.087)	32.991 (33.657)
-0.1	213.661 (210.378)	179.409 (177.378)	183.967 (183.990)
-0.08	251.301 (248.520)	233.205 (233.415)	254.553 (258.357)
-0.04	252.823 (252.657)	307.411 (309.388)	327.791 (329.037)
-0.02	228.497 (233.517)	283.904 (278.001)	304.651 (300.923)
0	204.217 (203.632)	238.552 (239.031)	277.182 (273.787)
0.02	167.358 (172.887)	176.706 (178.378)	243.192 (240.835)
0.04	142.135 (141.693)	140.615(137.063)	190.432 (187.359)
0.06	121.480 (119.899)	101.555 (102.258)	143.980 (143.508)
0.08	101.719 (99.834)	75.899 (76.290)	112.152 (115.584)
0.1	82.816 (83.910)	58.432 (58.148)	84.175 (82.994)
0.2	36.061 (35.892)	16.787 (17.355)	26.634 (26.932)
0.3	18.181(18.611)	6.237(6.747)	6.687(7.171)
	$\Delta\theta = -0.04$		
-0.3	8.116(8.717)	5.438(5.981)	5.666(6.221)
-0.2	39.919 (40.091)	24.881 (25.368)	27.687 (27.828)
-0.1	202.228 (202.461)	144.353 (148.195)	156.115 (156.972)
-0.08	240.958 (237.041)	197.099 (201.085)	216.393 (215.429)
-0.06	287.695 (282.448)	261.544 (259.733)	282.848 (281.407)
-0.02	288.003 (288.531)	326.057 (325.667)	340.305 (337.266)
0	262.173 (263.314)	290.495 (288.586)	311.873 (306.750)
0.02	226.845 (226.034)	231.251 (233.209)	287.975 (287.018)
0.04	198.517 (189.025)	177.966 (179.320)	229.115 (232.436)
0.06	164.970 (164.305)	131.281 (131.383)	177.014 (175.740)
0.08	133.642 (135.438)	101.146 (100.476)	133.021 (131.784)
0.1	112.469 (112.791)	76.257 (77.329)	103.376 (101.785)
0.2	48.386 (48.633)	20.929 (21.263)	31.555 (32.532)
0.3	22.761(22.997)	7.283(7.664)	12.396(13.141)
	$\Delta\theta = -0.02$		
-0.3	7.164(7.811)	4.725(5.133)	4.939(5.471)
-0.2	33.497 (34.843)	21.208 (21.850)	22.540 (22.257)
-0.1	170.787 (172.114)	120.951 (120.508)	130.460 (131.825)
-0.08	224.713 (229.501)	170.848 (174.452)	181.880 (185.662)
-0.06	281.890 (282.828)	234.437 (236.225)	254.898 (254.178)
-0.04	322.881 (322.501)	296.366 (297.089)	311.082 (312.213)
0	316.098 (319.640)	326.129 (325.610)	353.417 (354.495)
0.02	290.143 (291.939)	281.609 (285.059)	329.318 (331.790)
0.04	254.654 (253.103)	236.123 (238.064)	270.141 (268.717)
0.06	208.181 (210.002)	178.408 (176.740)	211.014 (208.905)
0.08	182.292 (181.063)	129.337 (129.630)	159.043 (159.878)
0.1	151.083 (153.169)	99.703 (98.518)	122.928 (120.460)
0.2	64.170 (62.063)	26.619 (27.183)	37.580 (37.805)
0.3	28.934(29.525)	8.812(9.258)	13.903(14.351)
	$\Delta\theta = 0$		
-0.3	5.146 (5.685)	4.026 (4.498)	4.342 (4.885)
-0.2	23.873 (24.526)	17.853 (18.572)	19.796 (20.633)
-0.1	120.044 (121.699)	98.649 (98.686)	107.044 (109.599)
-0.08	197.698 (195.721)	142.851 (143.442)	149.982 (152.995)
-0.06	248.660 (249.337)	198.897 (200.854)	206.549 (205.732)
-0.04	318.115 (313.830)	267.130 (267.405)	287.480 (288.742)
-0.02	347.811 (346.613)	330.700 (327.919)	346.422 (342.225)
0.02	349.378 (341.547)	345.469 (343.1509)	347.475 (343.017)
0.04	307.966 (305.231)	292.367 (293.186)	315.871 (311.009)
0.06	285.495 (287.511)	224.780 (222.699)	255.153 (256.592)
0.08	245.649 (241.926)	170.219 (169.888)	195.335 (196.753)
0.1	228.174 (224.763)	127.124 (125.624)	146.773 (147.685)
0.2	93.332 (92.879)	33.757 (33.946)	43.583 (44.329)
0.3	42.378 (43.231)	11.149 (11.803)	16.315 (16.910)
	$\Delta\theta = 0.02$		
-0.3	2.014(5.482)	3.554(3.990)	3.646(4.134)
-0.2	22.453 (23.251)	15.312 (15.695)	16.197 (16.674)
-0.1	121.237 (121.867)	80.721 (81.123)	87.831 (89.568)

Continued on next page

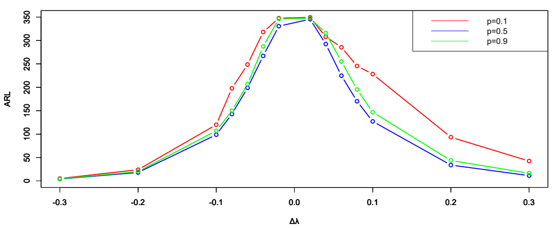
Table 2 – *Continued from previous page*

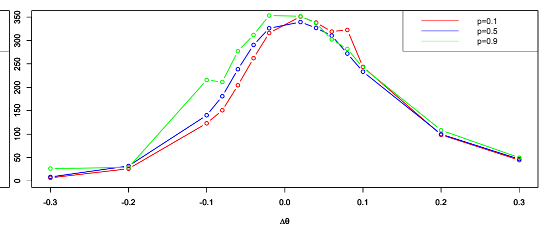
$\Delta\lambda$	$p = 0.1$	$p = 0.5$	0.9
-0.08	164.530 (165.194)	116.477 (117.060)	124.528 (122.930)
-0.06	225.273 (223.209)	167.112 (166.062)	174.944 (174.176)
-0.04	285.688 (285.807)	230.741 (230.871)	247.017 (243.818)
-0.02	325.712 (321.820)	309.026 (310.978)	316.239 (313.509)
0	351.165 (352.190)	339.507 (338.933)	351.835 (352.645)
0.04	327.298 (327.598)	341.754 (340.489)	358.973 (354.868)
0.06	284.917 (278.141)	292.467 (289.143)	298.720 (297.358)
0.08	244.925 (246.697)	219.153 (216.450)	238.672 (237.522)
0.1	216.756 (213.139)	168.603 (169.045)	178.548 (175.878)
0.2	110.682 (109.558)	42.547 (42.668)	50.968 (51.535)
0.3	49.894(49.325)	12.505(13.124)	18.327(18.749)
$\Delta\theta = 0.04$			
-0.3	4.258(4.744)	3.139(3.536)	3.334(3.829)
-0.2	19.298 (20.088)	2.681 (13.379)	13.767 (14.128)
-0.1	97.294 (96.922)	67.063 (67.514)	71.071 (70.970)
-0.08	134.830 (134.146)	97.933 (98.592)	104.091 (103.166)
-0.06	195.094 (195483)	138.501 (139.310)	147.901 (147.183)
-0.04	258.640 (254.946)	195.995 (196.845)	202.986 (202.735)
-0.02	324.457 (315.447)	269.969 (269.319)	290.090 (291.774)
0	338.298 (337.423)	326.874 (321.441)	337.103 (337.252)
0.02	353.916 (355.162)	307.010 (302.873)	348.082 (349.941)
0.06	312.641 (318.930)	262.881 (266.375)	315.847 (314.416)
0.08	283.480 (280.010)	234.701 (231.158)	276.028 (273.423)
0.1	246.262 (244.174)	216.532 (218.505)	214.942 (213.660)
0.2	148.436 (148.675)	52.733 (54.212)	61.271 (61.084)
0.3	63.508(62.827)	15.548(16.290)	20.046(20.429)
$\Delta\theta = 0.06$			
-0.3	3.773(4.181)	2.647(3.255)	2.825(3.151)
-0.2	15.951 (16.193)	10.821 (11.349)	12.082 (12.475)
-0.1	81.565 (79.756)	56.805 (57.194)	59.442 (60.425)
-0.08	113.241 (115.676)	78.945 (79.865)	87.668 (86.476)
-0.06	155.420 (154.542)	116.008 (117.311)	120.866 (120.387)
-0.04	215.970 (213.989)	163.720 (161.567)	176.411 (177.459)
-0.02	291.390 (290.645)	235.395 (235.929)	237.869 (239.999)
0	319.040 (314.315)	310.375 (307.344)	302.283 (304.516)
0.02	351.823 (344.020)	302.774 (301.931)	305.684 (305.013)
0.04	307.645 (309.847)	317.919 (319.292)	287.804 (281.186)
0.08	289.209 (282.092)	253.376 (249.784)	273.810 (275.690)
0.1	165.600 (166.456)	181.284 (183.759)	204.824 (206.420)
0.2	93.899 (86.272)	67.307 (67.272)	70.855 (70.129)
0.3	83.568(83.462)	18.510(18.518)	23.647(24.229)
$\Delta\theta = 0.08$			
-0.3	3.197(3.626)	2.413(2.916)	2.443(2.826)
-0.2	13.171 (13.351)	9.390 (9.708)	10.017 (10.386)
-0.1	65.757 (67.911)	47.064 (46.972)	50.248 (49.601)
-0.08	92.555 (93.534)	65.688 (65.594)	70.905 (70.764)
-0.06	130.089 (129.537)	93.709 (94.519)	101.872 (103.465)
-0.04	176.979 (177.393)	138.650 (138.043)	154.428 (149.648)
-0.02	245.615 (250.181)	198.299 (196.548)	213.642 (214.738)
0	322.647 (322.225)	272.080 (277.007)	281.983 (277.233)
0.02	316.544 (318.740)	323.037 (324.774)	333.435 (330.259)
0.04	289.769 (282.429)	248.173 (250.649)	314.347 (311.741)
0.06	161.657 (154.789)	174.933 (173.759)	224.01 (217.134)
0.1	125.323 (120.744)	102.847 (103.731)	195.939 (189.567)
0.2	107.973 (102.402)	84.612 (84.106)	82.274 (81.494)
0.3	25.917(26.392)	23.275(23.464)	26.788(27.545)
$\Delta\theta = 0.1$			
-0.3	2.582 (3.004)	2.060 (2.475)	2.144 (2.565)
-0.2	10.044 (10.573)	8.237 (8.744)	8.633 (9.220)
-0.1	45.458 (45.288)	40.190 (40.169)	42.671 (43.072)
-0.08	78.066 (79.923)	56.408 (56.222)	60.550 (59.510)
-0.06	108.639 (110.753)	78.858 (76.372)	85.276 (85.831)
-0.04	152.238 (151.485)	111.594 (114.535)	121.627 (121.197)
-0.02	211.178 (213.879)	166.236 (170.004)	180.452 (176.148)
0	243.698 (246.604)	233.343 (238.010)	242.020 (242.069)
0.02	304.725 (303.160)	305.640 (303.013)	307.912 (307.655)
0.04	272.858 (279.499)	280.166 (281.112)	289.241 (295.188)
0.06	229.642 (226.153)	192.379 (193.960)	218.105 (221.838)
0.08	196.232 (198.063)	151.472 (152.54)	202.449 (202.458)
0.2	101.995 (101.021)	107.061 (108.357)	97.898 (100.107)
0.3	31.119 (29.927)	28.886 (28.749)	31.125 (31.195)
$\Delta\theta = 0.2$			
-0.3	1.367 (1.777)	1.141 (1.568)	1.220 (1.666)
-0.2	4.940 (5.430)	4.284 (4.768)	4.579 (5.153)
-0.1	19.818 (20.438)	18.222 (18.748)	19.698 (20.689)
-0.08	32.992 (33.615)	24.992 (25.745)	27.607 (28.223)
-0.06	44.450 (45.491)	34.922 (35.687)	38.127 (38.019)
-0.04	61.030 (61.835)	48.214 (49.086)	52.721 (53.135)
-0.02	85.146 (84.163)	70.125 (70.402)	76.241 (76.271)
0	98.400 (98.306)	99.780 (101.924)	108.355 (107.993)
0.02	158.717 (154.999)	143.573 (142.778)	148.627 (151.554)
0.04	220.958 (221.866)	201.934 (200.050)	213.975 (213.585)
0.06	190.662 (190.574)	195.769 (190.288)	197.689 (189.725)

Continued on next page

Table 2 – *Continued from previous page*

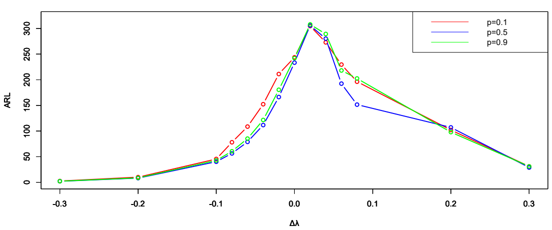
$\Delta\lambda$	$p = 0.1$	$p = 0.5$	0.9
0.08	136.692 (137.831)	111.273 (114.226)	175.503 (174.908)
0.1	125.508 (117.243)	134.163 (137.397)	153.334 (145.659)
0.3	581.559 (580.219)	82.061 (85.179)	60.863 (61.331)
	$\Delta\theta = 0.3$		
-0.3	0.733 (1.140)	0.614 (0.978)	0.666 (1.043)
-0.2	2.560 (2.991)	2.390 (2.762)	2.624 (3.131)
-0.1	10.291 (10.693)	9.490 (9.792)	10.162 (10.721)
-0.08	15.543(15.827)	13.437(13.687)	13.577(13.903)
-0.06	20.680(21.275)	18.087(18.454)	18.447(18.746)
-0.04	28.642(28.716)	24.397(24.426)	25.407(26.176)
-0.02	38.031(38.667)	34.034(34.785)	35.431(36.771)
0	44.358 (45.876)	46.537 (46.280)	49.652 (50.875)
0.1	186.297 (191.952)	277.517 (276.838)	277.816 (278.703)
0.02	69.029(67.574)	66.001(65.354)	68.319(68.212)
0.04	95.260(96.475)	97.923(96.799)	97.752(98.794)
0.06	77.063(75.622)	76.678(77.609)	70.614(73.677)
0.08	65.598(64.124)	63.965(62.746)	64.623(62.185)
0.2	663.674 (649.070)	885.480 (890.444)	433.937 (446.464)

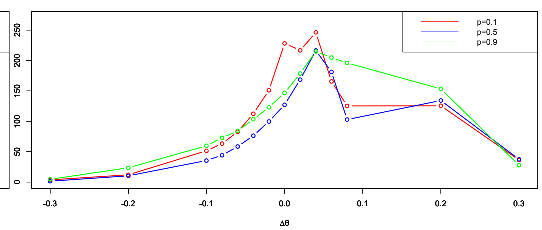




(a) ARL_1 for different choices of $\Delta\lambda$, when $\Delta\theta = 0$

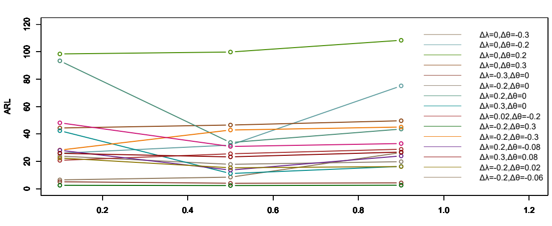
(b) ARL_1 for different choices of $\Delta\theta$, when $\Delta\lambda = 0$

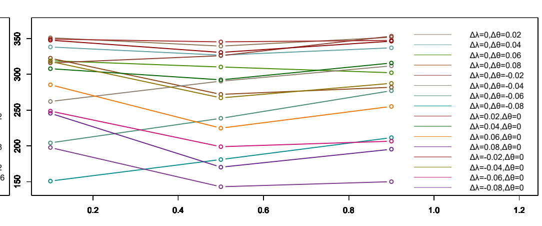




(c) ARL_1 for different choices of $\Delta\lambda$, when $\Delta\theta = 0.1$

(d) ARL_1 for different choices of $\Delta\theta$, when $\Delta\lambda = 0.1$





(e) ARL_1 for different choices of p

(f) ARL_1 for different choices of p

Figure 11: Graphs of ARL_1 for different choices of $\Delta\lambda$, $\Delta\theta$ and p