Computational Approach Test using Likelihood Based Tests for the Equality of Inverse Gaussian Means

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Abstract:

- In this study, we propose three different test procedures by plugging the Wald (W), score (S) and likelihood ratio (LR) statistics into the computational approach test (CAT) to test the equality of inverse Gaussian means when the scale parameters are unknown and arbitrary. Restricted maximum likelihood (RML) estimators are used in developing the proposed test procedures. Since the RML estimators cannot be derived in closed-form, the bisection method is used to obtain the numerical solutions. The motivation behind using the CAT procedure is that it can easily be implemented to hypothesis testing problems without knowing the sampling distributions of the test statistics. The proposed and existing procedures are compared in terms of type I error rates and powers via an extensive Monte Carlo simulation study. In addition, a real data set is analyzed for illustration.

Keywords:

- *computational approach test; score test; Wald test; likelihood ratio test; inverse Gaussian distribution.*

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1.   INTRODUCTION

The inverse Gaussian (IG) distribution was first introduced by Schrödinger [12] and Smoluchowsky [18] as the probability distribution of the first passage time in Brownian motion. This distribution was named as IG by Tweedie [17] since there exists inverse relationship between the cumulant generating function of the first passage time distribution and that of the normal distribution. The IG distribution is also known as Wald's distribution, especially in Russian literature.

The probability density function (pdf) of the $IG(\mu, \lambda)$ distribution is defined as

$$(1.1) \qquad f(x; \mu, \lambda) = \sqrt{\frac{\lambda}{2x^3\pi}} exp\left(-\frac{\lambda(x - \mu)^2}{2\mu^2 x}\right), \quad x > 0, \quad \mu > 0, \quad \lambda > 0,$$

where λ is a scale parameter and μ is the mean of the distribution.

The $IG(\mu, \lambda)$ distribution has the following attractive properties (see, Tian [16]; Shi and Lv [14]; Seshadri [13]):

- it is closed under convolution;
- it is suitable to model positively-skewed data sets;
- it is the only distribution that shares many elegant properties with Gaussian models among the distributions used to model positively skewed data, see for example Tian [16].

Due to the aforementioned properties, the IG distribution has been widely used in various fields such as cardiology, pharmacokinetics, linguistics, employment services, mathematical finance, demography, hydrology, management sciences, etc. For example, Chhikara and Folks [1] considered the use of the IG distribution for a lifetime model. Folks and Chhikara [5] reviewed the development of the IG distribution and of statistical methods based on it (Chhikara and Folks [2]). Doksum and Hbyland [3] developed models for variable-stress-accelerated life testing experiments based on Wiener processes and the IG distribution. Takagi *et al.* [15] calculated the percentiles of the IG distribution and considered the application of IG to occupational exposure data. Durham and Padgett [4] used IG models to develop a new general method based on cumulative damage for describing the failure of a system. Mudholkar and Tian [10] presented an entropy characterization of the IG family. Kara *et al.* [7] considered the statistical inference problem for the geometric process when the distribution of the first occurrence time is IG. Punzo [11] considered models based on the IG distribution for the problem of fitting the distribution of insurance and economic data.

Testing the equality of means of k independent IG populations is one of the common problems in statistics. The null and alternative hypotheses for this problem are defined as

$$(1.2) \qquad H_0 : \mu_1 = \mu_2 = \cdots = \mu_k \quad \text{and} \quad H_1 : \text{not all } \mu_i\text{'s are equal,}$$

respectively. Here, μ_i $(i = 1, ..., k)$ is the mean of the i-th population.

To test the null hypothesis against the alternative, analysis of reciprocals (ANORE) F test is used under the assumption of homogeneity of scale parameters. This assumption is

not always valid in most of the real-life problems, therefore, in recent years, there have been many studies testing the equality of several IG means under the assumption of heterogeneity of scale parameters. For example, Tian [16] developed a test based on the generalized p-value (GP) approach to test the equality of several IG means. Ma and Tian [9] proposed a parametric bootstrap (PB) approach and compared it with the GP approach in terms of the type I error rates. Shi and Lv [14] defined a new generalized pivot quantity and the generalized p-value based on this pivot. Gökpınar *et al.* [6] proposed a computational approach test (CAT) using a simple test statistic to assess the equality of several IG means under heterogeneity of scale parameters. Different from Gökpınar *et al.* [6], in this study, we propose three different test procedures by plugging the Wald (W), score (S) and likelihood ratio (LR) test statistics into CAT. W, S and LR test statistics are asymptotically equivalent. However, they differ in small samples and therefore their type I errors are different from the nominal level for small samples in general. Consequently, in this study we incorporate W, S and LR test statistics into CAT to improve their performances in terms of Type I error rates. Restricted maximum likelihood (RML) estimators are used in developing the proposed test procedures. In this study, in contrast to Ma and Tian [9] who derived RML estimators of the parameters by using expectation-maximization (EM) algorithm, the bisection method is used to obtain the numerical solutions for the RML estimators by maximizing the profile likelihood function under the null hypothesis.

This article is organized as follows. In Section 2, maximum likelihood (ML) and RML estimators for the parameters of the IG distribution are obtained. In Section 3, the PB approach proposed by Ma and Tian [9] and GP approach proposed by Tian [16] are briefly reviewed. In Section 4, W, S and LR test statistics are defined and the CAT procedure is explained for testing the equality of IG means under heterogeneity of scale parameters. In Section 5, simulated Type I error rates and powers of the existing and proposed tests are presented. Proposed tests are illustrated by using a real data set in Section 6. Concluding remarks are given in Section 7.

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2.   MAXIMUM LIKELIHOOD AND RESTRICTED MAXIMUM LIKELIHOOD ESTIMATORS

Let X_{ij}, $i = 1, ..., k; j = 1, ..., n_i$ be k independent random samples from IG(μ_i, λ_i). Here, X_{ij} represents the j-th observation of the i-th population and n_i denotes the number of observations in the i-th population. Then the log-likelihood function ($\ln L_1$) under the unrestricted model (H_1) is obtained as follows:

$$(2.1) \qquad \ln L_1 = \sum_{i=1}^{k} \frac{n_i}{2} (\ln \lambda_i - \ln(2\pi)) - \frac{3}{2} \sum_{i=1}^{k} \sum_{j=1}^{n_i} \ln X_{ij} - \sum_{i=1}^{k} \sum_{j=1}^{n_i} \frac{\lambda_i (X_{ij} - \mu_i)^2}{2\mu_i^2 X_{ij}}.$$

To obtain the ML estimators of the parameters μ_i and λ_i, we take the derivatives of $\ln L_1$ with respect to the unknown parameters and then equate them to zero as follows:

$$(2.2) \qquad \frac{\partial \ln L_1}{\partial \mu_i} = \sum_{j=1}^{n_i} \frac{\lambda_i (X_{ij} - \mu_i)}{\mu_i^3} = 0$$

and

$$(2.3) \qquad \frac{\partial \ln L_1}{\partial \lambda_i} = \frac{n_i}{\lambda_i} - \sum_{j=1}^{n_i} \frac{(X_{ij} - \mu_i)^2}{\mu_i^2 X_{ij}} = 0.$$

Solutions of these likelihood equations in (2.2) and (2.3) are the ML estimators of the parameters μ_i and λ_i. They are obtained as

$$(2.4) \qquad \hat{\mu}_i = \bar{X}_i, \quad i = 1, ..., k,$$

and

$$(2.5) \qquad \hat{\lambda}_i = \left(\frac{1}{n_i} \sum_{j=1}^{n_i} \left(1/X_{ij} - 1/\bar{X}_i \right) \right)^{-1} = \left(\overline{X_i^{-1}} - (\bar{X}_i)^{-1} \right)^{-1}, \quad i = 1, ..., k,$$

respectively. Here, $\overline{X_i^{-1}} = (1/n_i) \sum_{j=1}^{n_i} (1/X_{ij})$. Similarly, under the restricted model (H_0), the log-likelihood function ($\ln L_0$) is obtained as

$$(2.6) \qquad \ln L_0 = \sum_{i=1}^{k} \frac{n_i}{2} (\ln \lambda_i - \ln(2\pi)) - \frac{3}{2} \sum_{i=1}^{k} \sum_{j=1}^{n_i} \ln X_{ij} - \sum_{i=1}^{k} \sum_{j=1}^{n_i} \frac{\lambda_i (X_{ij} - \mu)^2}{2\mu^2 X_{ij}}.$$

By taking derivatives of $\ln L_0$ with respect to the parameters μ and λ_i and equating them to zero, the following likelihood equations are

$$(2.7) \qquad \frac{\partial \ln L_0}{\partial \mu} = \sum_{i=1}^{k} \sum_{j=1}^{n_i} \left(\frac{\lambda_i X_{ij}}{\mu^3} - \frac{\lambda_i}{\mu^2} \right) = 0$$

and

$$(2.8) \qquad \frac{\partial \ln L_0}{\partial \lambda_i} = \frac{n_i}{2\lambda_i} - \sum_{j=1}^{n_i} \frac{(X_{ij} - \mu)^2}{2\mu^2 X_{ij}} = 0,$$

respectively. Solutions of likelihood equations in (2.7) and (2.8) are called as RML estimators. RML estimators of the parameters λ_i and μ are

$$(2.9) \qquad \tilde{\lambda}_i = \frac{n_i \mu^2}{\sum_{j=1}^{n_i} \frac{(X_{ij} - \mu)^2}{X_{ij}}}, \quad i = 1, ..., k,$$

and

$$(2.10) \qquad \tilde{\mu} = \frac{\sum_{i=1}^{k} n_i \lambda_i \bar{X}_i}{\sum_{i=1}^{k} n_i \lambda_i},$$

respectively.

It is obvious from equations (2.9) and (2.10) that the estimators of the unknown parameters have no explicit analytical solutions since there exists μ in the estimator of the parameter λ_i and vice versa. Therefore, the profile likelihood method is used to eliminate the effect of nuisance parameters $(\lambda_1, ..., \lambda_k)$ and consequently to estimate the parameter μ.

The profile log-likelihood function ($\ln L^*$) is obtained by replacing λ_i in equation (2.6) with $\tilde{\lambda}_i$ given in (2.9) as

$$(2.11) \qquad \ln L^* = \frac{1}{2} \sum_{i=1}^{k} n_i \left(\ln(n_i \mu^2) - \ln \left(\sum_{j=1}^{n_i} (X_{ij} - \mu)^2 / X_{ij} \right) \right) + \text{constant}.$$

Then we take the derivative of the profile log-likelihood function with respect to the parameter μ as follows:

$$(2.12) \qquad \frac{\partial \ln L^*}{\partial \mu} = \sum_{i=1}^{k} n_i \left(\frac{1}{\mu} + \frac{\sum_{j=1}^{n_i} (X_{ij} - \mu)/X_{ij}}{\sum_{j=1}^{n_i} (X_{ij} - \mu)^2 / X_{ij}} \right).$$

After rearranging the (2.12) and equating it to zero, we obtain the following equality

$$(2.13) \qquad \sum_{i=1}^{k} \frac{n_i(\bar{X}_i - \mu)}{\mu(\bar{X}_i - 2\mu + \mu^2 \overline{X_i^{-1}})} = 0.$$

By solving (2.13), the restricted profile maximum likelihood (RPML) estimator of μ denoted by $\tilde{\mu}^*$ is obtained. Finally, by incorporating $\tilde{\mu}^*$ into (2.9), we obtain the RPML estimator of λ_i as

$$(2.14) \qquad \tilde{\lambda}_i^* = \frac{n_i(\tilde{\mu}^*)^2}{\sum_{j=1}^{n_i} \frac{(X_{ij} - \tilde{\mu}^*)^2}{X_{ij}}}, \quad i = 1, ..., k.$$

Unfortunately, the RPML estimator of μ has no closed form. Therefore, the bisection method is used to obtain a numerical solution for the estimate value of the parameter μ. It requires an interval in which a root of the given equation must lie. Since the parameter μ is the common mean, the root of the given equation, i.e., the estimate of the parameter μ, must lie between the smallest and the largest group means. Hence, the bisection method always converges to a root and the RPML estimate of the parameter μ is obtained.

3.   REVIEWING SOME EXISTING TESTS

In this section, we briefly describe the GP and PB approaches proposed by Tian[16] and Ma and Tian [9], respectively.

3.1. Generalized p-value (GP) approach

Tian [16] calculated the p-value based on the GP approach using the following algorithm.

Step 1: $R_{\lambda_i} = \lambda_i V_i / v_i \sim \chi^2_{(n_i-1)}/v_i$, $i = 1, ..., k$, which is a generalized pivot for λ_i is generated and $R_{\hat{\mu}} = \dfrac{\sum\limits_{i=1}^{k} n_i R_{\lambda_i} \bar{x}_i}{\sum\limits_{i=1}^{k} n_i R_{\lambda_i}}$ is calculated. Here, $V_i = \sum\limits_{j=1}^{n_i}(X_{ij}^{-1} - \bar{X}_i^{-1})$, v_i is the observed value of V_i and $\bar{x}_i = \sum\limits_{j=1}^{n_i} x_{ij}/n_i$.

Step 2: $q = \sum\limits_{i=1}^{k} n_i R_{\lambda_i}(\bar{x}_i^{-1} - R_{\hat{\mu}}^{-1})$ is calculated.

Step 3: $Q \sim \chi^2_{(k-1)}$ is generated.

Step 4: Step 2 and Step 3 are repeated m times.

Step 5: Monte Carlo estimate of the p-value for testing (1.2) is calculated as $\hat{p} = \#(Q \geq q)/m$.

Step 6: If $\hat{p} < \alpha$, then H_0 given in (1.2) is rejected.

3.2. Parametric Bootstrap (PB) approach

Ma and Tian [9] calculated the p-value based on the PB approach using the following algorithm.

Step 1: For a given data set x_{ij}, v_i and $\bar{x}_i$ are calculated. Here, v_i is the observed value of $V_i = \sum\limits_{j=1}^{n_i}(X_{ij}^{-1} - \bar{X}_i^{-1})$ and $\bar{X}_i = \sum\limits_{j=1}^{n_i} X_{ij}/n_i$, $i = 1, ..., k$; $j = 1, ..., n_i$. Then RML estimates $\tilde{\mu}, \tilde{\lambda}_1, \tilde{\lambda}_2, ..., \tilde{\lambda}_k$ are calculated via EM algorithm. Based on these estimate values $Q_{B0} = \sum\limits_{i=1}^{k} n_i \tilde{\lambda}_i (\bar{x}_i^{-1} - \tilde{\mu}^{-1})$ is computed.

Step 2: $\bar{X}_{Bi} \sim IG\left(\sum\limits_{i=1}^{k} n_i \hat{\lambda}_i \bar{x}_i / \sum\limits_{i=1}^{k} n_i \hat{\lambda}_i, n_i \hat{\lambda}_i\right)$ and $\lambda_{Bi} \sim \chi^2_{(n_i-1)}/v_i, i = 1, ..., k$ are generated independently. Here $\hat{\lambda}_i = n_i/v_i$.

Step 3: $Q_B = \sum\limits_{i=1}^{k} n_i \lambda_{Bi}\left(\frac{1}{\bar{X}_{Bi}} - \frac{1}{\hat{\mu}_B}\right)$ is computed. Here, $\hat{\mu}_B = \dfrac{\sum\limits_{i=1}^{k} n_i \lambda_{Bi} \bar{X}_{Bi}}{\sum\limits_{i=1}^{k} n_i \lambda_{Bi}}$.

Step 4: Step 2 and Step 3 are repeated m times and $Q_B^{(l)}$, $l = 1, ..., m$, are obtained.

Step 5: Monte Carlo estimate of the p-value for testing (1.2) is calculated as $\hat{p} = \#(Q_B^{(l)} \geq Q_{B0})/m$.

Step 6: If $\hat{p} < \alpha$, then H_0 given in (1.2) is rejected.

4. THE PROPOSED CAT PROCEDURE USING LIKELIHOOD BASED TEST STATISTICS

In this section, the likelihood based test statistics W, S and LR are derived for testing the equality of IG means and then they are plugged into the CAT procedure.

4.1. Wald (W) test statistic

The test statistic W is defined as

(4.1)
$$W = (\hat{\boldsymbol{\mu}} - \tilde{\boldsymbol{\mu}}^{*})' \left(I^{\mu,\mu}(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\lambda}}) \right)^{-1} (\hat{\boldsymbol{\mu}} - \tilde{\boldsymbol{\mu}}^{*}).$$

Here, $\hat{\boldsymbol{\mu}}$ and $\tilde{\boldsymbol{\mu}}^{*}$ are the ML and RPML estimators of mean vector $\boldsymbol{\mu} = (\mu_1, ..., \mu_k)$, respectively. Also,

$$(\hat{\boldsymbol{\mu}} - \tilde{\boldsymbol{\mu}}^{*})' = (\hat{\mu}_1 - \tilde{\mu}^{*}, .., \hat{\mu}_k - \tilde{\mu}^{*})$$

and

$$I^{\mu,\mu}(\boldsymbol{\mu}, \boldsymbol{\lambda}) = \left(I_{\mu,\mu}(\boldsymbol{\mu}, \boldsymbol{\lambda}) - I_{\mu,\lambda}(\boldsymbol{\mu}, \boldsymbol{\lambda}) I_{\lambda,\lambda}^{-1}(\boldsymbol{\mu}, \boldsymbol{\lambda}) I_{\lambda,\mu}(\boldsymbol{\mu}, \boldsymbol{\lambda}) \right)^{-1}.$$

To obtain (4.1) the following steps are considered:

(i) By taking the second partial derivatives of the $\ln L_1$ with respect to the parameters, expected information matrix

(4.2)
$$I = \begin{bmatrix} I_{\mu,\mu}(\boldsymbol{\mu}, \boldsymbol{\lambda}) & I_{\mu,\lambda}(\boldsymbol{\mu}, \boldsymbol{\lambda}) \\ I_{\lambda,\mu}(\boldsymbol{\mu}, \boldsymbol{\lambda}) & I_{\lambda,\lambda}(\boldsymbol{\mu}, \boldsymbol{\lambda}) \end{bmatrix}$$

is obtained. Here,

(4.3)
$$I_{\mu,\mu}(\boldsymbol{\mu}, \boldsymbol{\lambda}) = \begin{bmatrix} \frac{n_1 \lambda_1}{\mu_1^3} & 0 & \cdots & 0 \\ 0 & \frac{n_2 \lambda_2}{\mu_2^3} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{n_k \lambda_k}{\mu_k^3} \end{bmatrix},$$

(4.4)
$$I_{\mu,\lambda}(\boldsymbol{\mu}, \boldsymbol{\lambda}) = I_{\lambda,\mu}(\boldsymbol{\mu}, \boldsymbol{\lambda}) = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

and

(4.5)
$$I_{\lambda,\lambda}(\boldsymbol{\mu}, \boldsymbol{\lambda}) = \begin{bmatrix} \frac{n_1}{2\lambda_1^2} & 0 & \cdots & 0 \\ 0 & \frac{n_2}{2\lambda_2^2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{n_k}{2\lambda_k^2} \end{bmatrix}.$$

(ii) Using (4.3), (4.4) and (4.5),

(4.6)
$$I^{\mu,\mu}(\boldsymbol{\mu}, \boldsymbol{\lambda}) = \begin{bmatrix} \frac{\mu_1^3}{n_1 \lambda_1} & 0 & \cdots & 0 \\ 0 & \frac{\mu_2^3}{n_2 \lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\mu_k^3}{n_k \lambda_k} \end{bmatrix}$$

and

$$(4.7) \qquad (I^{\boldsymbol{\mu},\boldsymbol{\mu}}(\boldsymbol{\mu},\boldsymbol{\lambda}))^{-1} = \begin{bmatrix} \frac{n_1\lambda_1}{\mu_1^3} & 0 & \cdots & 0 \\ 0 & \frac{n_2\lambda_2}{\mu_2^3} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{n_k\lambda_k}{\mu_k^3} \end{bmatrix}$$

are obtained.

(iii) By substituting $(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\lambda}})$ into the (4.7), we get

$$(4.8) \qquad \left(I^{\boldsymbol{\mu},\boldsymbol{\mu}}(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\lambda}})\right)^{-1} = \operatorname{diag}(n_i\hat{\lambda}_i/\hat{\mu}_i^3) \quad i = 1, ..., k.$$

(iv) The test statistic W is derived as follows:

$$(4.9) \qquad W = \sum_{i=1}^{k} \frac{n_i\hat{\lambda}_i(\hat{\mu}_i - \tilde{\mu}^*)^2}{\hat{\mu}_i^3}.$$

4.2. Score (S) test statistic

The test statistic S is defined as

$$(4.10) \qquad S = \boldsymbol{U}'_{\boldsymbol{\mu}}(\tilde{\boldsymbol{\mu}}^*, \tilde{\boldsymbol{\lambda}}^*)\left(I^{\boldsymbol{\mu},\boldsymbol{\mu}}(\tilde{\boldsymbol{\mu}}^*, \tilde{\boldsymbol{\lambda}}^*)\right)\boldsymbol{U}_{\boldsymbol{\mu}}(\tilde{\boldsymbol{\mu}}^*, \tilde{\boldsymbol{\lambda}}^*).$$

Here, $\tilde{\boldsymbol{\mu}}^*$ and $\tilde{\boldsymbol{\lambda}}^*$ are the RPML estimators of the vector of means $\boldsymbol{\mu}=(\mu_1, ..., \mu_k)$ and the vector of scale parameters $\boldsymbol{\lambda}=(\lambda_1, ..., \lambda_k)$. $\boldsymbol{U}_{\boldsymbol{\mu}}(\tilde{\boldsymbol{\mu}}^*, \tilde{\boldsymbol{\lambda}}^*)$ is the value of the vector of score function $\boldsymbol{U}_{\boldsymbol{\mu}}(\boldsymbol{\mu}, \boldsymbol{\lambda})$ at the point $(\tilde{\boldsymbol{\mu}}^*, \tilde{\boldsymbol{\lambda}}^*)=(\tilde{\mu}^*, ..., \tilde{\mu}^*, \tilde{\lambda}_1^*, ..., \tilde{\lambda}_k^*)$. To obtain the (4.10), the following steps are considered.

(i) By taking the first partial derivatives of the $\ln L_1$ with respect to the parameters

$$(4.11) \qquad \boldsymbol{U}'_{\boldsymbol{\mu}}(\boldsymbol{\mu}, \boldsymbol{\lambda}) = \left[\frac{n_1\lambda_1(\bar{X}_1 - \mu_1)}{\mu_1^3}, ..., \frac{n_k\lambda_k(\bar{X}_k - \mu_k)}{\mu_k^3}\right]$$

is obtained.

(ii) By substituting $(\tilde{\boldsymbol{\mu}}^*, \tilde{\boldsymbol{\lambda}}^*)$ into the (4.11), we get

$$(4.12) \qquad \boldsymbol{U}'_{\boldsymbol{\mu}}(\tilde{\boldsymbol{\mu}}^*, \tilde{\boldsymbol{\lambda}}^*) = \left[\frac{n_1\tilde{\lambda}_1^*(\bar{X}_1 - \tilde{\mu}^*)}{\tilde{\mu}^{*3}}, ..., \frac{n_k\tilde{\lambda}_k^*(\bar{X}_k - \tilde{\mu}^*)}{\tilde{\mu}^{*3}}\right].$$

(iii) By substituting $(\tilde{\boldsymbol{\mu}}^*, \tilde{\boldsymbol{\lambda}}^*)$ into the (4.6), we get

$$(4.13) \qquad \left(I^{\boldsymbol{\mu},\boldsymbol{\mu}}(\tilde{\boldsymbol{\mu}}^*, \tilde{\boldsymbol{\lambda}}^*)\right) = \operatorname{diag}(\tilde{\mu}^{*3}/n_i\tilde{\lambda}_i^*), \quad i = 1, ..., k.$$

(iv) The test statistic S is derived as follows:

$$(4.14) \qquad S = \sum_{i=1}^{k} \frac{n_i\tilde{\lambda}_i^*(\hat{\mu}_i - \tilde{\mu}^*)^2}{\tilde{\mu}^{*3}}.$$

4.3. Likelihood ratio (LR) test statistic

The test statistic LR is derived as

(4.15)
$$LR = 2[\ln L_1(\hat{\mu}_1, ..., \hat{\mu}_k; \hat{\lambda}_1, ..., \hat{\lambda}_k) - \ln L_0(\tilde{\mu}^*, ..., \tilde{\mu}^*; \tilde{\lambda}_1^*, ..., \tilde{\lambda}_k^*)]$$
$$= \sum_{i=1}^{k} n_i \ln \frac{\hat{\lambda}_i}{\tilde{\lambda}_i^*}.$$

It should be noted that W, S and LR test statistics are approximated by a chi-square distribution with $(k-1)$ degrees of freedom under H_0 and their values are asymptotically equivalent to each other. However, the mentioned approximation to the distributions of the test statistics may not be accurate for small sample sizes. This problem is also valid even for moderately large sample sizes. To eliminate this problem, likelihood-based test statistics, which are the most appropriate for PB methods can be used. The CAT procedure is a special case of PB methods; therefore, we incorporate the likelihood based test statistics W, S and LR into the CAT and call these tests $CATW$, $CATS$ and $CATLR$, respectively.

The algorithm of the CAT procedure based on T (T is any of W, S or LR) test statistic is as follows:

Step 1: The observed value of the test statistic T, i.e., T_0, is calculated.

Step 2: RPML estimators $\tilde{\mu}^*$ and $\tilde{\lambda}_i^*$, $i = 1, ..., k$, are computed.

Step 3: Artificial samples X_{ij}, $i = 1, ..., k; j = 1, ..., n_i$ from $IG(\tilde{\mu}^*, \tilde{\lambda}_i^*)$ are generated under H_0.

Step 4: For a large number of times, say m, step 3 is repeated. For each of the replicated samples, the values of the test statistic $T^{(l)}, l = 1, ..., m$, are calculated.

Step 5: Monte Carlo estimate of the p-value for testing (1.2) is calculated as $\hat{p} = \#(T^{(l)} \geq T_0)/m$.

Step 6: If $\hat{p} < \alpha$, then H_0 given in (1.2) is rejected.

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5. SIMULATION STUDY

In this section, the performances of the proposed tests $CATW$, $CATS$ and $CATLR$ are compared with the GP approach proposed by Tian [16], the PB approach proposed by Ma and Tian [9] and the CAT approach proposed by Gökpınar *et al.* [6] with respect to the estimated Type I error rate and power criteria via the MATLAB environment under the specified nominal level $\alpha = 0.050$. In comparing the performances of the proposed tests and the existing tests, 5,000 random samples are generated from the $IG(\mu_i, \lambda_i)$, $i = 1, ..., k$ distribution and $m = 5,000$ Monte Carlo runs are used for each of the samples. In the simulation study the following setup is used.

		Sample Sizes	Parameter Values	
		(n_1, n_2, n_3)	$(\lambda_1, \lambda_2, \lambda_3)$	(μ_1, μ_2, μ_3)
	k=3	$(8, 8, 8)$ $(15, 15, 15)$ $(30, 30, 30)$ $(8, 10, 12)$ $(10, 20, 30)$	$(10, 11, 12)$ $(10, 12, 14)$ $(10, 15, 20)$	$(2, 2.25, 2.5)$ $(2, 2.5, 3)$ $(2, 3, 4)$
		$(n_1, n_2, n_3, n_4, n_5)$	$(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$	$(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5)$
Number of Groups	k=5	$(8, 8, 8, 8, 8)$ $(15, 15, 15, 15, 15)$ $(30, 30, 30, 30, 30)$ $(8, 8, 10, 12, 12)$ $(10, 10, 20, 30, 30)$	$(10, 10, 11, 12, 12)$ $(10, 10, 12, 14, 14)$ $(10, 10, 15, 20, 20)$	$(2, 2, 2.25, 2.5, 2.5)$ $(2, 2, 2.5, 3, 3)$ $(2, 2, 3, 4, 4)$
		$(n_1, n_2, n_3, n_4, n_5, n_6, n_7)$	$(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7)$	$(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7)$
	k=7	$(8, 8, 8, 8, 8, 8, 8)$ $(15, 15, 15, 15, 15, 15, 15)$ $(30, 30, 30, 30, 30, 30, 30)$ $(8, 8, 10, 10, 10, 12, 12)$ $(10, 10, 20, 20, 20, 30, 30)$	$(10, 10, 11, 11, 11, 12, 12)$ $(10, 10, 12, 12, 12, 14, 14)$ $(10, 10, 15, 15, 15, 20, 20)$	$(2, 2, 2.25, 2.25, 2.25, 3, 3)$ $(2, 2, 2.5, 2.5, 2.5, 3, 3)$ $(2, 2, 3, 3, 3, 4, 4)$

The estimated type I error rates for the proposed and the existing tests are presented in Tables 1–3.

Table 1:   Simulated Type I error rates for the $CATW$, $CATS$, $CATLR$, GP, PB and CAT tests when $k=3$.

$(\lambda_1, \lambda_2, \lambda_3)$	$CATW$	$CATS$	$CATLR$	GP	PB	CAT
			$(n_1, n_2, n_3) = (8, 8, 8)$			
$(10, 11, 12)$	0.0454	0.0486	0.0460	0.0400	0.0616	0.0382
$(10, 12, 14)$	0.0456	0.0454	0.0476	0.0404	0.0606	0.0398
$(10, 15, 20)$	0.0448	0.0440	0.0432	0.0374	0.0602	0.0358
			$(n_1, n_2, n_3) = (15, 15, 15)$			
$(10, 11, 12)$	0.0558	0.0558	0.0554	0.0524	0.0636	0.0522
$(10, 12, 14)$	0.0554	0.0582	0.0558	0.0528	0.0658	0.0506
$(10, 15, 20)$	0.0482	0.0504	0.0492	0.0444	0.0588	0.0460
			$(n_1, n_2, n_3) = (30, 30, 30)$			
$(10, 11, 12)$	0.0466	0.0464	0.0456	0.0458	0.0496	0.0456
$(10, 12, 14)$	0.0494	0.0498	0.0496	0.0468	0.0514	0.0504
$(10, 15, 20)$	0.0464	0.0448	0.0442	0.0436	0.0504	0.0442
			$(n_1, n_2, n_3) = (8, 10, 12)$			
$(10, 11, 12)$	0.0482	0.0514	0.0496	0.0460	0.0654	0.0446
$(10, 12, 14)$	0.0462	0.0478	0.0464	0.0442	0.0636	0.0380
$(10, 15, 20)$	0.0500	0.0472	0.0498	0.0466	0.0600	0.0398
			$(n_1, n_2, n_3) = (10, 20, 30)$			
$(10, 11, 12)$	0.0522	0.0522	0.0504	0.0494	0.0584	0.0454
$(10, 12, 14)$	0.0534	0.0546	0.0534	0.0522	0.0592	0.0498
$(10, 15, 20)$	0.0524	0.0534	0.0552	0.0542	0.0584	0.0452

Table 2: Simulated Type I error rates for the $CATW$, $CATS$, $CATLR$, GP, PB and CAT tests when k=5.

$(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$	$CATW$	$CATS$	$CATLR$	GP	PB	CAT
$(n_1, n_2, n_3, n_4, n_5) = (8, 8, 8, 8, 8)$						
$(10, 10, 11, 12, 12)$	0.0472	0.0486	0.0484	0.0676	0.0480	0.0322
$(10, 10, 12, 14, 14)$	0.0480	0.0486	0.0458	0.0614	0.0498	0.0310
$(10, 10, 15, 20, 20)$	0.0498	0.0514	0.0490	0.0610	0.0508	0.0338
$(n_1, n_2, n_3, n_4, n_5) = (15, 15, 15, 15, 15)$						
$(10, 10, 11, 12, 12)$	0.0536	0.0508	0.0506	0.0522	0.0524	0.0400
$(10, 10, 12, 14, 14)$	0.0460	0.0468	0.0450	0.0502	0.0470	0.0416
$(10, 10, 15, 20, 20)$	0.0530	0.0568	0.0564	0.0620	0.0572	0.0460
$(n_1, n_2, n_3, n_4, n_5) = (30, 30, 30, 30, 30)$						
$(10, 10, 11, 12, 12)$	0.0530	0.0546	0.0548	0.0570	0.0556	0.0496
$(10, 10, 12, 14, 14)$	0.0508	0.0482	0.0486	0.0510	0.0494	0.0452
$(10, 10, 15, 20, 20)$	0.0500	0.0550	0.0536	0.0556	0.0528	0.0466
$(n_1, n_2, n_3, n_4, n_5) = (8, 8, 10, 12, 12)$						
$(10, 10, 11, 12, 12)$	0.0502	0.0510	0.0508	0.0600	0.0538	0.0350
$(10, 10, 12, 14, 14)$	0.0466	0.0494	0.0496	0.0602	0.0500	0.0376
$(10, 10, 15, 20, 20)$	0.0482	0.0472	0.0472	0.0602	0.0468	0.0332
$(n_1, n_2, n_3, n_4, n_5) = (10, 10, 20, 30, 30)$						
$(10, 10, 11, 12, 12)$	0.0516	0.0522	0.0500	0.0602	0.0492	0.0408
$(10, 10, 12, 14, 14)$	0.0516	0.0522	0.0500	0.0612	0.0492	0.0408
$(10, 10, 15, 20, 20)$	0.0486	0.0480	0.0496	0.0634	0.0472	0.0364

Table 3: Simulated Type I error rates for the $CATW$, $CATS$, $CATLR$, GP, PB and CAT tests when k=7.

$(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7)$	$CATW$	$CATS$	$CATLR$	GP	PB	CAT
$(n_1, n_2, n_3, n_4, n_5, n_6, n_7) = (8, 8, 8, 8, 8, 8, 8)$						
$(10, 10, 11, 11, 11, 12, 12)$	0.0470	0.0510	0.0518	0.0774	0.0482	0.0298
$(10, 10, 12, 12, 12, 14, 14)$	0.0482	0.0482	0.0468	0.0716	0.0422	0.0272
$(10, 10, 15, 15, 15, 20, 20)$	0.0474	0.0550	0.0530	0.0770	0.0484	0.0350
$(n_1, n_2, n_3, n_4, n_5, n_6, n_7) = (15, 15, 15, 15, 15, 15, 15)$						
$(10, 10, 11, 11, 11, 12, 12)$	0.0518	0.0522	0.0526	0.0646	0.0516	0.0440
$(10, 10, 12, 12, 12, 14, 14)$	0.0506	0.0526	0.0534	0.0636	0.0516	0.0388
$(10, 10, 15, 15, 15, 20, 20)$	0.0488	0.0518	0.0498	0.0636	0.0492	0.0398
$(n_1, n_2, n_3, n_4, n_5, n_6, n_7) = (30, 30, 30, 30, 30, 30, 30)$						
$(10, 10, 11, 11, 11, 12, 12)$	0.0500	0.0536	0.0516	0.0610	0.0524	0.0462
$(10, 10, 12, 12, 12, 14, 14)$	0.0518	0.0554	0.0562	0.0620	0.0568	0.0494
$(10, 10, 15, 15, 15, 20, 20)$	0.0500	0.0532	0.0524	0.0624	0.0518	0.0454
$(n_1, n_2, n_3, n_4, n_5, n_6, n_7) = (8, 8, 10, 10, 10, 12, 12)$						
$(10, 10, 11, 11, 11, 12, 12)$	0.0434	0.0436	0.0442	0.0656	0.0436	0.0306
$(10, 10, 12, 12, 12, 14, 14)$	0.0486	0.0486	0.0508	0.0708	0.0444	0.0356
$(10, 10, 15, 15, 15, 20, 20)$	0.0516	0.0488	0.0502	0.0742	0.0458	0.0350
$(n_1, n_2, n_3, n_4, n_5, n_6, n_7) = (10, 10, 20, 20, 20, 30, 30)$						
$(10, 10, 11, 11, 11, 12, 12)$	0.0502	0.0576	0.0532	0.0670	0.0534	0.0442
$(10, 10, 12, 12, 12, 14, 14)$	0.0518	0.0500	0.0506	0.0678	0.0474	0.0408
$(10, 10, 15, 15, 15, 20, 20)$	0.0476	0.0484	0.0462	0.0604	0.0472	0.0400

Table 4:　Simulated power values for the $CATW$, $CATS$, $CATLR$, GP, PB and CAT tests when $k=3$.

$(\lambda_1, \lambda_2, \lambda_3)$	(μ_1, μ_2, μ_3)	$CATW$	$CATS$	$CATLR$	GP	PB	CAT
		$(n_1, n_2, n_3) = (8, 8, 8)$					
	$(2, 2.25, 2.5)$	0.1018	0.1028	0.1194	0.1066	***	0.1006
$(10, 11, 12)$	$(2, 2.5, 3)$	0.2120	0.1896	0.2392	0.2200	***	0.2276
	$(2, 3, 4)$	0.5226	0.3314	0.5486	0.5318	***	0.5660
	$(2, 2.25, 2.5)$	0.1096	0.0940	0.1128	0.1034	***	0.1036
$(10, 12, 14)$	$(2, 2.5, 3)$	0.2392	0.1940	0.2600	0.2402	***	0.2510
	$(2, 3, 4)$	0.5608	0.3254	0.5610	0.5430	***	0.5800
	$(2, 2.25, 2.5)$	0.1378	0.0914	0.1260	0.1138	***	0.1160
$(10, 15, 20)$	$(2, 2.5, 3)$	0.3110	0.1958	0.2952	0.2786	***	0.2846
	$(2, 3, 4)$	0.6432	0.3352	0.6238	0.6108	***	0.6222
		$(n_1, n_2, n_3) = (15, 15, 15)$					
	$(2, 2.25, 2.5)$	0.1806	0.1616	0.1800	0.1750	***	0.1848
$(10, 11, 12)$	$(2, 2.5, 3)$	0.4906	0.4180	0.4814	0.4694	***	0.4908
	$(2, 3, 4)$	0.9016	0.7620	0.8864	0.8834	***	0.9054
	$(2, 2.25, 2.5)$	0.2076	0.1840	0.2062	0.1968	***	0.2020
$(10, 12, 14)$	$(2, 2.5, 3)$	0.5220	0.4462	0.5156	0.5068	***	0.5308
	$(2, 3, 4)$	0.9174	0.7596	0.9010	0.8974	***	0.9228
	$(2, 2.25, 2.5)$	0.2528	0.1918	0.2300	0.2228	0.2546	0.2290
$(10, 15, 20)$	$(2, 2.5, 3)$	0.6060	0.4696	0.5748	0.5678	0.5966	0.5962
	$(2, 3, 4)$	0.9546	0.7968	0.9372	0.9332	0.9414	0.9510
		$(n_1, n_2, n_3) = (30, 30, 30)$					
	$(2, 2.25, 2.5)$	0.3708	0.3466	0.3656	0.3622	0.3792	0.3734
$(10, 11, 12)$	$(2, 2.5, 3)$	0.8344	0.8036	0.8292	0.8258	0.8410	0.8430
	$(2, 3, 4)$	0.9982	0.9964	0.9986	0.9982	0.9984	0.9982
	$(2, 2.25, 2.5)$	0.3984	0.3564	0.3796	0.3778	0.3966	0.3882
$(10, 12, 14)$	$(2, 2.5, 3)$	0.8678	0.8272	0.8534	0.8502	0.8610	0.8656
	$(2, 3, 4)$	0.9990	0.9952	0.9986	0.9980	0.9986	0.9986
	$(2, 2.25, 2.5)$	0.4872	0.4090	0.4462	0.4420	0.4622	0.4570
$(10, 15, 20)$	$(2, 2.5, 3)$	0.9174	0.8720	0.9024	0.8976	0.9080	0.9080
	$(2, 3, 4)$	0.9998	0.9980	0.9996	0.9998	0.9998	0.9998
		$(n_1, n_2, n_3) = (8, 10, 12)$					
	$(2, 2.25, 2.5)$	0.1402	0.0964	0.1302	0.1218	***	0.1188
$(10, 11, 12)$	$(2, 2.5, 3)$	0.3066	0.1904	0.2808	0.2668	***	0.2670
	$(2, 3, 4)$	0.6680	0.3410	0.6166	0.6032	***	0.6240
	$(2, 2.25, 2.5)$	0.1492	0.0912	0.1302	0.1240	***	0.1164
$(10, 12, 14)$	$(2, 2.5, 3)$	0.3560	0.2136	0.3212	0.3082	***	0.3094
	$(2, 3, 4)$	0.7032	0.3448	0.6480	0.6402	***	0.6564
	$(2, 2.25, 2.5)$	0.1782	0.0898	0.1392	0.1322	***	0.1262
$(10, 15, 20)$	$(2, 2.5, 3)$	0.3900	0.2064	0.3408	0.3304	***	0.3282
	$(2, 3, 4)$	0.7658	0.3682	0.6958	0.6892	***	0.7072
		$(n_1, n_2, n_3) = (10, 20, 30)$					
	$(2, 2.25, 2.5)$	0.2448	0.1264	0.1912	0.1878	0.1958	0.1778
$(10, 11, 12)$	$(2, 2.5, 3)$	0.5624	0.3250	0.4854	0.4824	0.4842	0.4698
	$(2, 3, 4)$	0.9112	0.5672	0.8634	0.8592	0.8502	0.8644
	$(2, 2.25, 2.5)$	0.2514	0.1406	0.2032	0.2016	0.2176	0.1886
$(10, 12, 14)$	$(2, 2.5, 3)$	0.5908	0.3352	0.5062	0.5016	0.5132	0.4956
	$(2, 3, 4)$	0.9360	0.5970	0.8832	0.8796	0.8690	0.8874
	$(2, 2.25, 2.5)$	0.2852	0.1472	0.2222	0.2244	0.2364	0.2040
$(10, 15, 20)$	$(2, 2.5, 3)$	0.6412	0.3972	0.5624	0.5626	0.5684	0.5394
	$(2, 3, 4)$	0.9564	0.6716	0.9162	0.9164	0.9066	0.9128

***: The estimated type I error rates of the tests which are greater than or equal to 0.060.

Table 5: Simulated power values for the $CATW$, $CATS$, $CATLR$, GP, PB and CAT tests when k=5.

$(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$	$(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5)$	$CATW$	$CATS$	$CATLR$	GP	PB	CAT
	$(n_1, n_2, n_3, n_4, n_5) = (8, 8, 8, 8, 8)$						
	$(2, 2, 2.25, 2.5, 2.5)$	0.1126	0.1266	0.1424	***	0.1462	0.1092
$(10, 10, 11, 12, 12)$	$(2, 2, 2.5, 3, 3)$	0.2800	0.2886	0.3530	***	0.3648	0.3104
	$(2, 2, 3, 4, 4)$	0.6882	0.5216	0.7842	***	0.8000	0.7744
	$(2, 2, 2.25, 2.5, 2.5)$	0.1300	0.1106	0.1414	***	0.1428	0.1126
$(10, 10, 12, 14, 14)$	$(2, 2, 2.5, 3, 3)$	0.3320	0.2964	0.3828	***	0.4018	0.3490
	$(2, 2, 3, 4, 4)$	0.7332	0.5032	0.7970	***	0.8110	0.7900
	$(2, 2, 2.25, 2.5, 2.5)$	0.1698	0.1172	0.1614	***	0.1704	0.1304
$(10, 10, 15, 20, 20)$	$(2, 2, 2.5, 3, 3)$	0.4120	0.2690	0.4224	***	0.4414	0.3868
	$(2, 2, 3, 4, 4)$	0.8282	0.4770	0.8436	***	0.8526	0.8390
	$(n_1, n_2, n_3, n_4, n_5) = (15, 15, 15, 15, 15)$						
	$(2, 2, 2.25, 2.5, 2.5)$	0.2640	0.2408	0.2600	0.2734	0.2642	0.2522
$(10, 10, 11, 12, 12)$	$(2, 2, 2.5, 3, 3)$	0.6962	0.6596	0.7048	0.7216	0.7186	0.7066
	$(2, 2, 3, 4, 4)$	0.9816	0.9676	0.9890	0.9900	0.9900	0.9906
	$(2, 2, 2.25, 2.5, 2.5)$	0.2790	0.2460	0.2778	0.2974	0.2876	0.2714
$(10, 10, 12, 14, 14)$	$(2, 2, 2.5, 3, 3)$	0.7140	0.6614	0.7232	0.7384	0.7310	0.7312
	$(2, 2, 3, 4, 4)$	0.9918	0.9602	0.9904	0.9922	0.9914	0.9936
	$(2, 2, 2.25, 2.5, 2.5)$	0.3378	0.2542	0.3128	***	0.3220	0.2982
$(10, 10, 15, 20, 20)$	$(2, 2, 2.5, 3, 3)$	0.8102	0.6916	0.7866	***	0.7998	0.7896
	$(2, 2, 3, 4, 4)$	0.9950	0.9682	0.9944	***	0.9954	0.9960
	$(n_1, n_2, n_3, n_4, n_5) = (30, 30, 30, 30, 30)$						
	$(2, 2, 2.25, 2.5, 2.5)$	0.5174	0.5080	0.5218	0.5356	0.5324	0.5226
$(10, 10, 11, 12, 12)$	$(2, 2, 2.5, 3, 3)$	0.9670	0.9626	0.9698	0.9708	0.9702	0.9714
	$(2, 2, 3, 4, 4)$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	$(2, 2, 2.25, 2.5, 2.5)$	0.5746	0.5482	0.5730	0.5822	0.5812	0.5740
$(10, 10, 12, 14, 14)$	$(2, 2, 2.5, 3, 3)$	0.9810	0.9728	0.9804	0.9804	0.9810	0.9808
	$(2, 2, 3, 4, 4)$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	$(2, 2, 2.25, 2.5, 2.5)$	0.6656	0.5830	0.6274	0.6366	0.6332	0.6196
$(10, 10, 15, 20, 20)$	$(2, 2, 2.5, 3, 3)$	0.9922	0.9832	0.9906	0.9912	0.9914	0.9896
	$(2, 2, 3, 4, 4)$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	$(n_1, n_2, n_3, n_4, n_5) = (8, 8, 10, 12, 12)$						
	$(2, 2, 2.25, 2.5, 2.5)$	0.1718	0.1184	0.1612	***	0.1616	0.1292
$(10, 10, 11, 12, 12)$	$(2, 2, 2.5, 3, 3)$	0.4234	0.2792	0.4190	***	0.4180	0.3956
	$(2, 2, 3, 4, 4)$	0.8606	0.4846	0.8500	***	0.8470	0.8520
	$(2, 2, 2.25, 2.5, 2.5)$	0.1942	0.1210	0.1726	***	0.1704	0.1420
$(10, 10, 12, 14, 14)$	$(2, 2, 2.5, 3, 3)$	0.4752	0.2800	0.4500	***	0.4508	0.4248
	$(2, 2, 3, 4, 4)$	0.8762	0.4718	0.8636	***	0.8566	0.8616
	$(2, 2, 2.25, 2.5, 2.5)$	0.2184	0.1098	0.1780	***	0.1708	0.1510
$(10, 10, 15, 20, 20)$	$(2, 2, 2.5, 3, 3)$	0.5506	0.2850	0.5000	***	0.4940	0.4550
	$(2, 2, 3, 4, 4)$	0.9298	0.4582	0.9026	***	0.8962	0.9020
	$(n_1, n_2, n_3, n_4, n_5) = (10, 10, 20, 30, 30)$						
	$(2, 2, 2.25, 2.5, 2.5)$	0.3274	0.1516	0.2532	***	0.2422	0.2304
$(10, 10, 11, 12, 12)$	$(2, 2, 2.5, 3, 3)$	0.7514	0.4418	0.6770	***	0.6580	0.6558
	$(2, 2, 3, 4, 4)$	0.9912	0.7222	0.9792	***	0.9740	0.9822
	$(2, 2, 2.25, 2.5, 2.5)$	0.3394	0.1642	0.2612	***	0.2502	0.2418
$(10, 10, 12, 14, 14)$	$(2, 2, 2.5, 3, 3)$	0.7550	0.4274	0.6740	***	0.6594	0.6650
	$(2, 2, 3, 4, 4)$	0.9930	0.7252	0.9848	***	0.9788	0.9870
	$(2, 2, 2.25, 2.5, 2.5)$	0.3688	0.1674	0.2928	***	0.2798	0.2680
$(10, 10, 15, 20, 20)$	$(2, 2, 2.5, 3, 3)$	0.8206	0.4834	0.7488	***	0.7340	0.7312
	$(2, 2, 3, 4, 4)$	0.9944	0.7646	0.9896	***	0.9852	0.9900

***: The estimated type I error rates of the tests which are greater than or equal to 0.060.

Table 6:   Simulated power values for the $CATW$, $CATS$, $CATLR$, GP,
PB and CAT tests when k=7.

$(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7)$	$(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7)$	$CATW$	$CATS$	$CATLR$	GP	PB	CAT
$(n_1, n_2, n_3, n_4, n_5, n_6, n_7) = (8, 8, 8, 8, 8, 8, 8)$							
$(10, 10, 11, 11, 11, 12, 12)$	$(2, 2, 2.25, 2.25, 2.25, 3, 3)$	0.1082	0.1154	0.1292	***	0.1246	0.0954
	$(2, 2, 2.5, 2.5, 2.5, 3, 3)$	0.2386	0.2548	0.2934	***	0.2856	0.2668
	$(2, 2, 3, 3, 3, 4, 4)$	0.5970	0.4976	0.6876	***	0.6808	0.6956
$(10, 10, 12, 12, 12, 14, 14)$	$(2, 2, 2.25, 2.25, 2.25, 3, 3)$	0.1122	0.1064	0.1234	***	0.1176	0.0960
	$(2, 2, 2.5, 2.5, 2.5, 3, 3)$	0.2570	0.2582	0.3110	***	0.3018	0.2844
	$(2, 2, 3, 3, 3, 4, 4)$	0.6514	0.5010	0.7162	***	0.7042	0.7330
$(10, 10, 15, 15, 15, 20, 20)$	$(2, 2, 2.25, 2.25, 2.25, 3, 3)$	0.1412	0.1082	0.1392	***	0.1312	0.1038
	$(2, 2, 2.5, 2.5, 2.5, 3, 3)$	0.3246	0.2614	0.3580	***	0.3540	0.3258
	$(2, 2, 3, 3, 3, 4, 4)$	0.7482	0.5286	0.7824	***	0.7782	0.8010
$(n_1, n_2, n_3, n_4, n_5, n_6, n_7) = (15, 15, 15, 15, 15, 15, 15)$							
$(10, 10, 11, 11, 11, 12, 12)$	$(2, 2, 2.25, 2.25, 2.25, 3, 3)$	0.1898	0.1958	0.2112	***	0.2044	0.1914
	$(2, 2, 2.5, 2.5, 2.5, 3, 3)$	0.5696	0.5600	0.6168	***	0.6128	0.6204
	$(2, 2, 3, 3, 3, 4, 4)$	0.9702	0.9168	0.9752	***	0.9746	0.9832
$(10, 10, 12, 12, 12, 14, 14)$	$(2, 2, 2.25, 2.25, 2.25, 3, 3)$	0.2094	0.1966	0.2176	***	0.2210	0.068
	$(2, 2, 2.5, 2.5, 2.5, 3, 3)$	0.6228	0.5796	0.6476	***	0.6504	0.6588
	$(2, 2, 3, 3, 3, 4, 4)$	0.9804	0.9308	0.9822	***	0.9826	0.9870
$(10, 10, 15, 15, 15, 20, 20)$	$(2, 2, 2.25, 2.25, 2.25, 3, 3)$	0.2812	0.2262	0.2672	***	0.2682	0.2462
	$(2, 2, 2.5, 2.5, 2.5, 3, 3)$	0.7222	0.6422	0.7256	***	0.7324	0.7340
	$(2, 2, 3, 3, 3, 4, 4)$	0.9928	0.9510	0.9896	***	0.9896	0.9946
$(n_1, n_2, n_3, n_4, n_5, n_6, n_7) = (30, 30, 30, 30, 30, 30, 30)$							
$(10, 10, 11, 11, 11, 12, 12)$	$(2, 2, 2.25, 2.25, 2.25, 3, 3)$	0.4520	0.4534	0.4722	***	0.4678	0.4692
	$(2, 2, 2.5, 2.5, 2.5, 3, 3)$	0.9400	0.9262	0.9404	***	0.9410	0.9474
	$(2, 2, 3, 3, 3, 4, 4)$	1.0000	0.9998	1.0000	***	1.0000	1.0000
$(10, 10, 12, 12, 12, 14, 14)$	$(2, 2, 2.25, 2.25, 2.25, 3, 3)$	0.4894	0.4600	0.4892	***	0.4876	0.4892
	$(2, 2, 2.5, 2.5, 2.5, 3, 3)$	0.9640	0.9464	0.9604	***	0.9618	0.9652
	$(2, 2, 3, 3, 3, 4, 4)$	1.0000	0.9998	1.0000	***	1.0000	1.0000
$(10, 10, 15, 15, 15, 20, 20)$	$(2, 2, 2.25, 2.25, 2.25, 3, 3)$	0.5968	0.5318	0.5752	***	0.5760	0.5670
	$(2, 2, 2.5, 2.5, 2.5, 3, 3)$	0.9790	0.9656	0.9752	***	0.9750	0.9794
	$(2, 2, 3, 3, 3, 4, 4)$	1.0000	1.0000	1.0000	***	1.0000	1.0000
$(n_1, n_2, n_3, n_4, n_5, n_6, n_7) = (8, 8, 10, 10, 10, 12, 12)$							
$(10, 10, 11, 11, 11, 12, 12)$	$(2, 2, 2.25, 2.25, 2.25, 3, 3)$	0.1366	0.1086	0.1316	***	0.1280	0.1112
	$(2, 2, 2.5, 2.5, 2.5, 3, 3)$	0.3630	0.7380	0.3652	***	0.3506	0.3298
	$(2, 2, 3, 3, 3, 4, 4)$	0.7752	0.5306	0.7788	***	0.7572	0.7912
$(10, 10, 12, 12, 12, 14, 14)$	$(2, 2, 2.25, 2.25, 2.25, 3, 3)$	0.1526	0.1020	0.1320	***	0.1288	0.1114
	$(2, 2, 2.5, 2.5, 2.5, 3, 3)$	0.3752	0.2830	0.3836	***	0.3702	0.3524
	$(2, 2, 3, 3, 3, 4, 4)$	0.8216	0.5490	0.8094	***	0.7850	0.8248
$(10, 10, 15, 15, 15, 20, 20)$	$(2, 2, 2.25, 2.25, 2.25, 3, 3)$	0.1860	0.1162	0.1634	***	0.1534	0.1378
	$(2, 2, 2.5, 2.5, 2.5, 3, 3)$	0.4652	0.3202	0.4510	***	0.4424	0.4148
	$(2, 2, 3, 3, 3, 4, 4)$	0.8720	0.6136	0.8608	***	0.8484	0.8690
$(n_1, n_2, n_3, n_4, n_5, n_6, n_7) = (10, 10, 20, 20, 20, 30, 30)$							
$(10, 10, 11, 11, 11, 12, 12)$	$(2, 2, 2.25, 2.25, 2.25, 3, 3)$	0.2714	0.1678	0.2230	***	0.2140	0.2008
	$(2, 2, 2.5, 2.5, 2.5, 3, 3)$	0.6850	0.4904	0.6396	***	0.6248	0.6198
	$(2, 2, 3, 3, 3, 4, 4)$	0.9818	0.8286	0.9698	***	0.9624	0.9722
$(10, 10, 12, 12, 12, 14, 14)$	$(2, 2, 2.25, 2.25, 2.25, 3, 3)$	0.2916	0.1768	0.2434	***	0.2352	0.2214
	$(2, 2, 2.5, 2.5, 2.5, 3, 3)$	0.7168	0.5176	0.6736	***	0.6576	0.6534
	$(2, 2, 3, 3, 3, 4, 4)$	0.9888	0.8438	0.9780	***	0.9680	0.9824
$(10, 10, 15, 15, 15, 20, 20)$	$(2, 2, 2.25, 2.25, 2.25, 3, 3)$	0.3482	0.2162	0.2938	***	0.2826	0.2522
	$(2, 2, 2.5, 2.5, 2.5, 3, 3)$	0.7792	0.6010	0.7482	***	0.7330	0.7146
	$(2, 2, 3, 3, 3, 4, 4)$	0.9966	0.9236	0.9904	***	0.9864	0.9902

***: The estimated type I error rates of the tests which are greater than or equal to 0.060.

It can be seen from Table 1 that the estimated type I error rates of the tests are close to the nominal level, except for the PB approach when the number of observations in the populations are equal to $(n_1, n_2, n_3) = (8, 8, 8)$, $(8, 10, 12)$ and $(15, 15, 15)$. It can be seen from Table 2 that if the sample sizes are small and/or different the estimated type I error rates for the GP approach are larger than the nominal level. As seen from Table 3 that the estimated type I error rates for the GP approach are always greater than 0.060. It should be noted that the estimated type I error rates of the $CATW$, $CATS$, $CATLR$, PB and CAT approaches are close to the nominal level $\alpha = 0.050$ in all cases when the number of groups is moderate to large. The estimated powers of the proposed and the existing tests are presented in Tables 4–6. It should be noted that in Tables 4–6 the estimated type I error rates of the tests which are greater or equal to 0.060 are denoted by ***.

It can be seen from Table 4 that $CATS$ shows the worst performance in all cases. In addition, the $CATW$ outperforms the other tests when the sample sizes are unequal. It can be seen from Table 5 that the performances of the $CATLR$ and PB tests are similar when the sample sizes are small and equal. The $CATW$ test is more powerful than the other tests when the sample sizes are different. Also, $CATW$ shows the best performance when the sample sizes are moderately large and unequal while $CATS$ shows the worst performance. Since the simulation results given in Table 6 show a similar behavior as in Table 5, the same conclusions are drawn from Table 6. Therefore, we do not repeat them for the sake of brevity.

6. REAL DATA EXAMPLE

In this section, the fatigue life data taken from Leiva *et al.* [8] is analyzed to illustrate the implementation of the proposed and the existing tests; see Table 7. It contains fatigue life (T) of 6061-T6 aluminum pieces which were cut parallel to the direction of rolling and oscillating at 18 cycles/s at maximum stress levels of $x_1 = 2.1$, $x_2 = 2.6$ and $x_3 = 3.1$ psi ($\times 10^4$); see Leiva *et al.* [8] for more detailed information. These stress levels denote the groups whose means are to be compared and the sample sizes of them are $n_1 = 101$, $n_2 = 102$ and $n_3 = 101$, respectively.

Firstly we calculate the values of Lilliefors goodness of fit test and the corresponding p-value for each of the three groups to test the assumption that the data are from an IG distribution; see Table 8.

It can be seen from Table 8 that the IG distribution provides a good fit for each group of the fatigue life data, since the corresponding p-values are all greater than the nominal level $\alpha = 0.050$. Then, we compute the ML estimates $(\hat{\mu}_i, \hat{\lambda}_i)$ of the parameters (μ_i, λ_i) and the RML estimates $(\tilde{\mu}, \tilde{\lambda}_i)$ of the parameters (μ, λ_i), $i = 1, 2, 3$, by using the equalities given in Section 2; see Table 9.

The values of the W, S and LR test statistics based on the ML and RML estimates are calculated as 300141.0259, 41.2081 and 645.9193 using the (4.9), (4.14) and (4.15), respectively. In addition, to test the hypothesis of equality of means, the p-values for the existing tests GP and PB and the proposed tests $CATW$, $CATS$ and $CATLR$ are calculated using the algorithms given in Sections 3 and 4, respectively. The p-value for the CAT is calculated

from the algorithm given in Gökpınar *et al.* [6]. m=5,000 Monte Carlo runs are used in calculating the p-values for the fatigue life data. p-values for the proposed and the existing tests are obtained to be very close to zero. Therefore, the hypothesis of equality of means is rejected for all tests at the significance level $\alpha = 0.050$. It should be noted that all the calculations are made using MATLAB software.

Table 7: Fatigue life of aluminum pieces submitted to the maximum indicated stress level.

stress levels (psi)	
I: 2.1×10^4	370, 706, 716, 746, 785, 797, 844, 855, 858, 886, 886, 930, 960, 988, 999, 1000, 1010, 1016, 1018, 1020, 1055, 1085, 1102, 1102, 1108, 1115, 1120, 1134, 1140, 1199, 1200, 1200, 1203, 1222, 1235, 1238, 1252, 1258, 1262, 1269, 1270, 1290, 1293, 1300, 1310, 1313, 1315, 1330, 1355, 1390, 1416, 1419, 1420, 1420, 1450, 1452, 1475, 1478, 1481, 1485, 1502, 1505, 1513, 1522, 1522, 1530, 1540, 1560, 1567, 1578, 1594, 1602, 1604, 1608, 1630, 1642, 1674, 1730, 1750, 1750, 1763, 1768, 1781, 1782, 1792, 1820, 1868, 1881, 1890, 1893, 1895, 1910, 1923, 1924, 1945, 2023, 2100, 2130, 2215, 2268, 2440
II: 2.6×10^4	233, 258, 268, 276, 290, 310, 312, 315, 318, 321, 321, 329, 335, 336, 338, 338, 342, 342, 342, 344, 349, 350, 350, 351, 351, 352, 352, 356, 358, 358, 360, 362, 363, 366, 367, 370, 370, 372, 372, 374, 375, 376, 379, 379, 380, 382, 389, 389, 395, 396, 400, 400, 400, 403, 404, 406, 408, 408, 410, 412, 414, 416, 416, 416, 420, 422, 423, 426, 428, 432, 432, 433, 433, 437, 438, 439, 439, 443, 445, 445, 452, 456, 456, 460, 464, 466, 468, 470, 470, 473, 474, 476, 476, 486, 488, 489, 490, 491, 503, 517, 540, 560
III: 3.1×10^4	70, 90, 96, 97, 99, 100, 103, 104, 104, 105, 107, 108, 108, 108, 109, 109, 112, 112, 113, 114, 114, 114, 116, 119, 120, 120, 120, 121, 121, 123, 124, 124, 124, 124, 124, 128, 128, 129, 129, 130, 130, 130, 131, 131, 131, 131, 131, 132, 132, 132, 133, 134, 134, 134, 134, 134, 136, 136, 137, 138, 138, 138, 139, 139, 141, 141, 142, 142, 142, 142, 142, 142, 144, 144, 145, 146, 148, 148, 149, 151, 151, 152, 155, 156, 157, 157, 157, 157, 158, 159, 162, 163, 163, 164, 166, 166, 168, 170, 174, 196, 212

Table 8: Lilliefors goodness of fit test for the stress levels (groups) and the corresponding p-values.

	stress levels (psi)	I: 2.1×10^4	II: 2.6×10^4	III: 3.1×10^4
Lilliefors test	Test statistic	0.0699	0.0452	0.0752
	p-value	0.6011	0.9483	0.5154

Table 9: The ML and RML estimates for the parameters μ_i, μ and λ_i.

stress levels	ML		RML
I	$\hat{\mu}_1 = 1400.8$	$\hat{\lambda}_1 = 14222.3$	$\tilde{\lambda}_1 = 13876.9$
II	$\hat{\mu}_2 = 397.9$	$\hat{\lambda}_2 = 15165.8$	$\tilde{\lambda}_2 = 766.8$
III	$\hat{\mu}_3 = 133.7$	$\hat{\lambda}_3 = 4573.4$	$\tilde{\lambda}_3 = 159.4$
common mean			$\tilde{\mu} = 1334.8$

7. CONCLUSION

In this study, Wald, score and likelihood ratio test statistics are defined for the problem of testing the equality of IG means. Then they are plugged into the computational approach test procedure. The proposed tests are compared with the existing tests according to the estimated type I error rate and power criteria via a Monte Carlo simulation study. The estimated type I error rates for the proposed tests are close to the nominal level $\alpha = 0.050$ in all cases considered in the study. The computational approach test based on Wald test statistic appears to be more powerful than the other tests especially when the sample sizes are not equal among groups.

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