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Estimation and Prediction for the Half-Normal Distribution based on Progressively Type-II Censored Samples

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Abstract:

- In this paper, estimation and prediction problems are discussed for the half-normal distribution under a progressively Type-II censoring scheme. This study focuses on two statistical inferential problems. In the first part of the study, several point estimators and confidence intervals are obtained for the scale parameter of the half-normal distribution. In the second part, several predictors and predictive intervals are derived for the removed failure times. A Monte Carlo simulation study is performed to discuss the mean squared error (mean squared prediction errors) and bias of estimates (predictors). The coverage probabilities and average length of the confidence and predictive intervals are simulated and a numerical example is provided.

Key-Words:

- *confidence intervals; half-normal distribution; maximum likelihood estimator; Monte*

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Carlo simulation; pivotal estimator; prediction; predictive interval; progressively censoring; uncorrected likelihood ratio.

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1. INTRODUCTION

In the statistical literature, numerous distributions exist with two or more parameters. However, multi-parameter distributions can have problems with estimation and prediction due to non-identification. Therefore, in practice it is sometimes convenient to work with one-parameter distributions. One of the most popular single-parameter distributions is the half-normal (HN) distribution. In a recent study, Huang and Roth [10] demonstrated that the HN distribution is not only used for lifetime data but also in pragmatic randomized trials proving the convenience of the HN distribution for modeling different real data sets. The probability density function (pdf) and cumulative distribution function (cdf) of the HN distribution are given, respectively, by

$$f(x;\theta) = \frac{2}{\Gamma\left(\frac{1}{2}\right)\theta} \exp\left\{-\left(\frac{x}{\theta}\right)^2\right\}, \; x > 0$$

and

$$F(x;\theta) = 1 - \frac{\Gamma\left(\frac{1}{2}, \left(\frac{x}{\theta}\right)^2\right)}{\Gamma\left(\frac{1}{2}\right)}, \tag{1.1}$$

where $\theta > 0$ is a scale parameter and $\Gamma(\alpha, z)$ is the upper incomplete gamma function defined by

$$\Gamma(\alpha, z) = \int_z^{\infty} e^{-y} y^{\alpha-1} dy, \; \alpha, z > 0.$$

The distribution given with a cdf (1.1) will be denoted by HN(θ) for the remainder of this study. Shanker et al.[23] derived some statistical properties of the HN distribution (they called it Quasi-Exponential), such as moments, hazard and the hazard rate function, survival function and mean residual function. The maximum likelihood estimator (MLE) of the scale parameter is also studied. They examined the real data modeling capability of the HN distribution using lifetime data from biomedical science.

In reliability applications, the data is generally collected under some censoring schemes when the lifetime of the products is too long. One of the most popular schemes is progressive censoring. It should be pointed out that progressive censoring is not only used for reliability applications but also quite common in clinical trials due to the staggered entry. We refer readers to [15], [16], [20] and [21] for progressive censoring with staggered entry. In this paper, we consider the point and interval estimation (prediction) of the HN distribution under the progressive censoring scheme. A progressively Type-II censoring scheme is well-discussed by Balakrishnan and Aggarwala [4]. Progressively Type-II censored samples can be explained as follows: Let n units are put on a life test. When the first failure is observed, randomly selected r_1 of the $n-1$ surviving units are removed (withdrawn or censored) from the test. When the second failure occurs, randomly selected r_2 of the $n-2-r_1$ surviving units are removed from the test and this process is repeated. At the time of the mth failure, the remaining $r_m = n - m - r_1 - \cdots - r_{m-1}$ surviving units are removed randomly from the test. It can be easily observed that $n = m + \sum_{i=1}^{m} r_i$. The progressively Type-II censored (PCII) failure times are denoted by $X^{\mathbf{r}}_{1:m:n} < X^{\mathbf{r}}_{2:m:n} < \cdots < X^{\mathbf{r}}_{m:m:n}$.

There are several studies discussing the estimation and prediction problem based on the PCII sample. Seo and Kang [22] discussed the problem of point and interval estimation for the scaled half-logistic distribution and proposed a method to estimate the scale parameter using the pivotal quantity method under PCII samples. They also tackled the problem of estimation and prediction for the two-parameter half-logistic distribution. Ma and Gui [17] used the pivotal quantity method to derive the estimator for the inverse Rayleigh distribution based on general PCII samples. They also derived an explicit estimator of the scale parameter by the approximation of the likelihood equation using Taylor expansion. Khan [11] studied on the predictive inference for the HN distribution under the Type II censoring scheme. In a more recent study, Sindhu and Hussain [24] used Bayesian methods and made predictive inference on the HN distribution for left-censored data. Asgharzadeh and Valiollahi [3] studied prediction intervals for the PCII from proportional hazard rate models. El-Din and Shafay [18] derived one-sample and two-sample Bayesian prediction intervals based on PCII using Exponential, Pareto, Weibull and Burr Type X-II models. Dey et al. [5] discussed the parameter estimation problem for generalized inverted exponential distribution under PCII. The studies conducted by Wang et al. [26], Hemmati [9] and Kinaci et al. [12] are also examples of studies on deriving exact confidence intervals under PCII. Recently, Ahmadi et al. [2] studied statistical inference for the two-parameter generalized half-normal distribution based on a PCII sample.

In this study, the point estimation, interval estimation and prediction intervals are discussed for the HN model under PCII. This paper is organized as follows: In Section 2, the maximum likelihood and an approximate maximum likelihood estimation are discussed. In Section 3, pivotal type estimation is studied with an approximate version. Interval estimation is also discussed through MLE, likelihood ratio statistic and a pivotal quantity in Section 4. In Section 5, the prediction of the removed failure times is discussed. The predictive intervals are derived in Section 6. In Section 7, a simulation study is performed to observe the behavior of the point and interval estimates. A simulation study is also conducted to compare the predictors and predictive intervals. In Section 8, a numerical example is presented for illustration. The concluding remarks are given in Section 9.

2. MLE AND APPROXIMATE MLE ESTIMATION

Let $X^{\mathbf{r}}_{1:m:n} < X^{\mathbf{r}}_{2:m:n} < \cdots < X^{\mathbf{r}}_{m:m:n}$ be the progressively censored order statistics from HN(θ). Then the log-likelihood function can be written by

$$\ell(\theta) \propto -m\log(\theta) - \sum_{i=1}^{m}\left(\frac{x_i}{\theta}\right)^2 + \sum_{i=1}^{m} r_i \log\left(\Gamma\left(\frac{1}{2}, \left(\frac{x_i}{\theta}\right)^2\right)\right), \tag{2.1}$$

where x_i is a realization of $X^{\mathbf{r}}_{i:m:n}$ for $i = 1, 2, \ldots, m$. The log-likelihood function (2.1) is non-linear in parameter θ and MLE can not be obtained, explicitly. Therefore, nonlinear optimization methods such as Nelder-Mead or BFGS should be applied to get the MLE of the scale parameter θ. Initial point selection is an important problem in nonlinear optimization methods. An arbitrary initial point may lead us to misinterpretation. Therefore, the analytically obtained approximate MLE (AMLE) estimator, which does not require a searching method, will be discussed below. Let us consider a first-order likelihood equation

$$\frac{d\ell(\theta)}{d\theta} = -\frac{m}{\theta} + \frac{2}{\theta^3}\sum_{i=1}^{m} x_i^2 + \frac{2}{\theta^2}\sum_{i=1}^{m} \frac{r_i x_i \exp\left(\left(\frac{x_i}{\theta}\right)^2\right)}{\Gamma\left(\frac{1}{2}, \left(\frac{x_i}{\theta}\right)^2\right)} = 0. \tag{2.2}$$

We consider the random variable $Z = X/\theta$, it is easy to know that Z has the standard HN distribution (with θ=1) since the θ is a scale parameter. After some algebra, the Eq (2.2) can be re-written by

$$-\frac{m}{\theta} + \frac{2}{\theta^3}\sum_{i=1}^{m} x_i^2 + \frac{2}{\theta^2}\sum_{i=1}^{m} r_i x_i \frac{\exp\left(-z_i^2\right)}{\Gamma\left(\frac{1}{2}, z_i^2\right)} = 0, \tag{2.3}$$

where z_i is a realization of $Z^{\mathbf{r}}_{i:m:n} = X^{\mathbf{r}}_{i:m:n}/\theta$ which is progressively censored order statistic from standard HN distribution for $i = 1, 2, \ldots, m$.

Since Eq. (2.3) is can not be solved analytically, we approximate the tricky part $\frac{\exp\left(-z_i^2\right)}{\Gamma\left(\frac{1}{2}, z_i^2\right)}$ by expanding it in Taylor series around $v_i = E\left(Z^{\mathbf{r}}_{i:m:n}\right)$. By the way, we can write

$$G\left(Z^{\mathbf{r}}_{i:m:n}\right) = U^{\mathbf{r}}_{i:m:n} \tag{2.4}$$

by using the probability integral transformation, where $Z^{\mathbf{r}}_{i:m:n}$ is the ith the progressively censored order statistic from standard HN distribution with cdf

$$G(z;\theta) = 1 - \frac{\Gamma\left(\frac{1}{2}, z^2\right)}{\Gamma\left(\frac{1}{2}\right)}, z > 0, \tag{2.5}$$

and $U^{\mathbf{r}}_{i:m:n}$ is the standard uniform progressively censored order statistic.

According to Balakrishnan and Aggarwala [4], and using transformation (2.4) one can write

$$v_i = E\left(Z^{\mathbf{r}}_{i:m:n}\right) \approx G^{-1}(p_i), \tag{2.6}$$

where

$$p_i = E\left(U^{\mathbf{r}}_{i:m:n}\right) = 1 - \prod_{j=m-i+1}^{m} \frac{j + r_{m-j+1} + \cdots + r_m}{j + 1 + r_{m-j+1} + \cdots + r_m}. \tag{2.7}$$

Using Eqs. (2.5)-(2.7), v_i can be determined by solving the equation

$$\Gamma\left(\frac{1}{2}, v_i^2\right) = \Gamma\left(\frac{1}{2}\right)(1 - p_i), \; i = 1, 2, \ldots, m. \tag{2.8}$$

Let us turn back to Eq. (2.3) and we now focus on the Taylor expansion on the part

$$H(z_i) = \frac{\exp\left(-z_i^2\right)}{\Gamma\left(\frac{1}{2}, z_i^2\right)}.$$

Let h be the first-order derivative of the function H which is given by

$$h(z_i) = -\frac{2z_i \exp\left(-z_i^2\right)}{\Gamma\left(\frac{1}{2}, z_i^2\right)} + \frac{\exp\left(-z_i^2\right)\exp\left(-z_i\right)}{\Gamma^2\left(\frac{1}{2}, z_i^2\right)\sqrt{z_i}}.$$

Then

$$\begin{aligned} H(z_i) &\approx H(v_i) + (z_i - v_i)\, h(v_i) \\ &= A_i + B_i z_i, \end{aligned} \tag{2.9}$$

where

$$\begin{aligned} A_i &= H(v_i) - v_i h(v_i) \\ &= \frac{\Gamma\left(\frac{1}{2}, v_i^2\right)\left(1 - 2v_i^2\right)\exp\left(v_i^2\right) - 2v_i}{\Gamma^2\left(\frac{1}{2}, v_i^2\right)} \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} B_i &= h(v_i) \\ &= \frac{2v_i \exp\left(v_i^2\right)}{\Gamma\left(\frac{1}{2}, v_i^2\right)} + \frac{2}{\Gamma^2\left(\frac{1}{2}, v_i^2\right)}. \end{aligned} \tag{2.11}$$

Eventually, using Eq. (2.9) in Eq. (2.3), we can reach to the approximate likelihood equation

$$-\frac{m}{\theta} + \frac{2}{\theta^3}\sum_{i=1}^{m} x_i^2 + \frac{2}{\theta^2}\sum_{i=1}^{m} r_i x_i \left(A_i + B_i \frac{x_i}{\theta}\right) = 0.$$

After some algebra, AMLE of the parameter θ can be obtained by

$$\widetilde{\theta} = \frac{c + \sqrt{c^2 + 4mb + 4md}}{2m}, \tag{2.12}$$

where

$$b = 2\sum_{i=1}^{m}\left(X_{i:m:n}^{\mathbf{r}}\right)^2,$$
$$c = 2\sum_{i=1}^{m} r_i X_{i:m:n}^{\mathbf{r}} A_i,$$

and

$$d = 2\sum_{i=1}^{m} r_i \left(X_{i:m:n}^{\mathbf{r}}\right)^2 B_i.$$

It is noticed that the estimate (2.12) is not novel but it is another form of the MMLE given in Ahmadi et al. [2]. It is important here that the following revision brings us to the original ML estimate of θ without any searching methods.

Revised estimates: Following a suggestion by Lee et al. [13], we now calculate A_i and B_i in Eqs. (2.10)-(2.11) with replacing v_i by

$$v_i = \frac{x_{i,}}{\widetilde{\theta}},\ 1 \le i \le m. \tag{2.13}$$

and calculate the revised estimate $\widetilde{\theta}_{revised}$ by using Eq. (2.12). This process should be repeated a few times until the coefficients stabilize sufficiently enough. The flowchart is given in Figure 1 to illustrate the revising process.

3. A PIVOTAL QUANTITY ESTIMATION

In the previous section, AMLE is obtained for the scale parameter θ. In this section, the pivotal quantity type inference is discussed. This method is adopted from the results in Ma and Gui [17]. Let $X_{1:m:n}^{\mathbf{r}} < X_{2:m:n}^{\mathbf{r}} < \cdots < X_{m:m:n}^{\mathbf{r}}$ be a PCII sample from the HN(θ). Let

$$Y_{i:m:n}^{\mathbf{r}} = -\log\left(\frac{\Gamma\left(\frac{1}{2}, \left(\frac{X_{i:m:n}^{\mathbf{r}}}{\theta}\right)^2\right)}{\Gamma\left(\frac{1}{2}\right)}\right),\ i = 1, 2, \ldots, m$$

It can be easily seen that, $Y_{1:m:n}^{\mathbf{r}} < Y_{2:m:n}^{\mathbf{r}} < \cdots < Y_{m:m:n}^{\mathbf{r}}$ are PCII samples from a standard exponential distribution. Let us consider the following transformations:

$$S_1 = nY_{1:m:n}^{\mathbf{r}},$$
$$S_i = \left[n - \sum_{j=1}^{i-1}(r_j+1)\right]\left(Y_{i:m:n}^{\mathbf{r}} - Y_{i-1:m:n}^{\mathbf{r}}\right),\ i = 2, ..., m.$$

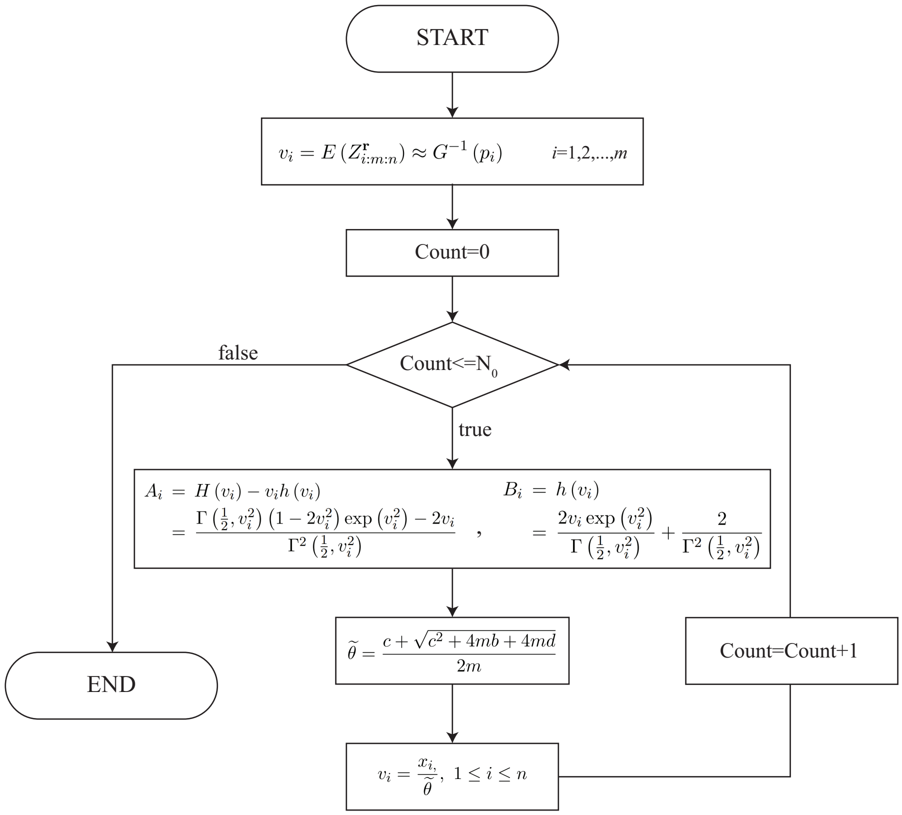

Figure 1: Flow chart for revised estimates

According to Thomas and Wilson [25], $S_1, S_2, ..., S_m$ are also independent and identically distributed from a standard exponential distribution. It is well-known that the pivotal quantity

$$
\begin{aligned}
W(\theta) &= 2\sum_{i=1}^{m} S_i \\
&= 2\sum_{i=1}^{m} (r_i+1)\, Y^{\mathbf{r}}_{i:m:n} \\
&= -2\sum_{i=1}^{m} (r_i+1) \log\left(\frac{\Gamma\left(\frac{1}{2}, \left(\frac{X^{\mathbf{r}}_{i:m:n}}{\theta}\right)^2\right)}{\Gamma\left(\frac{1}{2}\right)} \right), \qquad (3.1)
\end{aligned}
$$

is distributed χ^2 distribution with $2m$ degrees of freedom.

Ma and Gui [17] pointed out that the pivotal quantity $W(\theta)/(2m+2)$ converges to one in probability as $m \to \infty$. In this case, the pivotal type estimate

θ^* of θ can be proposed by solving the equation

$$\sum_{i=1}^{m}(r_i+1)\left[-\log\left(\frac{\Gamma\left(\frac{1}{2},\left(\frac{X_{i:m:n}}{\theta}\right)^2\right)}{\Gamma\left(\frac{1}{2}\right)}\right)\right]=m+1 \tag{3.2}$$

However, Eq. (3.2) does not allow for an explicit solution for θ. Since one-dimensional searching method can be used to get the pivotal type estimate of θ, the approximation method discussed in the previous section can also be applied to solve the Eq. (3.2). Let us start the re-write Eq. (3.2) by

$$\sum_{i=1}^{m}(r_i+1)\left\{-\log\left(\frac{\Gamma\left(\frac{1}{2},z_i^2\right)}{\Gamma\left(\frac{1}{2}\right)}\right)\right\}=m+1, \tag{3.3}$$

where z_i is a realization of $Z^{\mathbf{r}}_{i:m:n}=X^{\mathbf{r}}_{i:m:n}/\theta$ which is standard progressively censored order statistic from HN (1) for $i=1,2,\ldots,m$.

We expand the tricky part

$$K(z_i)=-\log\left(\frac{\Gamma\left(\frac{1}{2},z_i^2\right)}{\Gamma\left(\frac{1}{2}\right)}\right)$$

around the point $v_i=E\left(Z^{\mathbf{r}}_{i:m:n}\right)$ which is already defined in Eq. (2.8). Let k denotes the first-order derivative of K and it is given by

$$k(z_i)=\frac{2\exp\left(-z_i^2\right)}{\Gamma\left(\frac{1}{2},z_i^2\right)}.$$

Hence, we can write

$$\begin{aligned}K(z_i)&\approx K(v_i)+(z_i-v_i)\,k(v_i)\\&=C_i+D_iz_i,\ \ i=1,2,...,m,\end{aligned}$$

where

$$\begin{aligned}C_i&=K(v_i)-v_ik(v_i)\\&=-\log\left(\frac{\Gamma\left(\frac{1}{2},v_i^2\right)}{\Gamma\left(\frac{1}{2}\right)}\right)-\frac{2v_i\exp\left(-v_i^2\right)}{\Gamma\left(\frac{1}{2},v_i^2\right)},\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}D_i&=k(v_i)\\&=\frac{2\exp\left(-v_i^2\right)}{\Gamma\left(\frac{1}{2},v_i^2\right)}.\end{aligned} \tag{3.5}$$

Hence, the left-hand side of Eq. (3.3) can be approximated by

$$\sum_{i=1}^{m}(r_i+1)\left(C_i+D_i\frac{X^{\mathbf{r}}_{i:m:n}}{\theta}\right)=m+1$$

and the approximate pivotal quantity type estimate is obtained by

$$\theta^{*(a)} = \frac{\sum\limits_{i=1}^{m} (r_i + 1)\, X^{\mathbf{r}}_{i:m:n} D_i}{m + 1 - \sum\limits_{i=1}^{m} (r_i + 1)\, C_i}. \tag{3.6}$$

Revised estimates: We now use the method proposed by Lee et al. [13], and calculate C_i and D_i in the Eqs. (3.4)-(3.5) by replacing v_i by

$$v_i = \frac{x_{i,}}{\theta^*},\ 1 \le i \le m.$$

and calculate the revised estimate $\theta^{*(a)}_{revised}$ by using Eq. (3.6). This process should also be repeated a few times until the coefficients stabilize sufficiently enough.

4. INTERVAL ESTIMATIONS

In this section, we discuss the confidence interval estimation of the parameter θ based on progressively censored data $X^{\mathbf{r}}_{1:m:n} < X^{\mathbf{r}}_{2:m:n} < \cdots < X^{\mathbf{r}}_{m:m:n}$ from the HN(θ). In the ML theory, it is well-known that

$$\hat{\theta} \approx AN\left(\theta, I^{-1}(\theta)\right),$$

where

$$I(\theta) = -E\left(\frac{d^2}{d\theta^2}\ell(\theta)\right)$$

is the Fisher Information. It can be estimated by

$$I\left(\hat{\theta}\right) = -\left(\frac{d^2}{d\theta^2} l(\theta)\right)\Big|_{\hat{\theta}}$$

and standard error of $\hat{\theta}$ is estimated by $se\left(\hat{\theta}\right) = \sqrt{I^{-1}\left(\hat{\theta}\right)}$. Then we can write an approximate $(1-\alpha)\,100\%$ confidence interval for θ as follows:

$$\left(\hat{\theta} - z_{1-\frac{\alpha}{2}}\, se\left(\hat{\theta}\right), \hat{\theta} + z_{1-\frac{\alpha}{2}}\, se\left(\hat{\theta}\right)\right), \tag{4.1}$$

where z_a is the ath quantile of the standard normal distribution.

Let us define

$$W(\theta) = -2\sum_{i=1}^{m} (r_i + 1)\, Q(\theta),\ \theta > 0,$$

where

$$Q(\theta) = \log\left(\frac{\Gamma\left(\frac{1}{2}, \left(\frac{X^{\mathbf{r}}_{i:m:n}}{\theta}\right)^2\right)}{\Gamma\left(\frac{1}{2}\right)}\right).$$

It is well-known that the pivot $W(\theta)$ is distributed as χ^2 with $2m$ degrees of freedom. The following lemma is given before introducing a new confidence interval (CI) for the parameter θ.

Lemma 4.1. *Suppose that* $0 < a_1 < a_2 < \cdots < a_m < \infty$*. Then,* $W(\theta)$ *is strictly decreasing in* θ *for any* $\theta > 0$*. Furthermore, if* $t > 0$*, the equation* $W(\theta) = t$ *has a unique solution for any* $\theta > 0$*.*

Proof: Let us consider the first-order derivative of $Q(\theta)$ in θ which is given by

$$\frac{dQ(\theta)}{d\theta} = -\frac{2\exp(-\frac{a_i^2}{\theta^2})x}{\sqrt{\pi}\theta^2\left(erf(x)\left(\frac{a_i}{\theta}\right) - 1\right)},$$

where $erf(\cdot)$ is a well-known error function and it is defined as

$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp\left(-t^2\right) dt.$$

Since the $erf(x)\,(x) \in [0,1)$, it is observed that $\frac{dQ(\theta)}{d\theta} > 0$. This indicates that $\frac{dW(\theta)}{d\theta} < 0$ and $W(\theta)$ is decreasing function in θ. Furthermore, $\lim_{\theta\to 0} W(\theta) = \infty$ and $\lim_{\theta\to\infty} W(\theta) = 0$. Thus, if $t > 0$, $W(\theta) = t$ has a unique solution for any $\theta > 0$. $\square$

Let $\chi^2_{(a)2m}$ denotes the ath quantile of the χ^2 distribution with $2m$ degrees of freedom. The following theorem gives an exact CI for parameter θ.

Theorem 4.1. *A* $(1-\alpha)\,100\%$ *exact CI for* θ *is constructed by*

$$\left(W^{-1}\left(\chi^2_{(1-\alpha/2)2m}\right), W^{-1}\left(\chi^2_{(\alpha/2)2m}\right)\right), \tag{4.2}$$

where $W^{-1}(t)$ *is the solution of equation* $W(\theta) = t$*.*

Proof: The proof follows from Lemma 4.1 and the fact $W(\theta) \sim \chi^2_{(2m)}$ $\square$

Corollary 4.1. *An approximately* $(1-\alpha)\,100\%$ *CI for* θ *is constructed by*

$$\left(\frac{\sum_{i=1}^{m}(r_i+1)\,D_i X^{\mathbf{r}}_{i:m:n}}{\frac{\chi^2_{(1-\alpha/2)2m}}{2} - \sum_{i=1}^{m}(r_i+1)\,C_i}, \frac{\sum_{i=1}^{m}(r_i+1)\,D_i X^{\mathbf{r}}_{i:m:n}}{\frac{\chi^2_{(\alpha/2)2m}}{2} - \sum_{i=1}^{m}(r_i+1)\,C_i}\right), \tag{4.3}$$

where C_i *and* D_i *are defined as in Eqs. (3.4) and (3.5) respectively.*

Proof: The proof is analogous to that of Theorem 4.1 in Ma and Gui [17]. □

By the way, there is another method called the uncorrected likelihood ratio (ULR) interval, which has desirable properties. The ULR intervals are discussed in Doganaksoy and Schmee [6] and Doganaksoy [7]. The ULR interval can be described as follows: Under some mild regularity conditions, if the θ is the true parameter, then the likelihood ratio statistic $\Lambda = -2\left(\ell\left(\theta\right) - \ell\left(\hat{\theta}\right)\right)$ is distributed as χ^2 with degrees of freedom 1, where ℓ is the log-likelihood function as in (2.1) and $\hat{\theta}$ is the MLE of θ. The test statistic Λ can be used for testing $H_0 : \theta = \theta_0$ against $H_0 : \theta \neq \theta_0$ with critical region $\Lambda > \chi^2_{(1)}\left(1-\alpha\right)$. Then, we conclude that the ULR confidence interval for θ readily arises a nice set $\left\{\theta : \Lambda < \chi^2_{(1)}\left(\alpha\right)\right\}$. Using this fact, $100\left(1-\alpha\right)\%$ ULR CI limits

$$(\theta_L, \theta_U) \tag{4.4}$$

that satisfy

$$-2\left(\ell\left(\theta\right) - \ell\left(\hat{\theta}\right)\right) - \chi^2_{(1)}\left(1-\alpha\right) = 0 \tag{4.5}$$

with $\theta_L < \hat{\theta}$ and $\theta_U > \hat{\theta}$.

It is noticed by Fraser [8] that the ULR and asymptotically normal (AN) CIs are asymptotically equivalent. The ULR CIs are transformation invariant, unlike the AN method. Furthermore, the ULR CIs always produce limits inside of the parameter space.

In the following section, the prediction problem is discussed for the removed failure times within the PCII scheme.

5. PREDICTION

Let $X^{\mathbf{r}}_{1:m:n} < X^{\mathbf{r}}_{2:m:n} < \cdots < X^{\mathbf{r}}_{m:m:n}$ be progressively censored sample from the HN(θ) distribution and $Y = Y_{j:r_k}$ denotes the jth order statistics related to the removed sample of size r_k at the progressive stage k. Using the theorem in Ng et al. [19], the conditional pdf of $Y|X^{\mathbf{r}}_{k:m:n}$ can be written by

$$\begin{aligned} f_{Y|X^{\mathbf{r}}_{k:m:n}}\left(y|x_k\right) &= \frac{r_k!}{(j-1)!\,(r_k-j)!} f\left(y\right)\left(F\left(y\right) - F\left(x_k\right)\right)^{j-1} \\ &\quad \times \left(1-F\left(y\right)\right)^{r_k-j}\left(1-F\left(x_k\right)\right)^{-r_k} \quad \left(y > x_k\right), \end{aligned}$$

where y and x_k are the realizations of Y and $X^{\mathbf{r}}_{j:m:n}$. Then, the predictive log-likelihood function is given by

$$\begin{aligned}
\ell(y,\theta) \propto\ & \ell(\theta) - \log(\theta) - \left(\frac{y}{\theta}\right)^2 \\
& + (j-1)\log\left\{\Gamma\left(\frac{1}{2}, \left(\frac{x_k}{\theta}\right)^2\right) - \Gamma\left(\frac{1}{2}, \left(\frac{y}{\theta}\right)^2\right)\right\} \\
& + (r_k - j)\log\left\{\Gamma\left(\frac{1}{2}, \left(\frac{y}{\theta}\right)^2\right)\right\} \\
& - r_k \log\left\{\Gamma\left(\frac{1}{2}, \left(\frac{x_k}{\theta}\right)^2\right)\right\}. \qquad (5.1)
\end{aligned}$$

The predictive log-likelihood (5.1) is non-linear in y and parameter θ, and they can not be obtained explicitly. Therefore nonlinear optimization methods such as Nelder-Mead or BFGS should be applied to get the maximum likelihood predictor (MLP) of Y and predictive maximum likelihood estimate (PMLE) of the scale parameter θ. The MLP and PMLE are denoted by $\hat{Y}$ and $\hat{\theta}_P$, respectively. It is noted that the MLP of Y is the same as $X^{\mathbf{r}}_{k:m:n}$ for $j = 1$.

From Lemma 3.1 in Seo and Kang [22], the pivot

$$W_\theta(Y) = \frac{1 - F(Y)}{1 - F(x_k)} = \frac{\Gamma\left(\frac{1}{2}, \left(\frac{Y}{\theta}\right)^2\right)}{\Gamma\left(\frac{1}{2}, \left(\frac{x_k}{\theta}\right)^2\right)} \qquad (5.2)$$

has beta distribution with parameters $r_k - j + 1$ and j.

When the parameter θ is known or given, we can obtain a predictor for Y by solving the following equation:

$$W_\theta(Y) = \frac{\Gamma\left(\frac{1}{2}, \left(\frac{Y}{\theta}\right)^2\right)}{\Gamma\left(\frac{1}{2}, \left(\frac{x_k}{\theta}\right)^2\right)} \approx E\left(B_{r_k-j+1,j}\right) = \frac{r_k - j + 1}{r_k + 1}, \qquad (5.3)$$

where $B_{r_k-j+1,j}$ beta random variable with parameters $r_k - j + 1$ and j. This predictor is denoted by $\hat{Y}_2$. If θ is unknown, we can use $\hat{\theta}_P$ for θ in (5.3), then a new predictor of Y can be obtained by the solution of the equation

$$W_{\hat{\theta}_P}(Y) = \frac{r_k - j + 1}{r_k + 1},$$

where $\hat{\theta}_P$ is PMLE of θ and this predictor is denoted by $\hat{Y}_3$.

Presently, we aim to obtain some predictors which give explicit predictions. Let us define

$$
\begin{aligned}
p^* &= E\left(U_{j:r_k} | U_{k:m:n} > F\left(x_k\right)\right) \\
&= \frac{r_k!\left(1-F\left(x_k\right)\right)^{-r_k}}{(j-1)!\left(r_k-j\right)!} \int\limits_{F(x_k)}^{1} y\left(y-F\left(x_k\right)\right)^{j-1}(1-y)^{r_k-j} \, dy \\
&= \frac{r_k!\left(1-F\left(x_k\right)\right)^{-r_k}}{(j-1)!\left(r_k-j\right)!} \sum_{i_1=0}^{j-1} \sum_{i_2=0}^{r_k-j} \binom{j-1}{i_i} \binom{r_k-j}{i_2} \\
&\quad \times (-1)^{i_1+i_2} F^{i_1}\left(x_k\right)\left(\frac{1-F^{j-i_1+i_2+1}\left(x_k\right)}{j-i_1+i_2+1}\right),
\end{aligned} \tag{5.4}
$$

where $U_{k:m:n}$ is the kth standard uniform progressive censored statistic and $U_{j:r_k}$ is the standard uniform jth ordinary order statistics related to removed sample of size r_k at the stage k. Using the same methodology in Section 3, we can also write

$$
N_\theta(Y) = -\log\left(\frac{\Gamma\left(\frac{1}{2}, \left(\frac{Y}{\theta}\right)^2\right)}{\Gamma\left(\frac{1}{2}\right)}\right) \approx L + M\frac{Y}{\theta}, \tag{5.5}
$$

where

$$
L = -\log\left(\frac{\Gamma\left(\frac{1}{2}, \xi^2\right)}{\Gamma\left(\frac{1}{2}\right)}\right) - \frac{2\xi \exp\left(-\xi^2\right)}{\Gamma\left(\frac{1}{2}, \xi^2\right)},
$$

$$
M = \frac{2\exp\left(-\xi^2\right)}{\Gamma\left(\frac{1}{2}, \xi^2\right)},
$$

and

$$
\xi = E\left(Y_{j:R_k}\right) \approx G^{-1}\left(p^*\right).
$$

Using Eqs. (2.5) and (5.4), ξ can be determined by solving the following equation

$$
\Gamma\left(\frac{1}{2}, \xi^2\right) = \Gamma\left(\frac{1}{2}\right)\left(1-p^*\right).
$$

We are now ready to give a new explicit predictor. Using Eq. (5.5) in Eq. (5.3), a new predictor of Y is given by

$$
\hat{Y}_4 = \frac{\theta\left(-\log\left(\frac{\Gamma\left(\frac{1}{2}, \left(\frac{x_k}{\theta}\right)^2\right)}{\Gamma\left(\frac{1}{2}\right)} \frac{r_k-j+1}{r_k+1}\right) - L\right)}{M},
$$

where θ is given or known. If the θ is unknown, another predictor of Y can be defined as

$$
\hat{Y}_5 = \frac{\hat{\theta}_P\left(-\log\left(\frac{\Gamma\left(\frac{1}{2}, \left(\frac{x_k}{\hat{\theta}_P}\right)^2\right)}{\Gamma\left(\frac{1}{2}\right)} \frac{r_k-j+1}{r_k+1}\right) - L\right)}{M}.
$$

In the following section, the predictive intervals are discussed for failure times of progressively removed units.

6. PREDICTIVE INTERVALS

In this section, we discuss the predictive intervals (PIs) for Y. Using maximum likelihood theory, an approximately predictive interval for Y can be written by

$$\left(\hat{Y} - z_{1-\frac{\alpha}{2}} se\left(\hat{Y}\right), \hat{Y} + z_{1-\frac{\alpha}{2}} se\left(\hat{Y}\right)\right), \tag{6.1}$$

where $se\left(\hat{Y}\right)$ can be found in a similar way in Section 4 by using the negative Hessian matrix of predictive log-likelihood function (5.1).

Let us consider the pivot (5.2) to construct a new PI for Y. For this purpose, we need the following lemma.

Lemma 6.1. *Suppose that $0 < a_1 < a_2 < \cdots < a_m < \infty$. Then, $W_\theta(y)$ is strictly decreasing in y for any $y > 0$. Furthermore, if $t > 0$, the equation $W_\theta(y) = t$ has a unique solution for any $y > 0$.*

Proof: The proof is similar to the proof of Lemma 4.1 and it is omitted. □

Let $\beta_{(a)}$ be the ath quantile of the beta distribution with parameters $r_k - j + 1$ and j. Then the following theorem gives an exact PI of Y.

Theorem 6.1. *An exact $100(1-\alpha)\%$ predictive interval for Y can be constructed by*

$$\left(W_\theta^{-1}\left(\beta_{(1-\alpha/2)}\right), W_\theta^{-1}\left(\beta_{(\alpha/2)}\right)\right), \tag{6.2}$$

where scale parameter θ is known.

Proof: The proof follows by using Lemma 6.1 and Lemma 3.1 in Seo and Kang [22]. □

Corollary 6.1. *When the scale parameter θ is unknown, an approximately $100(1-\alpha)\%$ predictive interval for Y can be given by*

$$\left(W_{\hat{\theta}_P}^{-1}\left(\beta_{(1-\alpha/2)}\right), W_{\hat{\theta}_P}^{-1}\left(\beta_{(\alpha/2)}\right)\right), \tag{6.3}$$

where $\hat{\theta}_P$ is PMLE of θ.

However, PIs in Eqs. (6.2)-(6.3) cannot be obtained explicitly. In the following, we provide an explicit solution for the PI bounds. Using the pivot in Eq. (5.2), we have

$$\begin{aligned}
1-\alpha &\approx P\left(\beta_{(\alpha/2)} < W_\theta(Y) < \beta_{(1-\alpha/2)}\right) \\
&= P\left(\Gamma\left(\frac{1}{2},\left(\frac{x_k}{\theta}\right)^2\right)\beta_{(\alpha/2)} < \Gamma\left(\frac{1}{2},\left(\frac{Y}{\theta}\right)^2\right) < \Gamma\left(\frac{1}{2},\left(\frac{x_k}{\theta}\right)^2\right)\beta_{(1-\alpha/2)}\right) \\
&= P\left(-\log\left(\frac{\Gamma\left(\frac{1}{2},\left(\frac{x_k}{\theta}\right)^2\right)\beta_{(1-\alpha/2)}}{\Gamma\left(\frac{1}{2}\right)}\right) < N_\theta(Y) < -\log\left(\frac{\Gamma\left(\frac{1}{2},\left(\frac{x_k}{\theta}\right)^2\right)\beta_{(\alpha/2)}}{\Gamma\left(\frac{1}{2}\right)}\right)\right).
\end{aligned} \tag{6.4}$$

By substituting (5.5) in (6.4), we have the following corollaries.

Corollary 6.2. *An approximately* $100(1-\alpha)\%$ *predictive interval for* Y *can be constructed by*

$$\left(\frac{\theta\left(-\log\left(\frac{\Gamma\left(\frac{1}{2},\left(\frac{x_k}{\theta}\right)^2\right)\beta_{(1-\alpha/2)}}{\Gamma\left(\frac{1}{2}\right)}\right)-L\right)}{M},\frac{\theta\left(-\log\left(\frac{\Gamma\left(\frac{1}{2},\left(\frac{x_k}{\theta}\right)^2\right)\beta_{(\alpha/2)}}{\Gamma\left(\frac{1}{2}\right)}\right)-L\right)}{M}\right), \tag{6.5}$$

where the scale parameter θ *is known.*

Corollary 6.3. *When the scale parameter* θ *is unknown, an approximately* $100(1-\alpha)\%$ *predictive interval for* Y *can be constructed by*

$$\left(\frac{\hat{\theta}_P\left(-\log\left(\frac{\Gamma\left(\frac{1}{2},\left(\frac{x_k}{\hat{\theta}_P}\right)^2\right)\beta_{(1-\alpha/2)}}{\Gamma\left(\frac{1}{2}\right)}\right)-L\right)}{M},\frac{\hat{\theta}_P\left(-\log\left(\frac{\Gamma\left(\frac{1}{2},\left(\frac{x_k}{\hat{\theta}_P}\right)^2\right)\beta_{(\alpha/2)}}{\Gamma\left(\frac{1}{2}\right)}\right)-L\right)}{M}\right). \tag{6.6}$$

In the following section, all estimation and prediction methods are compared through the Monte Carlo simulation.

7. SIMULATION STUDY

In this section, we perform a simulation study to observe the performance of estimators, predictors, confidence intervals and predictive intervals discussed in Sections 2-6. Several censoring schemes are used in this study. 5000 trials are used in the simulation. The bias, variance and mean squared errors (MSEs) of the estimates $\hat{\theta}, \widetilde{\theta}, \widetilde{\theta}_{revised}, \theta^{*}, \theta^{*(a)}$ and $\theta^{*(a)}_{revised}$ are simulated. 100 iteration is performed to reach all the revised estimates. The coverage probabilities (CPs) and average lengths (ALs) of the CIs given in Eqs. (4.1)-(4.4) are calculated. Furthermore, the bias and mean squared prediction errors (MSPEs) of the predictors $\hat{Y}_1$, $\hat{Y}_2$, $\hat{Y}_3$, $\hat{Y}_4$ and $\hat{Y}_5$ are simulated. The CPs and ALs of the PIs given in Eqs. (6.1), (6.2), (6.3), (6.5) and (6.6) are also calculated. The nominal value is fixed at $\alpha = 0.05$ for all CIs and PIs.

In the following tables, the CIs has given in Eqs. (4.1)-(4.4) are denoted by CI1, CI2, CI3, CI4 and CI5 respectively. The PIs given in Eqs. (6.1), (6.2), (6.3), (6.5) and (6.6) are denoted by PI1, PI2, PI3, PI4 and PI5. A little part of the simulation results are presented in Tables 3-6, and the rest of the tables are included in the supplementary file.

According to Table 3 and the rest of the results in the supplementary file, it is concluded that $\hat{\theta}$ and $\widetilde{\theta}_{revised}$ have identical MSEs and bias. This result shows that the revised AMLE tends to MLE. θ^{*}, $\theta^{*(a)}$ and $\theta^{*(a)}_{revised}$ are worse than the others with a slight difference in terms of MSEs. It is observed that the censoring made at the first stage is a better choice to get low MSEs. It is also observed from Table 4 that the CI2, CI3 and CI4 have desired CPs even if small sample cases. However, the CPs of CI1 are not at the desired level for a small sample case but it reaches to the nominal value for the moderate size of m. It should be pointed out that CI4 (ULR) has the smallest average length in almost all censoring schemes.

According to Tables 5-6 and the rest of the results in the supplementary file, it is concluded that as j increases, the MSPEs of all predictors increases whereas the MSPEs of predictors decrease as k increases. The MSPEs of $\hat{Y}_2$ and $\hat{Y}_4$ are the same where $\hat{Y}_2$ is obtained by a numerical method and $\hat{Y}_4$ is obtained explicitly. Then we concluded that $\hat{Y}_4$ should be used to predict the Y instead of $\hat{Y}_2$ when the the θ is known. The MSPEs of $\hat{Y}_3$ and $\hat{Y}_5$ are almost the same where $\hat{Y}_3$ is obtained by a numerical method and $\hat{Y}_5$ is obtained explicitly. It is concluded that $\hat{Y}_5$ should be used to predict the Y instead of $\hat{Y}_3$ when the θ is unknown. The $\hat{Y}_5$ has better MSPEs than the $\hat{Y}_1$ has when the small values of j. This pattern is reversed for the large values of j.

The CPs of PI3 and PI5 are at the nominal level for small values of j. When j increases, CPs of PI3 and PI5 decrease from the nominal level 0.95. It should be pointed out that the as j increases, CPs are may decrease to 0.88, 0.91 and 0.93 for $m = 20, 40$ and 100, respectively. Fortunately, increasing m overcomes the low CPs problem. Also, the PI2 and PI4 keep the nominal level since the PI2 is an exact predictive interval and PI4 is an approximated version of PI2 with a Taylor expansion. The CPs of PI1 increase first to the nominal level and then decrease when the j increases. We conclude that PI3 should be used for small and large values of j, respectively. PI1 can be used for moderate values of j. That is, if $r_{10} = 10$, PI1 may be used to construct the PI of Y , for $j = 6, 7, 8$, otherwise, PI3 should be used instead of PI1. As j increases, the ALs of all predictive intervals increase whereas they get smaller for large values of k.

PI1 and PI3 have almost the same AL when they have the same CPs. PI2 (PI3) has smaller ALs than PI4 (PI5) for all cases discussed here.

8. ILLUSTRATIVE EXAMPLE

The real dataset represents the survival times of 121 patients with breast cancer obtained from a large hospital in a period from 1929 to 1938 in Lee [14]. For the complete data, the parameter estimation of the θ is obtained by ML methodology. The likelihood values, MLE of θ with the standard error, some goodness of fit tests such as Anderson-Darling (A), Cramer-von Mises (W) and Kolmogrov-Smirnov(K) test statistic, and corresponding p values (in parentheses) are given in Table 1. HN plot is also presented in Figure 2 which indicates the possibility to model for breast cancer data. In addition, since the p-values for the goodness of fit tests reported in Table 1 are greater than 0.05, the hypothesis that the data comes from the HN distribution cannot be rejected at a significance level 0.05.

ℓ	A	W	K	$\hat{\theta}$	se
-579.4310	0.6606 (0.5922)	0.0896 (0.6388)	0.0548 (0.8596)	82.2258	5.2856

Table 1: Some results for survival times of 121 patients data when complete data

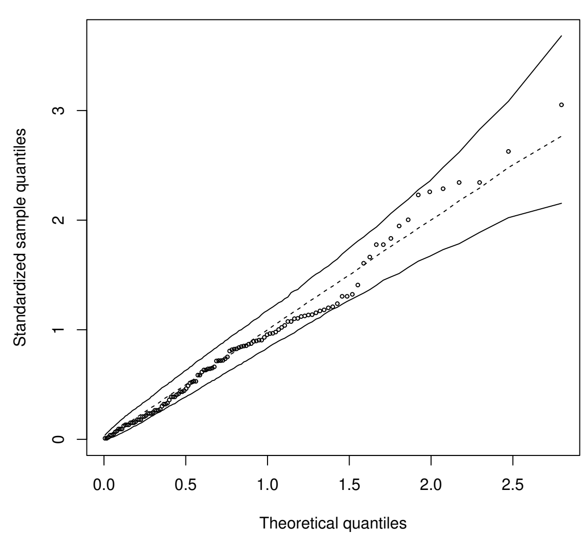

Figure 2: Half-Normal plot for the real data

Let us censoring the complete data with scheme r= $\left(\overbrace{109 * 0}^{109 \text{ times} \text{"0"}}, 5, 5\right)$.

Then the progressively censored data is produced by: 0.3, 0.3, 4.0, 5.0, 5.6, 6.2, 6.3, 6.6, 6.8, 7.4, 7.5, 8.4, 8.4, 10.3, 11.0, 11.8, 12.2, 12.3, 13.5, 14.4, 14.4, 14.8, 15.5, 15.7, 16.2, 16.3, 16.5, 16.8, 17.2, 17.3, 17.5, 17.9, 19.8, 20.4, 20.9, 21.0, 21.0, 21.1, 23.0, 23.4, 23.6, 24.0, 24.0, 27.9, 28.2, 29.1, 30.0, 31.0, 31.0, 32.0, 35.0, 35.0, 37.0, 37.0, 37.0, 38.0, 38.0, 38.0, 39.0, 39.0, 40.0, 40.0, 40.0, 41.0, 41.0, 41.0, 42.0, 43.0, 43.0, 43.0, 44.0, 45.0, 45.0, 46.0, 46.0, 47.0, 48.0, 49.0, 51.0, 51.0, 51.0, 52.0, 54.0, 55.0, 56.0, 57.0, 58.0, 59.0, 60.0, 60.0, 60.0, 61.0, 62.0, 65.0, 65.0, 67.0, 67.0, 68.0, 69.0, 78.0, 80.0, 83.0, 88.0, 89.0, 90.0, 93.0, 96.0, 103.0, 105.0, 109.0, 129.0
Using this progressively censored data, $\hat{\theta}$, $\widetilde{\theta}$, $\widetilde{\theta}_{revised}$, θ^*, $\theta^{*(a)}$ and $\theta^{*(a)}_{revised}$ give the estimates 87.1066, 97.7986, 87.1079, 84.9575, 85.1070 and 85.5067 respectively. CI1, CI2,CI3 and CI4 are also calculated as (75.3322, 98.8810), (75.7754, 99.3832), (74.9352, 98.5545) and (76.5309, 97.1066). Using this progressively censored data, predictions and predictive intervals are given in Table 2.

k	j	Y_j	$\hat{\theta}_P$	$\hat{Y}_1$	$\hat{Y}_3$	$\hat{Y}_5$		PI1	PI3	PI5
110	2	117	88.1947	117.5966	120.3328	121.6912		(101.1574, 134.0358)	(110.5625, 141.9586)	(110.9140, 147.9162)
	3	125	88.4368	128.4170	128.0046	129.1463		(102.7937, 154.0403)	(113.5498, 157.2334)	(113.6089, 164.7992)
	4	129	88.7842	140.7604	138.3815	139.2926		(106.9345, 174.5862)	(118.4837, 179.0106)	(118.5047, 189.3842)
	5	139	89.3816	160.4515	155.1442	155.7899		(112.6896, 208.2134)	(127.2086, 222.6391)	(127.6536, 241.9852)
111	2	111	88.7016	136.7600	139.1889	140.9083		(121.8496, 151.6704)	(130.3967, 158.9649)	(131.0636, 164.8641)
	3	126	88.9844	146.1106	146.1730	147.6439		(123.2522, 168.9689)	(133.0780, 173.1843)	(133.3209, 180.5102)
	4	127	89.3725	158.1813	155.712	156.9107		(126.6520, 189.7105)	(137.5327, 193.7159)	(137.5481, 203.5679)
	5	154	89.9751	175.3795	171.2774	172.1548		(131.3465, 219.4126)	(145.4661, 235.4339)	(145.6123, 253.4672)

Table 2: PMLEs, MLPs and PIs for the first real data

9. CONCLUDING REMARKS

This study addresses the problem of estimating the scale parameter of HN distribution under progressively Type-II censoring. An approximate maximum likelihood, pivotal type and approximate pivotal type estimators (predictors) are derived and confidence intervals (predictors) are constructed for the scale parameter. The performance of the derived estimators (predictors) is compared under different censoring schemes. In addition, a numerical example is presented. It is concluded that under progressive Type-II censoring, the above-mentioned estimators (predictors) and CIs (PIs) can be used in HN distribution and they are competitive with the MLEs (PMLEs). Furthermore, some of our estimators (predictors) are explicitly obtained unlike MLEs (PMLEs). Considering several methods and proving their superior performances with different criterias, this study contributes to estimation and prediction problems in different ways.

REFERENCES

[1] Aarset, M.V. (1987). How to identify a bathtub hazard rate, *IEEE Transactions on Reliability*, **36**, 1, 106–108.

[2] Ahmadi, K.; Rezaei, M. and Yousefzadeh, F. (2015). Estimation for the generalized half-normal distribution based on progressive type-II censoring, *Journal of Statistical Computation and Simulation*, **85**, 6, 1128–1150.

[3] Asgharzadeh, A. and Valiollahi, R. (2010). Prediction intervals for proportional hazard rate models based on progressively Type II censored samples, *Communications for Statistical Applications and Methods*, **17**, 1, 99–106.

[4] Balakrishnan, N. and Aggarwala, R. (2000). *Progressive Censoring: Theory, Methods, and Applications*, Springer Science & Business Media.

[5] Dey, S.; Singh, S.; Tripathi, Y. M. and Asgharzadeh, A. (2016). Estimation and prediction for a progressively censored generalized inverted exponential distribution, *Statistical Methodology*, **32**, 185–202.

[6] Doganaksoy, N. and Schmee, J. (1993). Comparisons of approximate confidence intervals for distributions used in life-data analysis, *Technometrics*, **35**, 2, 175–184.

[7] Doganaksoy, N. (2020). A Simplified Formulation of Likelihood Ratio Confidence Intervals Using a Novel Property, *Technometrics*, **63**, 127–135.

[8] Fraser, D.A.S. (1976). *Probability and Statistics: Theory and Applications*, North Scituate MA: Duxbury Press.

[9] Hemmati, F. and Khorram, E. (2017). On adaptive progressively Type-II censored competing risks data, *Communications in Statistics-Simulation and Computation*, **46**, 6, 4671–4693.

[10] Huang, J. and Roth, D.L. (2021). Using the half normal distribution to quantify covariate balance in cluster-randomized pragmatic trials, *Trials*, **22**, 1, 1–11.

[11] Khan, H.M.R. (2013). Predictive inference from the half-normal model given a type II censored sample, *Communications in Statistics-Theory and Methods*, **42**, 1, 42–55.

[12] Kinaci, I., Wu, S.J. and Kus, C. (2019). Confidence intervals and regions for the generalized inverted Exponential distribution based on progressively censored and upper records data, *REVSTAT–Statistical Journal*, **17**, 4, 429–448.

[13] Lee, K.R.; Kapadia, C.H. and Brock, D.B. (1980). On estimating the scale parameter of the Rayleigh distribution from doubly censored samples, *Springer*, **21**, 1, 14–29.

[14] Lee, E.T. (1992). *Statistical methods for survival data analysis*, John Wiley & Sons, New York.

[15] Majumdar, H. and Sen, P.K. (1978a). Nonparametric testing for simple regression under progressive censoring with staggering entry and random withdrawal, *Communication in Statistics-Theory and Methods*, **7**, 4, 349–371.

[16] Majumdar, H. and Sen, P.K. (1978b). Rank order tests for multiple regression under progressive censoring, *Journal of Multivariate Analysis*, **8**, 1, 73–95.

[17] Ma, Y. and Gui, W. (2019). Pivotal inference for the inverse Rayleigh distribution based on general progressively Type-II censored samples, *Journal of Applied Statistics*, **46**, 5, 771–797.

[18] Mohie El-Din, M.M. and Shafay, A.R. (2013). One-and two-sample Bayesian prediction intervals based on progressively Type-II censored data, *Statistical Papers*, **54**, 2, 287–307.

[19] Ng, H.K.T.; Chan, P.S. and Balakrishnan, N. (2002). Estimation of parameters from progressively censored data using EM algorithm, *Computational Statistics & Data Analysis*, **39**, 4, 371–386.

[20] Ng, H.K.T. (2021). Progressively Censored Data Analysis, *Wiley StatsRef: Statistics Reference Online*, 1–10.

[21] Sen, P.K. (2004). Progressively Censored Data Analysis, *Encyclopedia of Statistical Sciences*.

[22] Seo, J.I. and Kang, S.B. (2017). Inference for the two-parameter half-logistic distribution using pivotal quantities under progressively type-II censoring schemes, *Communications in Statistics-Simulation and Computation*, **46**, 7, 5462–5478.

[23] Shanker, R.; Shukla, K.K.; Sharma, S. and Shanker, R. (2018). A quasi exponential distribution, *Biometrics and Biostatistics International Journal*, **7**, 2, 90–94.

[24] Sindhu, T.N. and Hussain, Z. (2020). Predictive inference and parameter estimation from the half-normal distribution for the left censored data, *Annals of Data Science*, 1–15.

[25] Thomas, D.R. and Wilson, W.M. (1972). Linear order statistic estimation for the two-parameter Weibull and extreme-value distributions from type II progressively censored samples, *Technometrics*, **14**, 3, 679–691.

[26] Wang, B.X.; Yu, K. and Sheng, Z. (2014). New inference for constant-stress accelerated life tests with Weibull distribution and progressively type-II censoring, *IEEE Transactions on Reliability*, **63**, 3, 807–815.

n	m	Censoring Schemes	$\hat{\theta}$	$\widetilde{\theta}$	$\widetilde{\theta}_{revised}$	θ^*	$\theta^{*(a)}$	$\theta^{*(a)}_{revised}$
10	10	(10*0)	0.0487 (-0.0253) 0.0480	0.0487 (-0.0253) 0.0480	0.0487 (-0.0253) 0.0480	0.0565 (-0.1347) 0.0384	0.0518 (-0.0965) 0.0425	0.0498 (-0.0799) 0.0435
20	10	(10*1)	0.0583 (-0.0236) 0.0577	0.0621 (0.0021) 0.0621	0.0583 (-0.0236) 0.0577	0.0654 (-0.1437) 0.0448	0.0596 (-0.0955) 0.0505	0.0584 (-0.0838) 0.0513
	10	(5, 8*0, 5)	0.0584 (-0.0238) 0.0579	0.0600 (-0.0094) 0.0599	0.0584 (-0.0238) 0.0579	0.0655 (-0.1440) 0.0447	0.0588 (-0.0902) 0.0507	0.0583 (-0.0841) 0.0513
	10	(5, 5, 8*0)	0.0507 (-0.0252) 0.0501	0.0508 (-0.0230) 0.0503	0.0507 (-0.0252) 0.0501	0.0586 (-0.1368) 0.0399	0.0541 (-0.0986) 0.0443	0.0519 (-0.0807) 0.0454
	10	(8*0, 5, 5)	0.0689 (-0.0154) 0.0687	0.07001 (-0.0077) 0.0700	0.0689 (-0.0154) 0.0687	0.0724 (-0.1424) 0.0522	0.0667 (-0.0848) 0.0595	0.0664 (-0.0799) 0.0601
	10	(4*0, 5, 5, 4*0)	0.0560 (-0.0237) 0.0554	0.0562 (-0.0205) 0.0558	0.0560 (-0.0237) 0.0554	0.0634 (-0.1415) 0.0434	0.0591 (-0.1014) 0.0488	0.0565 (-0.0825) 0.0497
	20	(20*0)	0.0244 (-0.0119) 0.0243	0.0244 (-0.0119) 0.0243	0.0244 (-0.0119) 0.0243	0.0270 (-0.0716) 0.0219	0.0257 (-0.0496) 0.0232	0.0250 (-0.0404) 0.0234
40	20	(20*1)	0.0295 (-0.0113) 0.0293	0.0326 (0.0137) 0.0325	0.0295 (-0.0113) 0.0293	0.0319 (-0.0770) 0.0260	0.0303 (-0.0497) 0.0278	0.0298 (-0.0429) 0.0280
	20	(10, 18*0, 10)	0.0296 (-0.0116) 0.0294	0.0301 (-0.0028) 0.0301	0.0296 (-0.0116) 0.0294	0.0319 (-0.0772) 0.0259	0.0299 (-0.0457) 0.0278	0.0298 (-0.0431) 0.0279
	20	(10, 10, 18*0)	0.0249 (-0.0119) 0.0248	0.0250 (-0.0109) 0.0249	0.0249 (-0.0119) 0.0248	0.0276 (-0.0722) 0.0223	0.0263 (-0.0502) 0.0237	0.0256 (-0.0406) 0.0239
	20	(18*0, 10, 10)	0.0324 (-0.0106) 0.0323	0.0326 (-0.0068) 0.0325	0.0324 (-0.0106) 0.0323	0.0345 (-0.0798) 0.0281	0.0324 (-0.0460) 0.0303	0.0323 (-0.0442) 0.0304
	20	(9*0, 10, 10, 9*0)	0.0281 (-0.0112) 0.0279	0.0282 (-0.0095) 0.0281	0.0281 (-0.0112) 0.0279	0.0307 (-0.0755) 0.0250	0.0296 (-0.0529) 0.0268	0.0287 (-0.0420) 0.0269
50	50	(50*0)	0.0095 (-0.0039) 0.0095	0.0095 (-0.0039) 0.0095	0.0095 (-0.0039) 0.0095	0.0103 (-0.0300) 0.0094	0.0101 (-0.0206) 0.0097	0.0100 (-0.0165) 0.0097
100	50	(50*1)	0.0116 (-0.0038) 0.0116	0.0143 (0.0178) 0.0140	0.0116 (-0.0038) 0.0116	0.0123 (-0.0324) 0.0113	0.0121 (-0.0209) 0.0117	0.0120 (-0.0175) 0.0117
	50	(25, 48*0, 25)	0.0116 (-0.0040) 0.0116	0.0117 (0.0001) 0.0117	0.0116 (-0.0040) 0.0116	0.01236 (-0.0325) 0.0113	0.0119 (-0.0186) 0.0116	0.0119 (-0.0177) 0.0116
	50	(25, 25, 48*0)	0.0096 (-0.0039) 0.0096	0.0096 (-0.0036) 0.0096	0.0096 (-0.0039) 0.0096	0.0104 (-0.0301) 0.0095	0.0102 (-0.0207) 0.0098	0.0100 (-0.0165) 0.0098
	50	(48*0, 25, 25)	0.0129 (-0.0037) 0.0129	0.0129 (-0.0021) 0.0129	0.0129 (-0.0037) 0.0129	0.0135 (-0.0337) 0.0124	0.0131 (-0.0187) 0.0128	0.0131 (-0.0182) 0.0128
	50	(24*0, 25, 25, 24*0)	0.0110 (-0.0037) 0.0110	0.0110 (-0.0031) 0.0110	0.0110 (-0.0037) 0.0110	0.0118 (-0.0316) 0.0108	0.0117 (-0.0222) 0.0112	0.0114 (-0.0171) 0.0112

Table 3: MSEs, bias (in parenthesis) and variances for estimates of the scale parameter ($\hat{\theta}$=1)

			CPs				ALs			
n	m	Censoring Schemes	CI1	CI2	CI3	CI4	CI1	CI2	CI3	CI4
10	10	(10*0)	0.8988	0.9504	0.9516	0.9464	0.8530	1.0435	0.9810	0.9613
20	10	(10*1)	0.9014	0.9542	0.9518	0.9512	0.9427	1.1611	1.0788	1.0739
	10	$(5, 8^*0, 5)$	0.8918	0.9514	0.9438	0.9452	0.9417	1.1570	1.0658	1.0731
	10	$(5, 5, 8^*0)$	0.9052	0.9512	0.9540	0.9478	0.8798	1.0793	1.0167	0.9937
	10	$(8^*0, 5, 5)$	0.9010	0.9492	0.9462	0.9460	0.9852	1.2102	1.1193	1.1263
	10	$(4^*0, 5, 5, 4^*0)$	0.8976	0.9490	0.9518	0.9478	0.9236	1.1426	1.0762	1.0492
	20	(20*0)	0.9208	0.9514	0.9488	0.9448	0.6115	0.6791	0.6634	0.6488
40	20	(20*1)	0.9234	0.9486	0.9490	0.9464	0.6741	0.7504	0.7260	0.7191
	20	$(10, 18^*0, 10)$	0.9266	0.9518	0.9496	0.9510	0.6737	0.7479	0.7188	0.7188
	20	$(10, 10, 18^*0)$	0.9292	0.9488	0.9524	0.9468	0.6194	0.6899	0.6738	0.6576
	20	$(18^*0, 10, 10)$	0.9226	0.9488	0.9488	0.9490	0.7078	0.7852	0.7561	0.7566
	20	$(9^*0, 10, 10, 9^*0)$	0.9252	0.9452	0.9472	0.9454	0.6584	0.7359	0.7196	0.7013
50	50	(50*0)	0.9368	0.9580	0.9550	0.9530	0.3895	0.4088	0.4068	0.3988
100	50	(50*1)	0.9374	0.9490	0.9506	0.9490	0.4295	0.4509	0.4458	0.4407
	50	$(25, 48^*0, 25)$	0.9422	0.9508	0.9510	0.9518	0.4306	0.4507	0.4437	0.4418
	50	$(25, 25, 48^*0)$	0.9450	0.9510	0.9546	0.9498	0.3928	0.4128	0.4106	0.4022
	50	$(48^*0, 25, 25)$	0.9346	0.9482	0.9484	0.9476	0.4504	0.4707	0.4637	0.4625
	50	$(24^*0, 25, 25, 24^*0)$	0.9352	0.9500	0.9494	0.9466	0.4167	0.4393	0.4372	0.4273

Table 4: CPs and ALs of CI for the scale parameter $\hat{\theta}$ at the 0.95 confidence level ($\hat{\theta}$=1).

CSs	k	j		$\hat{Y}_1$ [PI1]	$\hat{Y}_2$ [PI2]	$\hat{Y}_3$ [PI3]	$\hat{Y}_4$[PI4]	$\hat{Y}_5$ [PI5]
(5*m)	k=5	j=2		0.0437(-0.1042)	0.0329(-0.0070)	0.0365(-0.0109)	0.0329(-0.0070)	0.0365(-0.0109)
			CPs	[0.8898]	[0.9518]	[0.9324]	[0.9296]	[0.9118]
			ALs	[0.6862]	[0.6940]	[0.6848]	[0.7609]	[0.7503]
		j=3		0.0631(-0.0940)	0.0544(-0.0273)	0.0642(-0.0307)	0.0544(-0.0273)	0.0642(-0.0307)
			CPs	[0.9184]	[0.9478]	[0.9124]	[0.9302]	[0.9010]
			ALs	[0.8560]	[0.8778]	[0.8713]	[0.9589]	[0.9513]
		j=4		0.0907(-0.0879)	0.0837(-0.0502)	0.1014(-0.0555)	0.0837(-0.0502)	0.1014(-0.0555)
			CPs	[0.9262]	[0.9486]	[0.9078]	[0.9348]	[0.9086]
			ALs	[1.1337]	[1.0959]	[1.0871]	[1.2107]	[1.2005]
		j=5		0.1785(-0.1140)	0.1737(-0.1392)	0.2122(-0.1444)	0.1737(-0.1392)	0.2123(-0.1444)
			CPs	[0.9244]	[0.9490]	[0.9012]	[0.9458]	[0.9172]
			ALs	[1.7537]	[1.5142]	[1.5060]	[1.7623]	[1.7522]
	k=10	j=2		0.0429(-0.1079)	0.0313(-0.0127)	0.0350(-0.0144)	0.0313(-0.0127)	0.0350(-0.0143)
			CPs	[0.8786]	[0.9538]	[0.9324]	[0.9358]	[0.9170]
			ALs	[0.6404]	[0.6712]	[0.6673]	[0.7336]	[0.7279]
		j=3		0.0592(-0.0947)	0.0505(-0.0276)	0.0598(-0.0313)	0.0505(-0.0276)	0.0598(-0.0312)
			CPs	[0.9156]	[0.9476]	[0.9184]	[0.9374]	[0.9122]
			ALs	[0.8970]	[0.8548]	[0.8480]	[0.9322]	[0.9232]
		j=4		0.0913(-0.0989)	0.0833(-0.0594)	0.1022(-0.0660)	0.0833(-0.0594)	0.1022(-0.0660)
			CPs	[0.9230]	[0.9480]	[0.9072]	[0.9398]	[0.9082]
			ALs	[1.0754]	[1.0738]	[1.0630]	[1.1846]	[1.1713]
		j=5		0.1719(-0.1189)	0.1664(-0.1390)	0.2071(-0.1518)	0.1664(-0.1390)	0.2071(-0.1517)
			CPs	[0.9294]	[0.9516]	[0.9120]	[0.9454]	[0.9170]
			ALs	[1.3226]	[1.4936]	[1.4737]	[1.7355]	[1.7109]
	k=15	j=2		0.0389(-0.1030)	0.0283(-0.0105)	0.0319(-0.0130)	0.0283(-0.0105)	0.0319(-0.0217)
			CPs	[0.8746]	[0.9480]	[0.9266]	[0.9301]	[0.9082]
			ALs	[0.6047]	[0.6382]	[0.6325]	[0.6942]	[0.6857]
		j=3		0.0561(-0.0936)	0.0475(-0.0265)	0.0567(-0.0306)	0.0475(-0.0265)	0.0567(-0.0303)
			CPs	[0.9132]	[0.9476]	[0.9140]	[0.9336]	[0.9042]
			ALs	[0.7431]	[0.8203]	[0.8129]	[0.8919]	[0.8813]
		j=4		0.0860(-0.0978)	0.0779(-0.0579)	0.0962(-0.0626)	0.0779(-0.0579)	0.0962(-0.0624)
			CPs	[0.9248]	[0.9482]	[0.9106]	[0.9410]	[0.9086]
			ALs	[1.0573]	[1.0395]	[1.0321]	[1.1441]	[1.1333]
		j=5		0.1550(-0.0988)	0.1508(-0.1202)	0.1866(-0.1280)	0.1508(-0.1202)	0.1866(-0.1278)
			CPs	[0.9336]	[0.9514]	[0.9112]	[0.9444]	[0.9160]
			ALs	[1.4495]	[1.4608]	[1.4490]	[1.6924]	[1.6756]

Table 5: MSEs, bias (in parenthesis) for $\hat{Y}_i$ and CPs [ ], ALs [ ] for PI_i when $\hat{\theta}$=1, m=*20*, $\mathbf{r}$=(5,5,…,5)

CSs	k	j		$\hat{Y}_1$ [PI1]	$\hat{Y}_2$ [PI2]	$\hat{Y}_3$ [PI3]	$\hat{Y}_4$[PI4]	$\hat{Y}_5$ [PI5]
(10*m)	k=5	j=2		0.0151(-0.0671)	0.0105(-0.0023)	0.0172(-0.0018)	0.0105(-0.0013)	0.0117(-0.0018)
			CPs	[0.8540]	[0.9528]	[0.9360]	[0.9404]	[0.9220]
			ALs	[0.3494]	[0.3881]	[0.3893]	[0.4114]	[0.4124]
		j=3		0.0193(-0.0589)	0.0159(-0.0048)	0.0184(-0.0061)	0.0159(-0.0048)	0.0184(-0.0060)
			CPs	[0.9040]	[0.9496]	[0.9218]	[0.9354]	[0.9130]
			ALs	[0.4719]	[0.4746]	[0.4721]	[0.5005]	[0.4976]
		j=4		0.0224(-0.0509)	0.0199(-0.0055)	0.0245(-0.0175)	0.0199(-0.0055)	0.0245(-0.0075)
			CPs	[0.9310]	[0.9488]	[0.9170]	[0.9418]	[0.9110]
			ALs	[0.5689]	[0.5484]	[0.5450]	[0.5763]	[0.5725]
		j=5		0.0284(-0.0488)	0.0261(-0.0111)	0.0338(-0.0130)	0.0261(-0.0111)	0.0338(-0.0130)
			CPs	[0.9344]	[0.9438]	[0.8986]	[0.9356]	[0.8926]
			ALs	[0.6365]	[0.6174]	[0.6147]	[0.6479]	[0.6447]
		j=6		0.0343(-0.0478)	0.0320(-0.0169)	0.0439(-0.0222)	0.0320(-0.0169)	0.0439(-0.0222)
			CPs	[0.9344]	[0.9500]	[0.8900]	[0.9420]	[0.8898]
			ALs	[0.6699]	[0.6886]	[0.6815]	[0.7229]	[0.7152]
		j=7		0.0409(-0.0465)	0.0389(-0.0230)	0.5556(-0.0293)	0.0389(-0.0230)	0.05556(-0.0293)
			CPs	[0.9426]	[0.9512]	[0.8878]	[0.9414]	[0.8894]
			ALs	[0.7445]	[0.7692]	[0.7616]	[0.8104]	[0.8021]
		j=8		0.0539(-0.0536)	0.0516(-0.0407)	0.0780(-0.0420)	0.0516(-0.0407)	0.0780(-0.0420)
			CPs	[0.9346]	[0.9512]	[0.8764]	[0.9482]	[0.8766]
			ALs	[0.8408]	[0.8733]	[0.8719]	[0.9276]	[0.9257]
		j=9		0.0754(-0.0553)	0.0731(-0.0594)	0.1128(-0.0641)	0.0731(-0.0594)	0.1128(-0.0641)
			CPs	[0.9420]	[0.9526]	[0.8710]	[0.9432]	[0.8798]
			ALs	[1.0038]	[1.0361]	[1.0310]	[1.1206]	[1.1148]
		j=10		0.1513(-0.0698)	0.1538(-0.1347)	0.2165(-0.1382)	0.1538(-0.1347)	0.2170(-0.1379)
			CPs	[0.9348]	[0.9480]	[0.8738]	[0.9482]	[0.8800]
			ALs	[1.5188]	[1.4091]	[1.4053]	[1.6079]	[1.6033]
	k=10	j=2		0.0137(-0.0634)	0.0097(0.0001)	0.0108(-0.0011)	0.0097(0.0001)	0.0108(-0.0010)
			CPs	[0.8544]	[0.9542]	[0.9370]	[0.9398]	[0.9258]
			ALs	[0.3423]	[0.3792]	[0.3762]	[0.4013]	[0.3975]
		j=3		0.0182(-0.0604)	0.0146(-0.0070)	0.0172(-0.0081)	0.0146(-0.0070)	0.0172(-0.0080)
			CPs	[0.9050]	[0.9540]	[0.9246]	[0.9454]	[0.9162]
			ALs	[0.4557]	[0.4647]	[0.4626]	[0.4895]	[0.4866]
		j=4		0.0222(-0.0539)	0.0193(-0.0089)	0.0244(-0.0102)	0.0193(-0.0089)	0.0244(-0.0101)
			CPs	[0.9266]	[0.9542]	[0.9162]	[0.9468]	[0.9138]
			ALs	[0.5367]	[0.5381]	[0.5359]	[0.5651]	[0.5621]
		j=5		0.0284(-0.0509)	0.0258(-0.0132)	0.0343(-0.0142)	0.0258(-0.0132)	0.0343(-0.0142)
			CPs	[0.9328]	[0.9462]	[0.9016]	[0.9394]	[0.8958]
			ALs	[0.5935]	[0.6073]	[0.6059]	[0.6369]	[0.6346]
		j=6		0.0333(-0.0476)	0.0310(-0.0164)	0.0435(-0.0217)	0.0310(-0.0164)	0.0435(-0.0217)
			CPs	[0.9366]	[0.9524]	[0.8880]	[0.9416]	[0.8860]
			ALs	[0.7046]	[0.6787]	[0.6716]	[0.7122]	[0.7043]
		j=7		0.0401(-0.0466)	0.0379(-0.0232)	0.0553(-0.0268)	0.0379(-0.0232)	0.0553(-0.0268)
			CPs	[0.9422]	[0.9546]	[0.8890]	[0.9462]	[0.8856]
			ALs	[0.7768]	[0.7597]	[0.7554]	[0.8000]	[0.7949]
		j=8		0.0535(-0.0557)	(0.0511)-0.0418	(0.0769)-0.0458	0.0511(-0.0418)	0.0769(-0.0458)
			CPs	[0.9348]	[0.9554]	[0.8762]	[0.9512]	[8784]
			ALs	[0.8871]	[0.8643]	[0.8598]	[0.9176]	[0.9121]
		j=9		0.0751(-0.0599)	0.0736(-0.0635)	0.1106(-0.0687)	0.0736(-0.0635)	0.1106(-0.0687)
			CPs	[0.9340]	[0.9498]	[0.8758]	[0.9474]	[0.8808]
			ALs	[1.1053]	[1.0274]	[1.0217]	[1.1106]	[1.1037]
		j=10		0.1401(-0.0639)	0.1413(-0.1266)	0.2050(-0.1345)	0.1413(-0.1266)	0.2054(-0.1343)
			CPs	[0.9372]	[0.9528]	[0.8748]	[0.9482]	[0.8874]
			ALs	[1.3007]	[1.4006]	[1.3915]	[1.5969]	[1.5859]
	k=15	j=2		0.0129(-0.0611)	0.0092(0.0004)	0.0103(-0.0005)	0.0092(0.0004)	0.0103(-0.0004)
			CPs	[0.8506]	[0.9516]	[0.9282]	[0.9340]	[0.9168]
			ALs	[0.3302]	[0.3653]	[0.3629]	[0.3856]	[0.3819]
		j=3		0.0174(-0.0580)	0.0140(-0.0056)	0.0166(-0.0079)	0.0140(-0.0056)	0.0166(-0.0077)
			CPs	[0.8992]	[0.9476]	[0.9184]	[0.9394]	[0.9144]
			ALs	[0.4316]	[0.4496]	[0.4451]	[0.4727]	[0.4670]
		j=4		0.0221(-0.0569)	0.0189(-0.0122)	0.0237(-0.0144)	0.0189(-0.0122)	0.0237(-0.0143)
			CPs	[0.9192]	[0.9444]	[0.9056]	[0.9380]	[0.9044]
			ALs	[0.5284]	[0.5224]	[0.5187]	[0.5479]	[0.5427]
		j=5		0.0253(-0.0499)	0.0229(-0.0124)	0.0300(-0.0142)	0.0229(-0.0124)	0.0300(-0.0141)
			CPs	[0.9348]	[0.9536]	[0.9032]	[0.9438]	[0.9034]
			ALs	[0.6051]	[0.5915]	[0.5891]	[0.6198]	[0.6159]
		j=6		0.0322(-0.0528)	0.0295(-0.0216)	0.0418(-0.0256)	0.0295(-0.0216)	0.0418(-0.0255)
			CPs	[0.9328]	[0.9528]	[0.8858]	[0.9464]	[0.8888]
			ALs	[0.6727]	[0.6632]	[0.6579]	[0.6955]	[0.6887]
		j=7		0.0400(-0.0496)	0.0378(-0.0254)	0.0549(-0.0329)	0.0378(-0.0254)	0.0549(-0.0328)
			CPs	[0.9356]	[0.9474]	[0.8764]	[0.9410]	[0.8788]
			ALs	[0.7739]	[0.7448]	[0.7358]	[0.7839]	[0.7734]
		j=8		0.0497(-0.0466)	0.0479(-0.0329)	0.0718(-0.0364)	0.0479(-0.0329)	0.0718(-0.0363)
			CPs	[0.9424]	[0.9486]	[0.8852]	[0.9432]	[0.8864]
			ALs	[0.8077]	[0.8496]	[0.8458]	[0.9014]	[0.8961]
		j=9		0.0734(-0.0581)	0.0715(-0.0619)	0.1099(-0.0646)	0.0715(-0.0619)	0.1099(-0.0645)
			CPs	[0.9378]	[0.9514]	[0.8686]	[0.9474]	[0.8682]
			ALs	[0.9801]	[1.0135]	[1.0107]	[1.0947]	[1.0902]
		j=10		0.1332(-0.0635)	0.1367(-0.1266)	0.1952(-0.1327)	0.1367(-0.1266)	0.1956(-0.1324)
			CPs	[0.9426]	[0.9556]	[0.8858]	[0.9504]	[0.8916]
			ALs	[1.4331]	[1.3869]	[1.3801]	[1.5792]	[1.5699]

Table 6: MSEs, bias (in parenthesis) for $\hat{Y}_i$ and CPs [ ], ALs [ ] for PI$_i$ when $\hat{\theta}$=1, m=*20*, $\mathbf{r}$=(10,10,...,10)