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DECOMPOSITIONS OF MARGINAL HOMOGENEITY MODEL USING CUMULATIVE LOGISTIC MODELS FOR MULTI-WAY CONTINGENCY TABLES

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Abstract:

- For square contingency tables with ordered categories, Agresti (1984, 2002) considered the marginal cumulative logistic (ML) model, which is an extension of the marginal homogeneity (MH) model. Miyamoto, Niibe and Tomizawa (2005) proposed the conditional marginal cumulative logistic (CML) model which is defined off the main diagonal cells, and gave the decompositions of the MH model using the ML (CML) model. This paper (1) considers the ML and CML models for multi-way tables, and (2) gives the decompositions of the MH model into the ML (CML) model and the model of the equality of marginal means for multi-way tables. An example is given.

Key-Words:

- *decomposition; marginal cumulative logistic model; marginal homogeneity; marginal mean; multi-way contingency table.*

AMS Subject Classification:

- 62H17.

1. INTRODUCTION

For an $R\times R$ square contingency table with ordered categories, let p_{ij} denote the probability that an observation will fall in the cell in row i and column j ($i=1,...,R$; $j=1,...,R$), and let X_1 and X_2 denote the row and column variables, respectively. The marginal homogeneity (MH) model is defined by

$$\Pr(X_1=i) \;=\; \Pr(X_2=i) \qquad \text{for} \quad i=1,...,R \;;$$

that is

$$p_{i\cdot} = p_{\cdot i} \qquad \text{for} \quad i=1,...,R \;,$$

where $p_{i\cdot} = \sum_{k=1}^{R} p_{ik}$ and $p_{\cdot i} = \sum_{k=1}^{R} p_{ki}$. This model indicates that the row marginal distribution is identical to the column marginal distribution (Stuart, 1955; Bhapkar, 1966; Bishop, Fienberg and Holland, 1975, p. 294; Tomizawa, 1991, 1993, 1998). Let $F_i^{(1)}$ and $F_i^{(2)}$ denote the marginal cumulative probabilities of X_1 and X_2, respectively. These are $F_i^{(1)} = \Pr(X_1 \le i) = \sum_{k=1}^{i} p_{k\cdot}$ and $F_i^{(2)} = \Pr(X_2 \le i) = \sum_{k=1}^{i} p_{\cdot k}$ for $i=1,...,R-1$. Then the MH model may also be expressed as

$$F_i^{(1)} = F_i^{(2)} \qquad \text{for} \quad i=1,...,R-1 \;.$$

Let $L_i^{(1)}$ and $L_i^{(2)}$ denote the marginal cumulative logit of X_1 and X_2, respectively. These are given as

$$L_i^{(1)} \;=\; \text{logit}\big[\Pr(X_1 \le i)\big] \;=\; \log\left[\frac{\Pr(X_1\le i)}{1-\Pr(X_1\le i)}\right] \;,$$

and

$$L_i^{(2)} \;=\; \text{logit}\big[\Pr(X_2 \le i)\big] \;=\; \log\left[\frac{\Pr(X_2\le i)}{1-\Pr(X_2\le i)}\right] \;,$$

for $i=1,...,R-1$. Then the MH model may be further expressed as

$$L_i^{(1)} = L_i^{(2)} \qquad \text{for} \quad i=1,...,R-1 \;.$$

As an extension of the MH model, Agresti (1984, p. 205; 2002, p. 420) considered the marginal cumulative logistic (ML) model defined by

$$L_i^{(1)} = L_i^{(2)} + \Delta \qquad \text{for} \quad i=1,...,R-1 \;.$$

This model states that the odds that X_1 is i or below instead of $i+1$ or above, is $\exp(\Delta)$ times higher than the odds that X_2 is i or below instead of $i+1$ or above, for every $i=1,...,R-1$. Note that the MH model implies the ML model. Consider the marginal mean equivalence (ME) model defined by

$$\sum_{i=1}^{R} i\, p_{i\cdot} = \sum_{i=1}^{R} i\, p_{\cdot i} \qquad (\text{i.e.,}\;\; E(X_1) = E(X_2)) \;.$$

Miyamoto, Niibe and Tomizawa (2005) gave the following theorem.

Theorem 1.1. *The MH model holds if and only if both the ML and ME models hold.*

Using the conditional probabilities, the MH model may also be expressed as

$$\Pr\Big(X_1 = i \mid X_1 \neq X_2\Big) \;=\; \Pr\Big(X_2 = i \mid X_1 \neq X_2\Big) \qquad \text{for} \quad i = 1, ..., R\ ;$$

that is

$$p^c_{i\cdot} = p^c_{\cdot i} \qquad \text{for} \quad i = 1, ..., R\ ,$$

where

$$\begin{aligned} p^c_{i\cdot} &= \frac{p_{i\cdot} - p_{ii}}{\delta} \;=\; \Pr\Big(X_1 = i \mid X_1 \neq X_2\Big)\ , \\ p^c_{\cdot i} &= \frac{p_{\cdot i} - p_{ii}}{\delta} \;=\; \Pr\Big(X_2 = i \mid X_1 \neq X_2\Big)\ , \\ \delta &= \sum\sum_{s \neq t} p_{st} \;=\; \Pr\big(X_1 \neq X_2\big)\ . \end{aligned}$$

Let $F_i^{c(1)}$ and $F_i^{c(2)}$ denote the conditional marginal cumulative probabilities of X_1 and X_2 given that $X_1 \neq X_2$, i.e.,

$$\begin{aligned} F_i^{c(1)} &= \Pr\Big(X_1 \leq i \mid X_1 \neq X_2\Big) \;=\; \sum_{k=1}^{i} p^c_{k\cdot}\ , \\ F_i^{c(2)} &= \Pr\Big(X_2 \leq i \mid X_1 \neq X_2\Big) \;=\; \sum_{k=1}^{i} p^c_{\cdot k}\ , \end{aligned}$$

for $i = 1, ..., R-1$. Then the MH model may be further expressed as $F_i^{c(1)} = F_i^{c(2)}$ for $i = 1, ..., R-1$. Miyamoto *et al.* (2005) also considered the conditional marginal cumulative logistic (CML) model defined by

$$L_i^{c(1)} = L_i^{c(2)} + \Delta^* \qquad \text{for} \quad i = 1, ..., R-1\ ,$$

where

$$\begin{aligned} L_i^{c(1)} &= \operatorname{logit}\Big[\Pr\Big(X_1 \leq i \mid X_1 \neq X_2\Big)\Big]\ , \\ L_i^{c(2)} &= \operatorname{logit}\Big[\Pr\Big(X_2 \leq i \mid X_1 \neq X_2\Big)\Big]\ . \end{aligned}$$

Miyamoto *et al.* (2005) also gave the following theorem.

Theorem 1.2. *The MH model holds if and only if both the CML and ME models hold.*

For analyzing the data of multi-way tables of the same classifications with ordered categories, the some models of symmetry, e.g., the symmetry model, the MH model (e.g., Bishop *et al.* 1975, pp. 300–307), and the ML model (Agresti, 2002, pp. 439–440) are applied. The symmetry and the MH models do not depend on the main diagonal cell probabilities, however, the ML model depends on them. So, we are now interested in the another ML model which does not depend on the main diagonal cell probabilities, namely, in the conditional ML model on condition that an observation will fall in one of off-diagonal cells of the table.

The purpose of this paper is (1) to extend the CML model into the multi-way tables (Section 2.4) and (2) to extend Theorems 1.1 and 1.2 into the multi-way tables (Section 3).

2. EXTENSION TO MULTI-WAY TABLES

2.1. The MH model

Consider an R^T table ($T \geq 3$) having ordered categories. Let X_t denote the t-th random variable for $t=1,...,T$ and let $\Pr(X_1 = i_1,...,X_T = i_T) = p_{i_1 \dots i_T}$ for $i_t = 1,...,R$. The marginal homogeneity (MH) model is defined by

$$\Pr(X_1 = i) = \cdots = \Pr(X_T = i) \qquad \text{for} \quad i=1,...,R\ ;$$

that is

$$p_i^{(1)} = \cdots = p_i^{(T)} \qquad \text{for} \quad i=1,...,R\ ,$$

where

$$p_i^{(t)} = \Pr(X_t = i) \qquad \text{for} \quad t=1,...,T\ .$$

Let $F_i^{(t)}$ denote the marginal cumulative probabilities and let $L_i^{(t)}$ denote the marginal cumulative logit of X_t for $i=1,...,R-1;\ t=1,...,T$. Namely, $F_i^{(t)} = \sum_{s=1}^{i} p_s^{(t)}$, and $L_i^{(t)} = \mathrm{logit}\big[\Pr(X_t \leq i)\big]$. Then the MH model may also be expressed as

$$F_i^{(k)} = F_i^{(1)} \qquad \text{for} \quad i=1,...,R-1; \quad k=2,...,T\ ,$$

or

$$L_i^{(k)} = L_i^{(1)} \qquad \text{for} \quad i=1,...,R-1; \quad k=2,...,T\ .$$

2.2. The ML model

Agresti (2002, p. 442) considered the marginal cumulative logistic (ML) model, defined by

$$L_i^{(k)} = L_i^{(1)} - \Delta_{k-1} \qquad \text{for} \quad i = 1, \ldots, R-1; \quad k = 2, \ldots, T\ .$$

By putting $L_i^{(1)} = \theta_i$, this model may be expressed as

$$F_i^{(k)} = \frac{\exp(\theta_i - \Delta_{k-1})}{1 + \exp(\theta_i - \Delta_{k-1})} \qquad \text{for} \quad i = 1, \ldots, R-1; \quad k = 1, \ldots, T\ ,$$

where $\Delta_0 = 0$. A special case of this model obtained by putting $\Delta_1 = \cdots = \Delta_{T-1} = 0$ is the MH model.

2.3. Other expressions of MH model

The MH model may also be expressed as

$$\begin{aligned} &\Pr\Big(X_k = i \mid (X_1, \ldots, X_T) \neq (s, \ldots, s),\ s = 1, \ldots, R\Big) = \\ &\qquad = \Pr\Big(X_1 = i \mid (X_1, \ldots, X_T) \neq (s, \ldots, s),\ s = 1, \ldots, R\Big)\ , \end{aligned}$$

for $i = 1, \ldots, R;\ k = 2, \ldots, T;$ that is

$$p_i^{c(k)} = p_i^{c(1)} \qquad \text{for} \quad i = 1, \ldots, R; \quad k = 2, \ldots, T\ ,$$

where, for $m = 1, \ldots, T$,

$$p_i^{c(m)} = \frac{p_i^{(m)} - p_{ii\cdots i}}{\delta} = \Pr\Big(X_m = i \mid (X_1, \ldots, X_T) \neq (s, \ldots, s),\ s = 1, \ldots, R\Big)\ ,$$

$$\delta = 1 - \sum_{i=1}^{R} p_{ii\cdots i} = \Pr\Big((X_1, \ldots, X_T) \neq (s, \ldots, s),\ s = 1, \ldots, R\Big)\ .$$

Let $F_i^{c(k)}$ denote the conditional marginal cumulative probabilities of X_k given that $(X_1, \ldots, X_T) \neq (s, \ldots, s),\ s = 1, \ldots, R$, i.e.,

$$F_i^{c(k)} = \Pr\Big(X_k \leq i \mid (X_1, \ldots, X_T) \neq (s, \ldots, s),\ s = 1, \ldots, R\Big) = \sum_{t=1}^{i} p_t^{c(k)}$$

for $i = 1, \ldots, R-1;\ k = 1, \ldots, T$. Then the MH model may be further expressed as

$$F_i^{c(k)} = F_i^{c(1)} \qquad \text{for} \quad i = 1, \ldots, R-1;\ k = 2, \ldots, T\ .$$

2.4. The CML model

Consider now a model defined by

$$L_i^{c(k)} = L_i^{c(1)} - \Delta_{k-1}^* \qquad \text{for} \quad i = 1, ..., R-1;\; k = 2, ..., T \ ,$$

where, for $m = 1, ..., T$,

$$\begin{aligned} L_i^{c(m)} &= \operatorname{logit}\Big[\Pr\Big(X_m \le i \mid (X_1, ..., X_T) \ne (s, ..., s),\; s = 1, ..., R\Big)\Big] \\ &= \log\left[\frac{\Pr\Big(X_m \le i \mid (X_1, ..., X_T) \ne (s, ..., s),\; s = 1, ..., R\Big)}{1 - \Pr\Big(X_m \le i \mid (X_1, ..., X_T) \ne (s, ..., s),\; s = 1, ..., R\Big)}\right] . \end{aligned}$$

We shall refer to this model as the conditional marginal cumulative logistic (CML) model. By putting $L_i^{c(1)} = \theta_i^*$, this model may be expressed as

$$F_i^{c(k)} = \frac{\exp(\theta_i^* - \Delta_{k-1}^*)}{1 + \exp(\theta_i^* - \Delta_{k-1}^*)} \qquad \text{for} \quad i = 1, ..., R-1;\; k = 1, ..., T \ ,$$

where $\Delta_0^* = 0$. A special case of the CML model obtained by putting $\Delta_1^* = \cdots = \Delta_{T-1}^* = 0$ is the MH model.

The CML model states that for $k = 2, ..., T$, on condition that the values of random variables are not all same, the odds that X_1 is i or below instead of $i+1$ or above, is $\exp(\Delta_{k-1}^*)$ times higher than the odds that X_k is i or below instead of $i+1$ or above, for every $i = 1, ..., R-1$. Thus, if $\Delta_{k-1}^* > 0$, on the same condition, X_1 rather than X_k tends to be i or below instead of $i+1$ or above for every $i = 1, ..., R-1$.

3. DECOMPOSITIONS OF THE MARGINAL HOMOGENEITY MODEL

We shall consider two kinds of decompositions of the MH model.

3.1. A decomposition of the MH model using the ML model

Consider a model defined as

$$\sum_{i=1}^{R} i\, p_i^{(1)} = \cdots = \sum_{i=1}^{R} i\, p_i^{(T)} \qquad \text{(i.e.,} \;\; E(X_1) = \cdots = E(X_T)) \ . \tag{3.1}$$

Namely, the means of variables X_k $(k=1,...,T)$ are equal. Note that the MH model implies model (3.1).

Consider a specified monotonic function $g(k)$ satisfying $g(1) \leq \cdots \leq g(R)$ or $g(1) \geq \cdots \geq g(R)$, where at least one strict inequality holds. Using the function $g(k)$, model (3.1) is generalized as

$$(3.2) \qquad \sum_{i=1}^{R} g(i)\, p_i^{(1)} = \cdots = \sum_{i=1}^{R} g(i)\, p_i^{(T)} \qquad (\text{i.e.,} \;\; E(g(X_1)) = \cdots = E(g(X_T)))\ .$$

We shall refer to (3.2) as the marginal mean equivalence (ME) model.

The $\{g(k)\}$ may be considered as the ordered scores $\{u_k\}$ assigned to the categories if it is possible to assign the scores; namely, $g(k) = u_k$ satisfying $u_1 \leq \cdots \leq u_R$ or $u_1 \geq \cdots \geq u_R$. In particular, when the scores are equal-interval; that is, when $u_2 - u_1 = u_3 - u_2 = \cdots = u_R - u_{R-1}$, then the ME model with $g(k) = u_k$ is equivalent to the model (3.1). We now obtain the following theorem.

Theorem 3.1. *For multi-way tables, the MH model holds if and only if both the ML and ME models hold.*

Proof: If the MH model holds, then both the ML and ME models hold. Therefore, assuming that both the ML and ME models hold, we shall show that the MH model holds. We have

$$\begin{aligned} E(g(X_1)) &= \sum_{k=1}^{R} g(k)\, p_k^{(1)} \\ &= g(1) + \sum_{k=2}^{R} \left(d_k \sum_{i=k}^{R} p_i^{(1)} \right) \\ &= g(1) + \sum_{k=2}^{R} d_k \big(1 - F_{k-1}^{(1)}\big) \\ &= g(R) - \sum_{k=2}^{R} d_k\, F_{k-1}^{(1)}\ , \end{aligned}$$

where

$$d_k = g(k) - g(k-1)\ .$$

Similarly, we have

$$E(g(X_2)) = g(R) - \sum_{k=2}^{R} d_k\, F_{k-1}^{(2)}\ .$$

This yields

$$E(g(X_2)) - E(g(X_1)) = \sum_{k=2}^{R} d_k \big(F_{k-1}^{(1)} - F_{k-1}^{(2)}\big)\ .$$

Since the ML and ME models hold, we obtain

$$\sum_{k=2}^{R} d_k \left(\frac{\exp(\theta_{k-1})}{1+\exp(\theta_{k-1})} - \frac{\exp(\theta_{k-1}-\Delta_1)}{1+\exp(\theta_{k-1}-\Delta_1)} \right) = 0 \ .$$

Thus

$$\Big(1-\exp(-\Delta_1)\Big) \sum_{k=2}^{R} d_k \frac{\exp(\theta_{k-1})}{\big(1+\exp(\theta_{k-1})\big)\,\big(1+\exp(\theta_{k-1}-\Delta_1)\big)} = 0 \ .$$

Then

$$\sum_{k=2}^{R} d_k \, \frac{\exp(\theta_{k-1})}{\big(1+\exp(\theta_{k-1})\big)\,\big(1+\exp(\theta_{k-1}-\Delta_1)\big)} \neq 0 \ ,$$

because $d_k \geq 0$ for all $k = 2, ..., R$ (or $d_k \leq 0$ for all $k = 2, ..., R$), with at least one of the d_k's being not equal to zero. Therefore we obtain $\Delta_1 = 0$. In the similar way, we obtain $\Delta_k = 0$ for $k = 2, ..., T-1$. Thus, the MH model holds. The proof is completed. □

3.2. A decomposition of the MH model using the CML model

We now obtain the following theorem.

Theorem 3.2. *For multi-way tables, the MH model holds if and only if both the CML and ME models hold.*

We omit the proof because it can be obtained in a similar way as the proof of Theorem 3.1.

Generally, consider a decomposition of model such that model M_1 holds if and only if both models M_2 and M_3 hold. When models M_1 and M_2 fit the data poorly but model M_3 fits the data well, we can then understand that the poor fit of model M_1 is caused by the lack of structure of model M_2 rather than the structure of model M_3. Thus, the decomposition of model M_1 may be useful to see the reason for the poor fit of model M_1.

Let $n_{i_1 \cdots i_T}$ denote the observed frequency in the $(i_1, ..., i_T)$ cell of the R^T table with $n = \sum \cdots \sum n_{i_1 \cdots i_T}$, and let $m_{i_1 \cdots i_T}$ denote the corresponding expected frequency. We assume that $\{n_{i_1 \cdots i_T}\}$ have a multinomial distribution. The maximum likelihood estimates (MLEs) of the expected frequencies under each model can be obtained using a Newton–Raphson method to solve the likelihood equation (see Appendix for the CML model). Denote the likelihood ratio chi-squared statistic for testing the goodness-of-fit of model M by $G^2(M)$. For testing that model M_1 holds assuming that model M_2 holds true, the likelihood ratio statistic is given as $G^2(M_1 \,|\, M_2) = G^2(M_1) - G^2(M_2)$ (≥ 0). The numbers of degrees of freedom (df) for testing the goodness-of-fit of the MH, ML (CML), and ME models are $(T-1)(R-1)$, $(T-1)(R-2)$, and $T-1$, respectively.

Table 1: Opinions about government spending; from Lang and Agresti (1994). The upper and lower paranthesized values are the MLEs of expected frequencies under the ML and CML models, respectively.

Cities	Health	Law Enforcement		
		(1)	(2)	(3)
(1)	(1)	76 (71.31) (76.00)	20 (17.03) (21.00)	5 (5.92) (5.66)
(1)	(2)	13 (12.29) (15.22)	11 (9.43) (12.59)	0 (0.00) (0.00)
(1)	(3)	4 (3.68) (3.31)	3 (2.51) (2.44)	2 (2.31) (1.72)
(2)	(1)	113 (122.83) (108.92)	56 (54.44) (52.96)	5 (7.16) (5.06)
(2)	(2)	30 (32.89) (31.29)	28 (27.43) (28.00)	1 (1.45) (1.10)
(2)	(3)	4 (4.25) (3.04)	1 (0.95) (0.75)	2 (2.78) (1.58)
(3)	(1)	103 (100.86) (103.88)	41 (36.28) (40.54)	15 (18.76) (15.92)
(3)	(2)	29 (28.61) (31.77)	21 (18.71) (22.51)	5 (6.32) (5.79)
(3)	(3)	6 (5.76) (4.73)	8 (6.95) (6.21)	5 (6.09) (5.00)

Note: (1) – too little; (2) – about right; (3) – too much.

4. EXAMPLE

The data in Table 1, taken directly from Lang and Agresti (1994), is the 1989 General Social Survey conducted by the National Opinion Research Center at the University of Chicago. Subjects in the sample were asked their opinion regarding government spending on the health (X_1), the law enforcement (X_2), and the assistance to big cities (X_3). The common response scale is (1) too little, (2) about right, and (3) too much. Table 2 presents the values of likelihood ratio statistic G^2 for each model.

The MH model fits these data very poorly. However the CML model fits these data well although the ML model does not fit so well. Also, the ME model with $g(k) = k$, $k = 1, 2, 3$, fits these data very poorly.

Consider the hypothesis that the MH model holds under the assumption that the ML (CML) model holds; namely, the hypothesis that $\Delta_1 = \Delta_2 = 0$ ($\Delta_1^* = \Delta_2^* = 0$) under the assumption. Because $G^2(\mathrm{MH}|\mathrm{ML}) = G^2(\mathrm{MH}) - G^2(\mathrm{ML}) = 328.57$ and $G^2(\mathrm{MH}|\mathrm{CML}) = G^2(\mathrm{MH}) - G^2(\mathrm{CML}) = 331.49$ with 2 df, we reject these hypotheses at the 0.05 level. These show the rejection of $\Delta_1 = \Delta_2 = 0$ ($\Delta_1^* = \Delta_2^* = 0$) in the ML (CML) model.

Under the CML model the MLEs of $\exp(\Delta_k^*)$ are $\exp(\hat{\Delta}_1^*) = 1.59$ and $\exp(\hat{\Delta}_2^*) = 17.3$ (i.e., $\hat{\Delta}_1^* = 0.46$ and $\hat{\Delta}_2^* = 2.85$). Thus, the CML model provides that (1) under the condition that the opinions are not all same, the odds that the opinion is 'too little' instead of not 'too little' are estimated to be 1.59 times higher in health than in law, and (2) the odds that the opinion is not 'too much' instead of 'too much' are estimated to be 1.59 times higher in health than in law, and similarly, (3) the odds that the opinion is 'too little' instead of not 'too little' are estimated to be 17.3 times higher in health than in cities, and (4) the odds that the opinion is not 'too much' instead of 'too much' are estimated to be 17.3 times higher in health than in cities.

Table 2: Likelihood ratio statistic G^2 for models applied to the data in Table 1.

Models	Table 1	
	df	G^2
MH	4	334.62*
ML	2	6.05*
CML	2	3.13
ME	2	316.01*

* means significant at 0.05 level.

Note: $g(k)$ for the ME model are the equal-interval scores.

5. CONCLUDING REMARKS

When the MH model fits the data poorly, the decompositions of the MH model may be useful for seeing the reason for its poor fit. Indeed, for the data in Table 1, the poor fit of the MH model is caused by the poor fit of the ME model rather than the ML (or CML) model.

Each of the MH, CML and ME models does not depend on the probabilities $\{p_{ii\cdots i}\}$ on the main diagonal of the table, but the ML model depends on them. Notice that the estimated expected frequencies on the main diagonal cells under the ML model are different from the observed frequencies on the main diagonal (see Table 1).

When the MH model does not hold, if we want to see the reason why the equalities of the conditional marginal cumulative probabilities $\{F_i^{c(k)}\}$ do not hold, the analyst would be interested in inferring the structure of only off-diagonal probabilities. In this case, the decomposition of the MH model into the CML and ME models may be preferable to that into the ML and ME models.

Also, the MH model indicates the equalities of marginal cumulative probabilities $\{F_i^{(k)}\}$, which include the probabilities $\{p_{ii\cdots i}\}$ on the main diagonal. Therefore, when the MH model does not hold, if we want to see the reason why the equalities of $\{F_i^{(k)}\}$ do not hold, the analyst would be interested in inferring the structure of $\{F_i^{(k)}\}$. Then, the decomposition of the MH model into the ML and ME models may be preferable to that into the CML and ME models.

The decompositions of the MH model described here should be considered for ordinal categorical data, because each of the decomposed models is not invariant under the same arbitrary permutations of all categories.

APPENDIX

We consider the MLEs of the expected frequencies $\{m_{ijt}\}$ under the CML model. We give the case of three-way table below and omit the case of more multi-way table because those are obtained in the similar way.

To obtain the MLEs under the CML model, we must maximize the Lagrangian

$$
\begin{aligned}
L \;=\; & \sum_{i=1}^{R}\sum_{j=1}^{R}\sum_{t=1}^{R} n_{ijt}\log p_{ijt} \;-\; \mu\left(\sum_{i=1}^{R}\sum_{j=1}^{R}\sum_{t=1}^{R} p_{ijt} \;-1\right)\\
& -\sum_{i=1}^{R-1}\lambda_{1i}\Big(F_i^{c(1)}\big(1-F_i^{c(2)}\big) - \exp(\Delta_1^*)\,\big(1-F_i^{c(1)}\big)\,F_i^{c(2)}\Big)\\
& -\sum_{i=1}^{R-1}\lambda_{2i}\Big(F_i^{c(1)}\big(1-F_i^{c(3)}\big) - \exp(\Delta_2^*)\,\big(1-F_i^{c(1)}\big)\,F_i^{c(3)}\Big)
\end{aligned}
$$

with respect to $\{p_{ijt}\}$, μ, $\{\lambda_{1i}\}$, $\{\lambda_{2i}\}$, Δ_1^*, and Δ_2^*. Setting the partial derivatives of L equal to zeros, we obtain the equations

$$
\begin{aligned}
m_{ijt} &= \frac{n_{ijt}}{1+\frac{1}{n}(T_{1ij}+T_{2it})} \qquad \text{for} \quad i,j,t=1,...,R; \quad (i,j,t)\neq(i,i,i)\ ,\\
m_{iii} &= n_{iii} \qquad \text{for} \quad i=1,...,R\ ,
\end{aligned}
$$

where

$$
\begin{aligned}
T_{1ij} \;=\; \delta^{-1}\sum_{u=1}^{R-1} \Bigg[& I_{(u\ge i)}\,I_{(u\ge j)}\,\lambda_{1u}\Big\{\big(1-F_u^{c(2)}\big) - \exp(\Delta_1^*)\,\big(1-F_u^{c(1)}\big)\Big\}\\
& + I_{(u<i)}\,I_{(u\ge j)}\,\lambda_{1u}\Big\{-\exp(\Delta_1^*)\Big(F_u^{c(2)} + \big(1-F_u^{c(1)}\big)\Big)\Big\}\\
& + I_{(u\ge i)}\,I_{(u<j)}\,\lambda_{1u}\Big\{\big(1-F_u^{c(2)}\big) + F_u^{c(1)}\Big\}\\
& + I_{(u<i)}\,I_{(u<j)}\,\lambda_{1u}\Big\{F_u^{c(1)} - \exp(\Delta_1^*)\,F_u^{c(2)}\Big\}\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
T_{2it} \;=\; \delta^{-1}\sum_{u=1}^{R-1} \Bigg[& I_{(u\ge i)}\,I_{(u\ge t)}\,\lambda_{2u}\Big\{\big(1-F_u^{c(3)}\big) - \exp(\Delta_2^*)\,\big(1-F_u^{c(1)}\big)\Big\}\\
& + I_{(u<i)}\,I_{(u\ge t)}\,\lambda_{2u}\Big\{-\exp(\Delta_2^*)\Big(F_u^{c(3)} + \big(1-F_u^{c(1)}\big)\Big)\Big\}\\
& + I_{(u\ge i)}\,I_{(u<t)}\,\lambda_{2u}\Big\{\big(1-F_u^{c(3)}\big) + F_u^{c(1)}\Big\}\\
& + I_{(u<i)}\,I_{(u<t)}\,\lambda_{2u}\Big\{F_u^{c(1)} - \exp(\Delta_2^*)\,F_u^{c(3)}\Big\}\Bigg],
\end{aligned}
$$

and

$$\sum_{i=1}^{R-1} \lambda_{k-1,i} \left(1 - F_i^{c(1)}\right) F_i^{c(k)} = 0 \qquad \text{for} \quad k = 2, 3 ,$$

$$F_i^{c(1)} \left(1 - F_i^{c(k)}\right) = \exp(\Delta_{k-1}^*) \left(1 - F_i^{c(1)}\right) F_i^{c(k)} \qquad \text{for} \quad i = 1, ..., R-1; \quad k = 2, 3 ,$$

where $m_{ijt} = n p_{ijt}$ and $I_{(\cdot)}$ is the indicator function. Using the Newton–Raphson method, we can solve the equations with respect to $\{p_{ijt}\}$, $\{\lambda_{1i}\}$, $\{\lambda_{2i}\}$, Δ_1^* and Δ_2^*. Therefore, we can obtain the MLEs of $\{m_{ijt}\}$, Δ_1^* and Δ_2^* under the CML model.

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