REVSTAT – Statistical Journal
Volume 22, Number 2, April 2024, 169–188
https://doi.org/10.57805/revstat.v22i2.468

A Novel Fractional Forecasting Model for Time Dependent Real World Cases

Authors: Ummugulsum Erdinc
– Department of Mathematics, Aksaray University,
Aksaray, Turkey
ummugulsumerdinc@aksaray.edu.tr

Halis Bilgil ✉
– Department of Mathematics, Aksaray University,
Aksaray, Turkey
halis@aksaray.edu.tr

Zafer Ozturk
– Institute of Science, Nevsehir Haci Bektas Veli University,
Turkey
zaferozturk@aksaray.edu.tr

Received: August 2021 Revised: May 2022 Accepted: May 2022

Abstract:

- The grey modelling in the prediction of time series has been one of the interesting study fields recently due to its efficiency and convenience. Fractional grey models, on the other hand, have become preferable, despite the difficulty in calculations, since they give more effective results than standard models. Difficulties in fractional accumulation and difference calculations have begun to be overcome thanks to new definitions and theorems made in recent years. The new trend in grey modelling is to compose models that are more useful than the previous ones and give results with less error. In this paper, a new grey model derived from the conformable fractional order is defined and it is shown that more effective estimates are made compared to the models created in recent years. The verification of the method is shown with real data set. The results show that the proposed conformable fractional grey model is more effective than the existing models.

Keywords:

- *grey systems; least square method; nonresponse; conformable fractional calculus; fractional grey model; ECFGM model.*

AMS Subject Classification:

- 26A33, 60G25, 81T80.

✉ Corresponding author

1. INTRODUCTION

The grey system is a discipline that studies the problem of uncertainty, which was first proposed by Deng in 1982, plays an important role in the grey system theory [4]. It has been a useful tool in processing uncertain or excursive systems with small samples and limited data set as distinct from the machine learning models, hybrid models, the empirical models, ..., etc. The grey prediction models have been widely and successfully applied to various fields, such as science and technology, energy, environmental problems, economy, health and other fields [24, 35, 36, 7, 1, 33, 5, 18].

Recent studies on grey modelling focus on two main purpose: practicality and prediction accuracy. For these purposes, important studies have been carried out in recent times. Ma and Lui [15] proposed a time-delayed polynomial grey prediction model called TDPGM$(1,1)$ model, the grey polynomial model with a tuned background coefficient was proposed by Wei *et al.* [20], Cui *et al.* [3] developed a parameter optimization method to improve the ONGM$(1,1,k)$ model, Bilgil [1] proposed an exponential grey model named EXGM$(1,1)$. Furthermore, Ma *et al.* [14] developed a novel nonlinear multivariate forecasting grey model based on the Bernoulli equation named NGBMC$(1,n)$, Wang *et al.* [19] introduced a seasonal grey model called SGM$(1,1)$, Wu *et al.* [25] proposed a new grey model called BNGM$(1,1,t^2)$ model, Liu and Wu [12] proposed the ANDGM model, Ma [13] proposed kernel-based KARGM$(1,1)$ model, Li *et al.* [10] developed structure-adaptive intelligent grey forecasting model, Wu *et al.* [26] developed a novel grey Riccati model (GRM), the modified grey prediction model with damping trend factor was proposed by Liu *et al.* [11], the nonlinear grey Bernoulli model with improved parameters, INGBM$(1,1)$, was proposed by Jiang *et al.* [8].

It is clear that most of the existing grey models are defined with a first-order whitening differential equation. If the original data is a disordered sequence, the characteristic features of the sequence may not be exactly found out by first-order accumulative generation operation (1-AGO) [32]. Moreover, first-order derivative models are ideal memory models, which are not suitable for describing irregular phenomena. As a result of this, the parameters of the model may not be compatible according to the data characteristics of the actual problem for a sequence with large data fluctuation. Therefore, fractional accumulation generating operation and fractional derivative should be introduced into the grey model to overcome this problem [32].

Wu *et al.* developed traditional GM$(1,1)$ with fractional order accumulated operator named FAGM$(1,1)$ [23]. Some researchers optimized the FAGM$(1,1)$ model and reached better prediction accuracy in recent years. Wu *et al.* suggested a fractional FAGMO$(1,1,k)$ model with linear grey input of time in lieu of constant grey input in the initial FAGM$(1,1)$ model and optimized it with optimal order and optimal parameters [27]. Besides, Mao *et al.* introduced the fractional grey model FGM$(q,1)$ [17], a power-driven fractional accumulated grey model named PFAGM is introduced by Zhang *et al.* [34], Yuxiao *et al.* proposed the multivariable Caputo fractional derivative grey model with convolution integral named CFGMC(q,N), Xie *et al.* developed a conformable fractional grey model in opposite direction CFGOM$(1,1)$ [30].

However, the definition is given by Wu [23] only show us a single situation of fractional order calculus and differencing. From the perspective of computational complexity, frequently operated descriptions of fractional accumulation and its suitable fractional differencing in the present grey models are not easy to apply, and it causes serious rigours to the more deeply theoretical analysis. Yang and Xue [31] submitted the fractional order calculus, but the exact solution of this kind of model include infinite series, and it is clearly difficult to use and analyse. Furthermore, this kind of complications would also impede the improvement of the fractional order grey models thanks to some new operators [22, 21].

Lately, Khalil *et al.* defined a limit-based fractional derivative in 2014 [9], which is named the conformable fractional derivative. The structure of this new definitions of fractional derivative is simpler than that of other popular fractional derivatives, such as the Caputo derivative and Riemann–Liouville derivative. Then, Ma *et al.* [16] introduced the new useful definitions of the fractional order difference and accumulation based on the conformable fractional derivative in which the computational complexity of accumulation is lower than that of the traditional fractional accumulated operator and they firstly proposed an improved fractional order grey model named CFGM$(1,1)$. Recently, a continuous grey model named CCFGM based on the conformable fractional derivative was described by Xie *et al.* [29].

In this paper, we introduced the novel exponential conformable fractional grey model (denoted as ECFGM$(1,1)$ for short) by using the new definitions of conformable fractional difference and conformable fractional accumulation by Ma *et al.* [16]. The structural order of ECFGM$(1,1)$ is sought out by using the Brute Force algorithm. The effectiveness of ECFGM$(1,1)$ is validated by real data sets in comparison with other used new grey models and it is seen that the performance of the ECFGM$(1,1)$ model is very successful.

The rest of this paper is organized as follows: Section 1 includes relevant literature. Some useful properties and definitions of the conformable fractional calculus are given in Section 2. The presentations and modelling mechanism of the ECFGM$(1,1)$ are introduced in Section 3. In Sections 4 and 5, we present a series of samples to validate ECFGM$(1,1)$. Finally, the conclusions of this study are given in Sections 6.

2. SOME DEFINITIONS AND PROPERTIES ON CONFORMABLE FRACTIONAL CALCULUS

In this section, some useful definions and properties of the conformable fractional derivative are summerized.

2.1. The conformable fractional derivative

Definition 2.1 (See [16]). If $f\colon [0,\infty) \to R$ is a differentiable function, the conformable fractional derivative of f with $\alpha \in (n, n+1]$ order is defined as

$$T_\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{\lceil \alpha \rceil - \alpha}) - f(t)}{\varepsilon} = t^{\lceil \alpha \rceil - \alpha} \frac{df(t)}{dt}, \tag{2.1}$$

where $\lceil \cdot \rceil$ is the ceil function, i.e. the $\lceil \alpha \rceil$ is the smallest integer no larger than α. It is clear that $\lceil \alpha \rceil = 1$ for $\alpha \in (0,1]$. Hence, equation (2.1) can be written as

$$T_\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon} = t^{1-\alpha}\frac{df(t)}{dt}$$

for all $t > 0$.

The following theorem gives the properties of the definition (Khalil *et al.* [9]).

Theorem 2.1 (See [9]). *If the functions f and g are differentiable, $\alpha \in (0,1]$, then we have:*

1. $T_\alpha(f)(t) = t^{1-\alpha}\frac{df(t)}{dt}$;
2. $T_\alpha(mf + ng) = mT_\alpha(f) + nT_\alpha(g)$ *for all* $m, n \in \mathbb{R}$;
3. $T_\alpha(f.g) = fT_\alpha(g) + gT_\alpha(f)$;
4. $T_\alpha\left(\frac{f}{g}\right) = \frac{gT_\alpha(f) - fT_\alpha(g)}{g^2}$;
5. $T_\alpha(c) = 0$ *for all constant* c;
6. $T_\alpha(t^p) = pt^{p-\alpha}$ *for all* $p \in \mathbb{R}$;
7. $T_\alpha(e^{cx}) = cx^{1-\alpha}e^{cx}$ *for all* $c \in \mathbb{R}$.

Proof: The proof is omitted. □

2.2. The conformable fractional accumulation and difference

New definitions to calculate the conformable fractional accumulation (CFA) and the conformable fractional difference (CFD) are given by Ma *et al.* [16] as follows.

Definition 2.2 (see [16, 28]). The conformable fractional difference (CFD) of f with α order is defined as

$$\Delta^\alpha f(k) = k^{1-\alpha}\Delta f(k) = k^{1-\alpha}[f(k) - f(k-1)] \tag{2.2}$$

for all $k \in N^+$, $\alpha \in (0,1]$, and

$$\Delta^\alpha f(k) = k^{\lceil \alpha \rceil - \alpha}\Delta^{n+1} f(k) = k^{\lceil \alpha \rceil - \alpha} \sum_{j=k-\lceil \alpha \rceil}^{k} (-1)^{k-j}\binom{\lceil \alpha \rceil}{k-j} f(j) \tag{2.3}$$

for all $k \in N^+$, $\alpha \in (n, n+1]$.

Definition 2.3 (see [16]). The conformable fractional accumulation (CFA) of f with α order is defined as

$$\nabla^\alpha f(k) = \nabla\left(\frac{f(k)}{k^{1-\alpha}}\right) = \sum_{j=1}^{k} \frac{f(j)}{j^{1-\alpha}} \tag{2.4}$$

for all $k \in N^+$, $\alpha \in (0,1]$, and

$$\nabla^\alpha f(k) = \nabla^{n+1}\left(\frac{f(k)}{k^{\lceil \alpha \rceil - \alpha}}\right) \tag{2.5}$$

for all $k \in N^+$, $\alpha \in (n, n+1]$.

3. PRESENTATION OF EXPONENTIAL CONFORMABLE FRACTIONAL GREY MODEL

In this section, a novel exponential conformable fractional grey model, named ECFGM$(1,1)$, is introduced, which optimizes the classical CFGM$(1,1)$ model with an exponential grey action quantity.

3.1. Formulation of proposed fractional grey model

The original series $X^{(0)} = \left(x^{(0)}(1), x^{(0)}(2), ..., x^{(0)}(n)\right)$ is given. CFA with α order is calculated as follows:

$$X^{(\alpha)} = \left(x^{(\alpha)}(1), x^{(\alpha)}(2), ..., x^{(\alpha)}(n)\right), \tag{3.1}$$

where

$$x^{(\alpha)}(k) = \nabla^{\alpha} x^{(0)}(k) = \sum_{i=1}^{k} \begin{bmatrix} \lceil \alpha \rceil \\ k-i \end{bmatrix} \frac{x^{(0)}(i)}{i^{\lceil \alpha \rceil - \alpha}}, \quad \alpha \in \mathbb{R}^{+}, \tag{3.2}$$

where $\begin{bmatrix} \lceil \alpha \rceil \\ k-i \end{bmatrix} = \frac{\Gamma(k-i+\lceil \alpha \rceil)}{\Gamma(k-i+1)\Gamma(\lceil \alpha \rceil)} = \frac{(k-i+\lceil \alpha \rceil -1)!}{(k-i)!(\lceil \alpha \rceil -1)!}$ (see [28]).

Definition 3.1. The first-order whitening differential equation of the ECFGM$(1,1)$ is defined as

$$\frac{dx^{(\alpha)}(t)}{dt} + a x^{(\alpha)}(t) = b + ce^{-t}, \tag{3.3}$$

where a is a development coefficient, b is called driving coefficient and ce^{-t} is an exponential grey action quantity. So that, the monotone decreasing term ce^{-t} will suppress the growth of the prediction error.

When $\alpha = 1$, the ECFGM$(1,1)$ model yields the EXGM$(1,1)$ [9]. In addition, the proposed model can be translated to the conventional CFGM$(1,1)$ model for $c = 0$ [23].

3.2. Parameters estimation

Theorem 3.1. *For the computed* CFA *and the value of fractional order, the system parameters a, b and c of the* ECFGM$(1,1)$ *satisfy the following equation:*

$$[a, b, c]^{\top} = (B^{\top} B)^{-1} B^{\top} Y, \tag{3.4}$$

where the matrix B and Y are

$$B = \begin{bmatrix} -0.5(x^{(\alpha)}(2) + x^{(\alpha)}(1)) & 1 & (e-1)e^{-2} \\ -0.5(x^{(\alpha)}(3) + x^{(\alpha)}(2)) & 1 & (e-1)e^{-3} \\ \vdots & \vdots & \vdots \\ -0.5(x^{(\alpha)}(n) + x^{(\alpha)}(n-1)) & 1 & (e-1)e^{-n} \end{bmatrix}, \quad Y = \begin{bmatrix} x^{(\alpha)}(2) - x^{(\alpha)}(1) \\ x^{(\alpha)}(3) - x^{(\alpha)}(2) \\ \vdots \\ x^{(\alpha)}(n) - x^{(\alpha)}(n-1) \end{bmatrix}. \tag{3.5}$$

Proof: Integrating both sides of the whitening equation (3.3) within the interval $[k-1,k]$ the discrete form of ECFGM(1,1) model is obtained as follows:

$$\int_{k-1}^{k}\frac{dx^{(\alpha)}(t)}{dt}dt+a\int_{k-1}^{k}x^{(\alpha)}(t)dt=\int_{k-1}^{k}(b+ce^{-t})dt. \tag{3.6}$$

According to the Newton–Leibniz formula, the first integral of (3.6) can be expressed as

$$\int_{k-1}^{k}\frac{dx^{(\alpha)}(t)}{dt}dt=x^{(\alpha)}(k)-x^{(\alpha)}(k-1). \tag{3.7}$$

It is clear that the integration term $\int_{k-1}^{k}x^{(\alpha)}(t)dt$ denotes the area between t-axis and the curve $x^{(\alpha)}(t)$ in the interval $[k-1,k]$. Then, using the generalized trapezoid formula as in many recent studies [23, 25, 16, 29, 26, 17, 10, 21], the second integral of (3.6) can be obtained as

$$a\int_{k-1}^{k}x^{(\alpha)}(t)dt=\frac{a}{2}\Big(x^{(\alpha)}(k)+x^{(\alpha)}(k-1)\Big) \tag{3.8}$$

and the right side of (3.6) is equal to

$$\int_{k-1}^{k}(b+ce^{-t})dt=b+c(e^{1-k}-e^{-k}). \tag{3.9}$$

Substituting equations (3.7)–(3.9) into equation (3.6), it can be written as

$$\Big(x^{(\alpha)}(k)-x^{(\alpha)}(k-1)\Big)+\frac{a}{2}\Big(x^{(\alpha)}(k)+x^{(\alpha)}(k-1)\Big)=b+c(e^{1-k}-e^{-k}), \tag{3.10}$$

where $k=2,3,...,n$.

The linear equations system (3.10) can be written as follows:

$$\begin{aligned} x^{(\alpha)}(2)-x^{(\alpha)}(1)&=-0.5a(x^{(\alpha)}(2)+x^{(\alpha)}(1))+b+c(e^{-1}-e^{-2})\\ x^{(\alpha)}(3)-x^{(\alpha)}(2)&=-0.5a(x^{(\alpha)}(3)+x^{(\alpha)}(2))+b+c(e^{-2}-e^{-3})\\ \vdots\quad&\vdots\quad\vdots\\ x^{(\alpha)}(n)-x^{(\alpha)}(n-1)&=-0.5a(x^{(\alpha)}(n)+x^{(\alpha)}(n-1))+b+c(e^{1-n}-e^{-n}) \end{aligned} \tag{3.11}$$

and system (3.11) can be written as

$$Y=B\rho, \tag{3.12}$$

where

$$B=\begin{bmatrix} -0.5(x^{(\alpha)}(2)+x^{(\alpha)}(1)) & 1 & (e-1)e^{-2}\\ -0.5(x^{(\alpha)}(3)+x^{(\alpha)}(2)) & 1 & (e-1)e^{-3}\\ \vdots & \vdots & \vdots\\ -0.5(x^{(\alpha)}(n)+x^{(\alpha)}(n-1)) & 1 & (e-1)e^{-n}\end{bmatrix},\quad Y=\begin{bmatrix} x^{(\alpha)}(2)-x^{(\alpha)}(1)\\ x^{(\alpha)}(3)-x^{(\alpha)}(2)\\ \vdots\\ x^{(\alpha)}(n)-x^{(\alpha)}(n-1)\end{bmatrix} \tag{3.13}$$

and $\rho=[a,b,c]^{\top}$ in which n is the number of samples used to construct the model.

The parameter estimation of the ECFGM$(1,1)$ model using the least squares method can be obtained. For the estimated value of the parameter sequence ρ, the $x^{(\alpha)}(k) - x^{(\alpha)}(k-1)$ on the left side of the equation (3.11) is replaced with $-0.5a(x^{(\alpha)}(k) + x^{(\alpha)}(k-1)) + b + c(e^{1-k} - e^{-k})$, the error sequence $\epsilon = Y - B\rho$ is obtained. Here,

$$\epsilon = [\epsilon_2, \epsilon_3, ..., \epsilon_n]^\top \tag{3.14}$$

and ϵ_k represents the error for each equation in the system (3.11) for $k = 2, 3, ..., n$.

Notice, $S(\rho)$ is defined as the sum of squares of errors, which yields

$$\begin{aligned} S(\rho) &= \sum_{k=2}^{n} \epsilon_k^2 \\ &= \epsilon^\top \epsilon \\ &= (Y - B\rho)^\top (Y - B\rho) \\ &= (Y^\top - \rho^\top B^\top)(Y - B\rho) \\ &= Y^\top Y - Y^\top B\rho - \rho^\top B^\top Y + \rho^\top B^\top B\rho \\ &= Y^\top Y - 2\rho^\top B^\top Y + \rho^\top B^\top B\rho. \end{aligned} \tag{3.15}$$

The parameter vector $\rho = [a, b, c]^\top$ that minimize $S(\rho)$ satisfies

$$\frac{\partial S}{\partial \rho} = -2B^\top Y + 2B^\top B\rho = 0, \tag{3.16}$$

so

$$B^\top Y = B^\top B\rho. \tag{3.17}$$

Thus

$$\rho = (B^\top B)^{-1} B^\top Y, \tag{3.18}$$

or

$$[a, b, c]^\top = \left(B^\top B\right)^{-1} B^\top Y. \tag{3.19}$$

Thence the proof is completed by using the least square estimation method. □

3.3. Response function and restored values

Theorem 3.2. *The discrete form of the response function of* ECFGM$(1,1)$ *model is given as*

$$\hat{x}^{(\alpha)}(k) = \left(x^{(0)}(1) - \frac{b}{a} - \frac{c}{a-1}e^{-1}\right)e^{a(1-k)} + \frac{b}{a} + \frac{c}{a-1}e^{-k}, \tag{3.20}$$

where $k = 2, 3, ..., n$.

Proof: It is clear that the solution of the first order linear *whitening differential equation* (3.3) can be obtained as

$$x^{(\alpha)}(t) = \frac{b}{a} + \frac{c}{a-1}e^{-t} + de^{-at}, \tag{3.21}$$

where d is integral constant. By using the initial condition $x^{(\alpha)}(1) = x^{(0)}(1)$, the constant d can be found as

$$d = \left(x^{(0)}(1) - \frac{b}{a} - \frac{c}{a-1}e^{-1}\right)e^{a}.$$

Therefore the grey prediction model equation (3.21) can be obtained as

$$\hat{x}^{(\alpha)}(t) = \left(x^{(0)}(1) - \frac{b}{a} - \frac{c}{a-1}e^{-1}\right)e^{a(1-t)} + \frac{b}{a} + \frac{c}{a-1}e^{-t}$$

and the discrete form of the response function can be written as

$$\hat{x}^{(\alpha)}(k) = \left(x^{(0)}(1) - \frac{b}{a} - \frac{c}{a-1}e^{-1}\right)e^{a(1-k)} + \frac{b}{a} + \frac{c}{a-1}e^{-k}.$$

The proof is completed. □

Theorem 3.3. *Then, restored values can be given as*

$$\hat{x}^{(0)}(k) = \Delta^{\alpha}\hat{x}^{(\alpha)}(k) = k^{\lceil\alpha\rceil-\alpha}\Delta^{n+1}\hat{x}^{(\alpha)}(k), \quad \alpha \in (n, n+1], \tag{3.22}$$

where $k = 2, 3, ..., n.$

Proof: From (3.2) it can be seen that $\hat{x}^{(\alpha)}(k) = \nabla^{\alpha}\hat{x}^{(0)}(k)$. If we apply the inverse operator Δ^{α}, it is obtained as

$$\hat{x}^{(0)}(k) = \Delta^{\alpha}\hat{x}^{(\alpha)}(k)$$

and from Definition 2.2 it can be written as

$$\hat{x}^{(0)}(k) = \Delta^{\alpha}\hat{x}^{(\alpha)}(k) = k^{\lceil\alpha\rceil-\alpha}\Delta^{n+1}\hat{x}^{(\alpha)}(k), \quad \alpha \in (n, n+1].$$

This completes the proof. □

It is clear that, the restored values for $\alpha \in (0, 1]$ can be written as

$$\hat{x}^{(0)}(k) = k^{1-\alpha}(\hat{x}^{(\alpha)}(k) - \hat{x}^{(\alpha)}(k-1)). \tag{3.23}$$

3.4. Evaluative accuracy of the forecasting model

The relative percentage error (RPE) and the mean absolute percentage error (MAPE) are used to evaluate the fitting and predicting performance of ECFGM(1, 1). The lowest MAPE value indicates the best prediction model. They are defined as follows:

$$\mathrm{RPE}(k) = \left|\frac{\hat{x}^{(0)}(k) - x^{(0)}(k)}{x^{(0)}(k)}\right| \times 100\%, \tag{3.24}$$

$$\mathrm{MAPE} = \frac{1}{n}\sum_{k=1}^{n}\mathrm{RPE}(k), \tag{3.25}$$

where $x^{(0)}(k)$ is the original series, and $\hat{x}^{(0)}(k)$ is the predicted series.

For a raw sequence with n samples, the metric MAPE_{fit} is defined as the fitting performance metric while MAPE_{pre} is defined as a prediction performance metric. Mathematically, they can be formulated as:

$$\text{MAPE}_{\text{fit}} = \frac{1}{p}\sum_{k=1}^{p}\text{RPE}(k), \tag{3.26}$$

$$\text{MAPE}_{\text{pre}} = \frac{1}{n-p}\sum_{k=p+1}^{n}\text{RPE}(k), \tag{3.27}$$

where p represents the number of samples used for fitting a model while the rest of the raw sequence is used to examine the prediction accuracy of the model. The total MAPE is given in (3.25) and it is used to evaluate the whole performance of a model.

3.5. Computation steps

The computation steps of ECFGM$(1,1)$ model with given sample and α can be summarized as follows:

Step 1: Create a raw data set $\left(x^{(0)}(1), x^{(0)}(2), ..., x^{(0)}(n)\right)$;

Step 2: Take as $\alpha = 0.01$ to the initial value and designate an initial value of MAPE_{min};

Step 3: Compute the CFA series with α order of the given raw data set by using (3.2);

Step 4: Build the matrix B and Y using (3.5);

Step 5: Calculate the parameters a, b and c using (3.4);

Step 6: Calculate the predicted values and the response function using (3.20) and (3.22);

Step 7: Calculate the mean absolute percentage error (MAPE) using (3.25);

Step 8: If MAPE is greater than MAPE_{min}, set α as $\alpha = \alpha + h$ (where h is the step size), otherwise take as $\text{MAPE}_{\text{min}} = \text{MAPE}$ and go to Step 3; where step 8 is continued until α reaches the predetermined value.

Brute Force is a straightforward approach, which is also known as the Naive algorithm, for solving optimization problems that rely on sheer computing power and trying every possibility rather than advanced techniques to improve efficiency. Unlike some of the other popular swarm intelligence algorithms, Brute Force is applicable to a very wide variety of problems and it is an effective and easy method to find the optimum value in the solution interval.

Despite the convergence speed advantages of other algorithms, it is a disadvantage that they may focus on the local extremum point rather than the global extremum. However, the Brute Force algorithm scans the whole domain, evaluates each point, then calculates the MAPE's on these points and reaches optimum parameters without any delusion.

In this paper, our purpose is to find the optimum parameter α that minimizes the model's mean absolute percentage error (MAPE). Therefore we enumerate all the values in an interval with step 0.01. For suitability, in the next section, α will be generated in $(0,1]$. In this way, all the α points in the whole interval and the MAPE's at these points are calculated.

Hence α, which give the minimum of the calculated MAPE's, is determined as the optimum parameters. The computational steps mentioned above are employed for all the α points.

From equation (2.5), as the α value approaches 1, the CFA series approaches the 1-AGO series, and as the α value approaches 2, the CFA series approaches the 2-AGO series. Obviously, it can be seen that for each point the CFA value $x^{(\alpha)}(k)$ becomes larger with larger α, and the growing speed also increases with larger α [16].

According to the Theorem 3.3, if $\alpha \in (0,1]$ the following first-order fractional difference must be calculated to evaluate the $\hat{x}^{(0)}(k)$ values:

$$\hat{x}^{(0)}(k) = k^{1-\alpha}(\hat{x}^{(\alpha)}(k) - \hat{x}^{(\alpha)}(k-1)), \tag{3.28}$$

where $k = 2, 3, ..., n$. This shows that the error in $\hat{x}^{(0)}(k)$ values is due to two points of the CFA series. If $\alpha \in (1, 2]$, the following second-order fractional difference must be calculated to evaluate the $\hat{x}^{(0)}(k)$ values:

$$\hat{x}^{(0)}(k) = k^{2-\alpha}\Delta^2\hat{x}^{(\alpha)}(k) = k^{2-\alpha}(\hat{x}^{(\alpha)}(k) - 2\hat{x}^{(\alpha)}(k-1) + \hat{x}^{(\alpha)}(k-2)), \tag{3.29}$$

where $k = 3, ..., n$. If $k = 2$, $\hat{x}^{(\alpha)}(0)$, $\hat{x}^{(\alpha)}(1)$, and $\hat{x}^{(\alpha)}(2)$ values are needed to determine the value of $\hat{x}^{(0)}(2)$. However, according to equation (3.20), since there is no $\hat{x}^{(\alpha)}(0)$ value in the CFA series, the second-order difference can only be calculated for $k = 3, 4, ..., n$. In this case, although $\hat{x}^{(\alpha)}(1)$ is known as $x^{(0)}(1)$ according to the initial value, the value of $\hat{x}^{(\alpha)}(2)$ can only be calculated with the help of a first-order fractional difference. This will increase the total error rate. According to equation (3.29), the error of restored value is effected by the errors of the CFA series with three points, which will be another factor that increases the error rate. Moreover, 99% optimal α values in the CFGM model are in the range of $[0, 1)$. In the FGM model, 72% optimal α values are in the range of $(0, 1)$ (for more detailed information, see [16]). In the ECFGM model, 89% optimal α values are in the range of $(0, 1)$ and 10% optimal α are obtained at 1.

All these processes are completed in about five seconds by writing a simple FORTRAN code.

4. VALIDATION OF THE ECFGM(1,1)

In this section, two instructive examples are given to demonstrate the efficacy of the proposed model.

4.1. Example A

In this subsection, a numerical example is presented to show the computational steps of the ECFGM(1,1) model with the raw data $X^{(0)}(k) = (13.21, 18.82, 26.45, 36.04, 42.34, 51.00, 59.12)$. In this example, we select the raw data as a monotone increasing series.

4.1.1. Selecting the optimal α

By using the Brute Force strategy, calculated MAPEs with α in the interval $(0, 1]$ with step 0.01 are given in Figure 1. It can be seen that the values of MAPE increase when the α moves away from its optimum value. For this example, the optimal α is obtained at $\alpha = 0.14$, and the MAPE is calculated for the optimal α as $\text{MAPE} = 0.9843$. It is seen that the optimal α is easily obtained by using the Brute Force strategy. Furthermore, it is clear that the performance of the proposed ECFGM(1, 1) model is respectable.

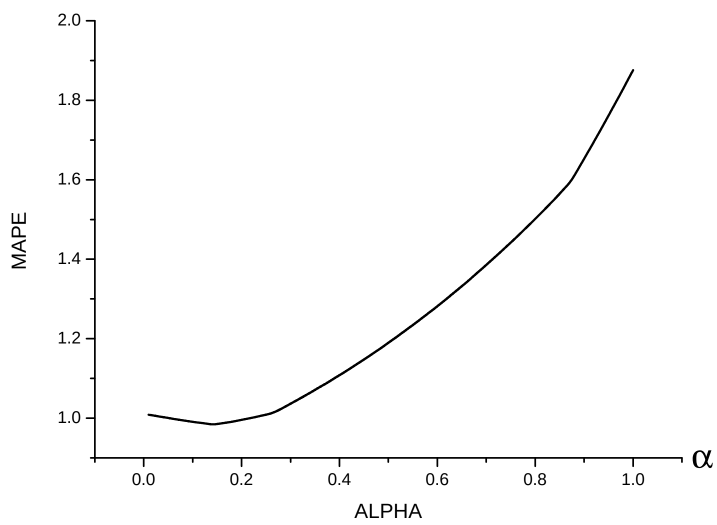


Figure 1: MAPEs of ECFGM model with α in $(0, 1]$ for example A.

4.1.2. Computing the CFA of the original series and modelling the ECFGM(1,1)

Computation of the CFA of the original series is the first step to build the ECFGM(1, 1) model. For the optimum $\alpha = 0.14$, the conformable fractional accumulation series can be obtained using (3.2) as $X^{(0.14)}(k) = (13.2100, 23.5789, 33.8615, 44.8014, 55.4095, 66.3329, 77.4234)$.

The matrices B and Y can be constructed as

$$B = \begin{bmatrix} -18.3945 & 1 & 0.2325 \\ -28.7202 & 1 & 0.0855 \\ -39.3314 & 1 & 0.0315 \\ -50.1054 & 1 & 0.0116 \\ -60.8712 & 1 & 0.0043 \\ -71.8781 & 1 & 0.0016 \end{bmatrix}, \qquad Y = \begin{bmatrix} 10.3689 \\ 10.2826 \\ 10.9399 \\ 10.6081 \\ 10.9235 \\ 11.0905 \end{bmatrix}.$$

Then we obtain the parameters $\hat{a}$, $\hat{b}$ and $\hat{c}$ using the least squares solution as

$$[a, b, c]^\top = \left(B^\top B\right)^{-1} B^\top Y$$

and

$$a = -0.0136199261,$$
$$b = 10.0950218700,$$
$$c = -0.0670244100.$$

By substituting the parameters into the response function equation (3.20) we have

$$\hat{x}^{(\alpha)}(k) = 754.3807\ e^{-0.0136(1-k)} - 741.1951 + 0.06612\ e^{-k}. \tag{4.1}$$

Then the restored can be obtained using (4.1) by k from 1 to 7 as

$$\hat{X}^{(0.14)}(k) = (13.2100,\ 23.5396,\ 34.0206,\ 44.6491,\ 55.4247,\ 66.3485,\ 77.4223).$$

Then the restored values can be obtained using the CFD in (3.23) as

$$\hat{X}^{(0)}(k) = (13.2100, 18.7485,\ 26.9607,\ 35.0141,\ 43.0085,\ 51.0018,\ 59.0312).$$

The original raw series $X^{(0)}(k)$ and predicted values $\hat{X}^{(0)}(k)$ are plotted in Figure 2.

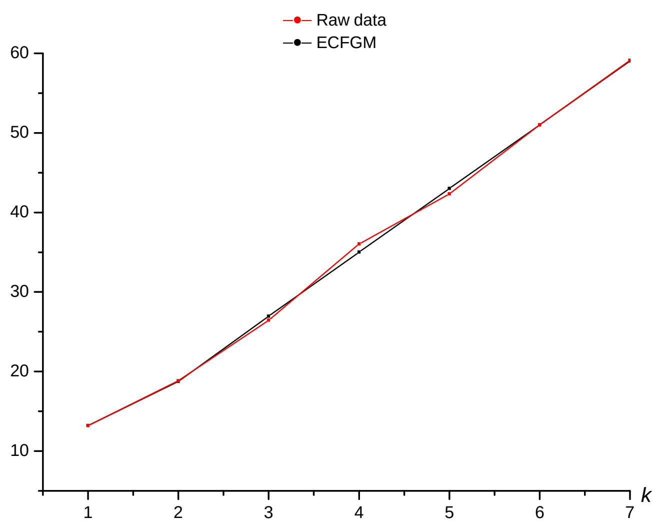


Figure 2: Actual values and forecasting values of example A.

4.2. Example B

Different from example A, the raw data are not a monotone increasing series here. The computational mechanism is similar to example A. In this example, the raw data select as $X^{(0)}(k) = (120.21, 131.83,\ 143.45,\ 150.02, 134.34,\ 121.04,\ 110.15)$.

4.2.1. Selecting the optimal α

The calculated MAPEs with α in the interval $(0, 1]$ with step 0.01 is given in Figure 3. For this example, the minimum MAPE is calculated at optimal α as $\alpha = 0.89$. The MAPE is calculated for the optimal α as MAPE $= 1.1836$. It is seen that the prediction performance of the proposed ECFGM$(1, 1)$ model is successful again.

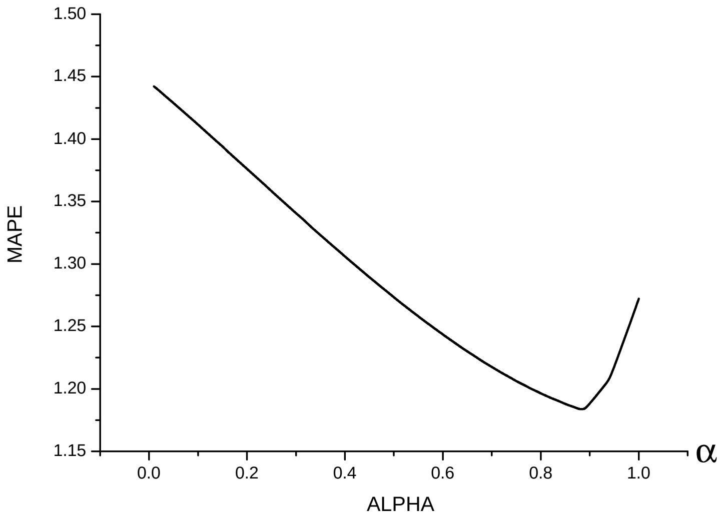


Figure 3: MAPEs of ECFGM model with α in $(0, 1]$ for example B.

4.2.2. Computing the CFA of the original series and modelling the ECFGM

Computation of the conformable fractional accumulation series of the original series is the first step to build the ECFGM model. For $\alpha = 0.89$, the CFA series can be obtained using (3.2) as $X^{(0.89)}(k) = (120.2100,\ 242.3621,\ 369.4831,\ 498.2851,\ 610.8281,\ 710.2157,\ 799.1407)$.

The matrices B and Y can be constructed as

$$B = \begin{bmatrix} -181.2861 & 1 & 0.2325 \\ -305.9223 & 1 & 0.0855 \\ -433.8841 & 1 & 0.0315 \\ -554.5566 & 1 & 0.0116 \\ -660.5219 & 1 & 0.0043 \\ -754.6782 & 1 & 0.0016 \end{bmatrix}, \qquad Y = \begin{bmatrix} 122.1521 \\ 127.1210 \\ 127.1210 \\ 112.5430 \\ 99.3876 \\ 88.9250 \end{bmatrix}.$$

Then we obtain the parameters $\hat{a}$, $\hat{b}$ and $\hat{c}$ using the least squares solution as

$$[a, b, c]^\top = \left(B^\top B\right)^{-1} B^\top Y$$

and

$$a = 0.1266697621,$$
$$b = 184.88810740,$$
$$c = -174.9875634.$$

By substituting the parameters into the response function equation (3.20) we have

$$\hat{x}^{(\alpha)}(k) = -1413.1086\ e^{0.12667(1-k)} + 1459.6073 + 200.3681\ e^{-k}. \tag{4.2}$$

Then the restored can be obtained using (4.2) by k from 1 to 7 as

$$\hat{X}^{(0.89)}(k) = (120.2100,\ 241.7407,\ 372.7221, 496.9157,\ 609.5692,\ 710.0100,\ 798.9388).$$

Then the restored values can be obtained using the CFD in (3.23) as

$$\hat{X}^{(0)}(k) = (120.2100, 131.1594,\ 147.8062,\ 144.6525,\ 134.4718,\ 122.3227,\ 110.1547).$$

The original raw series $X^{(0)}(k)$ and predicted values $\hat{X}^{(0)}(k)$ are plotted in Figure 4.

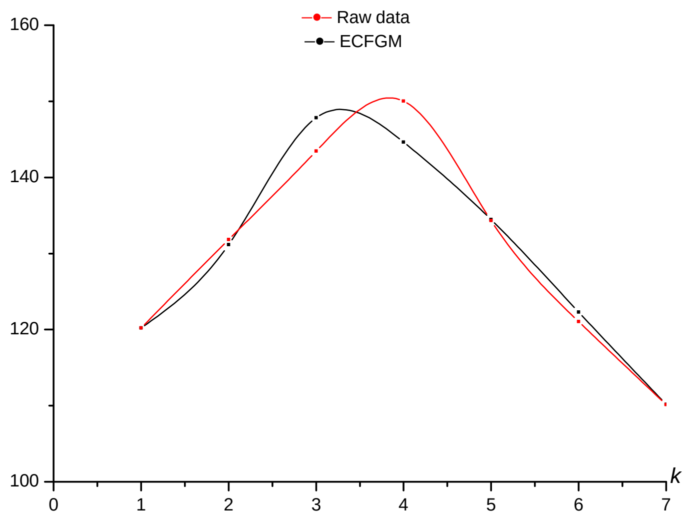


Figure 4: Actual values and forecasting values of example B.

5. VALIDATION OF ECFGM(1,1) WITH A REAL CASE

One of the renewable clean energy sources is wind. Generally, comparing with the many other energy sources, producing energy using wind has fewer effects on the environment. There is no released emission that can pollute the air or water and they do not require water for cooling. Wind turbines may also have the benefit that reduce the amount of power generation from fossil fuels, which outcomes in lower total air pollution and carbon dioxide emissions that could help solve the shortage problem of energy. From prehistoric to today, human beings have used wind energy for sailing, windmills, and wind turbines. Electric generators convert wind energy to electrical energy [16].

One of the biggest countries with large land mass and coastline is China has rich wind resources. With regard to the evaluations by China Meteorological Administration, based on the relatively low height of 10 m above ground, the total theoretical wind power reserves in China 4350 GW, while the technically exploitable wind resources estimated at 297 GW [9].

In this section, we use the novel ECFGM$(1,1)$ model to predict wind energy consumption in China. The data set from [34, 2, 6] is used to test for the efficacy and applicability of the proposed grey model. Furthermore, the ECFGM model compared with the six effective models, including the GM$(1,1)$, EXGM$(1,1)$, FAGM$(1,1)$, FAGMO$(1,1,k)$, PFAGM$(1,1)$ and CFGM$(1,1)$. The response function of the grey models can be given as follows.

The response function of standard GM$(1,1)$ model is obtained as

$$x^{(1)}(k) = 56.12166e^{0.22483(k-1)} - 49.87166. \tag{5.1}$$

The response function of EXGM$(1,1)$ is

$$x^{(1)}(k) = 67.78567e^{-0.20588(1-k)} - 70.97226. \tag{5.2}$$

For FAGM$(1,1)$ model, the response function is

$$x^{(0.36872)}(k) = 37.55924e^{0.17072(k-1)} - 31.35924. \tag{5.3}$$

The response function of FAGMO$(1,1,k)$ is

$$x^{(1.13366)}(k) = 283.1282e^{0.13851(k-1)} - 30.4471k - 246.4811. \tag{5.4}$$

The response function of PFAGM$(1,1)$ is,

$$x^{(0.17874)}(k) = 19.69290e^{0.17874k} - 14.47094e^{-0.08124(k-1)} - 2.87604. \tag{5.5}$$

The response function of CFGM$(1,1)$ model is obtained as

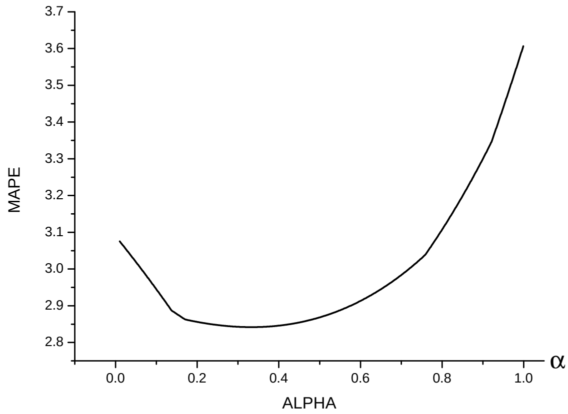
$$x^{(0.07)}(k) = 81.63469e^{0.06355(k-1)} - 75.38469. \tag{5.6}$$

Here, the optimal value of α is obtained as 0.07 by using the Brute Force strategy for the CFGM$(1,1)$.

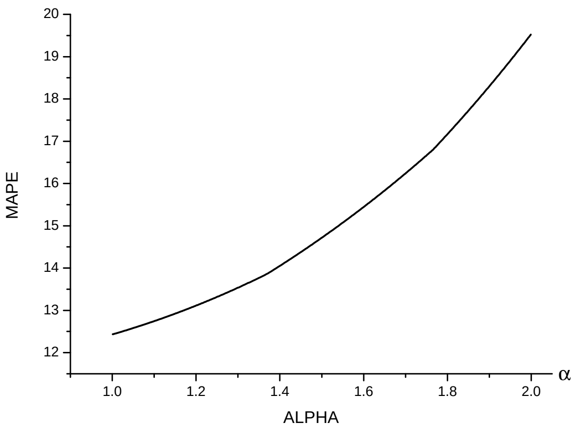
The response function of ECFGM$(1,1)$ model is obtained as

$$x^{(0.3319)}(k) = 66.64788e^{0.10331(k-1)} + 3.98468e^{-k} - 61.86377. \tag{5.7}$$

In addition, the mean absolute percentage error (MAPE) is used to assess the prediction accuracy of these grey models. Firstly, we split the raw sequence into two groups to build a model and test the model. The first group, including the consumption from 2009 to 2017, is used to build models for the seven grey models separately. The second group, including wind energy consumption from 2018 to 2020, is used to verify the prediction accuracy of these grey models. In this section we enumerate all the values in the interval $[0,2]$ with step 0.0001, then use the computational steps presented in Section 3.4 and select the α corresponding to the minimum $\mathrm{MAPE}_{\mathrm{fit}}$ as the optimal value. Optimum α is found as $\alpha = 0.3319$ by using the Brute Force strategy and values of α and calculated MAPEs are shown in Figure 5.



(a)



(b)

Figure 5: MAPEs of ECFGM model for the real data set. (a) $\alpha \in (0, 1]$; (b) $\alpha \in (1, 2]$.

From Figure 5 it is clear that the values of MAPE increase when the α moves away from its optimum value. The optimal parameters are calculated as $\alpha = 0.3319$, $a = -0.10331$, $b = 6.39111$ and $c = -4.39633$. The prediction results and the mean absolute percentage errors of the recent models are shown in Table 1.

Table 1: The results generated by the proposed model and other comparative grey models for forecasting values of China's wind energy consumption (million tonnes oil equivalent).

Year	Actual Value	GM(1, 1)	EXGM(1, 1)	FAGM(1, 1)	FAGMO(1, 1, k)	PFAGM	CFGM	ECFGM
2009	6.25	6.2500	6.2500	6.2500	6.2500	6.2500	6.2500	6.2500
2010	10.10	14.1489	9.5314	10.9059	10.7861	10.8294	10.2050	10.0534
2011	15.91	17.7160	16.8448	15.9000	15.9521	15.9001	15.8554	16.0464
2012	21.72	22.1824	22.5845	21.4733	21.7572	21.5803	22.0785	22.2010
2013	31.95	27.7749	28.4425	27.8495	28.3244	28.0628	28.9531	28.8472
2014	35.32	34.7772	35.2004	35.2395	35.7864	35.5571	36.5538	36.2435
2015	42.03	43.5450	43.3416	43.8718	44.2900	44.3005	44.9565	44.5937
2016	53.64	54.5231	53.2845	54.0032	53.9997	54.5677	54.2405	54.0796
2017	66.75	68.2690	65.4787	65.9310	65.1024	66.6817	64.4903	64.8821
MAPE_{fit}		8.4111	3.6099	3.1595	3.1901	3.1104	3.0416	**2.8418**
2018	82.82	85.4804	80.4526	80.0030	77.8110	81.0239	75.7962	77.1931
2019	93.31	107.0310	98.8466	96.6289	92.3686	98.0478	88.2555	91.2232
2020	107.30	134.0148	121.4447	116.2921	109.0538	118.2925	101.9728	107.2063
MAPE_{pre}		14.2714	7.3248	5.1128	2.8971	5.8303	6.2875	**3.0392**
MAPE		9.8762	4.5386	3.6478	3.1169	3.7904	3.8531	**2.8912**

According to the Table 1, the essential conclusions can be drawn as follows:

i. Table 1 reveals that, the seven grey models' $\mathrm{MAPE}_{\mathrm{fit}}$ values are 8.4111%, 3.6099%, 3.1595%, 3.1901%, 3.1104%, 3.0416% and **2.8418**%, respectively. So that, in the fitting period, the fitting performance of the ECFGM$(1,1)$ model is best.

ii. The $\mathrm{MAPE}_{\mathrm{pre}}$ values of seven models are calculated as 14.2714%, 7.3248%, 5.1128%, 2.8971%, 5.8303%, 6.2875% and **3.0392**%, respectively. At this point, the FAGMO$(1,1,k)$ model has the smallest $\mathrm{MAPE}_{\mathrm{pre}}$ value while the ECFGM$(1,1)$ model has the second smallest $\mathrm{MAPE}_{\mathrm{pre}}$.

iii. It is observed from Table 1 that for the whole period, the total MAPEs of seven models are 9.8762%, 4.5386%, 3.6478%, 3.1169%, 3.7904%, 3.8531% and **2.8912**%, respectively. Thence, in the whole period, the performance of the ECFGM$(1,1)$ model is the best (see Figure 6).

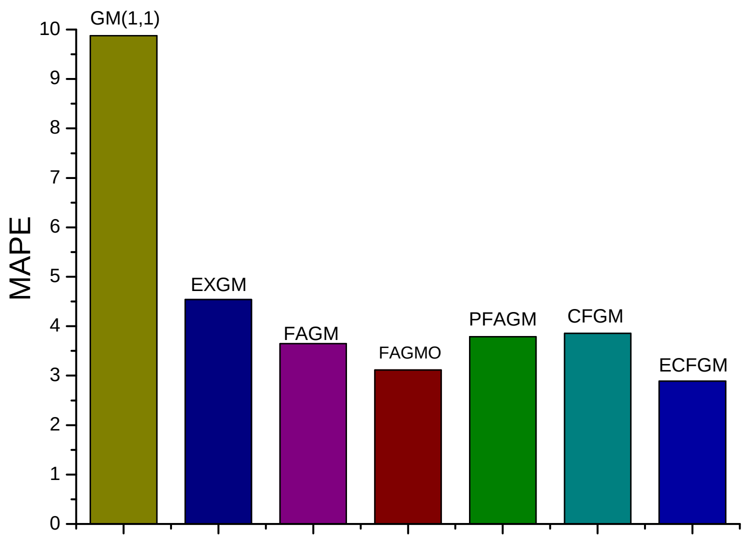


Figure 6: Total MAPE values of the models.

iv. The wind energy consumption of China can be estimated properly with the proposed ECFGM$(1,1)$ model. Hence, the forecasting values of wind energy consumption of China are given in Table 2.

Table 2: Forecasted wind energy consumption of China.

Years	Forecasting values of wind energy consumption
2021	125.4042
2022	146.1105
2023	169.6550
2024	196.4082
2025	226.7862

It is seen from Figure 7 that the actual (red line) and forecasted (black line) values of wind energy consumption of China are matched to each other.

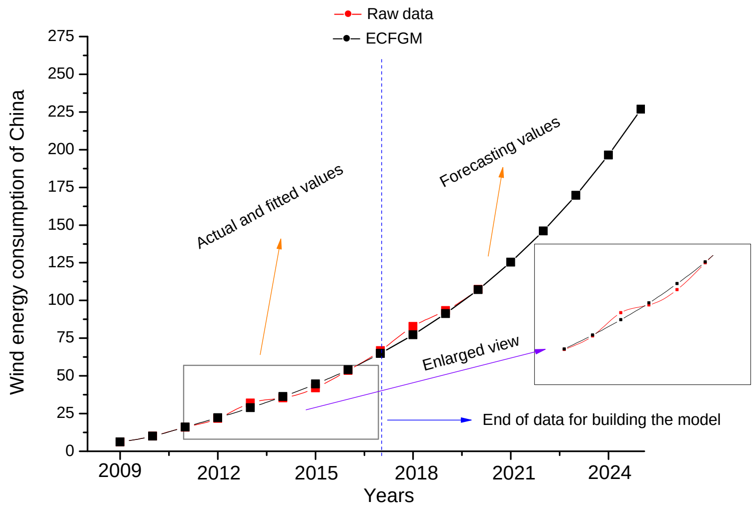


Figure 7: Actual, fitting and forecasting values of wind energy consumption of China.

6. CONCLUSION

Since the fractional calculations are most important to the grey prediction model, there are many scholars proposing new methods on the fractional grey models. Hence, a novel optimization for the CFGM$(1,1)$ and EXGM$(1,1)$ models have been developed in this study. The results of the numerical examples indicated that the proposed grey prediction model aims to achieve very effective performance. The structural parameters (a, b and c) of the model can be dynamically adjusted according to the real world systems. The optimal value of fractional order, α, is calculated by using the brute-force approach. The proposed ECFGM$(1,1)$ model is suitable for predicting the data sequence with the characteristics of non-homogeneous exponential law. Comparison results indicate, ECFGM$(1,1)$ performs better than those achieved by the other grey models such as GM$(1,1)$, EXGM$(1,1)$, FAGM$(1,1)$, FAGMO$(1,1,k)$, PFAGM$(1,1)$ and CFGM$(1,1)$. However, they can all be employed for estimations.

Wind energy with cleanliness and pollution-free will have a positive attitude on global energy transformation. Because of this, research on more accurate prediction of wind energy consumption is quite important for wind power generation. Therefore, the wind energy consumption of China is predicted successfully by using a novel proposed fractional grey model based on the conformable fractional difference and conformable fractional accumulation in the paper. The forecasting results show that wind energy consumption of China will develop rapidly in recent years, and will reach approximately 200 million tones oil equivalent by 2023.

Using two examples and a case study in Sections 4 and 5, we show that the MAPE of the ECFGM$(1,1)$ model is very low.

The proposed ECFGM$(1,1)$ model may play an important role in enriching the theoretical system of grey forecasting theory and it can be used for other real cases of small sample forecasting in the future. Besides, the combination of other fractional forecasting models, especially for the time series with highly effective, is also an interesting direction for next studies.

ACKNOWLEDGMENTS

The authors would like to thank the reviewers for their valuable comments and suggestions which have improved many aspects of this article.

Conflict of interest

The author declares no conflict of interest in this paper.

REFERENCES

[1] Bilgil, H. (2021). New grey forecasting model with its application and computer code, *AIMS Mathematics*, **6**, 1497–1514.

[2] Cao, J.; Liu, L.; Yang, L. and Xie, S. (2020). Application of a novel fractional order grey support vector regression model to forecast wind energy consumption in China, *Journal of Advances in Mathematics and Computer Science*, **35**(2), 58–69.

[3] Cui, J.; Liu, S.; Zeng, B. and Xie, N. (2013). A novel grey forecasting model and its optimization, *Appl. Math. Model.*, **37**, 4399–4406.

[4] Deng, J.L. (1982). Control problems of grey systems, *Syst. Control Lett.*, **1**, 288–294.

[5] Ene, S. and Ozturk, N. (2018). Grey modelling based forecasting system for return flow of end-of-life vehicles, *Technol. Forecast. Soc. Change*, **117**, 155–166.

[6] China Energy Portal (2021). 2020 electricity and other energy statistics (preliminary), *China Energy Portal*. https://chinaenergyportal.org/2020-electricity-other-energy-statistics-preliminary/

[7] Javed, S.A. and Liu, S. (2018). Predicting the research output/growth of selected countries: application of even GM$(1,1)$ and NDGM models, *Scientometrics*, **115**, 395–413.

[8] Jiang, J.; Feng, T. and Liu, C. (2021). An improved nonlinear grey Bernoulli model based on the whale optimization algorithm and its application, *Mathematical Problems in Engineering*, **2021**, 6691724.

[9] Khalil, R.; Al Horani, M.; Abdelrahman, Y. and Mohammad, S. (2014). A new definition of fractional derivative, *J. Comput. Appl. Math.*, **264**, 65–70.

[10] Li, S.; Ma, X. and Yang, C. (2018). A novel structure-adaptive intelligent grey forecasting model with full-order time power terms and its application, *Computers & Industrial Engineering*, **120**, 53–67.

[11] Liu, L.; Chen, Y. and Wu, L. (2021). The damping accumulated grey model and its application, *Communications in Nonlinear Science and Numerical Simulation*, **95**, 105665.

[12] Liu, L. and Wu, L. (2021). Forecasting the renewable energy consumption of the european countries by an adjacent non-homogeneous grey model, *Appl. Math. Modell.*, **89**, 1932–1948.

[13] Ma, X. (2016). Research on a novel kernel based grey prediction model and its applications, *Mathematical Problems in Engineering*, **2016**, 1–9.

[14] Ma, X.; Liu, Z. and Wang, Y. (2019). Application of a novel nonlinear multivariate grey Bernoulli model to predict the tourist income of China, *J. Comput. Appl. Math.*, **347**, 84–94.

[15] Ma, X. and Liu, Z.B. (2017). Application of a novel time-delayed polynomial grey model to predict the natural gas consumption in China, *J. Comput. Appl. Math.*, **324**, 17–24.

[16] Ma, X.; Wu, W.; Zeng, B.; Wang, Y. and Wu, X. (2020). The conformable fractional grey system model, *ISA Transactions*, **96**, 255–271.

[17] Mao, S.; Gao, M.; Xiao, X. and Zhu, M. (2016). A novel fractional grey system model and its application, *Appl. Math. Model*, **40**, 5063–5076.

[18] Sahin, U. and Sahin, T. (2020). Forecasting the cumulative number of confirmed cases of COVID-19 in Italy, UK and USA using fractional nonlinear grey Bernoulli model, *Chaos, Solitons & Fract.*, **138**, 109948.

[19] Wang, Z.X.; Li, Q. and Pei, L.L. (2018). A seasonal $GM(1,1)$ model for forecasting the electricity consumption of the primary economic sectors, *Energy*, **154**, 522–534.

[20] Wei, B.; Xie, N. and Hu, A. (2018). Optimal solution for novel grey polynomial prediction model, *Applied Mathematical Modelling*, **62**, 717–727.

[21] Wei, M.; Li, Q. and Zeng, B. (2016). Study on fractional order grey reducing generation operator, *Grey Syst Theory Application*, **6**, 80–95.

[22] Wu, L.; Liu, S.; Yang, Y.; Ma, L. and Liu, H. (2016). Multivariable weakening buffer operator and its application, *Inform. Sci.*, **339**, 98–107.

[23] Wu, L.; Liu, S.; Yao, L.; Yan, S. and Liu, D. (2013). Grey system model with the fractional order accumulation, *Commun. Nonlinear Sci. Numer. Simul.*, **18**, 1775–1785.

[24] Wu, L.F.; Liu, S.F.; Chen, D.; Yao, L. and Cui, W. (2014). Using gray model with fractional order accumulation to predict gas emission, *Nat. Hazards*, **71**, 2231–2236.

[25] Wu, L.Z.; Li, S.H.; Huang, R.Q. and Xi, Q. (2020). A new grey prediction model and its application to predicting landslide displacement, *Appl. Soft Comput.*, **95**, 106543.

[26] Wu, W.; Ma, X.; Wang, Y.; Cai, W. and Zeng, B. (2020). Predicting China's energy consumption using a novel grey Riccati model, *Appl. Soft Comput.*, **95**, 106555.

[27] Wu, W.; Ma, X.; Zeng, B.; Wang, Y. and Cai, W. (2018). Application of the novel fractional grey model $FAGMO(1,1,k)$ to predict China's nuclear energy consumption, *Energy*, **165**, 223–234.

[28] Wu, W.; Ma, X.; Zhang, Y.; Li, W. and Wang, Y. (2020). A novel conformable fractional non-homogeneous grey model for forecasting carbon dioxide emissions of BRICS countries, *Sci. Total Environ.*, **707**, 135447.

[29] Xie, W.; Caixia, L.; Wu, W.; Weidong, L. and Chong, L. (2020). Continuous grey model with conformable fractional derivative, *Chaos, Solitons & Fractals*, **139**, 110285.

[30] Xie, W.; Wu, W.Z.; Liu, C. and Zhao, J. (2020). Forecasting annual electricity consumption in China by employing a conformable fractional grey model in opposite direction, *Energy*, **202**, 117682.

[31] Yang, Y. and Xue, D. (2016). Continuous fractional-order grey model and electricity prediction research based on the observation error feedback, *Energy*, **115**, 722–733.

[32] Yuxiao, K.; Shuhua, M.; Yonghong, Z. and Huimin, Z. (2020). Fractional derivative multivariable grey model for nonstationary sequence and its application, *Journal of Systems Engineering and Electronics*, **31**, 1009–1018.

[33] Zeng, B.; Tan, Y.; Xu, H.; Quan, J.; Wang, L. and Zhou, X. (2018). Forecasting the electricity consumption of commercial sector in Hong Kong using a novel grey dynamic prediction model, *J. Grey Syst.*, **30**, 157–172.

[34] Zhang, P.; Ma, X. and She, K. (2019). A novel power-driven fractional accumulated grey model and its application in forecasting wind energy consumption of China, *Plos One*, **14**, e0225362.

[35] Zhang, Y.G.; Xu, Y. and Wang, Z.P. (2009). $GM(1,1)$ grey prediction of Lorenz chaotic system, *Chaos, Solitons & Fract.*, **42**, 1003–1009.

[36] Zhou, W. and He, J.M. (2013). Generalized $GM(1,1)$ model and its application in forecasting of fuel production, *Appl. Math. Modell.*, **37**, 6234–6243.