Non-parametric Inference of Window-Observation Recurrent Event Data with Multiple Causes of Failure

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Abstract:

- Recurrent event data with multiple causes of failure are common in survival studies. There are situations where the recurrence history of some systems is available in disconnected observation windows with gaps in between. Such data are called window-observation recurrence data, and no events can be observed during these gaps. This article discusses the non-parametric estimation of cause specific mean function for window-observation recurrence data with multiple failure modes. A test statistic is proposed to compare the effect of different causes. Simulation studies are performed to evaluate the performance of the proposed methods. An automobile warranty database is analyzed to illustrate the suggested techniques.

Keywords:

- *Counting process; window observation; recurrent events; mean function; multiple causes.*

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1. INTRODUCTION

In survival studies, the event of interest can occur multiple times on the same subject. Such outcomes are termed as recurrent events. Recurrent event data are common in fields such as public health, medicine, reliability, social sciences, and insurance. Examples in biomedicine include the recurrence of tumors in cancer patients, repeated hospitalizations of patients with a specific disease, and recurrence of caries in oral health studies. In engineering and reliability, the recurrence of a crack in concrete structures, failure of an electronic system, and software bugs are examples. Various methods for analyzing recurrent event data are available in many literature, examples include [6], [8], and [9]. A comprehensive review of recurrent event data analysis is given in [2].

There are situations in which a study unit may experience an event due to multiple causes. For example, an electronic system may experience a breakdown due to the failure of any of the components in that system. To model and characterize the marginal event processes that generate the particular type of events, the cause specific sub distribution functions and the cause specific hazard rate functions are generally used. The cause specific hazard rate function gives the instantaneous probability of failure due to a specific cause, while the cause specific sub-distribution function measures the probability of failure due to a specific cause before a given time. [11] and [12] have studied semi-parametric inferences of recurrent event data with multiple causes. A non-parametric method for estimating the system lifetime distribution with recurrent competing risks data was studied by [15]. [3] discussed the non-parametric tests in the context of the recurrent event competing risks based on the mean number of events, and [13] proposed a non-parametric test for comparing cause specific cumulative incidence functions.

In many lifetime studies, the recurrence of events may not be recorded throughout the study period. Even though the event process is continuous in time, the recurrence status of some systems is available in separate observation windows with gaps in between. For example, in a clinical trial study, patients may enter and leave the study multiple times, resulting in gaps in their follow-up tenure. No events are recorded during these gaps in the follow-up window, and the exact recurrence times are recorded when the patient is under monitoring. Such data is referred to as window-observation recurrence data. [17] provides an example of window observation data utilizing the U.S. Army's Field Exercise Data Collection (FEDC) program, which is overseen by the Army Materiel Systems Analysis Activity (AMSAA). The FEDC program tracks the replacement rates of parts for military vehicles during field training exercises, where distinct vehicles within the fleet engage in various exercises with intervals in between. Some vehicles may not participate in every FEDC exercise, and during the intervals between exercises, they accumulate mileage from non-exercise activities and other unobserved field exercises. Consequently, recurrences may occur in these intervals but remain unrecorded. The observation windows and the gaps can vary for different observational units, and also, the length of each observation window and the gaps between the windows need not be the same.

In typical recurrence data with left and/or right censoring, each system is associated with a single observation window. However, for window-observation recurrence data, multiple disjoint observation windows can exist for each system. These windows can have random lengths, and the length of the gaps between them may also vary randomly. Additionally, the beginning or ending time points across windows for different observational units can differ. Window-observation data differ from interval-grouped recurrent event data, where the number of events in time intervals is recorded but the exact times of recurrences are not specified, and there are no gaps between intervals. [17] analyzed such data in terms of mean cumulative function. They proposed non-parametric estimation methods for handling window-observation recurrence data. The asymptotic properties of the non-parametric estimator for the mean function are studied in [18]. [5] considered the case when information on the observation gap is incomplete, that is, the starting time of intermittent dropout is known, but the terminating time is not available, and they modeled it in terms of an interval-censored mechanism. Regression analysis of recurrent event data with repeated observation gaps with unknown termination times of observation gaps was studied by [4]. [16] focused on window-censored data, where only events within a specific interval are documented. They developed the likelihood function for a model where the distributions of inter-recurrence intervals within a single path are not required to be identical and may be linked to covariate information.

In this study, we focus on the situation in which a unit under observation can experience more than one type of failure, and also, there are intermittent dropouts that result in observation gaps during which no recurrent events are observed. The mean cumulative function corresponding to different causes is used to model this problem, and a test is proposed to compare the effect of various causes of failures on the recurrence process.

The article is organized as follows. In Section 2, we introduce cause specific mean function for the window-observation recurrence data. We propose a non-parametric estimator for the cause specific mean function, and the asymptotic properties of the estimator are discussed. The test procedure for the mean functions due to different causes is discussed in Section 3. Section 4 discusses the simulation studies performed for assessing the finite sample behaviors of the proposed estimator. We put the empirical power to check the efficiency of the test. The methods are then illustrated using a real life data set in Section 5. Finally, Section 6 presents the major conclusion of the study.

2. NON-PARAMETRIC INFERENCE PROCEDURES

Consider a study on n individuals exposed to the recurrent events due to $\{1,2,\ldots,K\}$ different causes. Let $N_k(t)$ denote the cumulative number of events due to cause k up to time t, $k=1,2,\ldots,K$. Then the cause specific mean function is defined as

$$\mu_k(t) = E[N_k(t)]; \quad k = 1,2,\ldots,K. \tag{2.1}$$

Then $\mu_k(t)$ in (2.1) is interpreted as the expected number of cumulative events due to cause k up to time t, $k=1,2,\ldots,K$. Assumptions required for the non-parametric estimation of mean cumulative function were described in [9] and the same for the window-observation scenario were covered in [17]. These assumptions are briefed below.

i) Target population should be clearly specified and the sample units are a simple random sample from this population.

ii) The population cause specific mean function exists up to the greatest censoring and is zero at time zero.

iii) The stochastic process that generates the recurrences and the observation windows are independent.

iv) The size of the risk set must be positive up to the maximum study time.

Let m_k denote the number of unique event times due to cause k. Thus $t_1, t_2, \ldots,$ and t_{m_k} are the unique event times for cause k, $k = 1, 2, \ldots, K$. Then a non-parametric estimator of the cause specific mean function is given by

$$\hat{\mu}_k(t_j) = \sum_{l=1}^{j} \left[\frac{\sum_{i=1}^{n} \delta_i(t_l) d_{ki}(t_l)}{\sum_{i=1}^{n} \delta_i(t_l)} \right]; \quad j = 1, 2, \ldots, m_k,\ k = 1, 2, \ldots, K, \tag{2.2}$$

where

$$\delta_i(t_l) = \begin{cases} 1, & \text{if individual } i \text{ is under observation in a time window at } t_l \\ 0, & \text{otherwise} \end{cases},$$

and $d_{ki}(t_l)$ is the number of events due to cause k at time t_l for the individual i. Denote $d_{k\cdot}(t_l) = \sum_{i=1}^{n} \delta_i(t_l) d_{ki}(t_l)$, total number of events at t_l due to cause k and $\delta_{\cdot}(t_l) = \sum_{i=1}^{n} \delta_i(t_l)$, total number of units at risk at time t_l. Then (2.2) can be written as

$$\begin{aligned} \hat{\mu}_k(t_j) &= \sum_{l=1}^{j} \frac{d_{k\cdot}(t_l)}{\delta_{\cdot}(t_l)} \\ &= \sum_{l=1}^{j} \bar{d}_k(t_l); \quad j = 1, 2, \ldots, m_k,\ k = 1, 2, \ldots, K. \end{aligned} \tag{2.3}$$

The size of the risk set at time t_l is calculated by taking into account the gaps between observation windows and the censoring.

2.1. Asymptotic properties

Assume that the cause specific mean function $\mu_k(t)$ is differentiable. Then the recurrence rate at time t due to cause k is given by $\nu_k(t) = \frac{d\mu_k(t)}{dt}$, $k = 1, 2, \ldots, K$. Let there be w_i observation windows for the individual i, $i = 1, 2, \ldots, n$. If t_{max} denotes the largest observation time among all n units, then the number of units at risk, $\delta_{\cdot}(t) = \sum_{i=1}^{n} \delta_i(t)$, is a piece-wise constant over $(0, t_{max}]$. To ensure a non zero risk set throughout the study period, we assume that the overlap of the set of observation windows of all the n units leaves no complete gaps in the study timeline. If $(t^*_{1L}, t^*_{1U}], (t^*_{2L}, t^*_{2U}], \ldots, (t^*_{bL}, t^*_{bU}]$ be the b intervals in which $\delta.(t)$ is constant, then $t^*_{1L} = 0$, $t^*_{bU} = t_{max}$ and $t^*_{jL} = t^*_{(j-1)U}$ for $j = 2, 3, \ldots, b$. Let $\tau < \infty$ be the ending time for all the windows with $t_{max} \leq \tau$ for all n and $t_{max} \to \tau$ as $n \to \infty$. Then for this window-observation recurrence data, [18] showed that $\delta.(t)$ follows a $Binomial(n, p(t))$ distribution, where $p(t)$ is the probability that a unit is observed at time t, with $p(t) > 0$ for all $t \in (0, t_{max}]$.

To establish the uniform consistency and the asymptotic normality of the estimator, Theorem IV.1.1 and Theorem IV.1.2 of [1] are employed.

Theorem 2.1. Let $t \in (0, \tau]$ and assume that, as $n \to \infty$,

$$\int_0^t \frac{I[\delta.(u) > 0]}{\delta.(u)} \nu_k(u) du \xrightarrow{\mathbf{P}} 0 \tag{2.4}$$

and

$$\int_0^t \{1 - I[\delta.(u) > 0]\} \nu_k(u) du \xrightarrow{\mathbf{P}} 0. \tag{2.5}$$

Then, as $n \to \infty$,

$$\sup_{t \in [0,\tau]} |\hat{\mu}_k(t) - \mu_k(t)| \xrightarrow{\mathbf{P}} 0, \tag{2.6}$$

where $\hat{\mu}_k(t)$ is the non-parametric estimator of $\mu_k(t)$, $k = 1, 2, \ldots, K$.

Theorem 2.1 can be used to show that the estimator of cause specific mean function given in (2.2) is uniformly consistent on compact intervals. It can be verified that the conditions (2.4) and (2.5) are satisfied by the estimator and hence the uniform consistency can be established. The proof for the same is similar to the one given in [18] and can be referred.

Theorem 2.2. Let $t \in (0, \tau]$ and assume that there exist a sequence of positive constants $\{a_n\}$, increasing to infinity as $n \to \infty$, and non-negative function $g(\cdot)$ such that $\nu_k(u)/g(u)$ is integrable over $[0, t]$ for $k = 1, 2, \ldots, K$. Assume the following.

(A) For each $s \in [0, t]$ and $k = 1, 2, \ldots, K$,

$$a_n^2 \int_0^s \frac{I[\delta.(u) > 0]}{\delta.(u)} \nu_k(u) du \xrightarrow{\mathbf{P}} \sigma_k^2(s) \quad as \quad n \to \infty, \tag{2.7}$$

where

$$\sigma_k^2(s) = \int_0^s \frac{\nu_k(u)}{g(u)} du.$$

(B) For all $\varepsilon > 0$ and $k = 1, 2, \ldots, K$,

$$(2.8) \qquad a_n^2 \int_0^t \frac{I[\delta.(u) > 0]}{\delta.(u)} \nu_k(u) I\left\{a_n \frac{I[\delta.(u) > 0]}{\delta.(u)} > \varepsilon\right\} du \xrightarrow{\mathbf{P}} 0 \quad as \quad n \to \infty.$$

(C) For $k = 1, 2, \ldots, K$,

$$(2.9) \qquad a_n \int_0^t \{1 - I[\delta.(u) > 0]\} \nu_k(u) du \xrightarrow{\mathbf{P}} 0 \quad as \quad n \to \infty.$$

Then

$$a_n(\hat{\mu}_k(t) - \mu_k(t)) \xrightarrow{\mathbf{D}} U_k \quad as \quad n \to \infty$$

on $D[0,\tau]$, where $U_1, U_2, \ldots, U_K$ are independent Gaussian martingales with $U_k(0) = 0$ and $cov(U_k(s_1), U_k(s_2)) = \sigma_k^2(s_1 \wedge s_2)$. Here $D[0,\tau]$ is the Skorohod space on $[0,\tau]$, that is, the space of right-continuous functions with left-hand limits on $[0,t]$; $s_1 \wedge s_2$ is the smaller of s_1 and s_2.

The conditions $(A), (B)$ and (C) of Theorem 2.2 can be verified using the methods given in [18] and hence the asymptotic normality of the estimator in (2.2) can be established.

2.2. Non-parametric estimation in the presence of size-zero risk set intervals

Sometimes there can be time intervals in which the number of subjects at risk is zero. Such intervals can be referred to as size-zero risk set intervals. This may lead to a downward bias in the non-parametric estimation method proposed. To overcome this, [17] proposed a hybrid estimator for mean cumulative function, which makes use of the information from the non-zero risk set interval to estimate the increase in the mean function over the size-zero risk set interval. A similar approach can be employed for the estimation of cause specific mean function and is briefed below.

Let $(t_{1L}, t_{1U}], (t_{2L}, t_{2U}], \ldots, (t_{rL}, t_{rU}]$ be the r size-zero risk set intervals with $t_{1L} \geq 0$, $t_{rU} < t_{max}$ and $t_{(j-1)U} < t_{jL}$ for $j = 2, 3, \ldots, r$. Then $(t_{1U}, t_{2L}], (t_{2U}, t_{3L}], \ldots, (t_{rU}, t_{max}]$ are the non-zero risk set interval. The estimated cause specific mean function of cause k increases only at time points that have events due to cause k and the increase at an event time t_l is given by $\bar{d}_k(t_l) = \frac{\sum_{i=1}^n \delta_i(t_l) d_{ki}(t_l)}{\sum_{i=1}^n \delta_i(t_l)}$ as defined in (2.3). Then the estimated increase in the cause-specific mean function over the non-zero risk set interval $(t_{jU}, t_{(j+1)L}]$ is,

$$(2.10) \qquad \bar{d}_{k\cdot}(t_{jU}, t_{(j+1)L}) = \sum_{l: t_{jU} < t_l \leq t_{(j+1)L}} \bar{d}_k(t_l).$$

An estimate of the cause specific recurrence rate in this interval is $\frac{\bar{d}_{k\cdot}(t_{jU}, t_{(j+1)L})}{(t_{(j+1)L} - t_{jU})}$.

To calculate the cause specific rate of a size-zero risk set interval, a weighted mean of the recurrence rates of the two neighboring non-zero risk set intervals, weighted by the length of those intervals, is taken. The recurrence rate due to cause k for the j^{th} size-zero risk set interval $(t_{jL}, t_{jU}]$ is given by,

$$\omega_{kj} = \frac{\bar{d}_{k\cdot}(t_{(j-1)U}, t_{jL}) + \bar{d}_{k\cdot}(t_{jU}, t_{(j+1)L})}{(t_{jL} - t_{(j-1)U}) + (t_{(j+1)L} - t_{jU})}. \tag{2.11}$$

If the data begin with a size-zero risk set interval, the recurrence rate is estimated as that of the first non-zero risk set interval.

Then for the j^{th} size-zero risk set interval, the estimated increase in the cause specific mean function is,

$$d^*_{kj} = \omega_{kj} \times (t_{jU} - t_{jL}). \tag{2.12}$$

Therefore, a non-parametric hybrid estimator for the cause specific mean function is given by,

(2.13)

$$\hat{\mu}^*_k(t) = \begin{cases} \sum_{l:t_l \le t} \bar{d}_k(t_l) + \sum_{j:t_{jU} \le t} d^*_{kj} \\ \qquad \text{if } t \text{ is in a non-zero risk set interval} \\ \sum_{l:t_l \le t} \bar{d}_k(t_l) + \sum_{j:t_{jU} \le t} d^*_{kj} + (\omega_{kj} \times (t - t_{iL}) \\ \qquad \text{if } t \text{ is in a size-zero risk set interval } (t_{iL}, t_{iU}] \end{cases} \quad ,k = 1,2,\ldots,K.$$

3. TEST STATISTIC

In the study of multiple modes of failure, it is always of greater interest to compare the effect of various causes on recurrence times. We can compare the cause specific mean functions to test whether or not the effect of all causes is identical on the recurrence process. Now consider the testing problem,

$$H_0 : \mu_k(t) = \mu_l(t) \text{ for all } t > 0 \text{ and } k \neq l = 1,2,\ldots,K$$

against

$$H_1 : \mu_k(t) \neq \mu_l(t) \text{ at least for some } t > 0 \text{ and for any } k \neq l = 1,2,\ldots,K. \tag{3.1}$$

If we denote the overall mean function as $\mu(t)$, the above hypothesis can be rewritten as

$$H_0 : \mu_k(t) = \frac{\mu(t)}{K} \text{ for all } t > 0 \text{ and } k = 1,2,\ldots,K$$

against

$$H_1 : \mu_k(t) \neq \frac{\mu(t)}{K} \text{ at least for some } t > 0 \text{ and for any } k = 1,2,\ldots,K. \tag{3.2}$$

To test H_0 against H_1, the cause specific mean function $\hat{\mu}_k(t)$ defined in (2.2) is taken as the estimator of $\mu_k(t)$. The estimator for the overall mean function $\mu(t)$ can be constructed by ignoring the cause of failure information as discussed in [17]. To develop the test statistic, consider the function

$$\eta_k(t) = \int_0^t w(u)\Big[\hat{\mu}_k(u) - \frac{\hat{\mu}(u)}{K}\Big]du \text{ for all } k = 1,2,\dots,K, \tag{3.3}$$

where $w(\cdot)$ is a data dependent weight function. The function $\eta_k(\cdot)$ is similar to the one proposed by [10] and [7] in the context of comparing two independent cumulative incidence functions. The weight function is used to compensate for the effect of censoring and is also employed to enhance the efficiency of the test statistic and ensure its asymptotic distribution-free nature, as suggested by [10]. The weight function should be chosen such that it maximizes the power of the test.

Now we propose a test statistic $Z(\tau)$ given by

$$Z(\tau) = \eta'(\tau)\hat{\Sigma}(\tau)^{-1}\eta(\tau), \tag{3.4}$$

where τ is the largest monitoring time in the study, $\eta(\tau) = (\eta_1(\tau),\dots,\eta_K(\tau))'$ and $\hat{\Sigma}(\tau)^{-1}$ is the generalized inverse of $\hat{\Sigma}(\tau)$, with $\hat{\Sigma}(\tau)$ as a consistent estimator of the variance-covariance matrix $\Sigma(\tau)$ of $\eta(\tau)$. The expression for the estimator $\hat{\Sigma}(\tau)$ is complex and thus generally requires smoothing. In practice, $\hat{\Sigma}(\tau)$ can be calculated using bootstrap resampling technique.

To find the asymptotic distribution of $Z(\tau)$, consider

$$\begin{aligned}\eta_k(t) &= \int_0^t w(u)\Big[\hat{\mu}_k(u) - \frac{\hat{\mu}(u)}{K}\Big]du \\ &= \int_0^t w(u)\Big[\hat{\mu}_k(u) - \mu_k(u)\Big]du + \int_0^t w(u)\Big[\mu_k(u) - \frac{\mu(u)}{K}\Big]du \\ &\quad + \int_0^t \frac{w(u)}{K}\Big[\mu(u) - \hat{\mu}(u)\Big]du.\end{aligned} \tag{3.5}$$

Under H_0, $\mu_k(u) = \frac{\mu(u)}{K}$ for all t. Then

$$\eta_k(t) = \int_0^t w(u)\Big[\hat{\mu}_k(u) - \mu_k(u)\Big]du + \int_0^t \frac{w(u)}{K}\Big[\mu(u) - \hat{\mu}(u)\Big]du. \tag{3.6}$$

Since for fixed t, $\hat{\mu}_k(t)$ and $\hat{\mu}(t)$ are asymptotically normal, η_k is asymptotically normal with mean zero and $\eta(\tau) = (\eta_1(\tau),\dots,\eta_K(\tau))'$ is asymptotically a k-variate normal with mean zero vector and variance-covariance matrix $\Sigma(\tau)$. Therefore, under H_0, the quadratic form $Z(\tau)$ given in (3.4) follows a chi-square distribution with $K-1$ degrees of freedom. The null hypothesis given in (3.2) is rejected when $Z(\tau) \geq \chi^2_{\alpha,K-1}$ where $\chi^2_{\alpha,K-1}$ is the ordinate value of chi-square distribution with $K-1$ degrees of freedom at α level.

4. SIMULATION STUDY

An extensive simulation study is conducted to evaluate the finite sample performance of the proposed estimator and the test statistic. The study is limited to two failure mode scenario, and recurrence times are generated using Weibull and Gompertz cause-specific hazard functions. The following algorithm outlines the data simulation process:

1. Two cases are considered for the generation of recurrence times:

   (a) Using the Weibull cause-specific hazard functions

   $$\lambda_k(t) = \theta_k \alpha_k t^{\alpha_k - 1}, k = 1, 2 \; with \; \theta_k, \alpha_k > 0.$$

   (b) Using the Gompertz cause-specific hazard functions

   $$\lambda_k(t) = \rho_k e^{\beta_k t}, k = 1, 2 \; with \; \rho_k, \beta_k > 0.$$

   For simplicity, $\alpha_1 = \alpha_2 = \alpha$ and $\beta_1 = \beta_2 = \beta$ are assumed in the simulation study.

2. Recurrence times are simulated from the distribution with the overall hazard function $\lambda_1(t) + \lambda_2(t)$.

3. To assign failure modes associated with a simulated event time T, a binomial experiment is run. The experiment decides with probability $\frac{\lambda_1(T)}{\lambda_1(T)+\lambda_2(T)}$ for failure mode 1 and $\frac{\lambda_2(T)}{\lambda_1(T)+\lambda_2(T)}$ for failure mode 2.

4. The maximum follow-up time for each unit is fixed at time 2.

5. It is assumed that the follow-up of each unit begins with an observation window, and this initiation is randomly generated from a $Uniform(0.08, 0.4)$ distribution. The lengths of the subsequent observation windows are also randomly generated from a $Uniform(0.08, 0.4)$ distribution, while the lengths of the gaps between the windows follow a $Uniform(0.08, 0.24)$ distribution.

6. Recurrences falling in gaps for each unit are not considered, and only recurrences in an observational window for each unit are included in the analysis.

We simulate 500 data with sample sizes n=50 and 100 each for different combinations of $(\theta_1, \theta_2, \alpha)$ and (ρ_1, ρ_2, β). The average of the cause specific mean functions estimated across different iterations are presented in Figure 1 and Figure 2 for different sample sizes along with the actual mean functions.The absolute bias and mean squared error (MSE) of the cause specific mean functions are shown in Table 1 and Table 2, which are small and decrease as the sample size increases.

The empirical type I error and power of the proposed test are calculated at both 1% and 5% levels of significance. The parameter combination with $\theta_1 = \theta_2$ and $\rho_1 = \rho_2$ gives the type I error of the proposed test and all other choices of parameter combinations

$(\theta_1,\theta_2,\alpha)$	Time	n=50				n=100			
		$\hat{\mu}_1(t)$		$\hat{\mu}_2(t)$		$\hat{\mu}_1(t)$		$\hat{\mu}_2(t)$	
		Absolute bias	MSE	Absolute bias	MSE	Absolute bias	MSE	Absolute bias	MSE
(0.2,0.2,2.2)	0.1	0.00232	0.00002	0.00303	0.00004	0.00277	0.00003	0.00285	0.00002
	0.4	0.02594	0.00096	0.02320	0.00079	0.01802	0.00050	0.01457	0.00034
	0.7	0.04400	0.00326	0.04249	0.00294	0.02755	0.00112	0.02993	0.00131
	1.0	0.05778	0.00462	0.07198	0.00720	0.03699	0.00230	0.04974	0.00410
	1.3	0.08815	0.01329	0.08670	0.01233	0.05409	0.00494	0.05948	0.00555
	1.6	0.12241	0.02142	0.12409	0.02347	0.06313	0.00592	0.06600	0.00817
	1.9	0.14646	0.03271	0.14965	0.03204	0.07500	0.00895	0.07438	0.00842
(0.2,0.4,2)	0.1	0.00337	0.00005	0.00488	0.00005	0.00292	0.00002	0.00493	0.00003
	0.4	0.02341	0.00077	0.04511	0.00292	0.01658	0.00045	0.02794	0.00121
	0.7	0.04669	0.00303	0.06383	0.00624	0.02952	0.00141	0.05062	0.00373
	1.0	0.07269	0.00782	0.10479	0.01570	0.04803	0.00375	0.06855	0.00737
	1.3	0.09093	0.01299	0.11231	0.01954	0.04789	0.00397	0.09759	0.01396
	1.6	0.10599	0.01801	0.14735	0.02900	0.06520	0.00718	0.09465	0.01402
	1.9	0.11490	0.01835	0.18701	0.05052	0.08615	0.01171	0.12482	0.02209
(0.2,0.5,2)	0.1	0.00264	0.00002	0.00722	0.00009	0.00294	0.00002	0.00561	0.00004
	0.4	0.02547	0.00104	0.04314	0.00289	0.01836	0.00054	0.02861	0.00135
	0.7	0.03862	0.00243	0.05972	0.00579	0.03343	0.00162	0.05475	0.00463
	1.0	0.06698	0.00658	0.08847	0.01269	0.04784	0.00342	0.07128	0.00770
	1.3	0.08988	0.01273	0.12212	0.02266	0.06528	0.00602	0.09917	0.01473
	1.6	0.10743	0.02123	0.16251	0.04193	0.07965	0.00878	0.11085	0.01989
	1.9	0.11565	0.01929	0.15434	0.04036	0.08725	0.01063	0.12895	0.02524
(0.4,0.7,2)	0.1	0.00618	0.00006	0.00961	0.00012	0.00553	0.00005	0.00706	0.00006
	0.4	0.03772	0.00216	0.05864	0.00491	0.03006	0.00133	0.04186	0.00271
	0.7	0.07467	0.00944	0.08599	0.01092	0.04562	0.00290	0.07009	0.00757
	1.0	0.10002	0.01520	0.11883	0.02211	0.07594	0.00737	0.09478	0.01445
	1.3	0.11585	0.02046	0.15592	0.03342	0.09810	0.01346	0.12758	0.02827
	1.6	0.15117	0.03388	0.19412	0.05414	0.12189	0.02208	0.15579	0.03901
	1.9	0.17583	0.04800	0.22552	0.07684	0.13617	0.02605	0.17251	0.04172

Table 1: Absolute bias and MSE of $\hat{\mu}_k; k = 1,2$ for the Weibull model.

give the power of the test. Three different weight functions are employed, which are $(i).\, w(\cdot) = 1$, $(ii).\, w(\cdot) = \sqrt{n}$ and $(iii).\, w(\cdot) = \hat{\mu}_k(\cdot)$. Results are presented in Table 3 and Table 4, and it is clear that the empirical type I error approaches the significance level when $\theta_1 = \theta_2$ and $\rho_1 = \rho_2$. The rejection probabilities of the test at the 5% level under the Weibull model, with $\theta_1 = 0.2$ and θ_2 increasing from 0.2 to 0.5, are plotted in Figure 3. Similarly, under the Gompertz model, with $\rho_1 = 0.3$ and ρ_2 increasing from 0.3 to 0.6, the rejection probabilities are shown in Figure 4. Also, the test has very good power that increases as the difference between the parameters increases. The weight function $w(\cdot) = \hat{\mu}_k(\cdot)$ gives better power compared to others.

(ρ_1,ρ_2,β)	Time	n=50				n=100			
		$\hat{\mu}_1(t)$		$\hat{\mu}_2(t)$		$\hat{\mu}_1(t)$		$\hat{\mu}_2(t)$	
		Absolute bias	MSE	Absolute bias	MSE	Absolute bias	MSE	Absolute bias	MSE
(0.1,0.1,1.5)	0.1	0.01201	0.00018	0.01207	0.00019	0.00886	0.00014	0.00721	0.00010
	0.4	0.03242	0.00163	0.03645	0.00201	0.02378	0.00094	0.02481	0.00089
	0.7	0.04957	0.00367	0.05051	0.00398	0.03410	0.00183	0.03873	0.00227
	1.0	0.07282	0.00804	0.06945	0.00747	0.05784	0.00472	0.04500	0.00330
	1.3	0.09463	0.01419	0.09130	0.01289	0.07351	0.00823	0.06289	0.00599
	1.6	0.11684	0.02130	0.11783	0.02205	0.08795	0.01231	0.08179	0.00992
	1.9	0.14055	0.03176	0.14181	0.03201	0.11550	0.01996	0.10227	0.01592
(0.3,0.5,1.1)	0.1	0.02066	0.00063	0.02681	0.00094	0.01293	0.00031	0.01927	0.00058
	0.4	0.05937	0.00474	0.07070	0.00825	0.03730	0.00206	0.04705	0.00385
	0.7	0.08685	0.01103	0.10038	0.01744	0.05888	0.00485	0.07130	0.00826
	1.0	0.10684	0.01600	0.15112	0.03776	0.08144	0.01007	0.10955	0.01745
	1.3	0.14177	0.03057	0.14024	0.03676	0.09694	0.01618	0.13714	0.02670
	1.6	0.15256	0.03867	0.17802	0.05433	0.11517	0.02219	0.15139	0.03627
	1.9	0.16660	0.04904	0.26433	0.09548	0.13815	0.02723	0.17773	0.04989
(0.3,0.6,1.1)	0.1	0.02073	0.00059	0.02829	0.00148	0.01433	0.00036	0.02333	0.00090
	0.4	0.05059	0.00426	0.07906	0.00921	0.03823	0.00244	0.05138	0.00429
	0.7	0.08411	0.01049	0.11910	0.02050	0.05953	0.00594	0.06841	0.00735
	1.0	0.09824	0.01711	0.14610	0.03745	0.07124	0.00850	0.10017	0.01430
	1.3	0.15494	0.03880	0.21100	0.06770	0.09728	0.01380	0.12914	0.02981
	1.6	0.17544	0.04808	0.25562	0.09666	0.11209	0.01869	0.17581	0.04214
	1.9	0.23005	0.07679	0.31608	0.14118	0.13288	0.02529	0.20030	0.06149
(0.4,0.6,1.1)	0.1	0.02242	0.00080	0.03360	0.00179	0.01661	0.00050	0.02032	0.00072
	0.4	0.06641	0.00712	0.06943	0.00705	0.04521	0.00326	0.05072	0.00427
	0.7	0.07941	0.00896	0.08502	0.01136	0.06455	0.00714	0.07980	0.00888
	1.0	0.10688	0.02057	0.10526	0.01834	0.07837	0.00997	0.09747	0.01348
	1.3	0.12311	0.02451	0.15443	0.03817	0.09829	0.01608	0.12627	0.02763
	1.6	0.17441	0.04299	0.25481	0.09228	0.11675	0.02344	0.13286	0.02868
	1.9	0.21571	0.07401	0.25492	0.10076	0.13504	0.03037	0.15555	0.04198

Table 2: Absolute bias and MSE of $\hat{\mu}_k; k = 1,2$ for the Gompertz model.

5. DATA ANALYSIS

The proposed inference procedures are illustrated using an automobile warranty data-base given in [14]. The data set consists of recurrent failure histories of 172 automobiles in which the outcome of interest is the repeated mileages at which the failures occur. The mileages are type I censored at 3000 miles. The failure modes are classified into three categories, FM1, FM2, and FM3, resulting in multiple modes of failure framework. We can observe from the data a total of 274 failures in which 76 failures are due to FM1, 87 are due to FM2, and 111 are due to FM3.

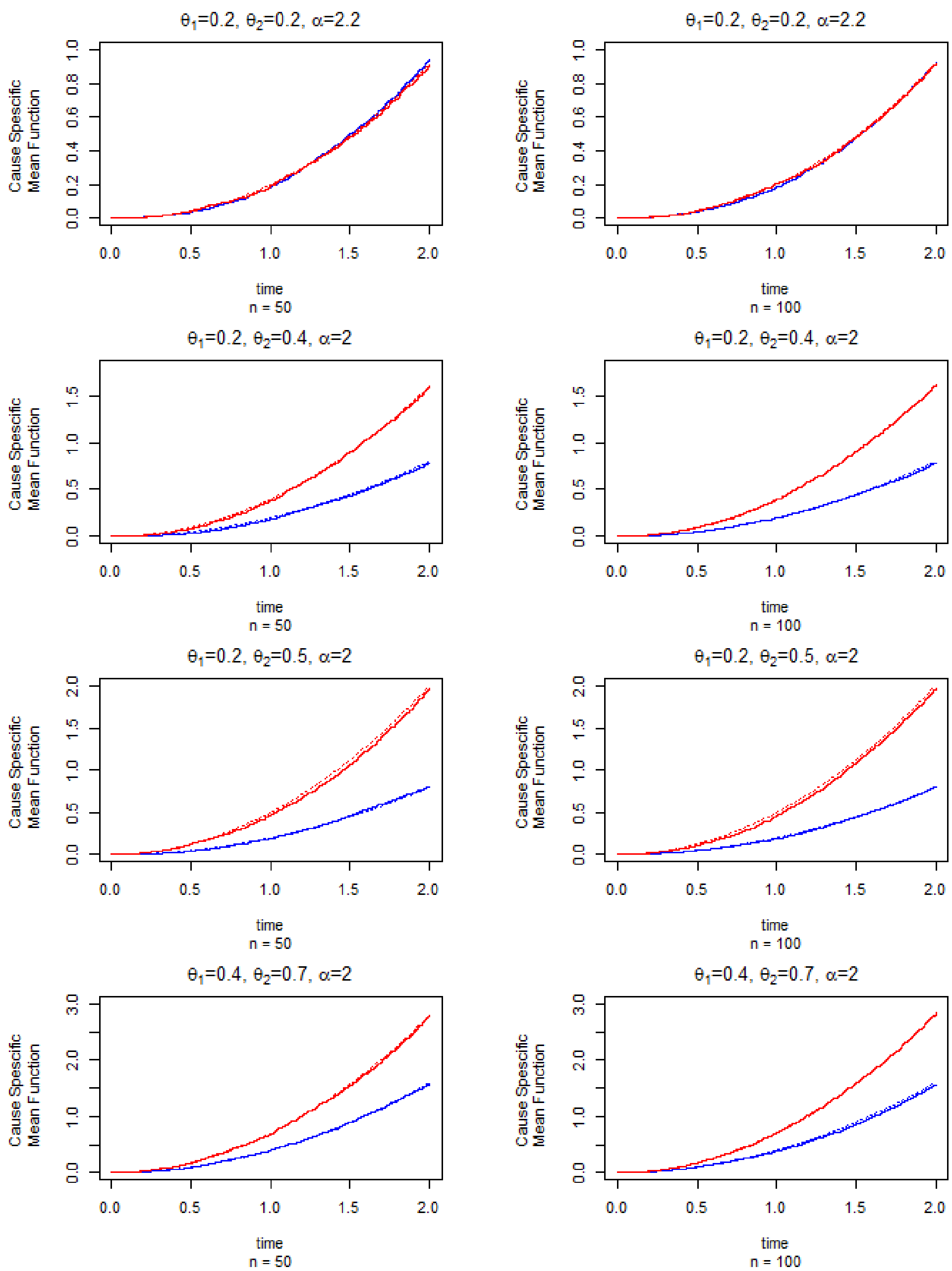

Figure 1: Estimates of cause specific mean functions of Weibull model for different values of $(\theta_1, \theta_2, \alpha)$. The blue color represents cause 1, and the red color represents cause 2. Solid lines are for the estimated values, and dotted lines are the true values.

Since there are no intermittent observation gaps in the original data set, we generate gaps in the data records to create an automobile warranty database with simulated windows. We modify the original data set by allowing each vehicle to have a minimum of four windows of observation gaps in their study tenure. The length of the observation

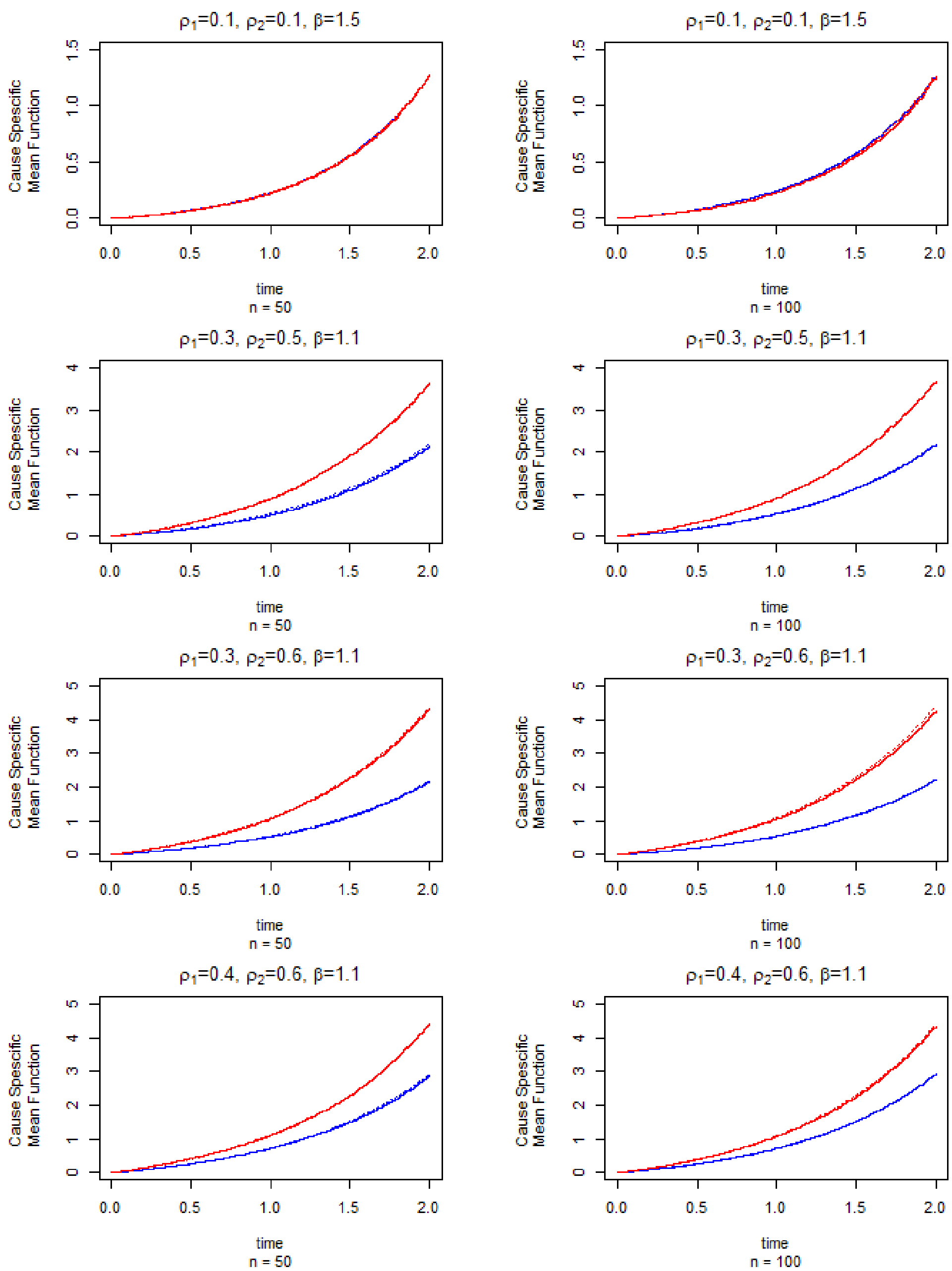

Figure 2: Estimates of cause specific mean functions of Gompertz model for different values of (ρ_1, ρ_2, β). The blue color represents cause 1, and the red color represents cause 2. Solid lines are for the estimated values, and dotted lines are the true values.

windows is randomly generated from a $Uniform(400, 600)$ distribution while the length of the gaps follows a $Uniform(50, 150)$ distribution. All the vehicles are assumed to begin with an observation window. In the transformed data, we observe a total of 245 failure recurrences, where 29 failures are missing due to gaps in the observation time-

$(\theta_1,\theta_2,\alpha)$	w(.)	1%		5%	
		n=50	n=100	n=50	n=100
(0.2,0.2,2.2)	1	1.8	1.2	5.6	5.2
	$\sqrt{n}$	1.8	1.4	5.8	5.4
	$\hat{\mu}_k(\cdot)$	1.6	1.2	5.6	4.8
(0.2,0.4,2)	1	73	90.8	88.6	96
	$\sqrt{n}$	67.6	92	88.4	95.2
	$\hat{\mu}_k(\cdot)$	74.4	94	91.4	100
(0.2,0.5,2)	1	92.4	100	98.2	100
	$\sqrt{n}$	91.2	100	98.8	100
	$\hat{\mu}_k(\cdot)$	93.6	100	100	100
(0.4,0.7,2)	1	82.4	98	95	100
	$\sqrt{n}$	80.8	96.4	94.4	98
	$\hat{\mu}_k(\cdot)$	82.8	98.4	97.2	100

Table 3: Empirical type I error and power of the test in percentage at an asymptotic level of 1% and 5% for the Weibull model.

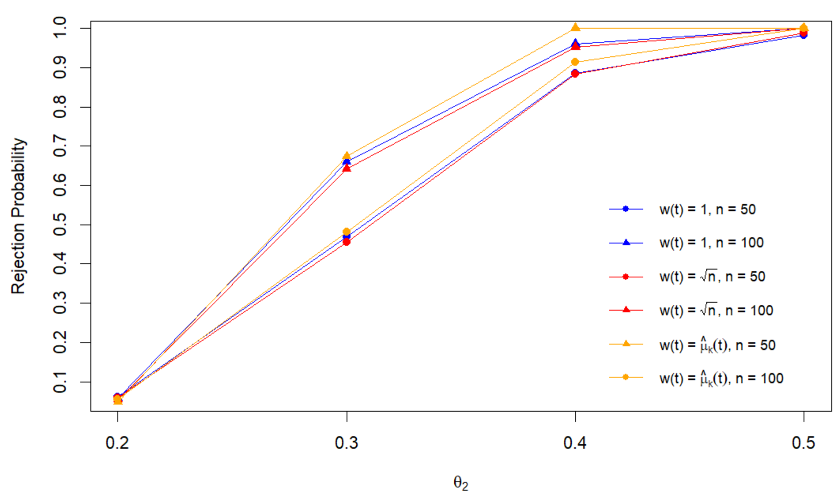

Figure 3: Rejection probabilities of the test at the 5% level under the Weibull model with $\theta_1 = 0.2$.

line. The new recurrences comprise 71 failures due to FM1, 78 due to FM2, and 96 due to FM3. Because of the presence of gaps in the follow-up, the number of vehicles at risk changes with time, and the size of the new risk set is presented in Figure 5.

(ρ_1,ρ_2,β)	w(.)	1%		5%	
		n=50	n=100	n=50	n=100
(0.1,0.1,1.5)	1	1.8	1.4	5.8	5.2
	$\sqrt{n}$	2	1.4	6.2	5.6
	$\hat{\mu}_k(\cdot)$	1.6	1.2	5.4	4.6
(0.3,0.5,1.1)	1	88.2	96.8	94.6	100
	$\sqrt{n}$	87.2	96.4	94.2	100
	$\hat{\mu}_k(\cdot)$	87.8	97	95.4	100
(0.3,0.6,1.1)	1	95.6	100	100	100
	$\sqrt{n}$	94.4	100	100	100
	$\hat{\mu}_k(\cdot)$	96.2	100	100	100
(0.4,0.6,1.1)	1	71.4	92.8	94.2	98.4
	$\sqrt{n}$	67.4	91.4	92.8	98.4
	$\hat{\mu}_k(\cdot)$	78.2	93.4	94.6	100

Table 4: Empirical type I error and power of the test in percentage at an asymptotic level of 1% and 5% for the Gompertz model.

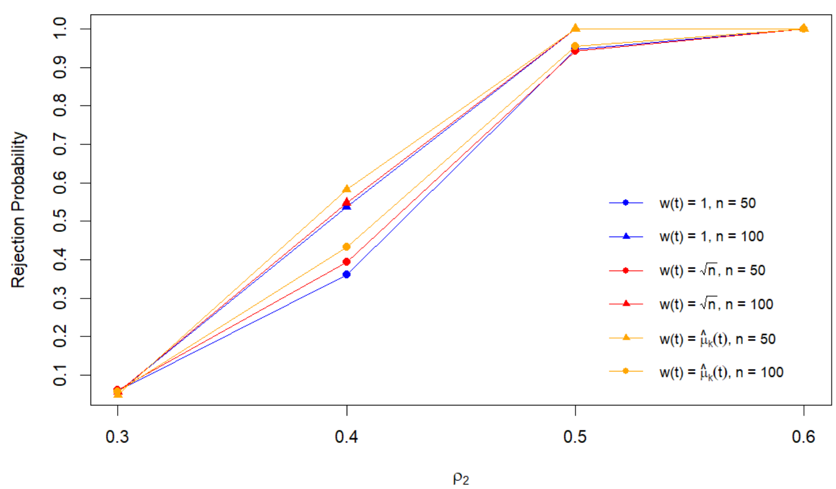

Figure 4: Rejection probabilities of the test at the 5% level under the Gompertz model with $\rho_1 = 0.3$.

Our goal here is to compare the effect of three modes of failures FM1, FM2, and FM3 on the failure recurrences of automobiles. The mean functions due to three different modes of failures are computed using the estimator proposed in (2.3) and the results are

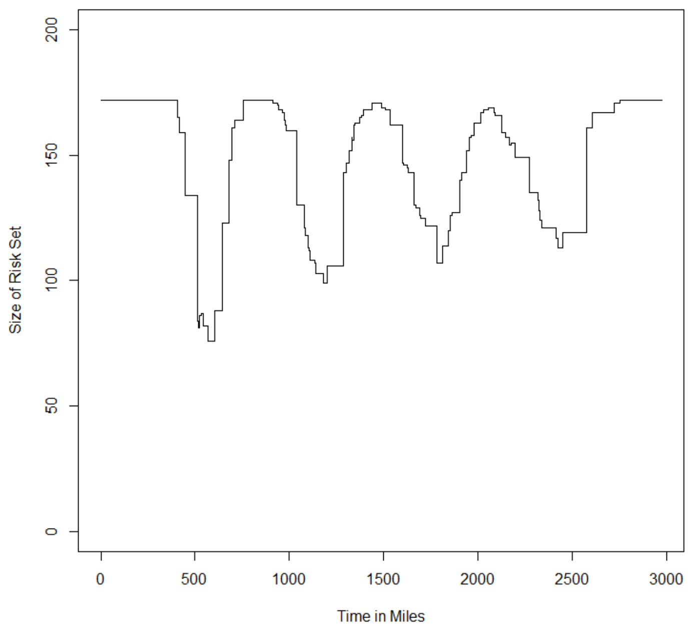

Figure 5: Risk set plot for the automobile data with observation gaps.

presented in Figure 6.

From Figure 6, we observe that the mean functions for three different causes FM1, FM2, and FM3 are different. The mean cumulative functions of the causes FM1 and FM2 show an almost similar pattern and compete, but failure mode 3 exhibits a different pattern. We can see a higher failure rate at the beginning of the observation time, somewhat up to 500 miles, due to causes FM1 and FM2. The rate becomes steady beyond 1500 miles for these two causes. The earlier failures of vehicles, somewhat up to 1500 miles, are more likely to be due to failure modes 1 or 2, and failure mode 3 causes more events beyond 1500 miles than the other two.

We test statistically whether these three modes of failures namely FM1, FM2, and FM3 have the same effect on the recurrence process or not by using the method proposed in Section 3. With the help of bootstrap techniques, the variance-covariance matrix in (3.4) is estimated. Table 5 presents the test statistic values obtained and the corresponding p-values for weight functions $w(\cdot) = 1$, $w(\cdot) = \sqrt{n}$ and $w(\cdot) = \hat{\mu}_k(\cdot)$.

From Table 5, we see that the p-value is small for each of the weight functions, and we conclude that the three cause specific mean functions are significantly different.

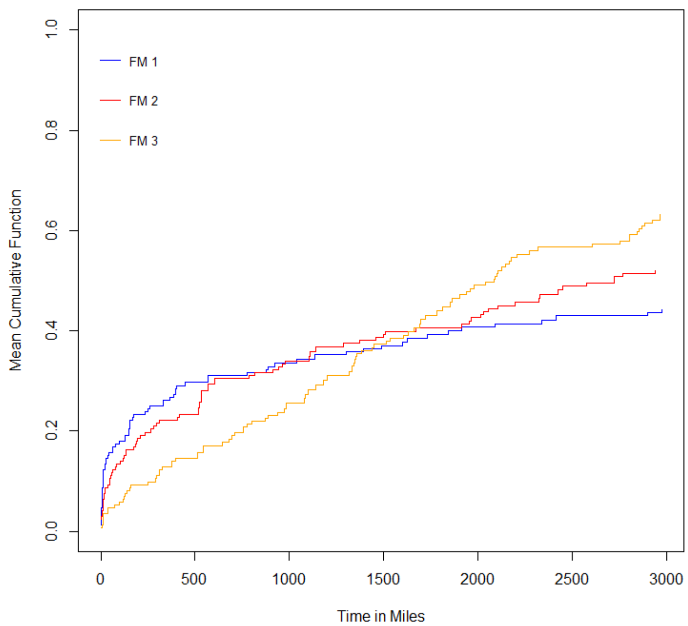

Figure 6: Estimates of cumulative cause specific mean functions of automobile data with observation gaps for different failure modes (FM1, FM2 and FM3).

Weight function	Test statistic	p-value
$w(\cdot) = 1$	15.857	0.00036
$w(\cdot) = \sqrt{n}$	15.856	0.00036
$w(\cdot) = \hat{\mu}_k(\cdot)$	9.577	0.00832

Table 5: Test statistic values and the corresponding p-values for different weight functions.

6. CONCLUSION

Recurrent event data with gaps in the follow-up windows are often found in survival and reliability studies. In this paper, we are extending the methods proposed by [17] for analyzing the window-observation recurrence data to the multiple causes of failure scenario. A non-parametric estimator for the cause specific mean function in the window-observation setup is studied and the asymptotic properties were discussed. A test statistic is proposed to test whether the mean functions due to different causes are identical or not. The results of the simulation study ensure that the proposed methods are efficient. A data analysis is also performed to illustrate the methods proposed in this paper.

In some applications, there will be additional information on the factors that may affect the recurrence process, such as medical history, demographic details, and vital signs of patients in clinical studies or model, make, and operating environment of systems in reliability studies. Allowing such covariate information to analyze multiple modes of failure in window-observation recurrent event data is an important research area, and works in these directions will be reported elsewhere.

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