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One Parameter Polynomial Exponential Distribution with Binomial Mixture

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Abstract:

- A further generalized version of one parameter polynomial exponential distribution with binomial probability mass as a mixture called a Binomial Mixture One Parameter Polynomial Exponential Distribution (BMOPPE) is proposed in the article. The moments and stochastic orderings are studied. Maximum likelihood estimator (MLE) and uniformly minimum variance unbiased estimator (UMVUE) of the probability density function and the cumulative distribution function have been derived and compared in the mean squared error sense. Estimation issues (both MLE and UMVUE) of reliability functions- mission time and stress-strength have been considered, and asymptotic variances of MLEs and variances of UMVUEs have been derived. UMVUEs of the variance of UMVUE of reliability functions have also been derived. Simulation study results have been reported to validate the theoretical findings. Few data sets have been fitted and compared.

Keywords:

- *Akaike's information criterion; asymptotic variance; gamma distribution; mixture distribution; reliability function.*

AMS Subject Classification:

- 60E05, 62E99.

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1. INTRODUCTION

The probability density function (PDF) of the Lindley distribution [See, Lindley ([8])] is specified by

$$f_X(x,\theta) = \frac{\theta^2}{1+\theta}(1+x)\exp(-\theta x), \qquad \theta > 0,\ x > 0, \tag{1.1}$$

and the corresponding cumulative density function (CDF) is given by

$$F_X(x,\theta) = 1 - \frac{1+\theta+\theta x}{1+\theta}\exp(-\theta x), \qquad \theta > 0,\ x > 0. \tag{1.2}$$

A distribution that is close in form to (1.1) is the well-known exponential distribution given by the PDF

$$f_X(x,\theta) = \theta\exp(-\theta x), \qquad \theta > 0,\ x > 0.$$

Ghitany *et al.* ([5]) showed that in many ways the Lindley distribution is a better model than one based on the exponential distribution. The distribution in (1.1) is a mixture of exponential and gamma distribution with shape parameter 2 and scale parameter θ with mixing proportions $\frac{\theta}{1+\theta}$ and $\frac{1}{1+\theta}$, respectively. So, it is a Bernoulli mixture of gamma distributions of the form $f_X(x;\theta) = \sum_{k=0}^{1}\binom{1}{k}\left(\frac{1}{\theta+1}\right)^k\left(\frac{\theta}{\theta+1}\right)^{1-k} f_{\text{GA}}(x;k+1,\theta)$, where, $f_{\text{GA}}(x;k+1,\theta) = \frac{\theta^{k+1}}{\Gamma(k+1)}x^k\exp(-\theta x)$ is the gamma PDF with shape parameter $k+1$ and scale parameter θ. We generalize this distribution to binomial mixing with parameter r and $\frac{1}{\theta+1}$ of the form $f_X(x;\theta) = \sum_{k=0}^{r}\binom{r}{k}\left(\frac{1}{\theta+1}\right)^k\left(\frac{\theta}{\theta+1}\right)^{r-k} f_{\text{GA}}(x;k+1,\theta)$.

It can be written in more generalized form of the PDF as

$$f_X(x,\theta) = \frac{\sum_{k=0}^{r} a_k p_k h_k(x;\theta)}{\sum_{k=0}^{r} a_k p_k}, \qquad \theta > 0,\ x > 0, \tag{1.3}$$

where, $p_k = \binom{r}{k}\left(\frac{1}{\theta+1}\right)^k\left(\frac{\theta}{\theta+1}\right)^{r-k}$, $h_k(x;\theta)) = \frac{\theta^{k+1}}{\Gamma(k+1)}x^{(k+1)-1}\exp(-\theta x)$ and a_k's are non-negative constants.

It can also be rewritten as

$$f_X(x,\theta) = h(\theta)p(x)\exp(-\theta x), \qquad \theta > 0,\ x > 0, \tag{1.4}$$

where, $h(\theta) = \frac{1}{\sum_{k=0}^{r} a_k\binom{r}{k}\frac{1}{\theta^{k+1}}}$, $p(x) = \sum_{k=0}^{r}\frac{a_k}{k!}\binom{r}{k}x^k$.

A random variable X is said to have a Binomial Mixture One Parameter Polynomial Exponential (BMOPPE) with parameter θ, if its probability density function (PDF) is given by

$$f_X(x,\theta) = \frac{\sum_{k=0}^{r} a_k\binom{r}{k}\left(\frac{1}{\theta+1}\right)^k\left(\frac{\theta}{\theta+1}\right)^{r-k}\frac{\theta^{k+1}}{\Gamma(k+1)}x^{(k+1)-1}\exp(-\theta x)}{\sum_{k=0}^{r} a_k\binom{r}{k}\left(\frac{1}{\theta+1}\right)^k\left(\frac{\theta}{\theta+1}\right)^{r-k}}, \qquad \theta > 0,\ x > 0. \tag{1.5}$$

The CDF of the random variable X is given by

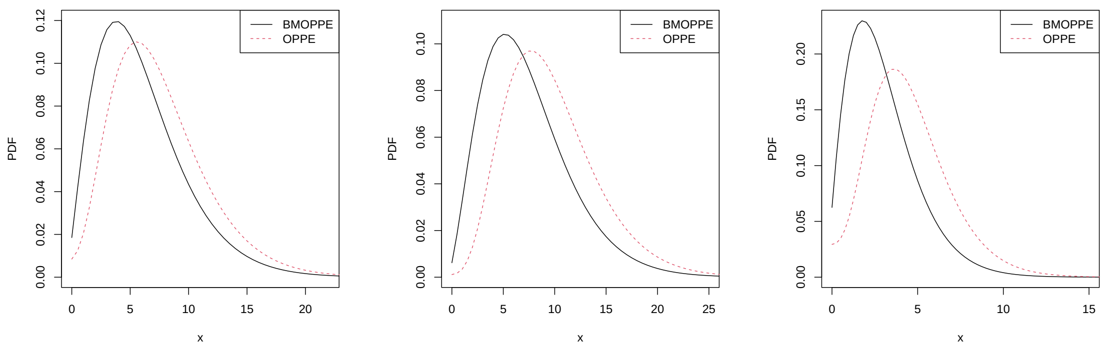
$$F(x) = \frac{\sum_{k=0}^{r} a_k \binom{r}{k} \left(\frac{1}{\theta+1}\right)^k \left(\frac{\theta}{\theta+1}\right)^{r-k} \gamma(k+1, \theta x)}{\sum_{k=0}^{r} a_k \binom{r}{k} \left(\frac{1}{\theta+1}\right)^k \left(\frac{\theta}{\theta+1}\right)^{r-k}}, \qquad \theta > 0,\ x > 0, \tag{1.6}$$

where $\gamma(s,t) = \frac{1}{\Gamma s} \int_0^t \exp(-x) x^{s-1} dx$ is the lower incomplete gamma function.

The CDF can also be written as

$$F(x) = 1 - \left(\frac{\sum_{k=0}^{r} a_k \binom{r}{k} \frac{1}{\theta^{k+1}} \Gamma(k+1, \theta x)}{\sum_{k=0}^{r} a_k \binom{r}{k} \frac{1}{\theta^{k+1}}} \right), \qquad x,\ \theta > 0, \tag{1.7}$$

where $\Gamma(m,x) = \frac{1}{\Gamma(m)} \int_x^\infty e^{-u} u^{m-1} du$, the upper incomplete gamma function.



(a) For $r = 3$ and $\theta = 0.5$. **(b)** For $r = 4$ and $\theta = 0.5$. **(c)** For $r = 4$ and $\theta = 1$.

Figure 1: Plot of PDF of BMOPPE and OPPE for different combinations.

Bouchahed and Zeghdoudi ([4]) proposed a new and unified approach to generalizing Lindley's distribution and investigated its properties. Mukherjee *et al.* ([11]) later called it the One Parameter Polynomial Exponential (OPPE) family of distributions and studied the estimation aspect of the PDF and the CDF of the distribution. The natural discrete version of the OPPE called the Natural Discrete One Parameter Polynomial Exponential (NDOPPE) family of distributions is studied by Maiti *et al.* ([9]), and the estimation aspect of the PMF and the CDF is discussed by Mukherjee *et al.* ([10]). The OPPE is a mixture of gamma distributions with some mixing probabilities. In contrast, the BMOPPE is revisited with a different look, and it is also a mixture of gamma distributions with binomial mixing probabilities, which is different from the previous one.

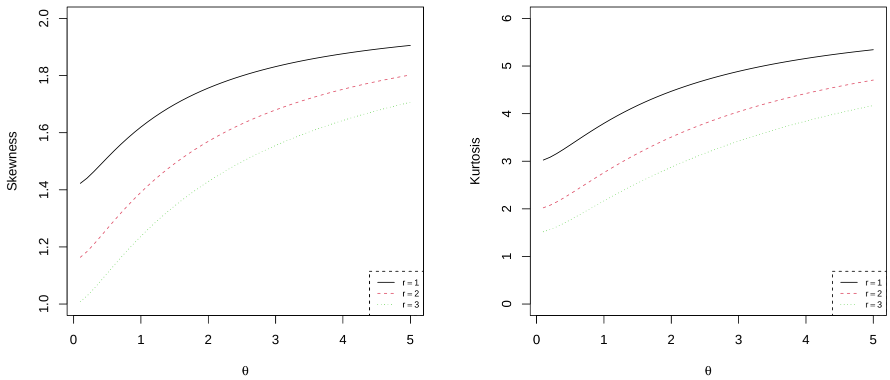
The article is organised as follows. Section 2 discusses different order moments and stochastic orderings of the random variable. In section 3, The maximum likelihood estimator (MLE) and uniformly minimum variance unbiased estimator (UMVUE) of the PDF and the CDF are discussed. The estimators are compared in the mean squared error (MSE) sense. This section also considers the estimation of both mission time and stress-strength reliability functions. Asymptotic variances of MLEs and variances of MVUEs are derived. UMVUEs of variances of UMVUE of reliability functions have been derived. Simulation study results have been reported to verify the theoretical findings in section 4. Three data sets have been analysed for illustration purposes in section 5. Section 6 is for making some concluding remarks.

2. MOMENTS AND STOCHASTIC ORDERINGS

The s-th raw moments for BMOPPE distribution is

$$\mu'_s = \frac{\sum_{k=0}^{r} a_k \binom{r}{k} \left(\frac{1}{\theta+1}\right)^k \left(\frac{\theta}{\theta+1}\right)^{r-k} \frac{\Gamma(k+s+1)}{\theta^s \Gamma(k+1)}}{\sum_{k=0}^{r} a_k \binom{r}{k} \left(\frac{1}{\theta+1}\right)^k \left(\frac{\theta}{\theta+1}\right)^{r-k}}. \tag{2.1}$$

The coefficient of skewness and kurtosis measures have been shown in Figure 2. These are shown for $r = 1,\ 2,\ 3$ and for different values of θ. The constants of polynomial are taken as $a_i = 1,\ i = 0,\ 1,\ 2,\ 3$. It is noticed that the distribution is positively skewed, and skewness decreases with the increment of the degree of the polynomial. Also, the distribution is leptokurtic, and it becomes long-tailed with the increment of the degree of the polynomial.



(a) Skewness. **(b)** Kurtosis.

Figure 2: Plot of Skewness and Kurtosis of BMOPPE for different θ and r.

The ordering relations between two BMOPPE random variables have been shown in the following Theorem.

Theorem 2.1. *Let $X_i \sim \text{BMOPPE}(\theta_i)$, $i = 1, 2$ be two random variables. If $\theta_2 \leq \theta_1$, then $X_1 \prec_{\text{lr}} X_2, X_1 \prec_{\text{hr}} X_2, X_1 \prec_{\text{st}} X_2$ and $X_1 \prec_{\text{cx}} X_2$.*

Proof: Since, Likelihood ratio order $\Longrightarrow$ Hazard rate order $\Longrightarrow$ Stochastic order and Convex order $\Longleftrightarrow$ Stochastic order, it is sufficient to show that Likelihood ratio order holds.

We have

$$L(x) = \ln\left(\frac{f_{X_1}(x)}{f_{X_2}(x)}\right) = (k+1)\ln\left(\frac{\theta_1}{\theta_2}\right) - (\theta_1 - \theta_2)x.$$

Now,

$$\triangle L(x) = \frac{d}{dx}\left[\ln\left(\frac{f_{X_1}(x)}{f_{X_2}(x)}\right)\right].$$

Clearly, it is evident that $\triangle L(x) \leq 0$, $\forall \theta_2 \leq \theta_1$. □

3. ESTIMATION OF PDF AND CDF AND THEIR APPLICATIONS IN RELIABILITY ESTIMATION

The PDF and the CDF estimates have immense importance in estimating reliability functions (both mission time and stress-strength), entropy functions, Kullback–Leibler divergence measure, Fisher information, cumulative residual entropy, quantile function, hazard rate function, etc. In this section, the MLE and UMVUE of the reliability functions are to be attempted. The asymptotic variances/variances of the estimators and their estimators are to be discussed.

First, we will discuss the MLE and UMVUE of the PDF and the CDF of BMOPPE family of distributions. Let $X_1, X_2, ..., X_n$ be random sample of size n drawn from the BMOPPE distribution in (1.5). The MLE of θ which is denoted as $\widetilde{\theta}$ is obtained by numerically solving the equation

$$\frac{\sum_{k=0}^{r} a_k \binom{r}{k} \frac{k+1}{\theta^{k+2}}}{\sum_{k=0}^{r} a_k \binom{r}{k} \frac{1}{\theta^{k+1}}} - \bar{X} \;=\; 0. \tag{3.1}$$

Using the invariance property of MLE, one can obtain the MLEs of the PDF and that of the CDF by substituting $\widetilde{\theta}$ in (1.5) and (1.6) respectively. Theoretical expressions for the MSE of the MLEs are not available. MSE is to be studied through simulation.

To derive the UMVUE of the PDF and that of the CDF (stated in Theorem 3.2), we will use the following Theorem 3.1 and Lemma 3.1.

Theorem 3.1. *Let $X_1, X_2, ..., X_n$ independently follow* BMOPPE(θ). *Then the distribution of $T = X_1 + X_2 + \cdots + X_n$ is*

$$f(t) \;=\; h^n(\theta) \sum_{y_0} \sum_{y_1} \cdots \sum_{y_r} c(n, y_0, y_1, ..., y_r) \exp(-\theta t)\; t^{\sum_{k=0}^{r}(k+1)y_k - 1}, \qquad t > 0,$$

with $y_0 + y_1 + \cdots + y_r = n$ and $c(n, y_0, y_1, ..., y_r) = \frac{n!}{y_0! y_1! \cdots y_r!} \frac{\prod_{k=0}^{r}[a_k \binom{r}{k}]^{y_k}}{\Gamma(\sum_{k=0}^{r}(k+1)y_k)}$.

Proof: Since X_i's are independent and identically distributed, the moment generating function (mgf) of T is

$$\begin{aligned}
M_T(t) \;&=\; h^n(\theta) \left[\sum_{k=0}^{r} a_k \binom{r}{k} \frac{1}{\theta^{k+1}} \left(1 - \frac{t}{\theta}\right)^{-(k+1)} \right]^n \\
&=\; h^n(\theta) \left[a_0 \binom{r}{0} \frac{1}{\theta} \left(1 - \frac{t}{\theta}\right)^{-1} + \cdots + a_r \binom{r}{r} \frac{1}{\theta^{r+1}} \left(1 - \frac{t}{\theta}\right)^{-(r+1)} \right]^n \\
&=\; h^n(\theta) \sum_{y_0} \sum_{y_1} \cdots \sum_{y_r} \frac{n!}{y_0! y_1! \cdots y_r!} \prod_{k=0}^{r} \left[a_k \binom{r}{k} \right]^{y_k} \theta^{-\sum_{k=0}^{r}(k+1)y_k} \left(1 - \frac{t}{\theta}\right)^{-\sum_{k=0}^{r}(k+1)y_k}.
\end{aligned}$$

Hence, the distribution of T is

$$\begin{aligned} f(t) &= h^n(\theta)\sum_{y_0}\sum_{y_1}\cdots\sum_{y_r}\frac{n!}{y_0!y_1!\cdots y_r!}\prod_{k=0}^{r}\left[a_k\binom{r}{k}\right]^{y_k}\theta^{-\sum_{k=0}^{r}(k+1)y_k}f_{\mathrm{GA}}(t,\sum_{k=0}^{r}(k+1)y_k,\theta)\\ &= h^n(\theta)\sum_{y_0}\sum_{y_1}\cdots\sum_{y_r}c(n,y_0,y_1,...,y_r)t^{\sum_{k=0}^{r}(k+1)y_k-1}\exp(-\theta t). \end{aligned}$$ □

Lemma 3.1. *The conditional distribution of X_1 given $T = X_1 + X_2 + \cdots + X_n$ is*

$$f_{X_1|T}(x|t) = \frac{p(x)}{A_n(t)}\sum_{q_0}\sum_{q_1}\cdots\sum_{q_r}c(n-1,q_0,q_1,...,q_r)\,(t-x)^{\sum_{k=0}^{r}(k+1)q_k-1},\qquad 0<x<t,$$

where

$$A_n(t) = \sum_{y_0}\sum_{y_1}\cdots\sum_{y_r}c(n,\ y_0,\ y_1,\ ...,\ y_r)t^{\sum_{k=0}^{r}(k+1)y_k-1},$$

and

$$c(n-1,q_0,q_1,...,q_r) = \frac{(n-1)!}{q_0!q_1!\cdots q_r!}\prod_{k=0}^{r}\left[a_k\binom{r}{k}\right]^{q_k}\frac{1}{\Gamma(\sum_{k=0}^{r}(k+1)q_k)},$$

with $q_0+q_1+q_2+\cdots+q_r = n-1$.

Proof: The proof is obviously conducted from

$$\begin{aligned} f_{X_1|T}(x|t) &= \frac{f_{X_1}(x)f(t-x)}{f(t)}\\ &= \frac{p(x)}{A_n(t)}\sum_{q_0}\sum_{q_1}\cdots\sum_{q_r}c(n-1,q_0,q_1,...,q_r)\,(t-x)^{\sum_{k=0}^{r}(k+1)q_k-1}. \end{aligned}$$ □

Theorem 3.2. *Let $T = t$ be given. Then*

$$\widehat{f}(x) = \frac{p(x)}{A_n(t)}\sum_{q_0}\sum_{q_1}\cdots\sum_{q_r}c(n-1,q_0,q_1,...,q_r)\,(t-x)^{\sum_{k=0}^{r}(k+1)q_k-1},\qquad 0<x<t, \tag{3.2}$$

is UMVUE for $f(x)$ and

$$\begin{aligned} \widehat{F}(x) &= 1-\frac{1}{A_n(t)}\sum_{q_0}\sum_{q_1}\cdots\sum_{q_r}c(n-1,q_0,q_1,...,q_r)\\ &\quad\times\sum_{k=0}^{r}a_k\binom{r}{k}\frac{1}{\Gamma(k+1)}t^{\sum_{k=0}^{r}(k+1)q_k+k}\\ &\quad\times B\left((k+1),\sum_{k=0}^{r}(k+1)y_k\right)I_{x/t}\left((k+1),\sum_{k=0}^{r}(k+1)q_k\right),\qquad 0<x<t, \end{aligned} \tag{3.3}$$

is UMVUE for $F(x)$, where $I_x(\alpha,\beta) = \frac{1}{B(\alpha,\beta)}\int_x^1 u^{\alpha-1}(1-u)^{\beta-1}du$ is an incomplete beta function and $B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$.

Proof: The BMOPPE in (1.4) is a member of one parameter exponential family with $T = \sum_{i=1}^{n} X_i$ as complete sufficient statistic. Therefore, by the use of Lehmann–Scheffe theorem, we get the UMVUE of the PDF from Lemma 3.1.

$$\begin{aligned}
\widehat{F}(x) &= 1 - \int_x^t \widehat{f}(w)dw \\
&= 1 - \int_x^t \frac{p(w)}{A_n(t)} \sum_{q_0}\sum_{q_1}\cdots\sum_{q_r} c(n-1, q_0, q_1, ..., q_r)\,(t-w)^{\sum_{k=0}^{r}(k+1)q_k - 1} dw \\
&= 1 - \frac{1}{A_n(t)} \sum_{q_0}\sum_{q_1}\cdots\sum_{q_r} c(n-1, q_0, q_1, ..., q_r) \\
&\quad \times \sum_{k=0}^{r} a_k \binom{r}{k} \frac{1}{\Gamma(k+1)} t^{\sum_{k=0}^{r}(k+1)q_k + k} \\
&\quad \times B\left((k+1), \sum_{k=0}^{r}(k+1)y_k\right) I_{x/t}\left((k+1), \sum_{k=0}^{r}(k+1)q_k\right). \qquad \square
\end{aligned}$$

3.1. Mission Time Reliability

Suppose the life length of a component X is distributed as BMOPPE(θ). Then the reliability of that component for a fixed mission time t_0 (> 0) is

$$\begin{aligned}
\bar{F}_X(t_0) &= P(X \geq t_0) \\
&= h(\theta) \sum_{k=0}^{r} \frac{a_k \binom{r}{k}}{\theta^{k+1}} \Gamma(k+1,\ \theta t_0).
\end{aligned}$$

By using the relation between incomplete gamma function and Poisson probability, $\bar{F}_X(t_0)$ can be written as

$$\bar{F}_X(t_0) = h(\theta) \sum_{k=0}^{r} \frac{a_k \binom{r}{k}}{\theta^{k+1}} \sum_{j=0}^{k} \frac{e^{-\theta t_0}(\theta t_0)^j}{j!}.$$

3.1.1. The MLE

The MLE of $\bar{F}_X(t_0)$ based on a random sample of size n, is

$$\widetilde{\bar{F}}_X(t_0) = h(\widetilde{\theta}) \sum_{k=0}^{r} \frac{a_k \binom{r}{k}}{\widetilde{\theta}^{k+1}} \sum_{j=0}^{k} \frac{e^{-\widetilde{\theta} t_0}(\widetilde{\theta} t_0)^j}{j!},$$

where $\widetilde{\theta}$ is the solution of the equation (3.1). Since MSE($\widetilde{\bar{F}}_X(t_0)$) has no closed form expression, we give the asymptotic distribution of $\widetilde{\bar{F}}_X(t_0)$ by using delta theorem as follows:

$$\sqrt{n}\left(\widetilde{\bar{F}}_X(t_0) - \bar{F}_X(t_0)\right) \xrightarrow{d} N\left(0, \frac{1}{I(\theta)}\left[\frac{d\bar{F}_X(t_0)}{d\theta}\right]^2\right),$$

where

$$I(\theta) = \frac{\left[\sum_{k=0}^{r} \frac{a_k\binom{r}{k}}{\theta^{k+1}}\right]\left[\sum_{k=0}^{r} \frac{a_k\binom{r}{k}(k+2)(k+1)}{\theta^{k+3}}\right] - \left[\sum_{k=0}^{r} \frac{a_k\binom{r}{k}(k+1)}{\theta^{k+2}}\right]^2}{\left[\sum_{k=0}^{r} \frac{a_k\binom{r}{k}}{\theta^{k+1}}\right]^2}$$

is the Fisher information about parameter θ for a single observation X.

3.1.2. The UMVUE

Application of incomplete beta function and binomial probability gives the UMVUE of $\bar{F}_X(t_0)$ as

$$\begin{aligned}\widehat{\bar{F}}_X(t_0) &= \frac{1}{A_n(t)} \sum_{x_0}\sum_{x_1}\cdots\sum_{x_r} {}_{\sum_{i=0}^{r} x_i = n-1}\, c(n-1,\ x_0,\ x_1,\ ...,\ x_r) \\ &\times \sum_{k=0}^{r} \frac{a_k\binom{r}{k}}{\Gamma(k+1)} t_0^{\sum_{k=0}^{r}(k+1)x_k + k} B\left((k+1),\ \sum_{k=0}^{r}(k+1)x_k\right) \\ &\times \sum_{j=0}^{k} \binom{\sum_{l=0}^{r}(l+1)x_l + k}{j} \left(\frac{t_0}{t}\right)^j \left(1 - \frac{t_0}{t}\right)^{\sum_{l=0}^{r}(l+1)x_l + k - j}.\end{aligned}$$

For exponential family of distribution, Blight and Rao ([3]) considered Bhattacharya bounds to calculate the variance of UMVUE of parametric function $\psi(\theta)$. So, variance of $\widehat{\bar{F}}_X(t_0)$ can be expressed as

$$\mathrm{Var}(\widehat{\bar{F}}_X(t_0)) = \sum_{l=1}^{\infty} \frac{[\bar{F}_X^{(l)}(t_0)]^2}{[J_l^*(\theta)]^2},$$

where

$$\begin{aligned}[J_l^*(\theta)]^2 &= [h(\theta)]^n \sum_{j=0}^{l}\sum_{i=0}^{l} (-1)^{i+j} \binom{l}{i}\binom{l}{j} [h^{(l-i)}(\theta)]^n [h^{(l-j)}(\theta)]^n \\ &\times \sum_{x_0}\sum_{x_1}\cdots\sum_{x_r} {}_{\sum_{i=0}^{r} x_i = n-1}\, c(n-1,\ x_0,\ x_1,\ ...,\ x_r) \\ &\times \frac{\Gamma(i+j+\sum_{l=0}^{r}(l+1)x_l)}{\theta^{i+j+\sum_{l=0}^{r}(l+1)x_l}}\end{aligned}$$

is the Bhattacharya function and $h^{(i)}(\theta)$ denotes the i-th derivative of h with respect to θ.

Now, for the derivation of UMVUE of $\mathrm{Var}(\widehat{\bar{F}}_X(t_0))$, we consider the representation

$$\begin{aligned}\mathrm{Var}(\widehat{\bar{F}}_X(t_0)) &= E(\widehat{\bar{F}}_X^2(t_0)) - [E(\widehat{\bar{F}}_X(t_0))]^2 \\ &= E(\widehat{\bar{F}}_X^2(t_0)) - \bar{F}_X^2(t_0).\end{aligned}$$

If $\widehat{Q}_1(t_0)$ is the UMVUE of $Q_1(t_0) = \bar{F}_X^2(t_0)$, we get the UMVUE of $\mathrm{Var}(\widehat{\bar{F}}_X(t_0))$ as $\widehat{\bar{F}}_X^2(t_0) - \widehat{Q}_1(t_0)$. We start from the fact that

$$I[X_1 \geq t_0, X_2 \geq t_0] = I[X_1 \geq t_0]I[X_2 \geq t_0],$$

which implies that $I[X_1 \geq t_0, X_2 \geq t_0]$ is unbiased for $Q_1(t_0)$. Then, we get the UMVUE by using Rao–Blackwell theorem as

$$\begin{aligned}
\widehat{Q}_1(t_0) \;=\; & \frac{1}{A_n(t)} \sum_{x_0}\sum_{x_1}\cdots\sum_{x_r} \; c(n-2,\; x_0,\; x_1,\; ...,\; x_r)_{\sum_{i=0}^{r} x_i = n-2} \\
& \times \sum_{k=0}^{r}\sum_{l=0}^{r} \frac{a_k a_l \binom{r}{k}\binom{r}{l}}{\Gamma(k+1)\Gamma(l+1)} \sum_{i=0}^{\sum_{m=0}^{r}(m+1)x_m-1} \frac{(-1)^i \binom{\sum_{m=0}^{r}(m+1)x_m-1}{i}}{l+i+1} \\
& \times \Bigg[t^{\sum_{m=0}^{r}(m+1)x_m+l} \sum_{u=0}^{\sum_{m=0}^{r}(m+1)x_m+l} \frac{(-1)^u \binom{\sum_{m=0}^{r}(m+1)x_m+l}{u}}{t^u(k+u+1)} \left(t^{k+u+1} - t_0^{k+u+1}\right) \\
& - t_0^{l+i+1} t^{\sum_{m=0}^{r}(m+1)x_m-i-1} \sum_{v=0}^{\sum_{m=0}^{r}(m+1)x_m-i-1} \frac{(-1)^v \binom{\sum_{m=0}^{r}(m+1)x_m-i-1}{v}}{t^v(k+v+1)} \\
& \times \left(t^{k+v+1} - t_0^{k+v+1}\right) \Bigg].
\end{aligned}$$

3.2. Stress Strength Reliability

For estimation of stress strength reliability, we assume the stress random variable X follows BMOPPE(θ_1) and strength random variable Y follows BMOPPE(θ_2) and they are independently distributed. Then the expression of stress strength reliability becomes

$$\begin{aligned}
R \;&=\; P(X < Y) \\
&= \int_0^\infty \bar{F}_Y(x) f(x)dx \\
&= h(\theta_1)h(\theta_2) \sum_{k_2=0}^{r_2} \frac{b_{k_2}\binom{r_2}{k_2}}{\theta_2^{k_2+1}} \sum_{k_1=0}^{r_1} \frac{a_{k_1}\binom{r_1}{k_1}}{\theta_1^{k_1+1}} \sum_{j=0}^{k_2} \binom{k_1+j}{j} \left(\frac{\theta_2}{\theta_1+\theta_2}\right)^j \left(\frac{\theta_1}{\theta_1+\theta_2}\right)^{k_1+1}.
\end{aligned}$$

3.2.1. The MLE of R

Let $(x_1, x_2, ..., x_{n_1})$ and $(y_1, y_2, ..., y_{n_2})$ be independent samples drawn from BMOPPE(θ_1) and BMOPPE(θ_2), respectively. Let the MLEs of θ_1 and θ_2 be $\widetilde{\theta}_1$ and $\widetilde{\theta}_2$, respectively. By putting the values of $\widetilde{\theta}_1$ and $\widetilde{\theta}_2$ in the expression of R, we get $\widetilde{R}$ by its invariance property as

$$\widetilde{R} \;=\; h(\widetilde{\theta}_1)h(\widetilde{\theta}_2) \sum_{k_2=0}^{r_2} \frac{b_{k_2}\binom{r_2}{k_2}}{\widetilde{\theta}_2^{k_2+1}} \sum_{k_1=0}^{r_1} \frac{a_{k_1}\binom{r_1}{k_1}}{\widetilde{\theta}_1^{k_1+1}} \sum_{j=0}^{k_2} \binom{k_1+j}{j} \left(\frac{\widetilde{\theta}_2}{\widetilde{\theta}_1+\widetilde{\theta}_2}\right)^j \left(\frac{\widetilde{\theta}_1}{\widetilde{\theta}_1+\widetilde{\theta}_2}\right)^{k_1+1}.$$

Similarly as in mission time reliability, we derive the asymptotic distribution of $\widetilde{R}$ in the following theorem.

Theorem 3.3. *If the ratio $\frac{n_1}{n_2}$ converges to a positive number κ when both n_1 and n_2 tends to infinity, then*

$$\sqrt{n}(\widetilde{R} - R) \xrightarrow{d} N(0, \sigma^2),$$

where $\sigma^2 = \frac{1}{I_1(\theta_1)}[\frac{\partial R}{\partial \theta_1}]^2 + \frac{\kappa}{I_2(\theta_2)}[\frac{\partial R}{\partial \theta_2}]^2$ and $I_1(\theta_1)$ is a Fisher information about parameter θ_1 in a single observation X, $I_2(\theta_2)$ is a Fisher information about θ_2 in a single observation Y.

Proof: Since log-likelihood equations satisfies all regularity conditions of asymptotic normality for MLE, then we have

$$\sqrt{n_j}(\widetilde{\theta_j} - \theta_j) \xrightarrow{d} N(0, [I_j(\theta_j)]^{-1}),$$

where

$$I_1(\theta_1) \;=\; \frac{\left[\sum_{k=0}^{r_1} \frac{a_k\binom{r_1}{k}}{\theta_1^{k+1}}\right]\left[\sum_{k=0}^{r_1} \frac{a_k\binom{r_1}{k}(k+2)(k+1)}{\theta_1^{k+3}}\right] - \left[\sum_{k=0}^{r_1} \frac{a_k\binom{r_1}{k}(k+1)}{\theta_1^{k+2}}\right]^2}{\left[\sum_{k=0}^{r_1} \frac{a_k\binom{r_1}{k}}{\theta_1^{k+1}}\right]^2}$$

and

$$I_1(\theta_2) \;=\; \frac{\left[\sum_{k=0}^{r_2} \frac{b_k\binom{r_2}{k}}{\theta_2^{k+1}}\right]\left[\sum_{k=0}^{r_2} \frac{b_k\binom{r_2}{k}(k+2)(k+1)}{\theta_2^{k+3}}\right] - \left[\sum_{k=0}^{r_2} \frac{b_k\binom{r_2}{k}(k+1)}{\theta_2^{k+2}}\right]^2}{\left[\sum_{k=0}^{r_2} \frac{b_k\binom{r_2}{k}}{\theta_2^{k+1}}\right]^2}.$$

Again, from the fact of independence of $\widetilde{\theta_1}$ and $\widetilde{\theta_2}$, we get

$$\sqrt{n_1}(\widetilde{\theta_1} - \theta_1, \widetilde{\theta_2} - \theta_2) \xrightarrow{d} N_2(0, J(\theta_1, \theta_2)),$$

where

$$J(\theta_1, \theta_2) = \begin{bmatrix} [I_1(\theta_1)]^{-1} & 0 \\ 0 & \kappa[I_2(\theta_2)]^{-1} \end{bmatrix}.$$

Now application of the transformation $R = R(\theta_1, \theta_2)$ together with Delta theorem conclude the proof. □

3.2.2. The UMVUE of R

By using the UMVUE of mission time reliability and the PDF of the BMOPPE distribution, the UMVUE of R can be expressed as

$$\widehat{R} \;=\; \int_{x=0}^{\mathrm{Min}(t_1,t_2)} \widehat{\overline{F}}_Y(x)\widehat{f}_X(x)dx,$$

where $t_1 = \sum_{i=1}^{n_1} x_i$ and $t_2 = \sum_{i=1}^{n_2} y_i$, respectively. Evaluation of the integral gives the final form of the UMVUE of R as

$$
\begin{aligned}
\widehat{R} \;=\; & \frac{1}{A_{n_1}(t_1)A_{n_2}(t_2)} \sum_{q_0}\sum_{q_1}\cdots\sum_{q_{r_1}} \; c(n_1-1,\; q_0,\; q_1,\; ...,\; q_{r_1})_{\sum_{i=0}^{r_1} q_i = n_1-1} \\
& \times\, t_1^{\sum_{l=0}^{r_1}(l+1)q_l} \sum_{w_0}\sum_{w_1}\cdots\sum_{w_{r_2}} \; c(n_2-1,\; w_0,\; w_1,\; ...,\; w_{r_2})_{\sum_{i=0}^{r_2} w_i = n_2-1} \\
& \times\, t_2^{\sum_{m=0}^{r_2}(m+1)w_m} J_1\left(\sum_{l=0}^{r_1}(l+1)q_l,\; \sum_{m=0}^{r_2}(m+1)w_m\right),
\end{aligned}
$$

where

$$
\begin{aligned}
J_1(\alpha,\; \beta) \;=\; & \sum_{k_1=0}^{r_1} \frac{a_{k_1}\binom{r_1}{k_1} t_1^{k_1}}{\Gamma(k_1+1)} \sum_{k_2=0}^{r_2} \frac{b_{k_2}\binom{r_2}{k_2} t_2^{k_2}}{\Gamma(k_2+1)} \\
& \times \sum_{j=0}^{k_2} \sum_{i=0}^{\beta+k_2-j} (-1)^i \left(\frac{t_1}{t_2}\right)^{i+j} \binom{\beta+k_2}{j}\binom{\beta+k_2-j}{i} B(k_2+1,\; \beta)\, B(i+j+k_1+1,\; \alpha) \\
& \times \sum_{l=i+j+k_1+1}^{\alpha+i+j+k_1} \binom{\alpha+i+j+k_1}{l} \left(\frac{\mathrm{Min}(t_1,t_2)}{t_1}\right)^l \left(1-\frac{\mathrm{Min}(t_1,t_2)}{t_1}\right)^{(\alpha+i+j+k_1-l)}.
\end{aligned}
$$

Under certain regularity conditions, variance of $\widehat{R}$ takes the form [See, Bartoszewicz ([1])] as follows:

$$
\mathrm{Var}(\widehat{R}) \;=\; \sum_{k=1}^{\infty}\sum_{j=0}^{k}\left[\frac{\partial^k R}{\partial\theta_1^j \partial\theta_2^{k-j}} \frac{1}{I_j(\theta_1)J_{k-j}(\theta_2)}\right]^2,
$$

where

$$
\begin{aligned}
[I_l(\theta_1)]^2 \;=\; & [h(\theta_1)]^{n_1} \sum_{j=0}^{l}\sum_{i=0}^{l}(-1)^{i+j}\binom{l}{i}\binom{l}{j}\; [h^{(l-i)}(\theta_1)]^{n_1}\, [h^{(l-j)}(\theta_1)]^{n_1} \\
& \times \sum_{q_0}\sum_{q_1}\cdots\sum_{q_{r_1}} c(n_1-1,\; q_0,\; q_1,\; ...,\; q_{r_1})_{\sum_{i=0}^{r_1} q_i = n_1-1}\; \frac{\Gamma(i+j+\sum_{l=0}^{r_1}(l+1)q_l)}{\theta_1^{i+j+\sum_{l=0}^{r_1}(l+1)q_l}},
\end{aligned}
$$

$$
\begin{aligned}
[J_l(\theta_2)]^2 \;=\; & [h(\theta_2)]^{n_2} \sum_{j=0}^{l}\sum_{i=0}^{l}(-1)^{i+j}\binom{l}{i}\binom{l}{j}\; [h^{(l-i)}(\theta_2)]^{n_2}\, [h^{(l-j)}(\theta_2)]^{n_2} \\
& \times \sum_{w_0}\sum_{w_1}\cdots\sum_{w_{r_2}} c(n_2-1,\; w_0,\; w_1,\; ...,\; w_{r_2})_{\sum_{i=0}^{r_2} w_i = n_2-1}\; \frac{\Gamma(i+j+\sum_{l=0}^{r_2}(l+1)w_l)}{\theta_2^{i+j+\sum_{l=0}^{r_2}(l+1)w_l}}.
\end{aligned}
$$

As earlier, the following representation

$$
\begin{aligned}
\mathrm{Var}(\widehat{R}) \;&=\; E(\widehat{R^2}) - [E(\widehat{R})]^2 \\
&=\; E(\widehat{R^2}) - R^2
\end{aligned}
$$

gives the UMVUE of $\mathrm{Var}(\widehat{R})$ as $\widehat{R^2} - \widehat{Q}_2$, where $\widehat{Q}_2$ is the UMVUE of $Q_2 = R^2$. Similarly, we give the final expression of $\widehat{Q}_2$ in the following equation:

$$\begin{aligned}
\widehat{Q}_2 \;=\; & \frac{1}{A_{n_1}(t_1)A_{n_2}(t_2)} \underset{\sum_{i=0}^{r_1} q_i = n_1-2}{\sum_{q_0}\sum_{q_1}\cdots\sum_{q_{r_1}}} c(n_1-2,\ q_0,\ q_1,\ \ldots,\ q_{r_1}) \\
& \times \underset{\sum_{i=0}^{r_2} w_i = n_2-1}{\sum_{w_0}\sum_{w_1}\cdots\sum_{w_{r_2}}} c(n_2-1,\ w_0,\ w_1,\ \ldots,\ w_{r_2}) \\
& \times \sum_{k_1=0}^{r_1}\sum_{l_1=0}^{r_1} \frac{a_{k_1}a_{l_1}}{\Gamma(k_1+1)} \frac{\binom{r_1}{k_1}\binom{r_1}{l_1}}{\Gamma(l_1+1)} \sum_{k_2=0}^{r_2}\sum_{l_2=0}^{r_2} \frac{b_{k_2}b_{l_2}}{\Gamma(k_2+1)} \frac{\binom{r_2}{k_2}\binom{r_2}{l_2}}{\Gamma(l_2+1)} \\
& \times \sum_{i=0}^{\sum_{m=0}^{r_2}(m+1)w_m-1} \frac{(-1)^i \binom{\sum_{m=0}^{r_2}(m+1)w_m-1}{i}}{l_2+i+1} \\
& \times \Bigg[t_2^{\sum_{m=0}^{r_2}(m+1)w_m+l_2} \sum_{u=0}^{\sum_{m=0}^{r_2}(m+1)w_m+l_2} \frac{(-1)^u \binom{\sum_{m=0}^{r_2}(m+1)w_m+l_2}{u}}{t_2^u(k_2+u+1)} \\
& \quad \times \left[t_2^u\, J_2\left(k_1,\ l_1,\ \sum_{m=0}^{r_1}(m+1)q_m-1\right) - J_2\left(k_1+k_1+u+1,\ l_1,\ \sum_{m=0}^{r_1}(m+1)q_m-1\right)\right] \\
& \quad - \ t_2^{\sum_{m=0}^{r_2}(m+1)w_m-i-1} \sum_{v=0}^{\sum_{m=0}^{r_2}(m+1)w_m-i-1} \frac{(-1)^v \binom{\sum_{m=0}^{r_2}(m+1)w_m-i-1}{v}}{t_2^v(k_2+v+1)} \\
& \quad \times \Bigg[t_2^{k_2+v+1} J_2\left(k_1,\ l_1+l_2+i+1,\ \sum_{m=0}^{r_1}(m+1)q_m-1\right) \\
& \qquad - \ J_2\left(k_1+k_2+v+1,\ l_1+l_2+i+1,\ \sum_{m=0}^{r_1}(m+1)q_m-1\right)\Bigg]\Bigg],
\end{aligned}$$

where

$$\begin{aligned}
J_2(a,\ b,\ \beta) \;=\; & \sum_{i=0}^{\beta-1} (-1)^i \binom{\beta-1}{i} \frac{t_1^{a+b+\beta+1}}{b+\beta+1}\ B(a+1,\ b+\beta+1) \\
& \times \sum_{j=a+1}^{a+b+\beta+1} \binom{a+b+\beta+1}{j} \left(\frac{\mathrm{Min}(t_1,t_2)}{t_1}\right)^j \left(1-\frac{\mathrm{Min}(t_1,t_2)}{t_1}\right)^{a+b+\beta+1-j}.
\end{aligned}$$

4. SIMULATION STUDY

Monte Carlo Simulation technique will not be helpful in generating random samples from the BMOPPE distribution, since the equation

$$F(x) = u, \qquad u \in (0,1),$$

cannot explicitly be solved in x. However, since the distribution is a mixture of gamma distributions given in (1.5), one can utilize this fact. For the BMOPPE distribution, the generation of a random sample $X_1,\ X_2,\ \ldots,\ X_n$ is made using the following algorithm:

1. Generate $U_i \sim \text{Uniform}(0,1), \; i = 1(1)n$.

2. If $\frac{\sum_{k=0}^{j-1} a_k \binom{r}{k} \frac{1}{\theta^k}}{\sum_{k=0}^{r} a_k \binom{r}{k} \frac{1}{\theta^k}} < U_i \leq \frac{\sum_{k=0}^{j} a_k \binom{r}{k} \frac{1}{\theta^k}}{\sum_{k=0}^{r} a_k \binom{r}{k} \frac{1}{\theta^k}}, \; j = 1, 2, ..., r,$ then set $X_i = V_i$, where $V_i \sim$ gamma$(j+1, \theta)$ and if $U_i \leq \frac{a_0}{\sum_{k=0}^{r} a_k \binom{r}{k} \frac{1}{\theta^k}}$, then set $X_i = V_i$, where $V_i \sim \exp(\theta)$.

The graphical representation of mean squared error (MSE) of the MLE and UMVUE of the PDF and the CDF of some BMOPPE distributions for different values of parameter based on simulation data is shown in Figure 3.

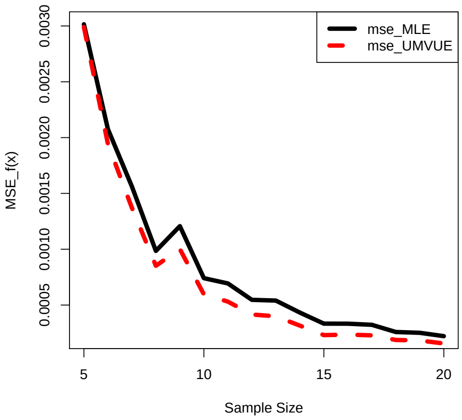
5. DATA ANALYSIS

In this section, we analyse three real data sets for comparing the performances of the MLE and the UMVUE of the PDF and the CDF. The probability model selection is made based on Akaike's Information Criterion (AIC $= -2$ log-likelihood $+ 2k$, where k is the number of parameters involved in the model) for each data set. Minimum is the AIC; better is the model fit. For our BMOPPE model, there is only one parameter θ; it is sufficient to select the model with a negative log-likelihood. But since this model is compared with other models with more than one parameter, the analyses are based on AIC. The AIC is calculated first using the MLE of the parameter(s), and the best model is selected. Then the AIC of the chosen model is compared with the AIC of this model calculated using the UMVUE of the PDF. It is mentioned that $a_0, \; a_1, \; ..., \; a_r$ and r are known constants in the model; in practice, these are chosen by trial and error such that AIC is at its minimum.

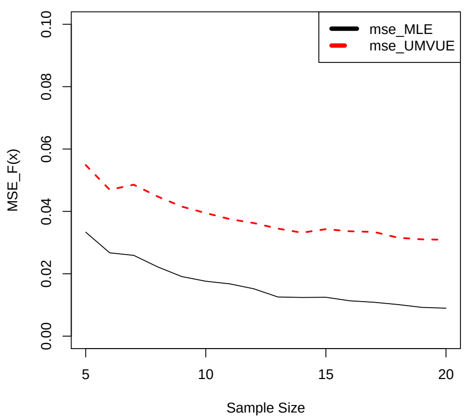
Data set-I, which is cited from Gross and Clark ([6]) and is given in Table 1, represents the relief times (in minutes) of 20 patients receiving an analgesic. We calculate the AIC of different standard distributions of One Parameter Polynomial Exponential (OPPE) family and few others from literature (presented in Table 4) and it is observed that BMOPPE family with $r = 2, \; a_0 = 0, \; a_1 = 0.1$ and $a_2 = 1$ is a better fit. The negative log-likelihood values of the selected model calculated using the MLE and the UMVUE of the PDF are presented in Table 7. Figures 4(a)–(b) show the histogram, the estimated PDF, and the CDF.

Data set-II represents the number of million revolutions before failure for each of the 23 ball bearings in the life test. It is obtained from Lawless ([7]) and shown in Table 2. For ease of calculation, we divide each observation into the data set by 2. The calculated AIC of different standard distributions of OPPE family and few others have been shown in Table 5 and it is noticed that BMOPPE family with $r = 2, \; a_0 = 0.2, \; a_1 = 0.1$ and $a_2 = 1$ is a better fit. The negative log-likelihood values of the selected model calculated using the MLE and the UMVUE of the PDF are also presented in Table 7. Figures 4(c)–(d) present the corresponding histogram, the estimated PDF, and the CDF.

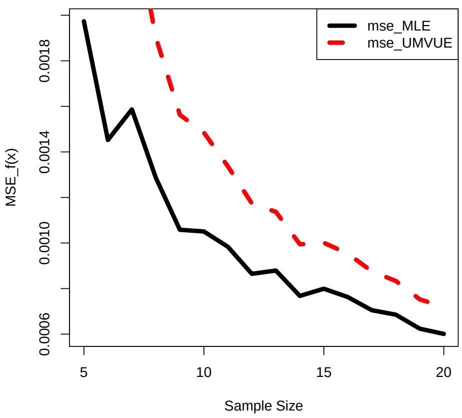
Data set-III is a collection from Bjerkedal ([2]), and Table 3 displays the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli. This data set has been fitted with a distribution of BMOPPE family with $r = 2, \; a_0 = 0.01, \; a_1 = 0.02$ and $a_2 = 4$ and it is found to be a good fit. AIC for this distribution and some other distributions available in the literature are listed in Table 6 that supports our claim. The negative log-likelihood values of the selected model calculated using the MLE and the UMVUE of the PDF are also presented in Table 7. The histogram, the estimated PDF, and the CDF have been shown in Figures 4(e)–(f).



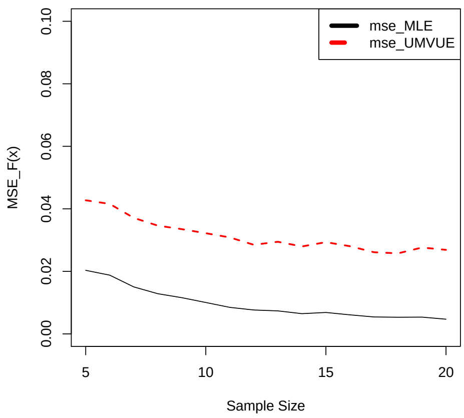
(a) For PDF at $\theta = 1.05$, $x = 2$, $a_0 = 0$, $a_1 = 0.1$, $a_2 = 1$ and $r = 2$.



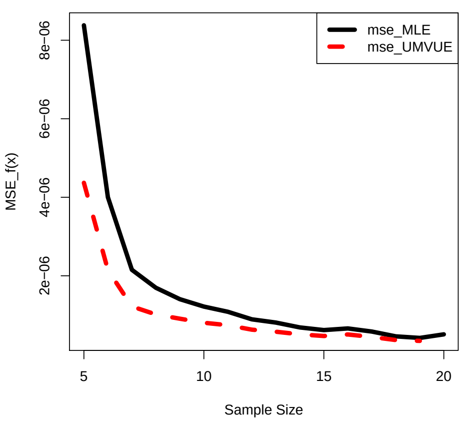
(b) For CDF at $\theta = 1.05$, $x = 2$, $a_0 = 0$, $a_1 = 0.1$, $a_2 = 1$ and $r = 2$.



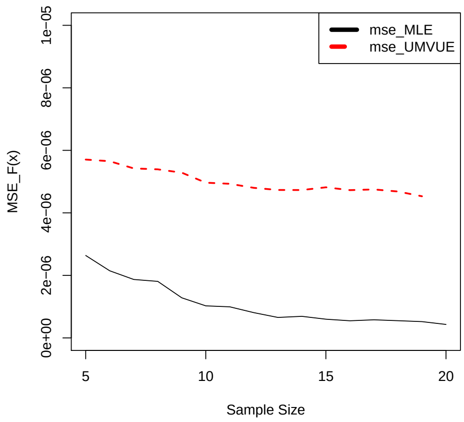
(c) For $\theta = 1.5$, $x = 2$, $a_0 = 0.01$, $a_1 = 0.02$, $a_2 = 4$ and $r = 2$.



(d) For $\theta = 1.5$, $x = 2$, $a_0 = 0.01$, $a_1 = 0.02$, $a_2 = 4$ and $r = 2$.

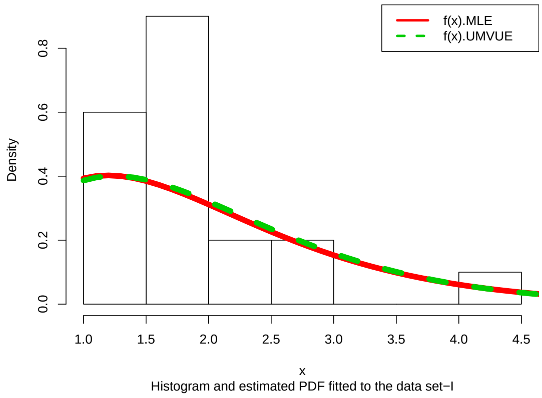


(e) For $\theta = 0.09$, $x = 2$, $a_0 = 0.2$, $a_1 = 0.1$, $a_2 = 1$ and $r = 2$.

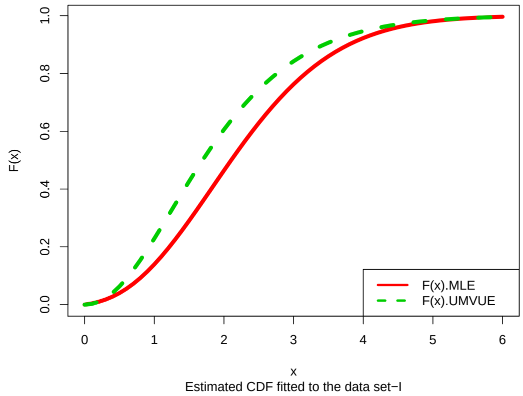


(f) For $\theta = 0.09$, $x = 2$, $a_0 = 0.2$, $a_1 = 0.1$, $a_2 = 1$ and $r = 2$.

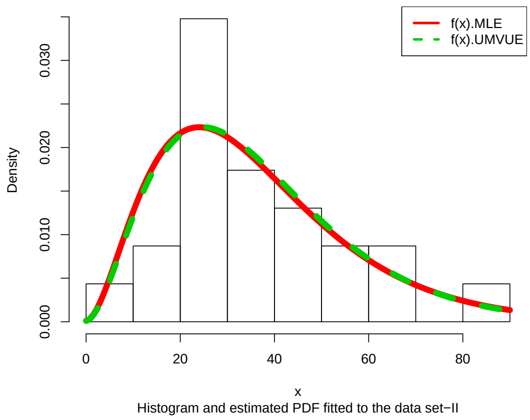
Figure 3: Graph of simulated MSE of the MLE and UMVUE of the PDF and the CDF of BMOPPE distribution.



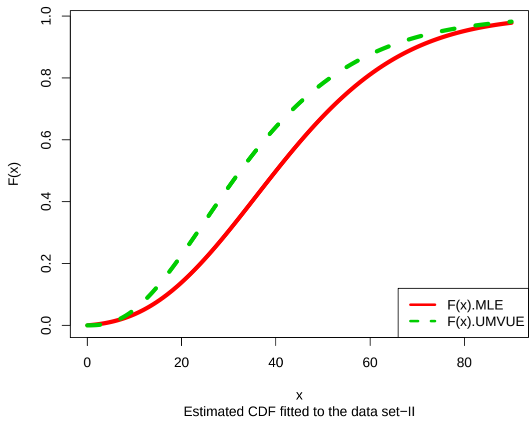
(a) Fitted PDF at $a_0 = 0.2$, $a_1 = 0.1$ and $a_2 = 1$ to the data set-I.



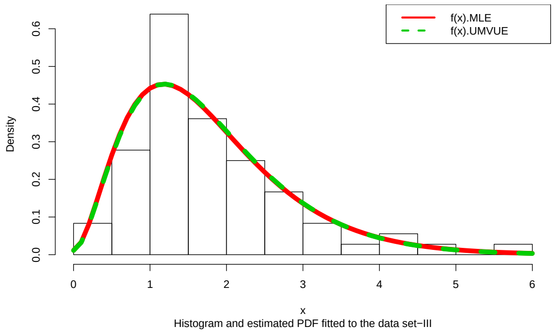
(b) Fitted CDF at $a_0 = 0.2$, $a_1 = 0.1$ and $a_2 = 1$ to the data set-I.



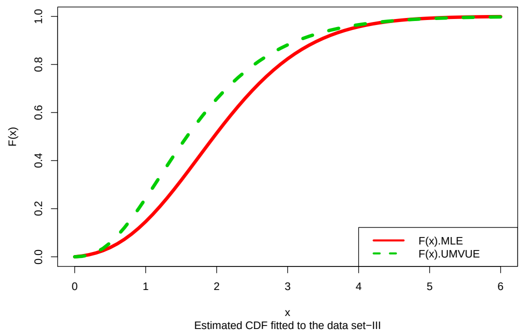
(c) Fitted PDF at $a_0 = 0$, $a_1 = 0.1$ and $a_2 = 1$ to the data set-II.



(d) Fitted CDF at $a_0 = 0$, $a_1 = 0.1$ and $a_2 = 1$ to the data set-II.



(e) Fitted PDF at $a_0 = 0.01$, $a_1 = 0.02$ and $a_2 = 4$ to the data set-III.



(f) Fitted CDF at $a_0 = 0.01$, $a_1 = 0.02$ and $a_2 = 4$ to the data set-III.

Figure 4: Graph of the estimated PDF and CDF of BMOPPE distribution for different data sets.

Table 1: Relief times (in minutes) of 20 patients receiving an analgesic.

1.1	1.4	1.3	1.7	1.9	1.8	1.6	2.2	1.7	2.7
4.1	1.8	1.5	1.2	1.4	3	1.7	2.3	1.6	2

Table 2: The number of million revolutions before failure for each of the 23 ball bearings in the life tests.

17.88	28.92	33.00	41.52	42.12	45.60	48.80	51.84
51.96	54.12	55.56	67.80	68.44	68.64	68.88	84.12
93.12	98.64	105.12	105.84	127.92	128.04	173.40	

Table 3: Survival times (in days) of 72 guinea pigs.

0.1	0.33	0.44	0.56	0.59	0.72	0.74	0.77	0.92
0.93	0.96	1	1	1.02	1.05	1.07	1.07	1.08
1.08	1.08	1.09	1.12	1.13	1.15	1.16	1.2	1.21
1.22	1.22	1.24	1.3	1.34	1.36	1.39	1.44	1.46
1.53	1.59	1.6	1.63	1.63	1.68	1.71	1.72	1.76
1.83	1.95	1.96	1.97	2.02	2.13	2.15	2.16	2.22
2.3	2.31	2.4	2.45	2.51	2.53	2.54	2.54	2.78
2.93	3.27	3.42	3.47	3.61	4.02	4.32	4.58	5.55

Table 4: Model selection criterion for data set-I.

Model	-2 log-likelihood value	AIC
BMOPPE ($a_0 = 0$, $a_1 = 0.1$ and $a_2 = 1$)	48.10	50.10
Length-biased Lindley	49.70	51.70
Akash	59.52	61.52
Shanker	59.78	61.78
Lindley	60.50	62.50
Moment Exponential	52.32	54.32
Exponential	65.67	67.67

Table 5: Model selection criterion for data set-II.

Model	-2 log-likelihood value	AIC
BMOPPE ($a_0 = 0.2$, $a_1 = 0.1$ and $a_2 = 1$)	195.26	197.26
Sujatha	195.38	197.38
Akash	227.06	229.06
Shanker	231.06	233.06
Lindley	231.47	233.47
Gamma	226.04	230.04
Weibull	232.27	236.27
Exponential	242.87	244.87

Table 6: Model selection criterion for data set-III.

Model	-2 log-likelihood value	AIC
BMOPPE ($a_0 = 0.01$, $a_1 = 0.02$ and $a_2 = 4$)	188.18	190.18
OPPE ($a_0 = 0.01$, $a_1 = 0.02$ and $a_2 = 4$)	188.27	190.27
Lindley	213.85	215.85
New Generalized Lindley	188.36	194.36
Moment Exponential	208.40	210.40
Marshall–Olkin Exponential	206.36	210.36

Table 7: Negative log-likelihood value using MLE and UMVUE fitted in data set I–III.

Data Set	Model	Negative log-likelihood value	
		MLE	UMVUE
I	BMOPPE ($a_0 = 0$, $a_1 = 0.1$ and $a_2 = 1$)	24.05	23.71
II	BMOPPE ($a_0 = 0.2$, $a_1 = 0.1$ and $a_2 = 1$)	97.63	97.52
III	BMOPPE ($a_0 = 0.01$, $a_1 = 0.02$ and $a_2 = 4$)	94.09	94.09

6. CONCLUDING REMARKS

The article searches for a more generalised version of Lindley distribution. Starting with the Lindley distribution as a Bernoulli mixture of gamma distributions, a generalised binomial mixture of gamma distributions called the BMOPPE family of distributions has been derived. It is a revisit of the Lindley distribution from a different angle. As a result, the generalised version of the Lindley, the OPPE family of distributions, got mixed with the binomial probabilities, and therefore, the BMOPPE is an improvement. Moments and stochastic orderings are discussed. The process of generation of observations is pointed out, and the results are summarised. Estimations of the PDF and the CDF are discussed. The MLEs and UMVUEs are derived and compared. We have the estimators in biased (i.e., MLE) and unbiased (i.e., UMVUE) classes. Estimators of reliability functions are derived. Asymptotic variances of MLEs and variances of UMVUEs have been derived. The UMVUEs of the variance of UMVUEs of reliability functions have also been derived. These may be helpful in data analysis and comparison. Few data sets have been fitted, and it is found that the proposed distribution fits well in AIC sense. Even though the gain in AIC is minimal compared to the OPPE family of distributions, the BMOPPE is an improvement and is preferred.

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